Introduction to

STATICS
and
DYNAMICS

Chapters 1-10

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Summary of Mechanics

0) The laws of mechanics apply to any collection of material or 'body.' This body could be the overall system of study or any part of it. In the equations below, the forces and moments are those that show on a free body diagram. Interacting bodies cause equal and opposite forces and moments on each other.

I) Linear Momentum Balance (LMB)/Force Balance

Equation of Motion	$\sum \vec{F}_i = \dot{\vec{L}}$	The total force on a body is equal to its rate of change of linear momentum.	(I)
Impulse-momentum (integrating in time)	$\int_{t_1}^{t_2} \sum \vec{F}_i \cdot dt = \Delta \vec{L}$	Net impulse is equal to the change in momentum.	(Ia)
Conservation of momentum (if $\sum \vec{F}_i = \vec{0}$)	$\dot{\vec{L}} = \vec{0} \Rightarrow$ $\Delta \vec{L} = \vec{L}_2 - \vec{L}_1 = \vec{0}$	When there is no net force the linear momentum does not change.	(Ib)
Statics (if $\dot{\vec{L}}$ is negligible)	$\sum \vec{F}_i = \vec{0}$	If the inertial terms are zero the net force on system is zero.	(Ic)

II) Angular Momentum Balance (AMB)/Moment Balance

Equation of motion	$\sum \vec{M}_{\mathrm{C}} = \dot{\vec{H}}_{\mathrm{C}}$	The sum of moments is equal to the rate of change of angular momentum.	(II)
Impulse-momentum (angular) (integrating in time)	$\int_{t_1}^{t_2} \sum \vec{M}_{\mathrm{C}} dt = \Delta \vec{H}_{\mathrm{C}}$	The net angular impulse is equal to the change in angular momentum.	(IIa)
Conservation of angular momentum (if $\sum \vec{M}_{\mathrm{C}} = \vec{0}$)	$\dot{\vec{H}}_{\mathrm{C}} = \vec{0} \Rightarrow$ $\Delta \vec{H}_{\mathrm{C}} = \vec{H}_{\mathrm{C}2} - \vec{H}_{\mathrm{C}1} = \vec{0}$	If there is no net moment about point C then the angular momentum about point C does not change.	(IIb)
Statics (if $\dot{\vec{H}}_{\mathrm{C}}$ is negligible)	$\sum \vec{M}_{\mathrm{C}} = \vec{0}$	If the inertial terms are zero then the total moment on the system is zero.	(IIc)

III) Power Balance (1st law of thermodynamics)

Equation of motion	$\dot{Q} + P = \underbrace{\dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} + \dot{E}_{\mathrm{int}}}_{\dot{E}}$	Heat flow plus mechanical power into a system is equal to its change in energy (kinetic + potential + internal).	(III)
for finite time	$\int_{t_1}^{t_2} \dot{Q} dt + \int_{t_1}^{t_2} P dt = \Delta E$	The net energy flow going in is equal to the net change in energy.	(IIIa)
Conservation of Energy (if $\dot{Q} = P = 0$)	$\dot{E} = 0 \Rightarrow$ $\Delta E = E_2 - E_1 = 0$	If no energy flows into a system, then its energy does not change.	(IIIb)
Statics (if $\dot{E}_{\mathrm{K}}$ is negligible)	$\dot{Q} + P = \dot{E}_{\mathrm{P}} + \dot{E}_{\mathrm{int}}$	If there is no change of kinetic energy then the change of potential and internal energy is due to mechanical work and heat flow.	(IIIc)
Pure Mechanics (if heat flow and dissipation are negligible)	$P = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}}$	In a system well modeled as purely mechanical the change of kinetic and potential energy is due to mechanical work.	(IIId)

Some Definitions

(Please also look at the tables inside the back cover.)

$\vec{r}$ or $\vec{x}$	Position	(*e.g.*, $\vec{r}_i \equiv \vec{r}_{i/\mathrm{O}}$ is the position of a point i relative to the origin, O)
$\vec{v} \equiv \dfrac{d\vec{r}}{dt}$	Velocity	(*e.g.*, $\vec{v}_i \equiv \vec{v}_{i/\mathrm{O}}$ is the velocity of a point i relative to O, measured in a non-rotating reference frame)
$\vec{a} \equiv \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$	Acceleration	(*e.g.*, $\vec{a}_i \equiv \vec{a}_{i/\mathrm{O}}$ is the acceleration of a point *i* relative to O, measured in a Newtonian frame)
$\vec{\omega}$	Angular velocity	A measure of rotational velocity of a rigid body.
$\vec{\alpha} \equiv \dot{\vec{\omega}}$	Angular acceleration	A measure of rotational acceleration of a rigid body.
$\vec{L} \equiv \begin{cases} \sum \mathrm{m}_i \vec{v}_i & \text{discrete} \\ \int \vec{v}\, dm & \text{continuous} \end{cases}$ $= m_{\mathrm{tot}} \vec{v}_{\mathrm{cm}}$	Linear momentum	A measure of a system's net translational rate (weighted by mass).
$\dot{\vec{L}} \equiv \begin{cases} \sum \mathrm{m}_i \vec{a}_i & \text{discrete} \\ \int \vec{a}\, dm & \text{continuous} \end{cases}$ $= m_{\mathrm{tot}} \vec{a}_{\mathrm{cm}}$	Rate of change of linear momentum	The aspect of motion that balances the net force on a system.
$\vec{H}_{\mathrm{C}} \equiv \begin{cases} \sum \vec{r}_{i/\mathrm{C}} \times \mathrm{m}_i \vec{v}_i & \text{discrete} \\ \int \vec{r}_{/C} \times \vec{v}\, dm & \text{continuous} \end{cases}$	Angular momentum about point C	A measure of the rotational rate of a system about a point C (weighted by mass and distance from C).
$\dot{\vec{H}}_{\mathrm{C}} \equiv \begin{cases} \sum \vec{r}_{i/\mathrm{C}} \times \mathrm{m}_i \vec{a}_i & \text{discrete} \\ \int \vec{r}_{/C} \times \vec{a}\, dm & \text{continuous} \end{cases}$	Rate of change of angular momentum about point C	The aspect of motion that balances the net torque on a system about a point C.
$E_{\mathrm{K}} \equiv \begin{cases} \frac{1}{2} \sum \mathrm{m}_i v_i^2 & \text{discrete} \\ \frac{1}{2} \int v^2 dm & \text{continuous} \end{cases}$	Kinetic energy	A *scalar* measure of net system motion.
$E_{int} =$ (heat-like terms)	Internal energy	The non-kinetic non-potential part of a system's total energy.
$P \equiv \sum \vec{F}_i \cdot \vec{v}_i + \sum \vec{M}_i \cdot \vec{\omega}_i$	Power of forces and torques	The mechanical energy flow into a system. Also, $P \equiv \dot{W}$, rate of work.
$[\boldsymbol{I}^{\mathrm{cm}}] \equiv \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix}$	Moment of inertia matrix about cm	A measure of how mass is distributed in a rigid body.

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Preface

This is a statics and dynamics text for second or third year engineering students with an emphasis on vectors, free body diagrams, the basic momentum balance principles, and the utility of computation. Students often start a course like this thinking of mechanics reasoning as being vague and complicated. Our aim is to replace this loose thinking with concrete and simple mechanics problem-solving skills that live harmoniously with a useful mechanical intuition.

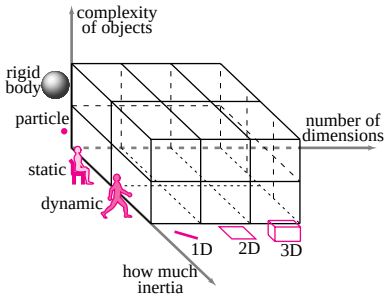
Knowledge of freshman calculus is assumed. Although most students have seen vector dot and cross products, vector topics are introduced from scratch in the context of mechanics. The use of matrices (to tidily set up systems of equations) and of differential equations (for describing motion in dynamics) are presented to the extent needed. The set up of equations for computer solutions is presented in a pseudo-language easily translated by the student into one or another computation package that the student knows.

Organization

We have aimed here to better unify the subject, in part, by an improved organization. Mechanics can be subdivided in various ways: statics *vs* dynamics, particles *vs* rigid bodies, and 1 *vs* 2 *vs* 3 spatial dimensions. Thus a 12 chapter mechanics table of contents could look like this

I. Statics
- A. particles
  - 1) 1D
  - 2) 2D
  - 3) 3D
- B. rigid bodies
  - 4) 1D
  - 5) 2D
  - 6) 3D

II. Dynamics
- C. particles
  - 7) 1D
  - 8) 2D
  - 9) 3D
- D. rigid bodies
  - 10) 1D
  - 11) 2D
  - 12) 3D



However, these topics are far from equal in their difficulty or in the number of subtopics they contain. Further, there are various concepts and skills that are common to many of the 12 sub-topics. Dividing mechanics into these bits distracts from the unity of the subject. Although some vestiges of the scheme above remain, our book has evolved to a different organization through trial and error, thought and rethought, review and revision, and nine semesters of student testing.

The first four chapters cover the basics of statics. Dynamics of particles and rigid bodies, based on progressively more difficult motions, is presented in chapters five to twelve. Relatively harder topics, that might be skipped in quicker courses, are identifiable by chapter, section or subsection titles containing words like “three dimensional” or “advanced”. In more detail:

Chapter 1 **defines mechanics** as a subject which makes predictions about forces and motions using models of mechanical behavior, geometry, and the basic balance laws. The **laws of mechanics** are informally summarized.

Chapter 2 introduces **vector skills** in the context of mechanics. Notational clarity is emphasized because correct calculation is impossible without distinguishing vectors from scalars. Vector addition is motivated by the need to add forces and relative positions, dot products are motivated as the tool which reduces vector equations to scalar equations, and cross products are motivated as the formula which correctly calculates the heuristically motivated concept of moment and moment about an axis.

Chapter 3 is about **free body diagrams**. It is a separate chapter because, in our experience, good use of free body diagrams is almost synonymous with correct mechanics problem solution. To emphasize this to students we recommend that, to get any credit for a problem that uses balance laws, a free body diagram must be drawn.

Chapter 4 makes up a short course in **statics** including an introduction to trusses, mechanisms, beams and hydrostatics. The emphasis is on two-dimensional problems until the last, more advanced section. Solution methods that depend on kinematics (*i.e.*, work methods) are deferred until the dynamics chapters. But for the stretch of linear springs, deformations are not covered.

Chapter 5 is about **unconstrained motion of one or more particles**. It shows how far you can go using $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and Cartesian coordinates in 1, 2 and 3 dimensions in the absence of kinematic constraints. The first five sections are a thorough introduction to motion of one particle in one dimension, so called scalar physics, namely the equation $F(x, v, t) = ma$. This involves review of freshman calculus as well as an introduction to energy methods. A few special cases are emphasized, namely, constant acceleration, force dependent on position (thus motivating energy methods), and the harmonic oscillator. After one section on coupled motions in 1 dimension, sections seven to ten discuss motion in two and three dimensions. The easy set up for computation of trajectories, with various force laws, and even with multiple particles, is emphasized. The chapter ends with a mostly theoretical section on the center-of-mass simplifications for systems of particles.

Chapter 6 is the first chapter that concerns **kinematic constraint** in its simplest context, systems that are constrained to move without rotation in a **straight line**. In one dimension pulley problems provide the main example. Two and three dimensional problems are covered, such as finding structural support forces in accelerating vehicles and the slowing or incipient capsize of a braking car. Angular momentum balance is introduced as a needed tool but without the usual complexities of curvilinear motion.

Chapter 7 treats **pure rotation** about a fixed axis in two dimensions. Polar coordinates and base vectors are first used here in their simplest possible context. The primary applications are pendulums, gear trains, and rotationally accelerating motors or brakes.

Chapter 8 extends chapter 7 to **fixed axis rotation in three dimensions**. The key new kinematic tool here is the non-trivial use of the cross product. Fixed axis rotation is the simplest motion with which one can introduce the full moment of inertia matrix, where the diagonal terms are analogous to the scalar 2D moment of inertia and the off-diagonal terms have a "centripetal" interpretation. The main new application is dynamic balance.

Chapter 9 treats **general planar motion** of a (planar) rigid body including rolling, sliding and free flight. Multi-body systems are also considered so long as they do not involve constraint (*i.e.*, collisions and spring connections but not hinges or prismatic joints).

Chapter 10 is entirely about kinematics of particle motion. The over-riding theme is the use of base vectors which change with time. First, the discussion of polar coordinates started in chapter 7 is completed. Then path coordinates are introduced. The kinematics of relative motion, a topic that many students find difficult, is treated carefully but not elaborately in two stages. First using rotating base vectors connected to a moving rigid body and then using the more abstract notation associated with the famous "five term acceleration formula."

Chapter 11 is about the mechanics of 2D mechanisms using the kinematics from chapter 10.

Chapter 12 pushes some of the contents of chapter 9 into three dimensions. In particular, the **three dimensional motion of a single rigid body** is covered. Rather than emphasize the few problems that are amenable to pencil and paper solution, emphasis is on the basic principles and on the setup for numerical solution.

Chapter 13 on **contact laws** (friction, collisions, and rolling) will probably serve only as a reference for most courses. Because elementary reference material on these topics is so lacking, these topics are covered here with more depth than can be found in any modern text at any level.

Chapter 14 on **units** and dimensions is placed at the end for reference. Because students are immune to preaching about units out of context, such as in an early or late chapter like this one, the main messages are presented by example throughout the book:

- All engineering calculations using dimensional quantities must be dimensionally 'balanced'.
- Units are 'carried' from one line of calculation to the next by the same rules as go numbers and variables.

A leisurely one semester statics course, or a more fast-paced half semester prelude to strength of materials should use chapters 1-4. A typical one semester dynamics course should cover about two thirds of chapters 5-12 preceded by topics from chapters 1-4, as needed. A one semester statics and dynamics course should cover about two thirds of chapters 1-6 and 8. A full year statics and dynamics course should cover most of the book.

Organization and formatting

Each subject is covered in various ways.

- Every section starts with **descriptive text** and short *examples* motivating and describing the theory;
- More detailed explanations of the **theory** are in boxes interspersed in the text. For example, one box explains the common derivation of angular momentum balance form linear momentum balance, one explains the genius of the wheel, and another connects $\vec{\omega}$ based kinematics to $\hat{e}_r$ and $\hat{e}_\theta$ based kinematics;
- **Sample problems** (marked with a gray border) at the end of most sections show how to do homework-like calculations. These are meticulous in their use of free body diagrams, systematic application of basic principles, vector notation, units, and checks against intuition and special cases;
- **Homework problems** at the end of each chapter give students a chance to practice mechanics calculations. The first problems for each section build a student's confidence with the basic ideas. The problems are ranked in approximate order of difficulty, with theoretical questions last. Problems marked with an * have an answer at the back of the book;

- **Reference tables** on the inside covers and end pages concisely summarize much of the content in the book. These tables can save students the time of hunting for formulas and definitions. They also serve to visibly demonstrate the basically simple structure of the whole subject of mechanics.

Notation

Clear vector notation helps students do problems. Students sometimes mistakenly transcribe a conventionally printed bold vector $\boldsymbol{F}$ the same way they transcribe a plain-text scalar F. To help minimize this error we use a redundant vector notation in this book (bold and harpooned $\vec{\boldsymbol{F}}$).

As for all authors and teachers concerned with motion in two and three dimensions we have struggled with the tradeoffs between a precise notation and a simple notation. Beautifully clear notations are intimidating. Perfectly simple notations are ambiguous. Our attempt to find clarity without clutter is summarized in the box on page 9.

Relation to other mechanics books

This book is in some ways original in organization and approach. It also contains some important but not sufficiently well known concepts, for example that angular momentum balance applies relative to any point, not just an arcane list of points. But there is little mechanics here that cannot be found in other books, including freshman physics texts, other engineering texts, and hundreds of classics.

Mastery of freshman physics (*e.g.*, from Halliday & Resnick, Tipler, or Serway) would encompass some part of this book's contents. However freshman physics generally leaves students with a vague notion of what mechanics is, and how it can be used. For example many students leave freshman physics with the sense that a free body diagram (or 'force diagram') is an vague conceptual picture with arrows for various forces and motions drawn on it this way and that. Even the book pictures sometimes do not make clear what force is acting on what body. Also, because freshman physics tends to avoid use of college math, many students end up with no sense of how to use vectors or calculus to solve mechanics problems. This book aims to lead students who may start with these fuzzy freshman physics notions into a world of intuitive yet precise mechanics.

There are many statics and dynamics textbooks which cover about the same material as this one. These textbooks have modern applications, ample samples, lots of pictures, and lots of homework problems. Many are good (or even excellent) in their own ways. Most of today's engineering professors learned from one of these books. We wrote this book because the other books do not adequately convey the simple network of ideas that makes up the whole of Newtonian mechanics. We intend that through this book book students will come to see not mechanics as a coherent network of basic ideas rather than a collection of ad-hoc recipes and tricks that one need memorize or hope to discover by divine inspiration.

There are hundreds of older books with titles like *statics*, *engineering mechanics*, *dynamics*, *machines*, *mechanisms*, *kinematics*, or *elementary physics* that cover aspects of the material here[①] Although many mechanics books written from 1689-1960, are amazingly thoughtful and complete, none are good modern textbooks. They lack an appropriate pace, style of speech, and organization. They are too reliant on geometry skills and not enough on vectors and numerical computation skills. They lack sufficient modern applications, sample calculations, illustrations, and homework problems for a modern text book.

① One near-classic that we have especially enjoyed is J.P. Den Hartog's *Mechanics* originally published in 1948 but still available as an inexpensive reprint.

Thank you

We have attempted to write a book which will help make the teaching and learning of mechanics more fun and more effective. We have tried to present the truth as we know it and as we think it is most effectively communicated. But we have undoubtedly left various technical and strategic errors. We thank you in advance for letting us know your thoughts.

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To the student

Mother nature is so strict that, to the extent we know her rules, we can make reliable predictions about the behavior of her children, the world of physical objects. In particular, for essentially all practical purposes all objects that engineers study strictly follow the laws of Newtonian mechanics. So, if you learn the laws of mechanics, as this book should help you do, you will gain intuition about how the world works and you will be able to make quantitative calculations that predict how things stand, move, and fall.

How to use this book

Most of you will naturally get help with homework by looking at similar examples and samples in the text or lecture notes, by looking up formulas in the front and back covers, or by asking questions of friends, teaching assistants and professors. What good are books, notes, classmates or teachers if they don't help you do homework problems? All the examples and sample problems in this book, for example, are just for this purpose. But too-much use of these resources while solving problems can lead to self deception. To see if you have learned to do a problem, do it again, justifying each step, *without looking up even one small thing*. If you can't do this, you have a new opportunity to learn at two levels. First, you can learn the missing skill or idea. More deeply, by getting stuck after you have been able to get through a problem with guidance, you can learn things about your learning process. Often the real source of difficulty isn't a key formula or fact, but something more subtle. We have tried to bring out some of these more subtle ideas in the text discussions which we hope you read, sooner or later.

Some of you are science and math school-smart, mechanically inclined, or are especially motivated to learn mechanics. Others of you are reluctantly taking this class to fulfil a requirement. We have written this book with both of you in mind. The sections start with generally accessible introductory material and include simple examples. The early sample problems in each section are also easy. But we also have discussions of the theory and other more advanced asides to challenge more motivated students.

Calculation strategies and skills

We try here to show you a systematic approach to solving problems. But it is not possible to reduce the world of mechanics problem solutions to one clear set of steps to follow. There is an art to solving problems, whether homework problems or engineering design problems. Art and human insight, as opposed to precise algorithm or recipe, is what makes engineering require humans and not just computers. Through discussion and examples, we will try to teach you some of this systematic art. Here are a few general guidelines that apply to many problems.

Understand the question

You may be tempted to start writing equations and quoting principles when you first see a problem. But it is generally worth a few minutes (and sometimes a few hours) to try to get an intuitive sense of a problem before jumping to equations. Before you draw any sketches or write equations, think: does the problem make sense? What information has been given? What are you trying to find? Is what you are trying to find determined by what is given? What physical laws make the problem solvable? What extra information do you think you need? What information have you been given that you don't need? Your general sense of the problem will steer you through the technical details.

Some students find they can read every line of sample problems yet cannot do test problems, or, later on, cannot do applied design work effectively. This failing may come from following details without spending time, thinking, gaining an overall sense of the problems.

Think through your solution strategy

For the problem solutions we present in this book or in class, there was a time when we had to think about the order of our work. You also have to think about the order of your work. You will find some tips in the text and samples. But it is your job to own the material, to learn how to think about it your own way, to become an expert in your own style, and to do the work in the way that makes things most clear to you and your readers.

What's in your toolbox?

In the toolbox of someone who can solve lots of mechanics problems are two well worn tools:

- A vector calculator that always keeps vectors and scalars distinct, and
- A reliable and clear free body diagram drawing tool.

Because many of the terms in mechanics equations are vectors, the ability to do vector calculations is essential. Because the concept of an isolated system is at the core of mechanics, every mechanics practitioner needs the ability to draw a good free body diagram. Would that we could write

> "Click on WWW.MECH.TOOL today and order your own professional vector calculator and expert free body diagram drawing tool!",

but we can't. After we informally introduce mechanics in the first chapter, the second and third chapters help you build your own set of these two most-important tools.

Guarantee: if you learn to do clear correct vector algebra and to draw good free body diagrams you will do well at mechanics.

Think hard

We do mechanics because we like mechanics. We get pleasure from thinking about how things work, and satisfaction from doing calculations that make realistic predictions. Our hope is that you also will enjoy idly thinking about mechanics and that you will be proud of your new modeling and calculation skills. You will get there if you think hard. And you will get there more easily if you learn to enjoy thinking hard. Often the best places to study are away from books, notes, pencil or paper.

A note on computation

Mechanics is a physical subject. The concepts in mechanics do not depend on computers. But mechanics is also a quantitative and applied subject described with numbers. Computers are very good with numbers. Thus the modern practice of engineering mechanics depends on computers. The most-needed computer skills for mechanics are:

- solution of simultaneous algebraic equations,
- plotting, and
- numerical solution of ODEs.

More basically, an engineer also needs the ability to routinely evaluate standard functions (x^3, $\cos^{-1}\theta$, *etc*.), to enter and manipulate lists and arrays of numbers, and to write short programs.

Classical languages, applied packages, and simulators

Programming in standard languages such as Fortran, Basic, C, Pascal, or Java probably take too much time to use in solving simple mechanics problems. Thus an engineer needs to learn to use one or another widely available computational package (*e.g.*, MATLAB, OCTAVE, MAPLE, MATHEMATICA, MATHCAD, TKSOLVER, LABVIEW, etc). We assume that students have learned, or are learning such a package. We also encourage the use of packaged mechanics simulators (*e.g.*, WORKING MODEL, ADAMS, DADS, etc) for building intuition, but none of the homework here depends on access to such a packaged simulator.

How we explain computation in this book.

Solving a mechanics problem involves these major steps

(a) Reducing a physical problem to a well posed mathematical problem;
(b) Solving the math problem using some combination of pencil and paper and numerical computation; and
(c) Giving physical interpretation of the mathematical solution.

This book is primarily about setup (a) and interpretation (c), which are the same, no matter what method is used to solve the equations. If a problem requires computation, the exact computer commands vary from package to package. So we express our computer calculations in this book using an informal pseudo computer language. For reference, typical commands are summarized in box on page xii.

Required computer skills.

Here, in a little more detail, are the primary computer skills you need.

- Many mechanics problems are statics or 'instantaneous mechanics' problems. These problems involve trying to find some forces or accelerations at a given configuration of a system. These problems can generally be reduced to the **solution of linear algebraic equations** of this general type: solve

$$\begin{array}{rcrcrcl} 3 & x & + & 4 & y & = & 8 \\ -7 & x & + & \sqrt{2} & y & = & 3.5 \end{array}$$

  for x and y. Some computer packages will let you enter equations almost as written above. In our pseudo language we would write:

```
set =  {  3*x +       4*y = 8
         -7*x + sqrt(2)*y = 3.5  }
solve set for x and y
```

Other packages may require you to write the equations in matrix form something like this (see, or wait for, page **??** for an explanation of the matrix form of algebraic equations):

```
A =  [ 3      4
       -7  sqrt(2) ]
b =  [ 8 3.5 ]'
solve  A*z=b for z
```

where `A` is a 2×2 matrix, `b` is a column of 2 numbers, and the two elements of `z` are x and y. For systems of two equations, like above, a computer is hardly needed. But for systems of three equations pencil and paper work is sometimes error prone. Most often pencil and paper solution of four or more equations is too tedious and error prone.

- In order to see how a result depends on a parameter, or to see how a quantity varies with position or time, it is useful to see a **plot**. Any plot based on more than a few data points or a complex formula is far more easily drawn using a computer than by hand. Most often you can organize your data into a set of (x, y) pairs stored in an `X` list and a corresponding `Y` list. A simple computer command will then plot *x vs y*. The pseudo-code below, for example, plots a circle using 100 points

  ```
  npoints = [1 2 3 ... 100]
  theta   = npoints * 2 * pi / 100
  X       = cos(theta)
  Y       = sin(theta)
  plot Y vs X
  ```

  where `npoints` is the list of numbers from 1 to 100, `theta` is a list of 100 numbers evenly spaced between 0 and 2π and `X` and `Y` are lists of 100 corresponding x, y coordinate points on a circle.
- The result of using the laws of dynamics is often a set of differential equations which need to be solved. A simple example would be:

  Find x at $t = 5$ given that $\dfrac{dx}{dt} = x$ and that at $t = 0$, $x = 1$.

  The solution to this problem can be found easily enough by hand to be e^5. But often the differential equations are just too hard for pencil and paper solution. Fortunately the **numerical solution of ordinary differential equations** is already programmed into scientific and engineering computer packages. The simple problem above is solved with computer code analogous to this:

  ```
  ODES =  { xdot  = x }
  ICS  =  { xzero = 1 }
  solve ODES with ICS until t=5
  ```

Examples of many calculations of these types will shown, starting on page **??**.

0.1 Summary of informal computer commands

Computer commands are given informally and descriptively in this book. The commands below are not as precise as any real computer package. You should be able to use your package's documentation to translate the informal commands below. Many of the commands below depend on mathematical ideas which are introduced in the text. At first reading a student is not expected to absorb this table.

`x=7`	Set the variable x to 7.
`omega=13`	Set ω to 13.
`u=[1 0 -1 0]` `v=[2 3 4 pi]`	Define u and v to be the lists shown.
`t= [.1 .2 .3 ... 5]`	Set t to the list of 50 numbers implied by the expression.
`y=v(3)`	sets y to the third value of v (in this case 4).
`A=[1 2 3 6.9` `5 0 1 12 ]`	Set A to the array shown.
`z= A(2,3)`	Set z to the element of A in the second row and third column.
`w=[3` `4` `2` `5]`	Define w to be a column vector.
`w = [3 4 2 5]'`	Same as above. `'` means transpose.
`u+v`	Vector addition. In this case the result is $[3\ 3\ 3\ \pi]$.
`u*v`	Element by element multiplication, in this case $[2\ 0\ -4\ 0]$.
`sum(w)`	Add the elements of w, in this case 14.
`cos(w)`	Make a new list, each element of which is the cosine of the corresponding element of $[w]$.
`mag(u)`	The square root of the sum of the squares of the elements in $[u]$, in this case 1.41421...
`u dot v`	The vector dot product of component lists $[u]$ and $[v]$, (we could also write `sum(A*B)`.
`C cross D`	The vector cross product of $\vec{C}$ and $\vec{D}$, assuming the three element component lists for $[C]$ and $[D]$ have been defined.
`A matmult w`	Use the rules of matrix multiplication to multiply $[A]$ and $[w]$.
`eqset = {3x + 2y = 6` `6x + 7y = 8}`	Define 'eqset' to stand for the set of 2 equations in braces.
`solve eqset` `for x and y`	Solve the equations in 'eqset' for x and y.
`solve Ax=b for x`	Solve the matrix equation $[A][x] = [b]$ for the list of numbers x. This assumes A and b have already been defined.
`for i = 1 to N` `such and such` `end`	Execute the commands 'such and such' N times, the first time with $i = 1$, the second with $i = 2$, etc
`plot y` *vs* `x`	Assuming x and y are two lists of numbers of the same length, plot the y values *vs* the x values.
`solve ODEs` `with ICs` `until t=5`	Assuming a set of ODEs and ICs have been defined, use numerical integration to solve them and evaluate the result at $t = 5$.

With an informality consistent with what is written above, other commands are introduced here and there as needed.

1 Mechanics

1.1 What is mechanics

Mechanics is the study of force, deformation, and motion, and the relations between them. We care about forces because we want to know how hard to push something to move it or whether it will break when we push on it for other reasons. We care about deformation and motion because we want things to move or not move in certain ways. Towards these ends we are confronted with this general mechanics problem:

> Given some (possibly idealized) information about the properties, forces, deformations, and motions of a mechanical system, make useful predictions about other aspects of its properties, forces, deformations, and motions.

By *system*, we mean a tangible thing such as a wheel, a gear, a car, a human finger, a butterfly, a skateboard and rider, a quartz timing crystal, a building in an earthquake, a piano string, and a space shuttle. Will a wheel slip? a gear tooth break? a car tip over? What muscles are used when you hit a key on your computer? How do people balance on skateboards? Which buildings are more likely to fall in what kinds of earthquakes? Why are low pitch piano strings made with helical windings instead of straight wires? How fast is the space shuttle moving when in low earth orbit?

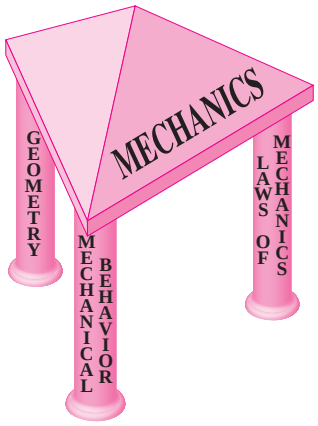
In mechanics we try to solve special cases of the general mechanics problem above by idealizing the system, using classical Euclidean geometry to describe deformation and motion, and assuming that the relation between force and motion is described

with Newtonian mechanics, or "Newton's Laws". Newtonian mechanics has held up, with minor refinement, for over three hundred years. Those who want to know how machines, structures, plants, animals and planets hold together and move about need to know mechanics. In another two or three hundred years people who want to design robots, buildings, airplanes, boats, prosthetic devices, and large or microscopic machines will probably still use the equations and principles we now call Newtonian mechanics①

① The laws of classical mechanics, however expressed, are named for Isaac Newton because his theory of the world, the *Principia* published in 1689, contains much of the still-used theory. Newton used his theory to explain the motions of planets, the trajectory of a cannon ball, why there are tides, and many other things.

Any mechanics problem can be divided into 3 parts which we think of as the 3 pillars that hold up the subject:

1. the mechanical behavior of objects and materials (*constitutive* laws);
2. the geometry of motion and distortion (kinematics); and
3. the laws of mechanics ($\vec{F} = m\vec{a}$, etc.).



Let's discuss each of these ideas a little more, although somewhat informally, so you can get an overview of the subject before digging into the details.

Mechanical behavior

The first pillar of mechanics is mechanical behavior. The *Mechanical behavior* of something is the description of how loads cause deformation (or visa versa). When something carries a force it stretches, shortens, shears, bends, or breaks. Your finger tip squishes when you poke something. Too large a force on a gear in an engine causes it to break. The force of air on an insect wing makes it bend. Various geologic forces bend, compress and break rock.

This relation between force and deformation can be viewed in a few ways. First, it gives us a definition of force. In fact, force can be defined by the amount of spring stretch it causes. Thus most modern force measurement devices measure force indirectly by measuring the deformation it causes in a calibrated spring. This is one justification for calling 'mechanical behavior' the first pillar. It gives us a notion of force even before we introduce the laws of mechanics.

Second, a piece of steel distorts under a given load differently than a same-sized piece of chewing gum. This observation that different objects deform differently with the same loads implies that the properties of the object affect the solution of mechanics problems. The relations of an object's deformations to the forces that are applied are called the *mechanical properties* of the object. Mechanical properties

are sometimes called *constitutive laws* because the mechanical properties describe how an object is constituted (at least from a mechanics point of view). The classic example of a constitutive law is that of a linear spring which you remember from your elementary physics classes: '$F = kx$'. When solving mechanics problems one has to make assumptions and idealizations about the constitutive laws applicable to the parts of a system. How stretchy (elastic) or gooey (viscous) or otherwise deformable is an object? The set of assumptions about the mechanical behavior of the system is sometimes called the *constitutive model*.

Distortion in the presence of forces is easy to see on squeezed fingertips, or when thin pieces of wood bend. But with pieces of rock or metal the deformation is essentially invisible and sometimes hard to imagine. With the exceptions of things like rubber, flesh, or compliant springs, solid objects that are not in the process of breaking typically change their dimensions much less than 1% when loaded. Most structural materials deform less than one part per thousand with working loads. But even these small deformations can be important because they are enough to break bones and collapse bridges.

When deformations are not of consequence engineers often idealize them away. Mechanics, where deformation is neglected, is called *rigid body* mechanics because a rigid (infinitely stiff) solid would not deform. Rigidity is an extreme constitutive assumption. The assumption of rigidity greatly simplifies many calculations while generating adequate predictions for many practical problems. The assumption of rigidity also simplifies the introduction of more general mechanics concepts. Thus for understanding the steering dynamics of a car we might model it as a rigid body, whereas for crash analysis where rigidity is clearly a poor approximation, we might model a car as a large collection of point masses connected by linear springs.

Most constitutive models describe the material inside an object. But to solve a mechanics problem involving friction or collisions one also has to have a constitutive model for the contact interactions. The standard friction model (or idealization) '$F \leq \mu N$' is an example of a contact constitutive model.

In all of mechanics, one needs constitutive models of a system and its components before one can make useful predictions.

The geometry of deformation and motion

The second pillar of mechanics concerns the geometry of deformation and motion. Classical Greek (Euclidean) geometry concepts are used. Deformation is defined by changes of lengths and angles between sets of points. Motion is defined by the changes of the position of points in time. Concepts of length, angle, similar triangles, the curves that particles follow and so on can be studied and understood without Newton's laws and thus make up an independent pillar of the subject.

We mentioned that understanding small deformations is often important to predict when things break. But large motions are also of interest. In fact many machines and machine parts are designed to move something. Bicycles, planes, elevators, and hearses are designed to move people; a clockwork, to move clock hands; insect wings, to move insect bodies; and forks, to move potatoes. A connecting rod is designed to move a crankshaft; a crankshaft, to move a transmission; and a transmission, to move a wheel. And wheels are designed to move bicycles, cars, and skateboards.

The description of the motion of these things, of how the positions of the pieces change with time, of how the connections between pieces restrict the motion, of the curves traversed by the parts of a machine, and of the relations of these curves to each other is called *kinematics*. Kinematics is the study of the geometry of motion (or geometry in motion).

For the most part we think of deformations as involving small changes of distance between points on one body, and of net motion as involving large changes of distance

between points on different bodies. Sometimes one is most interested in deformation (you would like the stretch between the two ends of a bridge brace to be small) and sometimes in the net motion (you would like all points on a plane to travel about the same large distance from Chicago to New York). Really, deformation and motion are not distinct topics, both involve keeping track of the positions of points. The distinction we have made is for simplicity. Trying to simultaneously describe deformations and large motions is just too complicated for beginners. So the ideas are kept (somewhat artificially) distinct in elementary mechanics courses such as this one. As separate topics, both the geometry needed to understand small deformations and the geometry needed to understand large motions of rigid bodies are basic parts of mechanics.

Relation of force to motion, the laws of mechanics

The third pillar of mechanics is loosely called *Newton's laws*. One of Newton's brilliant insights was that the same intuitive 'force' that causes deformation also causes motion, or more precisely, acceleration of mass. Force is related to deformation by material properties (elasticity, viscosity, etc.) and to motion by the laws of mechanics summarized in the front cover. In words and informally, these are:①

① Isaac Newton's original three laws are: 1) an object in motion tends to stay in motion, 2) $\vec{F} = m\vec{a}$ for a particle, and 3) the principle of action and reaction. These could be used as a starting point for study of mechanics. The more modern approach we take here leads to the same end.

0) The laws of mechanics apply to any system (rigid or not):
   a) Force and moment are *the* measure of mechanical interaction; and
   b) Action = minus reaction applies to all interactions, ('every action has an equal and opposite reaction');

I) The net force on a system causes a net linear acceleration (*linear momentum balance*),

II) The net turning effect of forces on system causes it to rotationally accelerate (*angular momentum balance*), and

III) The change of energy of a system is due to the energy flow into the system (*energy balance*).

The principles of action and reaction, linear momentum balance, angular momentum balance, and energy balance, are actually redundant various ways. Linear momentum balance can be derived from angular momentum balance and, sometimes (see section **??**), vice-versa. Energy balance equations can often be derived from the momentum balance equations. The principle of action and reaction can also be derived from the momentum balance equations. In the practice of solving mechanics problems, however, the ideas are generally considered independently without much concern for which idea could be derived from the others for the problem under consideration. That is, the four assumptions in O-III above are not a mathematically minimal set, but they are all accepted truths in Newtonian mechanics.

A lot follows from the laws of Newtonian mechanics, including the contents of this book. When these ideas are supplemented with models of particular systems (e.g., of machines, buildings or human bodies) and with Euclidean geometry, they lead to predictions about the motions of these systems and about the forces which act upon them. There is an endless stream of results about the mechanics of one or another special system. Some of these results are classified into entire fields of research such as 'fluid mechanics,' 'vibrations,' 'seismology,' 'granular flow,' 'biomechanics,' or 'celestial mechanics.'

The four basic ideas also lead to other more mathematically advanced formulations of mechanics with names like ‘Lagrange’s equations,’ ‘Hamilton’s equations,’ ‘virtual work’, and ‘variational principles.’ Should you take an interest in theoretical mechanics, you may learn these approaches in more advanced courses and books, most likely in graduate school.

Statics, dynamics, and strength of materials

Elementary mechanics is traditionally partitioned into three courses named ‘statics’, ‘dynamics’, and ‘strength of materials’. These subjects vary in how much they emphasize material properties, geometry, and Newton’s laws.

Statics is mechanics with the idealization that the acceleration of mass is negligible in Newton’s laws. The first four chapters of this book provide a thorough introduction to statics. Strictly speaking things need not be standing still to be well idealized with statics. But, as the name implies, statics is generally about things that don’t move much. The first pillar of mechanics, constitutive laws, is generally introduced without fanfare into statics problems by the (implicit) assumption of rigidity. Other constitutive assumptions used include inextensible ropes, linear springs, and frictional contact. The material properties used as examples in elementary statics are very simple. Also, because things don’t move or deform much in statics, the geometry of deformation and motion are all but ignored. Despite the commonly applied vast simplifications, statics is useful, for example, for the analysis of structures, slow machines or the light parts of fast machines, and the stability of boats.

Dynamics concerns motion associated with the non-negligible acceleration of mass. Chapters 5-12 of this book introduce dynamics. As with statics, the first pillar of mechanics, constitutive laws, is given a relatively minor role in the elementary dynamics presented here. For the most part, the same library of elementary properties properties are used with little fanfare (rigidity, inextensibility, linear elasticity, and friction). Dynamics thus concerns the two pillars that are labelled by the confusingly similar words kinematics and kinetics. *Kinematics* concerns geometry with no mention of force and *kinetics* concerns the relation of force to motion. Once one has mastered statics, the hard part of dynamics is the kinematics. Dynamics is useful for the analysis of, for example, fast machines, vibrations, and ballistics.

Strength of materials expands statics to include material properties and also pays more attention to distributed forces (traction and stress). This book only occasionally touches lightly on strength of materials topics like stress (loosely, force per unit area), strain (a way to measure deformation), and linear elasticity (a commonly used constitutive model of solids). Strength of materials gives equal emphasis to all three pillars of mechanics. Strength of materials is useful for predicting the amount of deformation in a structure or machine and whether or not it is likely to break with a given load.

How accurate is Newtonian mechanics?

In popular science culture we are repeatedly reminded that Newtonian ideas have been overthrown by relativity and quantum mechanics. So why should you read this book and learn ideas which are known to be wrong?

First off, this criticism is self contradictory because general relativity and quantum mechanics are inconsistent with each other, not yet united by a universally accepted deeper theory of everything. Lets look first at the size of the errors due to neglecting various modern physics theories.

- The errors from neglecting the effects of special relativity are on the order of v^2/c^2 where v is a typical speed in your problem and c is the speed of light. The biggest errors are associated with the fastest objects. For, say, calculating space shuttle trajectories this leads to an error of about

$$\frac{v^2}{c^2} \approx \left(\frac{5\,\text{mi/ s}}{3\times 10^8\,\text{m/s}}\right)^2 \approx .000000001 \approx \text{one millionth of one percent}$$

- In classical mechanics we assume we can know exactly where something is and how fast it is going. But according to quantum mechanics this is impossible. The product of the uncertainty δx in position of an object and the the uncertainty δp of its momentum must be greater than Planck's constant $\hbar$. Planck's constant is small; $\hbar \approx 1\times 10^{-34}$joule$\cdot$ s. The fractional error so required is biggest for small objects moving slowly. So if one measures the location of a computer chip with mass $m = 10^{-4}\,\text{kg}$ to within $\delta x = 10^{-6}\,\text{m} \approx$ a twenty fifth of a thousands of an inch, the uncertainty in its velocity $\delta v = \delta p/m$ is only

$$\delta x \delta p = \hbar \Rightarrow \delta v = m\hbar/\delta x \approx 10^{-24}\,\text{m/ s} \approx 10^{-12}\text{thousandths of an inch per year.}$$

- In classical mechanics we usually neglect fluctuations associated with the thermal vibrations of atoms. But any object in thermal equilibrium with its surroundings constantly undergoes changes in size, pressure, and energy, as it interacts with the environment. For example, the internal energy per particle of a sample at temperature T fluctuates with amplitude

$$\frac{\Delta E}{N} = \frac{1}{\sqrt{N}}\sqrt{k_B T^2 c_V},$$

  where k_B is Boltzmann's constant, T is the absolute temperature, N is the number of particles in the sample, and c_V is the specific heat. Water has a specific heat of 1 cal/K, or around 4 Joule/K. At room temperature of 300 Kelvin, for 10^{23} molecules of water, these values lead to an uncertainty of only 7.2×10^{-21} Joule in the the internal energy of the water. Thermal fluctuations are big enough to visibly move pieces of dust in an optical microscope, and to generate variations in electric currents that are easily measured, but for most engineering mechanics purposes they are negligible.
- general relativity errors having to do with the non-flatness of space are so small that the genius Einstein had trouble finding a place where the deviations from Newtonian mechanics could possibly be observed. Finally he predicted a small, barely measurable effect on the predicted motion of the planet Mercury.

On the other hand, the errors within mechanics, due to imperfect modeling or inaccurate measurement, are, except in extreme situations, far greater than the errors due to the imperfection of mechanics theory. For example, mechanical force measurements are typically off by a percent or so, distance measurements by a part in a thousand, and material properties are rarely known to one part in a hundred and often not one part in 10.

If your engineering mechanics calculations make inaccurate predictions it will surely be because of errors in modeling or measurement, not inaccuracies in the laws of mechanics. Newtonian mechanics, if not perfect, is still rather accurate while relatively much simpler to use than the theories which have 'overthrown' it. To seriously consider mechanics errors as due to neglect of relativity, quantum mechanics, or statistical mechanics, is to pretend to an accuracy that can only be obtained in the rarest of circumstances. You have trusted your life many times to engineers who treated classical mechanics as 'truth' and in the future, your engineering work will justly be based on these laws.

2 Vectors for mechanics

This book is about the laws of mechanics which were informally introduced in Chapter 1. The most fundamental quantities in mechanics, used to define all the others, are the two scalars, mass m and time t, and the two vectors, relative position $\vec{r}_{i/O}$, and force $\vec{F}$. Scalars are typed with an ordinary font (t and m) and vectors are typed in bold with a harpoon on top ($\vec{r}_{i/O}$, $\vec{F}$). All of the other quantities we use in mechanics are defined in terms of these four. A list of all the scalars and vectors used in mechanics are given in boxes 2 and 2.2 on pages 8 and page 9. Scalar arithmetic has already been your lifelong friend. For mechanics you also need facility with vector arithmetic. Lets start at the beginning.

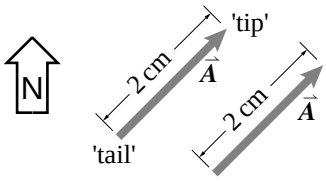


Figure 2.1: Vector $\vec{A}$ is 2 cm long and points Northeast. Two copies of $\vec{A}$ are shown.

(Filename:tfigure.northeast)

What is a vector?

> A vector is a (possibly dimensional) quantity that is fully described by its magnitude and direction.

Whereas scalars are just (possibly dimensional) single numbers. As a first vector example, consider a line segment with head and tail ends and a length (magnitude) of 2 cm and pointed Northeast. Lets call this vector $\vec{A}$ (see fig. 2.1).

$$\vec{A} \stackrel{def}{=} \text{2 cm long line segment pointed Northeast}$$

Every vector in mechanics is well visualized as an arrow. The direction of the arrow is the direction of the vector. The length of the arrow is proportional to the magnitude of the vector. The magnitude of $\vec{A}$ is a positive scalar indicated by $|\vec{A}|$. A vector does not lose its identity if it is picked up and moved around in space (so long as it is not rotated or stretched). Thus both vectors drawn in fig. 2.1 are $\vec{A}$.

Vector arithmetic makes sense

We have oversimplified. We said that a vector is something with magnitude and direction. In fact, by common modern convention, that's not enough. A one way street sign, for example, is not considered a vector even though is has a magnitude (its mass is, say, half a kilogram) and a direction (the direction of most of the traffic). A thing is only called a vector if elementary vector arithmetic, vector addition in particular, has a sensible meaning①.

① In abstract mathematics they don't even bother with talking about magnitudes and directions. All they care about is vector arithmetic. So, to the mathematicians, anything which obeys simple vector arithmetic is a vector, arrow-like or not. In math talk lots of strange things are vectors, like arrays of numbers and functions. In this book vectors always have magnitude and direction.

The following sentence summarizes centuries of thought and also motivates this chapter:

> The vectors in mechanics have magnitude and direction *and* elementary vector arithmetic with them has a sensible physical meaning.

This chapter is about vector arithmetic. In the rest of this chapter you will learn how to add and subtract vectors, how to stretch them, how to find their components, and how to multiply them with each other two different ways. Each of these operations has use in mechanics and, in particular, the concept of vector addition always has a physical interpretation.

2.1 Vector notation and vector addition

Facility with vectors has several aspects.

1. You must recognize which quantities are vectors (such as force) and which are scalars (such as length).
2. You have to use a notation that distinguishes between vectors and scalars using, for example, $\vec{a}$, or $\underline{\boldsymbol{a}}$ for acceleration and a or $|\vec{a}|$ for the magnitude of acceleration.

2.1 The scalars in mechanics

The scalar quantities used in this book, and their dimensions in brackets [ ], are listed below (M for mass, L for length, T for time, F for force, and E for energy).

- mass m, $[M]$;
- length or distance ℓ, w, x, r, ρ, d, or s, $[L]$;
- time t, $[T]$;
- pressure p, $[F/L^2] = [M/(L \cdot T^2)]$;
- angles θ 'theta', ϕ 'phi', γ 'gamma', and ψ 'psi', [dimensionless];
- energy E, kinetic energy E_{K}, potential energy E_{P}, $[E] = [F \cdot L] = [M \cdot L^2/T^2]$;
- work W, $[E] = [F \cdot L] = [M \cdot L^2/T^2]$;
- tension T, $[M \cdot L/T^2] = [F]$;
- power P, $[E/T] = [M \cdot L^2/T^3]$;
- the magnitudes of all the vector quantities are also scalars, for example
  - speed $|\vec{\boldsymbol{v}}|$, $[L/T]$;
  - magnitude of acceleration $|\vec{\boldsymbol{a}}|$, $[L/T^2]$;
  - magnitude of angular momentum $|\vec{\boldsymbol{H}}|$, $[M \cdot L^2/T]$;
- the components of vectors, for example
  - r_x (where $\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$), or
  - $L_{x'}$ (where $\vec{\boldsymbol{L}} = L_{x'}\hat{\boldsymbol{\imath}}' + L_{y'}\hat{\boldsymbol{\jmath}}'$);
- coefficient of friction μ 'mu', or friction angle ϕ 'phi';
- coefficient of restitution e;
- mass per unit length, area, or volume ρ;
- oscillation frequency β or λ.

3. You need skills in vector arithmetic, maybe a little more than you have learned in your previous math and physics courses.

In this first section (2.1) we start with notation and go on to the basics of vector arithmetic.

How to write vectors

A scalar is written as a single English or Greek letter. This book uses slanted type for scalars (*e.g.*, m for mass) but ordinary printing is fine for hand work (*e.g.*, m for mass). A vector is also represented by a single letter of the alphabet, either English or Greek, but ornamented to indicate that it is a vector and not a scalar. The common ornamentations are described below.

Use one of these vector notations in all of your work.

Various ways of representing vectors in printing and writing are described below.①.

$\vec{F}$ Putting a harpoon (or arrow) over the letter F is the suggestive notation used in in this book for vectors.

$\boldsymbol{F}$ In most texts a bold $\boldsymbol{F}$ represents the vector $\vec{F}$. But bold face is inconvenient for hand written work. The lack of bold face pens and pencils tempts students to transcribe a bold $\boldsymbol{F}$ as F. But F with no adornment represents a scalar and not a vector. Learning how to work with vectors and scalars is hard enough without the added confusion of not being able to tell at a glance which terms in your equations are vectors and which are scalars.

①**Caution:** Be careful to distinguish vectors from scalars all the time. Clear notation helps clear thinking and will help you solve problems. If you notice that you are not using clear vector notation, *stop*, determine which quantities are vectors and which scalars, and fix your notation.

2.2 The Vectors in Mechanics

The vector quantities used in mechanics and the notations used in this book are shown below. The dimensions of each are shown in brackets [ ]. Some of these quantities are also shown in figure **??**.

- position $\vec{\boldsymbol{r}}$ or $\vec{\boldsymbol{x}}$, $[L]$;
- velocity $\vec{\boldsymbol{v}}$ or $\dot{\vec{\boldsymbol{x}}}$ or $\dot{\vec{\boldsymbol{r}}}$, $[L/t]$;
- acceleration $\vec{\boldsymbol{a}}$ or $\dot{\vec{\boldsymbol{v}}}$ or $\ddot{\vec{\boldsymbol{r}}}$, $[L/t^2]$;
- angular velocity $\vec{\boldsymbol{\omega}}$ 'omega' (or, if aligned with the $\hat{\boldsymbol{k}}$ axis, $\dot{\theta}\hat{\boldsymbol{k}}$), $[1/t]$;
- rate of change of angular velocity $\vec{\boldsymbol{\alpha}}$ 'alpha' or $\dot{\vec{\boldsymbol{\omega}}}$ (or, if aligned with the $\hat{\boldsymbol{k}}$ axis, $\ddot{\theta}\hat{\boldsymbol{k}}$), $[1/t^2]$;
- force $\vec{\boldsymbol{F}}$ or $\vec{\boldsymbol{N}}$, $[m \cdot L/t^2] = [F]$;
- moment or torque $\vec{\boldsymbol{M}}$, $[m \cdot L^2/t^2] = [F \cdot L]$;
- linear momentum $\vec{\boldsymbol{L}}$, $[m \cdot L/t]$ and its rate of change $\dot{\vec{\boldsymbol{L}}}$, $[m \cdot L/t^2]$;
- angular momentum $\vec{\boldsymbol{H}}$, $[m \cdot L^2/t]$; and its rate of change $\dot{\vec{\boldsymbol{H}}}$, $[m \cdot L^2/t^2]$.
- unit vectors to help write other vectors [dimensionless]:
  - $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ for cartesian coordinates,
  - $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, and $\hat{\boldsymbol{k}}'$ for crooked cartesian coordinates,
  - $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$ for polar coordinates,
  - $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$ for path coordinates, and
  - $\hat{\boldsymbol{\lambda}}$ 'lambda' and $\hat{\boldsymbol{n}}$ as miscellaneous unit vectors.

Subscripts and superscripts are often added to indicate the point, points, body, or bodies the vectors are describing. Upper case letters (O, A, B, C,...) are used to denote points. Upper case calligraphic (or script if you are writing by hand) letters ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$...$\mathcal{F}$...) are for labeling rigid bodies or reference frames. $\mathcal{F}$ is the fixed, Newtonian, or 'absolute' reference frame (think of $\mathcal{F}$ as the ground if you are a first time reader).

For example, $\vec{\boldsymbol{r}}_{\mathrm{AB}}$ or $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ is the position of the point B relative to the point A. $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is the absolute angular velocity of the body called $\mathcal{B}$ ($\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is short hand for $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}}$). And $\vec{\boldsymbol{H}}_{\mathcal{A}/C}$ is the angular momentum of body $\mathcal{A}$ relative to point C.

The notation is further complicated when we want to take derivatives with respect to moving frames, a topic which comes up later in the book. For completeness: ${}^{\mathcal{B}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{E}}$ is the time derivative with respect to reference frame $\mathcal{B}$ of the angular velocity of body $\mathcal{D}$ with respect to body (or frame) $\mathcal{E}$. If this paragraph doesn't read like gibberish to you, you probably already know dynamics!

$\underline{\boldsymbol{F}}$ Underlining or undersquiggling ($\underline{\boldsymbol{F}}$) is an easy and unambiguous notation for hand writing vectors. A recent poll found that 14 out of 17 mechanics professors use this notation. These professors would copy a $\vec{\boldsymbol{F}}$ from this book by writing $\underline{\boldsymbol{F}}$. Also, in typesetting, an author indicates that a letter should be printed in bold by underlining.

$\bar{F}$ It is a stroke simpler to put a bar rather than a harpoon over a symbol. But the saved effort causes ambiguity since an over-bar is often used to indicate average. There could be confusion, say, between the velocity $\bar{v}$ and the average speed $\bar{v}$.

$\hat{\imath}$ Over-hat. Putting a hat on top is like an over-arrow or over-bar. In this book we reserve the hat for unit vectors. For example, we use $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, or $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_2$ for unit vectors parallel to the x, y, and z axis, respectively. The same poll of 17 mechanics professors found that 11 of them used no special notation for unit vectors and just wrote them like, *e.g.*, $\underline{\boldsymbol{i}}$.

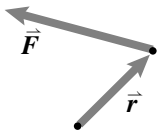


Figure 2.2: Position and force vectors are drawn with different scales.

(Filename:tfigure.posandforce)

Drawing vectors

In fig. 2.1, the magnitude of $\vec{A}$ was used as the drawing length. But drawing a vector using its magnitude as length would be awkward if, say, we were interested in vector $\vec{B}$ that points Northwest and has a magnitude of 2 m. To well contain $\vec{B}$ in a drawing would require a piece of paper about 2 meters square (each edge the length of a basketball player). This situation moves from difficult to ridiculous if the magnitude of the vector of interest is 2 km and it would take half an hour to stroll from tail to tip dragging a purple crayon. Thus in pictures we merely make scale drawings of vectors with, say, one centimeter of graph paper representing 1 kilometer of vector magnitude.

The need for scale drawings to represent vectors is apparent for a vector whose magnitude is not length. Force is a vector since it has magnitude and direction. Say $\vec{F}_{\text{gr}}$ is the 700 N force that the ground pushes up on your feet as you stand still. We can't draw a line segment with length 700 N for $\vec{F}_{\text{gr}}$ because a Newton is a unit of force not length. A scale drawing is needed.

One often needs to draw vectors with different units on the same picture, as for showing the position $\vec{r}$ at which a force $\vec{F}$ is applied (see fig. 2.2). In this case different scale factors are used for the drawing of the vectors that have different units.

Drawing and measuring are tedious and also not very accurate. And drawing in 3 dimensions is particularly hard (given the short supply of 3D graph paper now days). So the magnitudes and directions of vectors are usually defined with numbers and units rather than scale drawings. Nonetheless, the drawing rules, and the geometric descriptions in general, still define vector concepts.

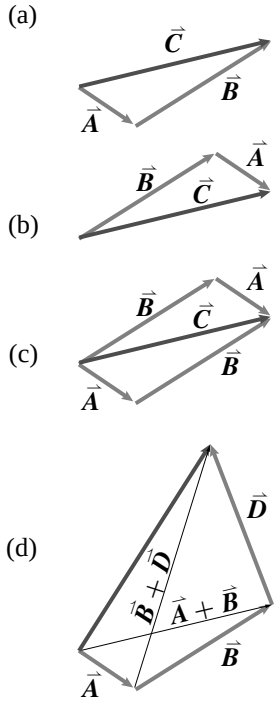


Figure 2.3: (a) tip to tail addition of $\vec{A}+\vec{B}$, (b) tip to tail addition of $\vec{B}+\vec{A}$, (c) the parallelogram interpretation of vector addition, and (d) The associative law of vector addition.

(Filename:tfigure.tiptotail)

Adding vectors

The sum of two vectors $\vec{A}$ and $\vec{B}$ is defined by the *tip to tail* rule of vector addition shown in fig. 2.3a for the sum $\vec{C} = \vec{A} + \vec{B}$. Vector $\vec{A}$ is drawn. Then vector $\vec{B}$ is drawn with its tail at the tip (or head) of $\vec{A}$. The sum $\vec{C}$ is the vector from the tail of $\vec{A}$ to the tip of $\vec{B}$.

The same sum is achieved if $\vec{B}$ is drawn first, as shown in fig. 2.3b. Putting both of ways of adding $\vec{A}$ and $\vec{B}$ on the same picture draws a parallelogram as shown in fig.2.3c. Hence the tip to tail rule of vector addition is also called the *parallelogram* rule. The parallelogram construction shows the *commutative* property of vector addition, namely that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. Note that you can view figs. 2.3a-c as 3D pictures. In 3D, the parallelogram will generally be on some tilted plane.

Three vectors are added by the same tip to tail rule. The construction shown in fig. 2.3d shows that $(\vec{A} + \vec{B}) + \vec{D} = \vec{A} + (\vec{B} + \vec{D})$ so that the expression $\vec{A} + \vec{B} + \vec{D}$

is unambiguous. This is the *associative* property of vector addition. This picture is also sensible in 3D where the 6 vectors drawn make up the edges of a tetrahedron which are generally not coplanar.

With these two laws we see that the sum $\vec{A} + \vec{B} + \vec{D} + \ldots$ can be permuted to $\vec{D} + \vec{A} + \vec{B} + \ldots$ or any which way without changing the result. So vector addition shares the associativity and commutivity of scalar addition that you are used to *e.g.*, that $3 + (7 + \pi) = (\pi + 3) + 7$.

We can reconsider the statement 'force is a vector' and see that it hides one of the basic assumptions in mechanics, namely:

> If forces $\vec{F}_1$ and $\vec{F}_2$ are applied to a point on a structure they can be replaced, for all mechanics considerations, with a single force $\vec{F} = \vec{F}_1 + \vec{F}_2$ applied to that point

as illustrated in fig. 2.4. The force $\vec{F}$ is said to be *equivalent* to the *concurrent* (acting at one point) force system consisting of $\vec{F}_1$ and $\vec{F}_2$.

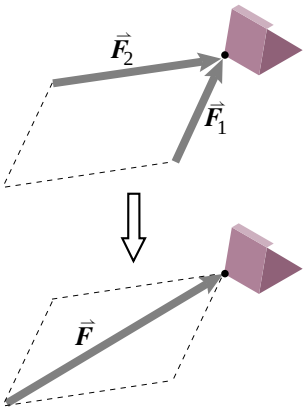


Figure 2.4: Two forces acting at a point may be replaced by their sum for all mechanics purposes.

(Filename:tfigure.forcesadd)

Note that two vectors with different dimensions cannot be added. Figure 2.2 on page 10 can no more sensibly be taken to represent meaningful vector addition than can the scalar sum of a length and a weight, "2 ft + 3 N", be taken as meaningful.

Subtraction and the zero vector $\vec{0}$

Subtraction is most simply defined by inverse addition. Find $\vec{C} - \vec{A}$ means find the vector which when added to $\vec{A}$ gives $\vec{C}$. We can draw $\vec{C}$, draw $\vec{A}$ and then find the vector which, when added tip to tail to $\vec{A}$ give $\vec{C}$. Fig. 2.3a shows that $\vec{B}$ answers the question. Another interpretation comes from defining the negative of a vector $-\vec{A}$ as $\vec{A}$ with the head and tail switched. Again you can see from fig. 2.3b, by imagining that the head and tail on $\vec{A}$ were switched that $\vec{C} + (-\vec{A}) = \vec{B}$. The negative of a vector evidently has the expected property that $\vec{A} + (-\vec{A}) = \vec{0}$, where $\vec{0}$ is the vector with no magnitude so that $\vec{C} + \vec{0} = \vec{C}$ for all vectors $\vec{C}$.

Relative position vectors

The concept of relative position permeates most mechanics equations. The position of point B relative to point A is represented by the vector $\vec{r}_{B/A}$ (pronounced 'r of B relative to A') drawn from A and to B (as shown in fig. 2.5). An alternate notation for this vector is $\vec{r}_{AB}$ (pronounced 'r A B' or 'r A to B'). You can think of the position of B relative to A as being the position of B relative to you if you were standing on A. Similarly $\vec{r}_{C/B} = \vec{r}_{BC}$ is the position of C relative to B.

Figure. 2.5a shows that relative positions add by the tip to tail rule. That is,

$$\vec{r}_{C/A} = \vec{r}_{B/A} + \vec{r}_{C/B} \qquad \text{or} \qquad \vec{r}_{AC} = \vec{r}_{AB} + \vec{r}_{BC}$$

so vector addition has a sensible meaning for relative position vectors.

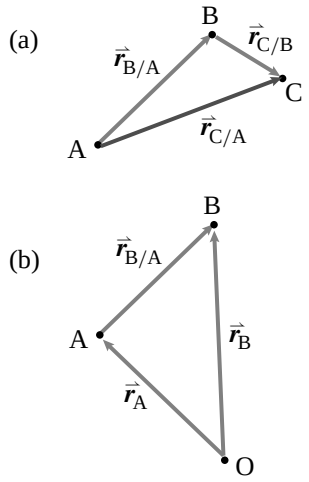


Figure 2.5: a) Relative position of points A, B, and C; b) Relative position of points O, A, and B.

(Filename:tfigure.relpos)

① For the first 7 chapters of this book you can just translate 'relative to' to mean 'minus' as in english. 'How much money does Rudra have *relative to* Andy?' means what is Rudra's wealth *minus* Andy's wealth? What is the position of B *relative to* A? It is the position of B *minus* the position of A.

Often when doing problems we pick a distinguished point in space, say a prominent point or corner of a machine or structure, and use it as the origin of a coordinate system O. The position of point A relative to O is $\vec{r}_{A/0}$ or $\vec{r}_{OA}$ but we often adopt the shorthand notation $\vec{r}_A$ (pronounced 'r A') leaving the reference point O as implied. Figure. 2.5b shows that

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

which rolls off the tongue easily and makes the concept of relative position easier to remember.①

Multiplying by a scalar stretches a vector

Naturally enough $2\vec{F}$ means $\vec{F} + \vec{F}$ (see fig. 2.6) and $127\vec{A}$ means $\vec{A}$ added to itself 127 times. Similarly $\vec{A}/7$ or $\frac{1}{7}\vec{A}$ means a vector in the direction of $\vec{A}$ that when added to itself 7 times gives $\vec{A}$. By combining these two ideas we can define any rational multiple of $\vec{A}$. For example $\frac{29}{13}\vec{A}$ means add 29 copies of the vector that when added 13 times to itself gives $\vec{A}$. We skip the mathematical fine point of extending the definition to $c\vec{A}$ for c that are irrational.

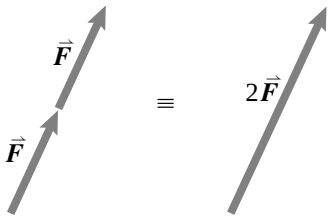


Figure 2.6: (Filename:tfigure.stretch)

We can define $-17\vec{A}$ as $17(-\vec{A})$, combining our abilities to negate a vector and multiply it by a positive scalar. In general, for any positive scalar c we define $c\vec{A}$ as the vector that is in the same direction as $\vec{A}$ but whose magnitude is multiplied by c. Five times a 5 N force pointed Northeast is a 25 N force pointed Northeast. If c is negative the direction is changed and the magnitude multiplied by $|c|$. Minus 5 times a 5 N force pointed Northeast is a 25 N force pointed SouthWest.

If you imagine stretching a vector addition diagram (*e.g.*, fig. 2.3a on page 10) equally in all directions the distributive rule for scalar multiplication is apparent:

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$$

Unit vectors have magnitude 1

Unit vectors are vectors with a magnitude of one. Unit vectors are useful for indicating direction. Key examples are the unit vectors pointed in the positive x, y and z directions $\hat{\imath}$ (called 'i hat' or just 'i'), $\hat{\jmath}$, and $\hat{k}$. We distinguish unit vectors by hatting them but any undistinguished vector notation will do (*e.g.*, using $\underline{\imath}$).

An easy way to find a unit vector in the direction of a vector $\vec{A}$ is to divide $\vec{A}$ by its magnitude. Thus

$$\hat{\lambda}_A \equiv \frac{\vec{A}}{|\vec{A}|}$$

is a unit vector in the $\vec{A}$ direction. You can check that this defines a unit vector by looking up at the rules for multiplication by a scalar. Multiplying $\vec{A}$ by the scalar $1/|\vec{A}|$ gives a new vector with magnitude $|\vec{A}|/|\vec{A}| = 1$.

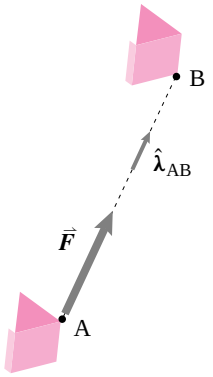


Figure 2.7: (Filename:tfigure.FAtoB)

A common situation is to know that a force $\vec{F}$ is a yet unknown scalar F multiplied by a unit vector pointing between known points A and B. (fig. 2.7). We can then write $\vec{F}$ as

$$\vec{F} = F\hat{\lambda}_{AB} = F\frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = F\frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}$$

where we have used $\hat{\lambda}_{AB}$ as the unit vector pointing from A to B.

Vectors in pictures and sketches.

Some options for drawing vectors are shown in sample **??** on page **??**. The two notations below are the most common.

Symbolic: labeling an arrow with a vector symbol. Indicate a vector, say a force $\vec{F}$, by drawing an arrow and then labeling it with one of the symbolic notations above as in figure 2.8a. *In this notation, the arrow is only schematic*, the magnitude and direction are determined by the algebraic symbol $\vec{F}$. It is sometimes helpful to draw the arrow in the direction of the vector and approximately to scale, but this is not necessary.

Graphical: a scalar multiplies an arrow. Indicate a vector's direction by drawing an arrow with direction indicated by marked angles or slopes. The scalar multiple with a nearby scalar symbol, say F, as shown in figure 2.8b. This means F times a unit vector in the direction of the arrow. (Because F might be negative, sign confusion is common amongst beginners. Please see sample 2.1.)

Combined: graphical representation used to define a symbolic vector. The full symbolic notation can be used in a picture with the graphical information as a way of defining the symbol. For example if the arrow in fig. 2.8b were labeled with an $\vec{F}$ instead of just F we would be showing that $\vec{F}$ is a scalar multiplied by a unit vector in the direction shown.

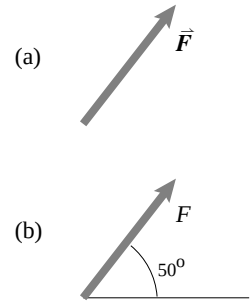


Figure 2.8: Two different ways of drawing a vector (a) shows a labeled arrow. The magnitude and direction of the vector is given by the symbol $\vec{F}$, the drawn arrow has no quantitative information. (b) shows an arrow with clearly indicated orientation next to the scalar F. This means a unit vector in the direction of the arrow multiplied by the scalar F.

(Filename:tfigure1.d)

The components of a vector

A given vector, say $\vec{F}$, can be described as the sum of vectors each of which is parallel to a coordinate axis. Thus $\vec{F} = \vec{F}_x + \vec{F}_y$ in 2D and $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$ in 3D. Each of these vectors can in turn be written as the product of a scalar and a unit vector along the positive axes, *e.g.*, $\vec{F}_x = F_x\hat{\imath}$ (see fig. 2.9). So

$$\vec{F} = \vec{F}_x + \vec{F}_y = F_x\hat{\imath} + F_y\hat{\jmath} \quad \text{(2D)}$$

or

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}. \quad \text{(3D)}$$

The scalars F_x, F_y, and F_z are called the components of the vector with respect to the axes xyz. The components may also be thought of as the orthogonal projections (the shadows) of the vector onto the coordinate axes.

Because the list of components is such a handy way to describe a vector we have a special notation for it. The bracketed expression $[\vec{F}]_{xyz}$ stands for the list of components of $\vec{F}$ presented as a horizontal or vertical array (depending on context), as shown below.

$$[\vec{F}]_{xyz} = [F_x, \quad F_y, \quad F_z] \quad \text{or} \quad [\vec{F}]_{xyz} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

If we had an xy coordinate system with x pointing East and y pointing North we could write the components of a 5 N force pointed Northeast as $[\vec{F}]_{xy} = [(5/\sqrt{2})\,\text{N}, \quad (5/\sqrt{2})\,\text{N}]$.

Note that the components of a vector in some crooked coordinate system $x'y'z'$ are different than the coordinates for the same vector in the coordinate system xyz because the projections are different. Even though $\vec{F} = \vec{F}$ it is *not true* that $[\vec{F}]_{xyz} = [\vec{F}]_{x'y'z'}$ (see fig. 2.19 on page 26). In mechanics we often make use of multiple coordinate systems. So to define a vector by its components the coordinate system used must be specified.

Rather than using up letters to repeat the same concept we sometimes label the coordinate axes x_1, x_2 and x_3 and the unit vectors along them $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ (thus freeing our minds of the silently pronounced letters y,z,j, and k).

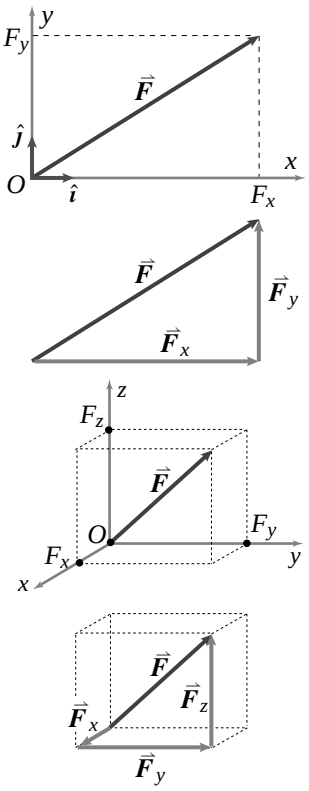


Figure 2.9: A vector can be broken into a sum of vectors, each parallel to the axis of a coordinate system. Each of these is a component multiplied by a unit vector along the coordinate axis, *e.g.*, $\vec{F}_x = F_x\hat{\imath}$.

(Filename:tfigure.vectproject)

Manipulating vectors by manipulating components

Because a vector can be represented by its components (once given a coordinate system) we should be able to relate our geometric understanding of vectors to their components. In practice, when push comes to shove, most calculations with vectors are done with components.

Adding and subtracting with components

Because a vector can be broken into a sum of orthogonal vectors, because addition is associative, and because each orthogonal vector can be written as a component times a unit vector we get the addition rule:

$$[\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}]_{xyz} = [(A_x + B_x), \quad (A_y + B_y), \quad (A_y + B_y)]$$

which can be described by the tricky words 'the components of the sum of two vectors are given by the sums of the corresponding components.' Similarly,

$$[\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}}]_{xyz} = [(A_x - B_x), \quad (A_y - B_y), \quad (A_y - B_y)]$$

Multiplying a vector by a scalar using components

The vector $\vec{\boldsymbol{A}}$ can be decomposed into the sum of three orthogonal vectors. If $\vec{\boldsymbol{A}}$ is multiplied by 7 than so must be each of the component vectors. Thus

$$[c\vec{\boldsymbol{A}}]_{xyz} = [cA_x, \quad cA_y, \quad cA_y].$$

The components of a scaled vector are the corresponding scaled components.

Magnitude of a vector using components

The Pythagorean theorem for right triangles ('$A^2 + B^2 = C^2$') tells us that

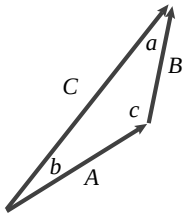
$$|\vec{\boldsymbol{F}}| = \sqrt{F_x^2 + F_y^2}, \tag{2D}$$

$$|\vec{\boldsymbol{F}}| = \sqrt{F_x^2 + F_y^2 + F_z^2}. \tag{3D}$$

To get the result in 3D the 2D Pythagorean theorem needs to be applied twice successively, first to get the magnitude of the sum $\vec{\boldsymbol{F}}_x + \vec{\boldsymbol{F}}_y$ and once more to add in $\vec{\boldsymbol{F}}_z$.

2.3 *THEORY*
Vector triangles and the laws of sines and cosines

The tip to tail rule of vector addition defines a triangle. Given some information about the vectors in this triangle how does one figure out the rest? One traditional approach is to use the laws of sines and cosines.



Consider the vector sum $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{C}}$ represented by the triangle shown with traditionally labeled sides A, B, and C and internal angles a, b, and c.

The sides and angles are related by

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C} \qquad \text{the law of sines, and}$$

$$C^2 = A^2 + B^2 - 2AB\cos c \qquad \text{the law of cosines.}$$

The first equality, say, in the law of sines can be proved by calculating the altitude from c two ways. The law of cosines can be proved by dropping one altitude from b and using the pythagorean theorem to calculate the lengths of the sides of the two right triangles.

If the vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ were known then we would know A, B and c and would want to know, perhaps, C, and b. We can find them using the laws of sines and cosines as:

$$C = \sqrt{A^2 + B^2 - 2AB\cos c} \quad \text{and}$$
$$b = \sin^{-1}\left(\frac{B\sin c}{C}\right).$$

A common situation in elementary mechanics is where one vector, say $\vec{\boldsymbol{C}}$, is known as well as the directions but not magnitudes of the other two vectors. Thus we might know C, a, and b but not A, B, and c. Well c is found easily enough as $c = \pi - a - b$ because the sums of the internal angles add up to $180^{\text{o}} = \pi$. The lengths of the unknown vectors are then

$$A = \frac{C\sin a}{\sin c} \quad \text{and}$$
$$B = \frac{C\sin b}{\sin c}.$$

In this era of vector algebra and vector components the laws of sines and cosines are seldom used. They are here for completeness.

SAMPLE 2.1 *Drawing vectors:* Draw the vector $\vec{r} = 3\,\text{ft}\hat{\imath} - 2\,\text{ft}\hat{\jmath}$ using

(a) its components and
(b) its magnitude and slope.

Solution

(a) **From its components:** To draw $\vec{r}$ using its components, we first draw the axes and measure 3 units (any units that we choose on the ruler) along the x-axis and 2 units along the negative y-axis. We mark this point as A (say) on the paper and draw a line from the origin to the point A. We write the dimensions '3 ft' and '2 ft' on the figure. Finally, we put an arrowhead on this line pointing towards A.

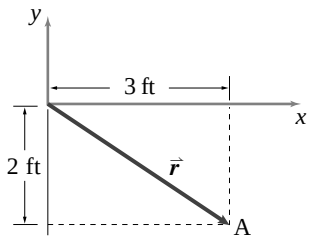


Figure 2.10: A vector $\vec{r} = 3\,\text{ft}\hat{\imath} - 2\,\text{ft}\hat{\jmath}$ is drawn by locating its end point which is 3 units away along the x-axis and 2 units away along the negative y-axis.

(Filename:sfig1.2.4a)

(b) **From its magnitude and slope:** First, we need to find the magnitude and the slope (angle, measured positive counterclockwise, that the vector makes with the positive x-axis).
For

$$\vec{r} = 3\,\text{ft}\hat{\imath} - 2\,\text{ft}\hat{\jmath} = r_x\hat{\imath} + r_y\hat{\jmath},$$

the magnitude or the length of $\vec{r}$ is

$$\begin{aligned} |\vec{r}| &= \sqrt{r_x^2 + r_y^2} \\ &= \sqrt{(3\,\text{ft})^2 + (-2\,\text{ft})^2} \\ &= 3.6\,\text{ft}. \end{aligned}$$

The slope angle of $\vec{r}$ is calculated as follows.

$$\begin{aligned} \tan\theta &= \frac{r_y}{r_x} \\ &= \frac{-2\,\text{ft}}{3\,\text{ft}} = -0.667 \\ \Rightarrow \quad \theta &= \tan^{-1}(-0.667) \\ &= -33.7^{\text{o}}. \end{aligned}$$

Now we draw a line from the origin at an angle -33.7^{o} from the x-axis (minus sign means measuring clockwise), measure 3.6 units (magnitude of $\vec{r}$) along this line and finally put an arrowhead pointing away from the origin.

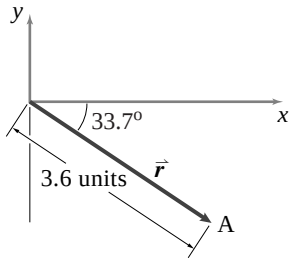


Figure 2.11: A vector $\vec{r} = 3\,\text{ft}\hat{\imath} - 2\,\text{ft}\hat{\jmath}$ is drawn using its magnitude $r = 3.6\,\text{ft}$ and its slope angle $\theta = -33.7^{\text{o}}$ by measuring 3.6 units along a line drawn at -33.7^{o} from the positive x-axis.

(Filename:sfig1.2.4b)

SAMPLE 2.2 *Various ways of representing a vector:* A vector $\vec{F} = 3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}$ is represented in various ways, some incorrect, in the following figures. The base vectors used are shown first. Comment on each representation, whether it is correct or incorrect, and why.

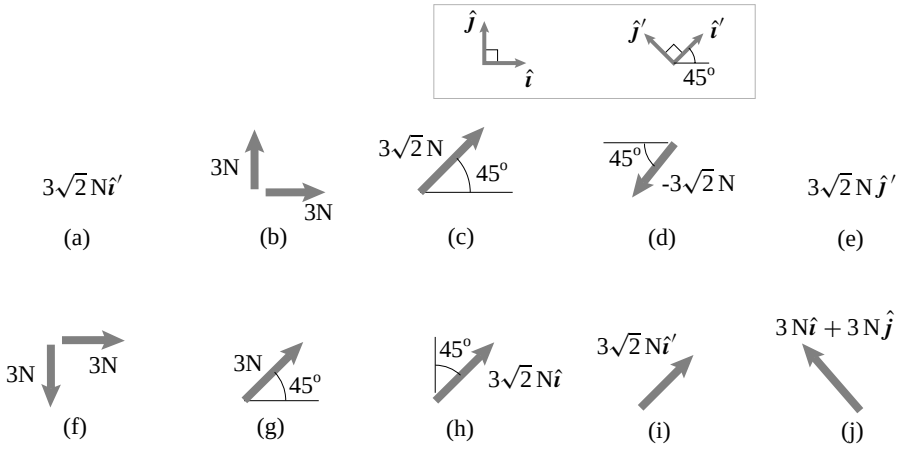


Figure 2.12: (Filename:sfig2.vectors.rep)

Solution The given vector is a force with components of 3 N each in the positive $\hat{\imath}$ and $\hat{\jmath}$ directions using the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ shown in the box above. The unit vectors $\hat{\imath}'$, and $\hat{\jmath}'$ are also shown.

a) Correct: $3\sqrt{2}\,\mathrm{N}\hat{\imath}'$. From the picture defining $\hat{\imath}'$, you can see that $\hat{\imath}'$ is a unit vector with equal components in the $\hat{\imath}$ and $\hat{\jmath}$ directions; *i.e.*, it is parallel to $\vec{F}$. So $\vec{F}$ is given by its magnitude $\sqrt{(3\,\mathrm{N})^2 + (3\,\mathrm{N})^2}$ times a unit vector in its direction, in this case $\hat{\imath}'$. It is the same vector.

b) Correct: Here two vectors are shown: one with magnitude 3 N in the direction of the horizontal arrow $\hat{\imath}$, and one with magnitude 3 N in the direction of the vertical arrow $\hat{\jmath}$. When two forces act on an object at a point, their effect is additive. So the net vector is the sum of the vectors shown. That is, $3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}$. It is the same vector.

c) Correct: Here we have a scalar $3\sqrt{2}$ N next to an arrow. The vector described is the scalar multiplied by a unit vector in the direction of the arrow. Since the arrow's direction is marked as the same direction as $\hat{\imath}'$, which we already know is parallel to $\vec{F}$, this vector represents the same vector $\vec{F}$. It is the same vector.

d) Correct: The scalar $-3\sqrt{2}$ N is multiplied by a unit vector in the direction indicated, $-\hat{\imath}'$. So we get $(-3\sqrt{2}\,\mathrm{N})(-\hat{\imath}')$ which is $3\sqrt{2}\,\mathrm{N}\hat{\imath}'$ as before. It is the same vector.

e) Incorrect: $3\sqrt{2}\,\mathrm{N}\hat{\jmath}'$. The magnitude is right, but the direction is off by 90 degrees. It is a different vector.

f) Incorrect: $3\,\mathrm{N}\hat{\imath} - 3\,\mathrm{N}\hat{\jmath}$. The ui component of the vector is correct but the $\hat{\jmath}$ component is in the opposite direction. The vector is in the wrong direction by 90 degrees. It is a different vector.

g) Incorrect: Right direction but the magnitude is off by a factor of $\sqrt{2}$.

h) Incorrect: The magnitude is right. The direction indicated is right. But, the algebraic symbol $3\sqrt{2}\,\mathrm{N}\hat{\boldsymbol{\imath}}$ takes precedence and it is in the wrong direction ($\hat{\boldsymbol{\imath}}$ instead of $\hat{\boldsymbol{\imath}}'$). It is a different vector.

i) Correct: A labeled arrow. The arrow is only schematic. The algebraic symbols $3\sqrt{2}\,\mathrm{N}\hat{\boldsymbol{\imath}}'$ define the vector. We draw the arrow to remind us that there is a vector to represent. The tip or tail of the arrow would be drawn at the point of the force application. In this case, the arrow is drawn in the direction of $\vec{\boldsymbol{F}}$ but it need not.

j) Correct: Like (i) above, the directional and magnitude information is in the algebraic symbols $3\,\mathrm{N}\hat{\boldsymbol{\imath}} + 3\,\mathrm{N}\hat{\boldsymbol{\jmath}}$. The arrow is there to indicate a vector. In this case, it points in the wrong direction so is not ideally communicative. But (j) still correctly represents the given vector. It is the same vector.

□

SAMPLE 2.3 *Adding vectors:* Three forces, $\vec{F}_1 = 2\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath}$, $\vec{F}_2 = -10\,\text{N}\hat{\jmath}$, and $\vec{F}_3 = 3\,\text{N}\hat{\imath} + 1\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}$, act on a partcile. Find the net force on the particle.

Solution The net force on the particle is the vector sum of all the forces, *i.e.*,

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\
&= (2\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath}) + (-10\,\text{N}\hat{\jmath}) + (3\,\text{N}\hat{\imath} + 1\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}) \\
&= \begin{array}{rcrcr} 2\,\text{N}\hat{\imath} & + & 3\,\text{N}\hat{\jmath} & + & 0\hat{k} \\ 0\hat{\imath} & - & 10\,\text{N}\hat{\jmath} & + & 0\hat{k} \\ 3\,\text{N}\hat{\imath} & + & 1\,\text{N}\hat{\jmath} & - & 5\hat{k} \end{array} \\
&= (2\,\text{N} + 3\,\text{N})\hat{\imath} + (3\,\text{N} - 10\,\text{N} + 1\,\text{N})\hat{\jmath} + (-5\,\text{N})\hat{k} \\
&= 5\,\text{N}\hat{\imath} - 6\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}.
\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = 5\,\text{N}\hat{\imath} - 6\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}}$$

Comments: In general, we do not need to write the summation so elaborately. Once you feel comfortable with the idea of summing only similar components in a vector sum, you can do the calculation in two lines.

SAMPLE 2.4 *Subtracting vectors:* Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a body. The net force on the body is $\vec{F}_{\text{net}} = 2\,\text{N}\hat{\imath}$. If $\vec{F}_1 = 10\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath}$, find the other force $\vec{F}_2$.

Solution

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 \\
\Rightarrow \quad \vec{F}_2 &= \vec{F}_{\text{net}} - \vec{F}_1 \\
&= 2\,\text{N}\hat{\imath} - (10\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath}) \\
&= (2\,\text{N} - 10\,\text{N})\hat{\imath} - (-10\,\text{N})\hat{\jmath} \\
&= -8\,\text{N}\hat{\imath} + 10\,\text{N}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{F}_2 = -8\,\text{N}\hat{\imath} + 10\,\text{N}\hat{\jmath}}$$

SAMPLE 2.5 *Scalar times a vector:* Two forces acting on a particle are $\vec{F}_1 = 100\,\text{N}\hat{\imath} - 20\,\text{N}\hat{\jmath}$ and $\vec{F}_2 = 40\,\text{N}\hat{\jmath}$. If $\vec{F}_1$ is doubled, does the net force double?

Solution

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 = (100\,\text{N}\hat{\imath} - 20\,\text{N}\hat{\jmath}) + (40\,\text{N}\hat{\jmath}) \\
&= 100\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath}
\end{aligned}$$

After $\vec{F}_1$ is doubled, the new net force $\vec{F}_{(\text{net})_2}$ is

$$\begin{aligned}
\vec{F}_{(\text{net})_2} &= 2\vec{F}_1 + \vec{F}_2 = 2(100\,\text{N}\hat{\imath} - 20\,\text{N}\hat{\jmath}) + (40\,\text{N}\hat{\jmath}) \\
&= 200\,\text{N}\hat{\imath} - 40\,\text{N}\hat{\jmath} + 40\,\text{N}\hat{\jmath} \\
&= 200\,\text{N}\hat{\imath} \neq 2\underbrace{(100\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath})}_{\vec{F}_{\text{net}}}
\end{aligned}$$

No, the net force does not double.

SAMPLE 2.6 *Magnitude and direction of a vector:* The velocity of a car is given by $\vec{v} = (30\hat{\imath} + 40\hat{\jmath})$ mph.

(a) Find the speed (magnitude of $\vec{v}$) of the car.
(b) Find a unit vector in the direction of $\vec{v}$.
(c) Write the velocity vector as a product of its magnitude and the unit vector.

Solution

(a) **Magnitude of $\vec{v}$:** The magnitude of a vector is the length of the vector. It is a scalar quantity, usually represented by the same letter as the vector but without the vector notation (i.e. no bold face, no underbar). It is also represented by the modulus of the vector (the vector written between two vertical lines). The length of a vector is the square root of the sum of squares of its components. Therefore, for

$$\begin{aligned} \vec{v} &= 30\,\text{mph}\hat{\imath} + 40\,\text{mph}\hat{\jmath}, \\ v = |\vec{v}| &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{(30\,\text{mph})^2 + (40\,\text{mph})^2} \\ &= 50\,\text{mph} \end{aligned}$$

which is the speed of the car.

$$\boxed{\text{speed} = 50\ \text{mph}}$$

(b) **Direction of $\vec{v}$ as a unit vector along $\vec{v}$:** The direction of a vector can be specified by specifying a unit vector along the given vector. In many applications you will encounter in dynamics, this concept is useful. The unit vector along a given vector is found by dividing the given vector with its magnitude. Let $\hat{\boldsymbol{\lambda}}_v$ be the unit vector along $\vec{v}$. Then,

$$\begin{aligned} \hat{\boldsymbol{\lambda}}_v &= \frac{\vec{v}}{|\vec{v}|} = \frac{30\,\text{mph}\hat{\imath} + 40\,\text{mph}\hat{\jmath}}{50\,\text{mph}} \\ &= 0.6\hat{\imath} + 0.8\hat{\jmath}. \qquad \text{(unit vectors have no units!)} \end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}}_v = 0.6\hat{\imath} + 0.8\hat{\jmath}}$$

(c) **$\vec{v}$ as a product of its magnitude and the unit vector $\hat{\boldsymbol{\lambda}}_v$:** A vector can be written in terms of its components, as given in this problem, or as a product of its magnitude and direction (given by a unit vector). Thus we may write,

$$\vec{v} = |\vec{v}|\hat{\boldsymbol{\lambda}}_v = 50\,\text{mph}(0.6\hat{\imath} + 0.8\hat{\jmath})$$

which, of course, is the same vector as given in the problem.

$$\boxed{\vec{v} = 50\,\text{mph}(0.6\hat{\imath} + 0.8\hat{\jmath})}$$

SAMPLE 2.7 *Position vector from the origin:* In the xyz coordinate system, a particle is located at the coordinate (3m, 2m, 1m). Find the position vector of the particle.

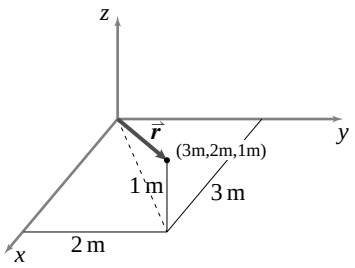


Figure 2.13: The position vector of the particle is a vector drawn from the origin of the coordinate system to the position of the particle.

(Filename:sfig2.vec1.6)

Solution The position vector of the particle at P is a vector drawn from the origin of the coordinate system to the position P of the particle. See Fig. 2.13. We can write this vector as

$$\begin{aligned} \vec{r}_{\rm P} &= (3\,{\rm m})\hat{\imath} + (2\,{\rm m})\,\hat{\jmath} + (1\,{\rm m})\hat{k} \\ \text{or} \qquad \vec{r}_{\rm P} &= (3\hat{\imath} + 2\hat{\jmath} + \hat{k})\,{\rm m}. \end{aligned}$$

$$\boxed{\vec{r}_{\rm P} = 3\,{\rm m}\hat{\imath} + 2\,{\rm m}\hat{\jmath} + 1\,{\rm m}\hat{k}}$$

SAMPLE 2.8 *Relative position vector:* Let A (2m, 1m, 0) and B (0, 3m, 2m) be two points in the xyz coordinate system. Find the position vector of point B with respect to point A, *i.e.*, find $\vec{r}_{\rm AB}$ (or $\vec{r}_{\rm B/A}$).

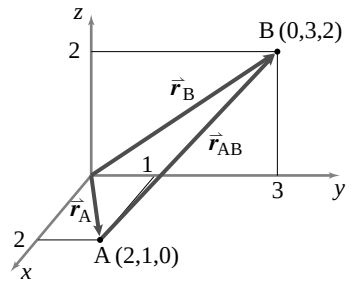


Figure 2.14: The position vector of B with respect to A is found from $\vec{r}_{\rm AB} = \vec{r}_{\rm B} - \vec{r}_{\rm A}$.

(Filename:sfig2.vec1.7)

Solution From the geometry of the position vectors shown in Fig. 2.14 and the rules of vector sums, we can write,

$$\begin{aligned} \vec{r}_{\rm B} &= \vec{r}_{\rm A} + \vec{r}_{\rm AB} \\ \Rightarrow \quad \vec{r}_{\rm AB} &= \vec{r}_{\rm B} - \vec{r}_{\rm A} \\ &= (0\hat{\imath} + 3\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k}) - (2\,{\rm m}\hat{\imath} + 1\,{\rm m}\hat{\jmath} + 0\hat{k}) \\ &= -2\,{\rm m}\hat{\imath} + 2\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k}. \end{aligned}$$

$$\boxed{\vec{r}_{\rm AB} \equiv \vec{r}_{\rm B/A} = -2\,{\rm m}\hat{\imath} + 2\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k}}$$

SAMPLE 2.9 *Finding a unit vector:* A string is pulled with a force $F = 100\,\mathrm{N}$ as shown in the Fig. 2.15. Write F as a vector.

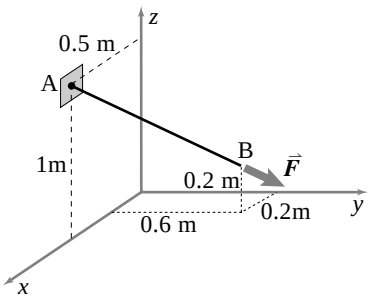


Figure 2.15: (Filename:sfig1.2.2)

Solution A vector can be written, as we just showed in the previous sample problem, as the product of its magnitude and a unit vector along the given vector. Here, the magnitude of the force is given and we know it acts along AB. Therefore, we may write

$$\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\lambda}}_{AB}$$

where $\hat{\boldsymbol{\lambda}}_{AB}$ is a unit vector along AB. So now we need to find $\hat{\boldsymbol{\lambda}}_{AB}$. We can easily find $\hat{\boldsymbol{\lambda}}_{AB}$ if we know vector AB. Let us denote vector AB by $\vec{\boldsymbol{r}}_{AB}$ (sometimes we will also write it as $\vec{\boldsymbol{r}}_{B/A}$ to represent the *position of B with respect to A* as a vector). Then,

$$\hat{\boldsymbol{\lambda}}_{AB} = \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|}.$$

To find $\vec{\boldsymbol{r}}_{AB}$, we note that (see Fig. 2.16)

$$\vec{\boldsymbol{r}}_A + \vec{\boldsymbol{r}}_{AB} = \vec{\boldsymbol{r}}_B$$

where $\vec{\boldsymbol{r}}_A$ and $\vec{\boldsymbol{r}}_B$ are the position vectors of point A and point B respectively. Hence,

$$\begin{aligned}
\vec{\boldsymbol{r}}_{B/A} &= \vec{\boldsymbol{r}}_{AB} = \vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A \\
&= (0.2\,\mathrm{m}\hat{\boldsymbol{\imath}} + 0.6\,\mathrm{m}\hat{\boldsymbol{\jmath}} + 0.2\,\mathrm{m}\hat{\boldsymbol{k}}) - (0.5\,\mathrm{m}\hat{\boldsymbol{\imath}} + 1.0\,\mathrm{m}\hat{\boldsymbol{k}}) \\
&= -0.3\,\mathrm{m}\hat{\boldsymbol{\imath}} + 0.6\,\mathrm{m}\hat{\boldsymbol{\jmath}} - 0.8\,\mathrm{m}\hat{\boldsymbol{k}}.
\end{aligned}$$

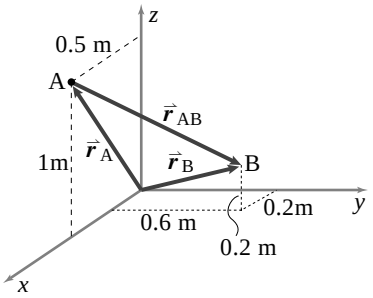


Figure 2.16: $\vec{\boldsymbol{r}}_{AB} = \vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A$.
(Filename:sfig1.2.2b)

Therefore,

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_{AB} &= \frac{-0.3\,\mathrm{m}\hat{\boldsymbol{\imath}} + 0.6\,\mathrm{m}\hat{\boldsymbol{\jmath}} - 0.8\,\mathrm{m}\hat{\boldsymbol{k}}}{\sqrt{(-0.3)^2 + (0.6)^2 + (-0.8)^2}\ \mathrm{m}} \\
&= -0.29\hat{\boldsymbol{\imath}} + 0.57\hat{\boldsymbol{\jmath}} - 0.77\hat{\boldsymbol{k}},
\end{aligned}$$

and, finally

$$\vec{\boldsymbol{F}} = (\underbrace{100\,\mathrm{N}}_{F})\hat{\boldsymbol{\lambda}}_{AB} = -29\,\mathrm{N}\hat{\boldsymbol{\imath}} + 57\,\mathrm{N}\hat{\boldsymbol{\jmath}} - 77\,\mathrm{N}\hat{\boldsymbol{k}}.$$

$$\boxed{\vec{\boldsymbol{F}} = -29\,\mathrm{N}\hat{\boldsymbol{\imath}} + 57\,\mathrm{N}\hat{\boldsymbol{\jmath}} - 77\,\mathrm{N}\hat{\boldsymbol{k}}}$$

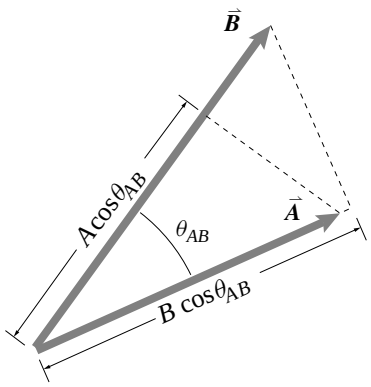


Figure 2.17: The dot product of $\vec{A}$ and $\vec{B}$ is a scalar and so is not easily drawn. It is given by $\vec{A}\cdot\vec{B} = AB\cos\theta_{AB}$ which is A times the projection of $\vec{B}$ in the A direction and also B times the projection of $\vec{A}$ in the B direction.

(Filename:tfigure1.11)

2.2 The dot product of two vectors

The dot product is used to project a vector in a given direction, to reduce a vector to components, to reduce vector equations to scalar equations, to define work and power, and to help solve geometry problems.

The *dot product* of two vectors $\vec{A}$ and $\vec{B}$ is written $\vec{A}\cdot\vec{B}$ (pronounced 'A dot B'). The dot product of $\vec{A}$ and $\vec{B}$ is the product of the magnitudes of the two vectors times a number that expresses the degree to which $\vec{A}$ and $\vec{B}$ are parallel: $\cos\theta_{AB}$, where θ_{AB} is the angle between $\vec{A}$ and $\vec{B}$. That is,

$$\vec{A}\cdot\vec{B} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\cos\theta_{AB}$$

which is sometimes written more concisely as $\vec{A}\cdot\vec{B} = AB\cos\theta$. One special case is when $\cos\theta_{AB} = 1$, $\vec{A}$ and $\vec{B}$ are parallel, and $\vec{A}\cdot\vec{B} = AB$. Another is when $\cos\theta_{AB} = 0$, $\vec{A}$ and $\vec{B}$ are perpendicular, and $\vec{A}\cdot\vec{B} = 0$.①

The dot product of two vectors is a scalar. So the dot product is sometimes called the *scalar product*. Using the geometric definition of dot product, and the rules for vector addition we have already discussed, you can convince yourself of (or believe) the following properties of dot products.

- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ — commutative law, $AB\cos\theta = BA\cos\theta$
- $(a\vec{A})\cdot\vec{B} = \vec{A}\cdot(a\vec{B}) = a(\vec{A}\cdot\vec{B})$ — a distributive law, $(aA)B\cos\theta = A(aB)\cos\theta$
- $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ — another distributive law, the projection of $\vec{B}+\vec{C}$ onto $\vec{A}$ is the sum of the two separate projections
- $\vec{A}\cdot\vec{B} = 0$ if $\vec{A}\perp\vec{B}$ — perpendicular vectors have zero for a dot product, $AB\cos\pi/2 = 0$
- $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|$ if $\vec{A}\parallel\vec{B}$ — parallel vectors have the product of their magnitudes for a dot product, $AB\cos 0 = AB$. In particular, $\vec{A}\cdot\vec{A} = A^2$ or $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$
- $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$, $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$ — The standard base vectors used with cartesian coordinates are unit vectors and they are perpendicular to each other. In math language they are 'orthonormal.'
- $\hat{\imath}'\cdot\hat{\imath}' = \hat{\jmath}'\cdot\hat{\jmath}' = \hat{k}'\cdot\hat{k}' = 1$, $\hat{\imath}'\cdot\hat{\jmath}' = \hat{\jmath}'\cdot\hat{k}' = \hat{k}'\cdot\hat{\imath}' = 0$ — The standard crooked base vectors are orthonormal.

The identities above lead to the following equivalent ways of expressing the dot product of $\vec{A}$ and $\vec{B}$ (see box 2.2 on page 25 to see how the component formula follows from the geometric definition above):

① If you don't know, almost without a thought, that $\cos 0 = 1, \cos\pi/2 = 0, \sin 0 = 0$, and $\sin\pi/2 = 1$ now is as good a time as any to draw as many triangles and unit circles as it takes to cement these special cases into your head.

$$\begin{aligned}
\vec{A}\cdot\vec{B} &= |\vec{A}||\vec{B}|\cos\theta_{AB} \\
&= A_x B_x + A_y B_y + A_z B_z \quad \text{(component formula for dot product)} \\
&= A_{x'} B_{x'} + A_{y'} B_{y'} + A_{z'} B_{z'} \\
&= |\vec{A}| \cdot [\text{projection of } \vec{B} \text{ in the } \vec{A} \text{ direction}] \\
&= |\vec{B}| \cdot [\text{projection of } \vec{A} \text{ in the } \vec{B} \text{ direction}]
\end{aligned}$$

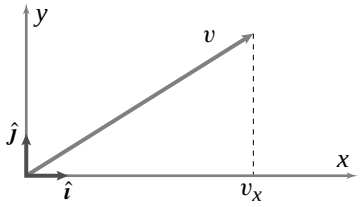


Figure 2.18: The dot product with unit vectors gives projection. For example, $v_x = \vec{v}\cdot\hat{i}$.

(Filename:tfigure1.3.dotprod)

Using the dot product to find components

To find the x component of a vector or vector expression one can use the dot product of the vector (or expression) with a unit vector in the x direction as in figure 2.18. In particular,

$$v_x = \vec{v}\cdot\hat{i}.$$

This idea can be used for finding components in any direction. If one knows the orientation of the crooked unit vectors $\hat{i}'$, $\hat{j}'$, $\hat{k}'$ relative to the standard bases $\hat{i}$, $\hat{j}$, $\hat{k}$ then all the angles between the base vectors are known. So one can evaluate the dot products between the standard base vectors and the crooked base vectors. In 2-D

2.4 *THEORY*

Using the geometric definition of the dot product to find the dot product in terms of components

Vectors are essentially a geometric concept and we have consequently defined the dot product geometrically as $\vec{A}\cdot\vec{B} = AB\cos\theta$. Almost 400 years ago René Descartes discovered that you could do geometry by doing algebra on the coordinates of points.

So we should be able to figure out the dot product of two vectors by knowing their components. The central key to finding this component formula is the distributive law $(\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C})$. If we write $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ then we just repeatedly use the distributive law as follows.

$$\begin{aligned}
\vec{A}\cdot\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\
&= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot B_x\hat{i} \; + \\
&\quad (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot B_y\hat{j} \; + \\
&\quad (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot B_z\hat{k} \\
&= A_x B_x\hat{i}\cdot\hat{i} + A_y B_x\hat{j}\cdot\hat{i} + A_z B_x\hat{k}\cdot\hat{i} \; + \\
&\quad A_x B_y\hat{i}\cdot\hat{j} + A_y B_y\hat{j}\cdot\hat{j} + A_z B_y\hat{k}\cdot\hat{j} \; + \\
&\quad A_x B_z\hat{i}\cdot\hat{k} + A_y B_z\hat{j}\cdot\hat{k} + A_z B_z\hat{k}\cdot\hat{k} \\
&= A_x B_x(1) + A_y B_x(0) + A_z B_x(0) \; + \\
&\quad A_x B_y(0) + A_y B_y(1) + A_z B_y(0) \; + \\
&\quad A_x B_z(0) + A_y B_z(0) + A_z B_z(1)
\end{aligned}$$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z \qquad \text{(3D)}.$$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = A_x B_x + A_y B_y \qquad \text{(2D)}.$$

The demonstration above could have been carried out using a different orthogonal coordinate system $x'y'z'$ that was crooked with respect to the xyz system. By identical reasoning we would find that $\vec{A}\cdot\vec{B} = A_{x'}B_{x'} + A_{y'}B_{y'} + A_{z'}B_{z'}$. Even though all of the numbers in the list $[A_x, A_y, A_z]$ might be different from the numbers in the list $[A_{x'}, A_{y'}, A_{z'}]$ and similarly all the list $[\vec{B}]_{xyz}$ might be different than the list $[\vec{B}]_{x'y'z'}$, so (somewhat remarkably),

$$A_x B_x + A_y B_y + A_z B_z = A_{x'}B_{x'} + A_{y'}B_{y'} + A_{z'}B_{z'}.$$

If we call our coordinate x_1, x_2, and x_3; and our unit base vectors $\hat{e}_1$,$\hat{e}_2$, and $\hat{e}_3$ we would have $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ and $\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3$ and the dot product has the tidy form: $\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_i B_i$.

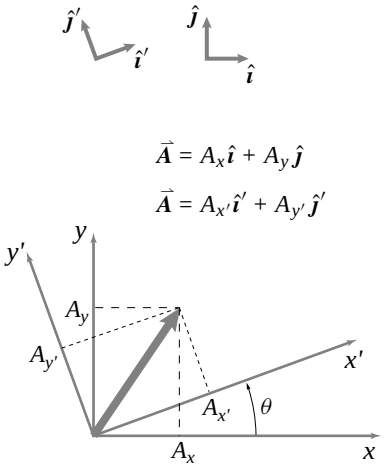


Figure 2.19: The dot product helps find components in terms of crooked unit vectors. For example, $A_{y'} = \vec{\boldsymbol{A}}\cdot\hat{\boldsymbol{j}}' = A_x(\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}}') + A_y(\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}}') = A_x(-\sin\theta) + A_y(\cos\theta)$.

(Filename:tfigure1.3.dotprod.a)

assume that the dot products between the standard base vectors and the vector $\hat{\boldsymbol{j}}'$ (*i.e.*, . $\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}}'$, $\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{j}}'$) are known. One can then use the dot product to find the $x'y'$ components $(A_{x'}, A_{y'})$ from the xy coordinates (A_x, A_y). For example, as shown in 2-D in figure 2.19, we can start with the obvious equation

$$\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}$$

and dot both sides with $\hat{\boldsymbol{j}}'$ to get:

$$\begin{aligned}
\vec{\boldsymbol{A}}\cdot\hat{\boldsymbol{j}}' &= \vec{\boldsymbol{A}}\cdot\hat{\boldsymbol{j}}' \\
\underbrace{(A_{x'}\hat{\boldsymbol{i}}' + A_{y'}\hat{\boldsymbol{j}}')}_{\vec{\boldsymbol{A}}}\cdot\hat{\boldsymbol{j}}' &= \underbrace{(A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}})}_{\vec{\boldsymbol{A}}}\cdot\hat{\boldsymbol{j}}' \\
A_{x'}\underbrace{\hat{\boldsymbol{i}}'\cdot\hat{\boldsymbol{j}}'}_{0} + A_{y'}\underbrace{\hat{\boldsymbol{j}}'\cdot\hat{\boldsymbol{j}}'}_{1} &= A_x\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}}' + A_y\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{j}}' \\
A_{y'} &= A_x\underbrace{(\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}}')}_{-\sin\theta} + A_y\underbrace{(\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{j}}')}_{\cos\theta}
\end{aligned}$$

Similarly, one could find the component $A_{x'}$ using a dot product with $\hat{\boldsymbol{i}}'$.

This technique of finding components is useful when one problem uses more than one base vector system.

Using dot products with other than $\hat{i}$, $\hat{j}$, or $\hat{k}$

It is often useful to use dot products to get scalar equations using vectors other than $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, and $\hat{\boldsymbol{k}}$.

Example: **Getting scalar equations without dotting with $\hat{i}$, $\hat{j}$, or $\hat{k}$**

Given the vector equation.

$$-mg\hat{\boldsymbol{j}} + N\hat{\boldsymbol{n}} = ma\hat{\boldsymbol{\lambda}}$$

where it is known that the unit vector $\hat{\boldsymbol{n}}$ is perpendicular to the unit vector $\hat{\boldsymbol{\lambda}}$, we can get a scalar equation by dotting both sides with $\hat{\boldsymbol{\lambda}}$ which we write as follows

$$\begin{aligned}
\Big\{\ (-mg\hat{\boldsymbol{j}} + N\hat{\boldsymbol{n}}) &= (ma\hat{\boldsymbol{\lambda}})\ \Big\}\cdot\hat{\boldsymbol{\lambda}} \\
(-mg\hat{\boldsymbol{j}} + N\hat{\boldsymbol{n}})\cdot\hat{\boldsymbol{\lambda}} &= (ma\hat{\boldsymbol{\lambda}})\cdot\hat{\boldsymbol{\lambda}} \\
-mg\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{\lambda}} + N\underbrace{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}}}_{0} &= ma\underbrace{\hat{\boldsymbol{\lambda}}\cdot\hat{\boldsymbol{\lambda}}}_{1} \\
-mg\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{\lambda}} &= ma.
\end{aligned}$$

Then we find $\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{\lambda}}$ as the cosine of the angle between $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{\lambda}}$. We have thus turned our vector equation into a scalar equation and eliminated the unknown N at the same time. □

Using dot products to solve geometry problems

We have seen how a vector can be broken down into a sum of components each parallel to one of the orthogonal base vectors. Another useful decomposition is this: Given any vector $\vec{A}$ and a unit vector $\hat{\lambda}$ the vector $\vec{A}$ can be written as the sum of two parts,

$$\vec{A} = \vec{A}^{\parallel} + \vec{A}^{\perp}$$

where $\vec{A}^{\parallel}$ is parallel to $\hat{\lambda}$ and $\vec{A}^{\perp}$ is perpendicular to $\hat{\lambda}$ (see fig. 2.20). The part parallel to $\hat{\lambda}$ is a vector pointed in the $\hat{\lambda}$ direction that has the magnitude of the projection of $\vec{A}$ in that direction,

$$\vec{A}^{\parallel} = (\vec{A} \cdot \hat{\lambda})\hat{\lambda}.$$

The perpendicular part of $\vec{A}$ is just what you get when you subtract out the parallel part, namely,

$$\vec{A}^{\perp} = \vec{A} - \vec{A}^{\parallel} = \vec{A} - (\vec{A} \cdot \hat{\lambda})\hat{\lambda}$$

The claimed properties of the decomposition can now be checked, namely that $\vec{A} = \vec{A}^{\parallel} + \vec{A}^{\perp}$ (just add the 2 equations above and see), that $\vec{A}^{\parallel}$ is in the direction of $\hat{\lambda}$ (its a scalar multiple), and that $\vec{A}^{\perp}$ is perpendicular to $\hat{\lambda}$ (evaluate $\vec{A}^{\perp} \cdot \hat{\lambda}$ and find 0).

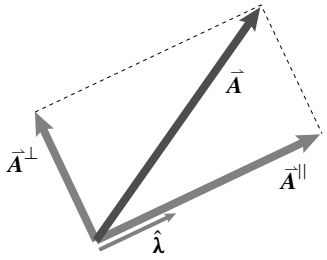
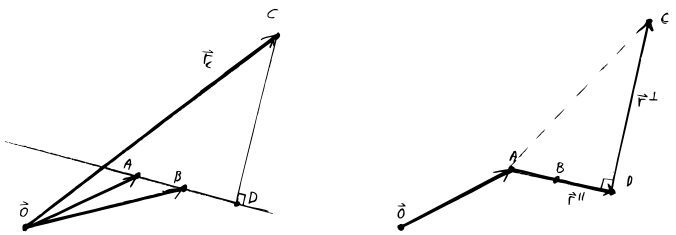


Figure 2.20: For any $\vec{A}$ and $\hat{\lambda}$, $\vec{A}$ can be decomposed into a part parallel to $\hat{\lambda}$ and a part perpendicular to $\hat{\lambda}$.

(Filename:tfigure.Graham1)



Example. Given the positions of three points $\vec{r}_A$, $\vec{r}_B$, and $\vec{r}_C$ what is the position of the point D on the line AB that is closest to C? The answer is,

$$\vec{r}_D = \vec{r}_A + \vec{r}_{C/A}{}^{\parallel}$$

where $\vec{r}_{C/A}{}^{\parallel}$ is the part of $\vec{r}_{C/A}$ that is parallel to the line segment AB. Thus,

$$\vec{r}_D = \vec{r}_A + (\vec{r}_C - \vec{r}_A) \cdot \frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}.$$

□

Likewise we could find the parts of a vector $\vec{A}$ in and perpendicular to a given plane. If the plane is defined by two vectors that are not necessarily orthogonal we could follow these steps. First find two vectors in the plane that are orthogonal, using the method above. Next subtract from $\vec{A}$ the part of it that is parallel to each of the two orthogonal vectors in the plane. In math lingo the execution of this process goes by the intimidating name 'Graham Schmidt orthogonalization.'

A Given vector can be written as various sums and products

A vector $\vec{A}$ has many representations. The equivalence of different representations of a vector is partially analogous to the case of a dimensional scalar which has the same value no matter what units are used (*e.g.*, the mass $m = 4.41\,\text{lbm}$ is equal to $m = 2\,\text{kg}$). Here are some common representations of vectors.

Scalar times a unit vector in the vector's direction. $\vec{F} = F\hat{\lambda}$ means the scalar F multiplied by the unit vector $\hat{\lambda}$.

Sum of orthogonal component vectors. $\vec{F} = \vec{F}_x + \vec{F}_y$ is a sum of two vectors parallel to the x and y axis, respectively. In three dimensions, $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$.

Components times unit base vectors. $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ or $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ in three dimensions. One way to think of this sum is to realize that $\vec{F}_x = F_x\hat{\imath}$, $\vec{F}_y = F_y\hat{\jmath}$ and $\vec{F}_z = F_z\hat{k}$.

Components times rotated unit base vectors. $\vec{F} = F'_x\mathbf{i}' + F'_y\mathbf{j}'$ or $\vec{F} = F'_x\mathbf{i}' + F'_y\mathbf{j}' + F'_z\mathbf{k}'$ in three dimensions. Here the base vectors marked with primes, $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$, are unit vectors parallel to some mutually orthogonal x', y', and z' axes. These x', y', and z' axes may be crooked in relation to the x, y, and z axis. That is, the x' axis need not be parallel to the x axis, the y' not parallel to the y axis, and the z' axis not parallel to the z axis.

Components times other unit base vectors. If you use polar or cylindrical coordinates the unit base vectors are $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_R$, so in 2-D , $\vec{F} = F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta$ and in 3-D, $\vec{F} = F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta + F_z\hat{k}$. If you use 'path' coordinates, you will use the path-defined unit vectors $\hat{\boldsymbol{e}}_t$, $\hat{\boldsymbol{e}}_n$, and $\hat{\boldsymbol{e}}_b$ so in 2-D $\vec{F} = F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n$. In 3-D $\vec{F} = F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n + F_b\hat{\boldsymbol{e}}_b$.

A list of components. $[\vec{F}]_{xy} = [F_x, F_y]$ or $[\vec{F}]_{xyz} = [F_x, F_y, F_z]$ in three dimensions. This form coincides best with the way computers handle vectors. The row vector $[F_x, F_y]$ coincides with $F_x\hat{\imath} + F_y\hat{\jmath}$ and the row vector $[F_x, F_y, F_z]$ coincides with $F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$.

In summary:

$$
\begin{array}{rcll}
\vec{A} & = & \vec{A} & \\
 & = & |\vec{A}|\hat{\boldsymbol{\lambda}}_A = A\hat{\boldsymbol{\lambda}}_A, & \text{where } \hat{\boldsymbol{\lambda}}_A \parallel \vec{A},\ A = |\vec{A}| \text{ and } |\hat{\boldsymbol{\lambda}}_A| = 1 \\
 & = & \vec{A}_x + \vec{A}_y + \vec{A}_z & \text{where } \vec{A}_x, \vec{A}_y, \vec{A}_z \text{ are parallel to the } x, y, z \text{ axis} \\
 & = & A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}, & \text{where } \hat{\imath}, \hat{\jmath}, \hat{k} \text{ are parallel to the } x, y, z \text{ axis} \\
 & = & A_{x'}\hat{\imath}' + A_{y'}\hat{\jmath}' + A_{z'}\hat{k}', & \text{where } \hat{\imath}', \hat{\jmath}', \hat{k}' \text{ are } \parallel \text{ to skewed } x', y', z' \text{ axes} \\
 & = & A_R\hat{\boldsymbol{e}}_R + A_\theta\hat{\boldsymbol{e}}_\theta + A_z\hat{k}, & \text{using polar coordinate basis vectors.} \\
[\vec{A}]_{xyz} & = & [A_x, A_y, A_z] & [\vec{A}]_{xyz} \text{ stands for the component list in } xyz \\
[\vec{A}]_{x'y'z'} & = & [A_{x'}, A_{y'}, A_{z'}] & [\vec{A}]_{x'y'z'} \text{ stands for the component list in } x'y'z'
\end{array}
$$

Vector algebra

Vectors are algebraic quantities and manipulated algebraically in equations. The rules for vector algebra are similar to the rules for ordinary (scalar) algebra. For example, if vector $\vec{A}$ is the same as the vector $\vec{B}$, $\vec{A} = \vec{B}$. For any scalar a and any vector $\vec{C}$, we then

$$
\begin{aligned}
\vec{A} + \vec{C} &= \vec{B} + \vec{C}, \\
a\vec{A} &= a\vec{B}, \text{ and} \\
\vec{A} \cdot \vec{C} &= \vec{B} \cdot \vec{C},
\end{aligned}
$$

because performing the same operation on equal quantities maintains the equality. The vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ might themselves be expressions involving other vectors.

The equations above show the allowable manipulations of vector equations: adding a common term to both sides, multiplying both sides by a common scalar, taking the dot product of both sides with a common vector.

All the distributive, associative, and commutative laws of ordinary addition and multiplication hold.①.

①**Caution:** But you cannot divide a vector by a vector or a scalar by a vector: $7/\hat{\imath}$ and $\vec{A}/\vec{C}$ are *nonsense* expressions. And it does not make sense to add a vector and a scalar, $7 + \vec{A}$ is a *nonsense* expression.

Vector calculations on the computer

Most computer programs deal conveniently with lists of numbers, but not with vector notation and units. Thus our computer calculations will be in terms of vector components with the units left off. For example, when we write on the computer

```
F = [ 3 5 -7]
```

we take that to be the plain computer typing for $[\vec{F}]_{xyz} = [3\,\text{N}, \quad 5\,\text{N}, -7\,\text{N}]$. This assumes that we are clear about what units and what coordinate system we are using. In particular, at this point in the course, you should only use one coordinate system in one problem in computer calculations.

Most computer languages will allow vector addition by a sequence of lines something like this:

```
A = [  1  2  5 ]
B = [ -2  4 19 ]
C = A + B
```

scaling (stretching) like this:

```
A = [  1  2  5 ]
C = 3*A
```

and dot products like this:

```
A   = [  1  2  5 ]
B   = [ -2  4 19 ]
D   = A(1)*B(1) + A(2)*B(2) + A(3)*B(3).
```

In our pseudo code we write `A dot B`. Many computer languages have a shorter way to write the dot product like `dot(A,B)`. In a language built for linear algebra `A*B'`① will work because the rules of matrix multiplication are then the same as the component formula for the dot product.

① `B'` is a common notation for the transpose of `B`, which means, in this case, to turn the row of numbers `B` into a column of numbers.

SAMPLE 2.10 *Calculating dot products:* Find the dot product of the two vectors $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{r} = 5\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}$.

Solution The dot product of the two vectors is

$$\begin{aligned}
\vec{a}\cdot\vec{r} &= (2\hat{\imath} + 3\hat{\jmath} - 2\hat{k})\cdot(5\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}) \\
&= (2\cdot 5\,\text{m})\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} - (2\cdot 2\,\text{m})\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} \\
&\quad + (3\cdot 5\,\text{m})\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0} - (3\cdot 2\,\text{m})\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} \\
&\quad - (2\cdot 5\,\text{m})\underbrace{\hat{k}\cdot\hat{\imath}}_{0} + (2\cdot 2\,\text{m})\underbrace{\hat{k}\cdot\hat{\jmath}}_{0} \\
&= 10\,\text{m} - 6\,\text{m} \\
&= 4\,\text{m}.
\end{aligned}$$

$$\boxed{\vec{a}\cdot\vec{r} = 4\,\text{m}}$$

Comments: Note that with just a little bit of foresight, we could totally ignore the $\hat{k}$ component of $\vec{a}$ since $\vec{r}$ has no $\hat{k}$ component, *i.e.*, $\hat{k}\cdot\vec{r} = 0$. Also, if we keep in mind that $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{\imath} = 0$, we could compute the above dot product in one line:

$$\vec{a}\cdot\vec{r} = (2\hat{\imath} + 3\hat{\jmath})\cdot(5\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}) = (2\cdot 5\,\text{m})\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} - (3\cdot 2\,\text{m})\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} = 4\,\text{m}.$$

SAMPLE 2.11 What is the y-component of $\vec{F} = 5\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$?

Solution Although it is perhaps obvious that the y-component of $\vec{F}$ is 3 N, the scalar multiplying the unit vector $\hat{\jmath}$, we calculate it below in a formal way using the dot product between two vectors. We will use this method later to find components of vectors in arbitrary directions.

$$\begin{aligned}
F_y &= \vec{F}\cdot(\text{a unit vector along } y\text{-axis}) \\
&= (5\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k})\cdot\hat{\jmath} \\
&= 5\,\text{N}\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} + 3\,\text{N}\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} + 2\,\text{N}\underbrace{\hat{k}\cdot\hat{\jmath}}_{0} \\
&= 3\,\text{N}.
\end{aligned}$$

$$\boxed{F_y = \vec{F}\cdot\hat{\jmath} = 3\,\text{N}.}$$

SAMPLE 2.12 *Finding angle between two vectors using dot product:* Find the angle between the vectors $\vec{r}_1 = 2\hat{\imath} + 3\hat{\jmath}$ and $\vec{r}_2 = 2\hat{\imath} - \hat{\jmath}$.

Solution From the definition of dot product between two vectors

$$\begin{aligned} \vec{r}_1 \cdot \vec{r}_2 &= |\vec{r}_1||\vec{r}_2|\cos\theta \\ \text{or} \quad \cos\theta &= \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} \\ &= \frac{(2\hat{\imath} + 3\hat{\jmath}) \cdot (2\hat{\imath} - \hat{\jmath})}{(\sqrt{2^2+3^2})(\sqrt{2^2+1^2})} \\ &= \frac{4-3}{\sqrt{13}\sqrt{5}} = 0.124 \\ \text{Therefore,} \quad \theta &= \cos^{-1}(0.124) = 82.87^{\text{o}}. \end{aligned}$$

$$\boxed{\theta = 83^{\text{o}}}$$

SAMPLE 2.13 *Finding angle information from unit vectors:* Find the angles between $\vec{F} = 4\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath} + 7\,\text{N}\hat{k}$ and each of the three axes.

Solution

$$\begin{aligned} \vec{F} &= F\hat{\lambda} \\ \hat{\lambda} &= \frac{\vec{F}}{F} \\ &= \frac{4\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath} + 7\,\text{N}\hat{k}}{\sqrt{4^2+6^2+7^2}\,\text{N}} \\ &= 0.4\hat{\imath} + 0.6\hat{\jmath} + 0.7\hat{k}. \end{aligned}$$

Let the angles between $\hat{\lambda}$ and the x, y, and z axes be θ, ϕ and ψ respectively. Then

$$\begin{aligned} \cos\theta &= \frac{\hat{\imath} \cdot \hat{\lambda}}{|\hat{\imath}||\hat{\lambda}|} = \frac{0.4}{|1||1|} = 0.4. \\ \Rightarrow \quad \theta &= \cos^{-1}(0.4) = 66.4^{\text{o}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \cos\phi &= 0.6 \quad \text{or} \quad \phi = 53.1^{\text{o}} \\ \cos\psi &= 0.7 \quad \text{or} \quad \psi = 45.6^{\text{o}}. \end{aligned}$$

$$\boxed{\theta = 66.4^{\text{o}},\ \phi = 53.1^{\text{o}},\ \psi = 45.6^{\text{o}}}$$

SAMPLE 2.14 *Projection of a vector in the direction of another vector:* Find the component of $\vec{F} = 5\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$ along the vector $\vec{r} = 3\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}$.

Solution The dot product of a vector $\vec{a}$ with a unit vector $\hat{\lambda}$ gives the projection of the vector $\vec{a}$ in the direction of the unit vector $\hat{\lambda}$. Therefore, to find the component of $\vec{F}$ along $\vec{r}$, we first find a unit vector $\hat{\lambda}_r$ along $\vec{r}$ and dot it with $\vec{F}$.

$$\begin{aligned}
\hat{\lambda}_r &= \frac{\vec{r}}{|\vec{r}|} = \frac{3\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}}{\sqrt{3^2+4^2}\,\text{m}} = 0.6\hat{\imath} - 0.8\hat{\jmath} \\
F_r &= \vec{F}\cdot\hat{\lambda}_r \\
&= (5\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k})\cdot(0.6\hat{\imath} - 0.8\hat{\jmath}) \\
&= 3.0\,\text{N} + 2.4\,\text{N} = 5.4\,\text{N}.
\end{aligned}$$

$$\boxed{F_r = 5.4\,\text{N}}$$

SAMPLE 2.15 Assume that after writing the equation $\sum \vec{F} = m\vec{a}$ in a particular problem, a student finds $\sum \vec{F} = (20\,\text{N} - P_1)\hat{\imath} + 7\,\text{N}\hat{\jmath} - P_2\hat{k}$ and $\vec{a} = 2.4\,\text{m/s}^2\hat{\imath} + a_3\hat{\jmath}$. Separate the scalar equations in the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ directions.

Solution

$$\sum \vec{F} = m\vec{a}$$

Taking the dot product of both sides of this equation with $\hat{\imath}$, we write

$$\begin{aligned}
\hat{\imath}\cdot\sum\vec{F} &= \hat{\imath}\cdot m\vec{a} \\
\hat{\imath}\cdot\Big[(20\,\text{N} - P_1)\hat{\imath} + 7\,\text{N}\hat{\jmath} - P_2\hat{k} &= m(2.4\,\text{m/s}^2\hat{\imath} + a_3\hat{\jmath})\Big] \\
\Rightarrow\quad \underbrace{(20\,\text{N} - P_1)}_{F_x}\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} + 7\,\text{N}\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0} - P - 2\underbrace{\hat{k}\cdot\hat{\imath}}_{0} &= m(\underbrace{2.4\,\text{m/s}^2}_{a_x}\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} + a_3\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0}) \\
\Rightarrow \sum F_x &= ma_x \\
\Rightarrow\quad 20\,\text{N} - P_1 &= m(2.4\,\text{m/s}^2)
\end{aligned}$$

Similarly,

$$\hat{\jmath}\cdot\Big[\sum\vec{F} = m\vec{a}\Big] \quad\Rightarrow\quad \sum F_y = ma_y \tag{2.1}$$

$$\hat{k}\cdot\Big[\sum\vec{F} = m\vec{a}\Big] \quad\Rightarrow\quad \sum F_z = ma_z. \tag{2.2}$$

Substituting the given components of $\vec{F}$ and $\vec{a}$ in the remaining Eqns. (2.1) and (2.2) we get

$$\begin{aligned}
7\,\text{N} &= ma_y \\
-P_2 &= 0.
\end{aligned}$$

Comments: As long as both sides of a vector equation are in the same basis, separating the scalar equations is trivial—simply equate the respective components from both sides. The technique of taking the dot product of both sides with a vector is quite general and powerful. It gives a scalar equation valid in any direction that one desires. You will appreciate this technique more if the vector equation uses more than one basis.

SAMPLE 2.16 *Adding vectors on computers:* The following six forces act at different points of a structure. $\vec{F}_1 = -3\,\text{N}\hat{\imath}$, $\vec{F}_2 = 20\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath}$, $\vec{F}_3 = 1\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}$, $\vec{F}_4 = 10\,\text{N}\hat{\imath}$, $\vec{F}_5 = 5\,\text{N}(\hat{\imath} + \hat{\jmath} + \hat{k})$, $\vec{F}_6 = -10\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$.

(a) Write all the force vectors in column form.
(b) Find the net force by hand calculation.
(c) Write a computer program to sum n vectors, each of length 3. Use your program to compute the net force.

Solution

(a) The 3-D vector $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ is represented as a column (or a row) as follows:

$$[\vec{F}] = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{xyz}$$

Following this convention, we write the given forces as

$$[\vec{F}_1] = \begin{pmatrix} 0 \\ -3\,\text{N} \\ 0 \end{pmatrix}_{xyz}, [\vec{F}_2] = \begin{pmatrix} 20\,\text{N} \\ -10\,\text{N} \\ 0 \end{pmatrix}_{xyz}, \cdots, [\vec{F}_6] = \begin{pmatrix} -10\,\text{N} \\ -10\,\text{N} \\ 2\,\text{N} \end{pmatrix}_{xyz}$$

(b) The net force, $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6$ or

$$\begin{aligned}[\vec{F}_{\text{net}}] &= \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 20 \\ -10 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 20 \\ -5 \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} + \begin{pmatrix} -10 \\ -10 \\ 2 \end{pmatrix} \text{N}_{xyz} \\ &= \begin{pmatrix} 26 \\ 2 \\ 2 \end{pmatrix} \text{N}_{xyz}\end{aligned}$$

(c) The steps to do this addition on computers are as follows.

- Enter the vectors as rows or columns:

```
F1 = [0  -3  0]
F2 = [20  -10  0]
F3 = [1  20  -5]
F4 = [10  0  0]
F5 = [5  5  5]
F6 = [-10  -10  2]
```

- Sum the vectors, using a summing operation that automatically does element by element addition of vectors:

```
Fnet = F1 + F2 + F3 + F4 + F5 + F6
```

- The computer generated answer is:

```
Fnet = [26  2  2].
```

$$\boxed{\vec{F}_{\text{net}} = 26\,\text{N}\hat{\imath} + 2\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}}$$

2.3 Cross product, moment, and moment about an axis

When you try to move something you can push and turn it. In mechanics, the measure of your pushing is the net force you apply. The measure of your turning is the net *moment*, also sometimes called the net *torque* or net *couple*. In this section we will define the moment of a force intuitively, geometrically, and finally using vector algebra. We will do this first in 2 dimensions and then in 3. The main mathematical tool here is the vector cross product, a second way of multiplying vectors together. The cross product is used to define (and calculate) moment and to calculate various quantities in dynamics. The cross product also sometimes helps solve three-dimensional geometry problems.

Although concepts involving moment (and rotation) are often harder for beginners than force (and translation), they were understood first. The ancient principle of the lever is the basic idea incorporated by moments. The principle of the lever can be viewed as the root of all mechanics.

Ultimately you can take on faith the vector definition of moment (given opposite the inside cover) and its role in eqs. II. But we can more or less deduce the definition by generalizing from common experience.

Teeter totter mechanics

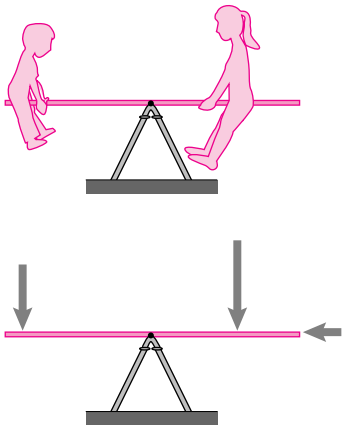

(not a free body diagram)

Figure 2.21: On a balanced teeter totter the bigger person gets the short end of the stick.

(Filename:tfigure.teeter)

The two people weighing down on the teeter totter in Fig. 2.21 tend to rotate it about its hinge, the right one clockwise the left one counterclockwise. We will now cook up a measure of the tendency of each force to cause rotation about the hinge and call it *the moment of the force about the hinge*.

As is verified a million times a year by young future engineering students, to balance a teeter-totter the smaller person needs to be further from the hinge. If two people are on one side then the teeter totter is balanced by two similar people on the other side. Two people can balance one similar person by scooting twice as close to the hinge. These proportionalities generalize to this: the tendency of a force to cause rotation is proportional to the size of the force and to its distance from the hinge (for forces perpendicular to the teeter totter).

If someone standing nearby adds a force that is directed towards the hinge it causes no tendency to rotate. Because any force can be decomposed into a sum of forces, one perpendicular to the teeter totter and the other towards the hinge, and because we assume that the affect of the sum of these forces is the sum of the affects of each separately, and because the force towards the hinge has no tendency to rotate, we have deduced:

> The moment of a force about a hinge is the product of its distance from the hinge and the component of the force perpendicular to the line from the hinge to the force.

Here then is the formula for 2D *moment about C* or *moment with respect to C*.[①]

$$M_{/C} = |\vec{r}|\,(|\vec{F}|\sin\theta) = (|\vec{r}|\sin\theta)\,|\vec{F}|. \tag{2.3}$$

Here, θ is the angle between $\vec{r}$ (the position of the point of force application relative to the hinge) and $\vec{F}$ (see fig. 2.22). This formula for moment has all the teeter totter deduced properties. Moment is proportional to r, and to the part of $\vec{F}$ that is perpendicular to $\vec{r}$. The re-grouping as $(|\vec{r}|\sin\theta)$ shows that a force $\vec{F}$ has the same effect if it is applied at a new location that is displaced in the direction of $\vec{F}$. That is,

① The '/' in the subscript of $\vec{M}$ reads as 'relative to' or 'about'. For simplicity we often leave the / out and just write $\vec{M}_C$.

the force $\vec{F}$ can slide along its length without changing its $M_{/C}$ and is *equivalent* in its effect on the teeter totter. The quantity $|\vec{r}|\sin\theta$ is sometimes called the *lever arm* of the force.

By common convention we define as positive a moment that causes a counterclockwise rotation. A moment that causes a clockwise rotation is negative. If we define θ appropriately then eqn. (2.3) obeys this sign convention. We define θ as the angle from the positive vector $\vec{r}$ to the positive vector $\vec{F}$ measured counterclockwise. Point the thumb of your right hand towards yourself. Point the fingers of your right hand along $\vec{r}$ and curl them towards the direction of $\vec{F}$ and see how far you have to rotate them. The force caused by the person on the left of the teeter totter has $\theta = 90^o$ so $\sin\theta = 1$ and the formula 2.3 gives a positive counterclockwise M. The force of the person on the right has $\theta = 270^o$ (3/4 of a revolution) so $\sin\theta = -1$ and the formula 2.3 gives a negative M.

In two dimensions moment is really a scalar concept, it is either positive or negative. In three dimensions moment is a vector. But even in 2D we find it easier to keep track of signs if we treat moment as a vector. In the xy plane, the 2D moment is a vector in the $\hat{k}$ direction (straight out of the plane). So eqn. 2.3 becomes

$$\vec{M}_{/C} = |\vec{r}|\,|\vec{F}|\sin\theta\,\hat{k}. \tag{2.4}$$

If you curl the fingers of your *right hand* in the direction of rotation caused by a force your thumb points in the direction of the moment vector.

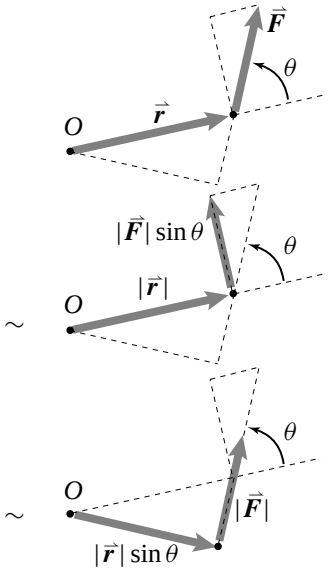


Figure 2.22: The moment of a force is either the product of its radius with its perpendicular component or of its lever arm and the full force. The $\sim$ indicates that the lower two forces and positions have the same moment.

(Filename:tfigure.slidevector)

The 2D cross product

The expression we have found for the right side of eqn. 2.4 is the 2D cross product of vectors $\vec{r}$ and $\vec{F}$. We can now apply the concept to any pair of vectors whether or not they represent force and position. The 2D cross product is defined as :

$$\underbrace{\vec{A}\times\vec{B}}_{\text{'A cross B'}} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\,\sin\theta\,\hat{k}. \tag{2.5}$$

where θ is the amount that $\vec{A}$ would need to be rotated counterclockwise to point in the same direction as $\vec{B}$. An equivalent alternative approach is to define the cross product as

$$\vec{A}\times\vec{B} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\,\sin\theta\,\hat{n}. \tag{2.6}$$

with θ defined to be less than 180^o and $\hat{n}$ defined as the unit vector pointing in the direction of the thumb when the fingers are curled from the direction of $\vec{A}$ towards the direction of $\vec{B}$. For the $\vec{r}$ and $\vec{F}$ on the right of the teeter totter this definition forces us to point our thumb into the plane (in the negative $\hat{k}$ direction). With this definition $\sin\theta$ is always positive and the negative moments come from $\hat{n}$ being in the $-\hat{k}$ direction.

With a few sketches you could convince yourself that the definition of cross product in eqn.2.5 obeys these standard algebra rules (for any 3 2D vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ and any scalar s):

$$\begin{aligned} d(\vec{A}\times\vec{B}) &= (d\vec{A})\times\vec{B} = \vec{A}\times(d\vec{B}) \\ \vec{A}\times(\vec{B}+\vec{C}) &= \vec{A}\times\vec{B}+\vec{A}\times\vec{C}. \end{aligned}$$

The important exception to the rules for scalar algebra is that $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$ because the definition of θ in eqn. 2.5 and $\hat{n}$ in 2.6 depends on order. In particular $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.

Qualitatively the cross product measures how much vectors are perpendicular because the magnitude of the cross product of $\vec{A}$ and $\vec{B}$ is the magnitude of $\vec{A}$ times the magnitude of the projection of $\vec{B}$ in the direction perpendicular to $\vec{A}$ (as shown in the top two illustrations of fig. 2.22). In particular

$$\begin{array}{rcrcl} \text{if } \vec{A} \perp \vec{B} & \Rightarrow & |\vec{A} \times \vec{B}| & = & |\vec{A}|\,|\vec{B}|, \text{ and} \\ \text{if } \vec{A} \parallel \vec{B} & \Rightarrow & |\vec{A} \times \vec{B}| & = & \vec{0}. \end{array}$$

For example, $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$, $\hat{\imath} \times \hat{\imath} = \vec{0}$, and $\hat{\jmath} \times \hat{\jmath} = \vec{0}$.

Component form for the 2D cross product

Just like the dot product, the cross product can be expressed using components. As can be verified by writing $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$, and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$ and using the distributive rules:

$$\vec{A} \times \vec{B} \quad = (A_x B_y - B_x A_y)\hat{k}. \tag{2.7}$$

Some people remember this formula by putting the components of $\vec{A}$ and $\vec{B}$ into a matrix and calculating the determinant $\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$. If you number the components of $\vec{A}$ and $\vec{B}$ (*e.g.*, $[\vec{A}]_{x_1x_2} = [A_1, A_2]$), the cross product is $\vec{A} \times \vec{B} \quad = (A_1B_2 - B_2A_1)\hat{e}_3$. This you might remember as "first times second minus second times first."

Example: Given that $\vec{A} = 1\hat{\imath} + 2\hat{\jmath}$ and $\vec{B} = 10\hat{\imath} + 20\hat{\jmath}$ then $\vec{A} \times \vec{B} = (1 \cdot 20 - 2 \cdot 10)\hat{k} = 0\hat{k} = \vec{0}$. □

For vectors with just a few components it is often most convenient to use the distributive rule directly.

Example: Given that $\vec{A} = 7\hat{\imath}$ and $\vec{B} = 37.6\hat{\imath} + 10\hat{\jmath}$ then $\vec{A} \times \vec{B} = (7\hat{\imath}) \times (37.6\hat{\imath} + 10\hat{\jmath}) = (7\hat{\imath}) \times (37.6\hat{\imath}) + (7\hat{\imath}) \times (10\hat{\jmath}) = \vec{0} + 70\hat{k} = 70\hat{k}$. □

There are many ways of calculating a 2D cross product

You have several options for calculating the 2D cross product. Which you choose depends on taste and convenience. You can use the geometric definition directly, the first times the perpendicular part of the second (distance times perpendicular component of force), the second times the perpendicular part of the first (lever arm times the force), components, or break each of the vectors into a sum of vectors and use the distributive rule.

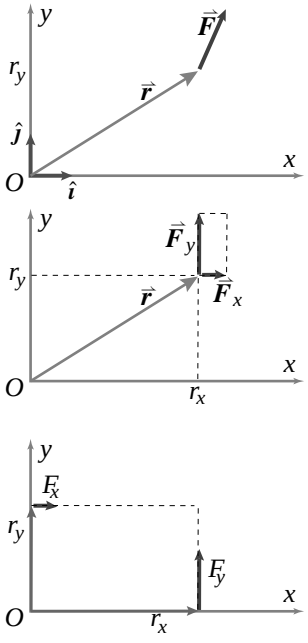


Figure 2.23: The component form of the 2D moment can be found by sequentially breaking the force into components, sliding each component along its line of action to the x and y axis, and adding the moments of the two components.

(Filename:tfigure1.2Dcrosscomps)

2D moment by components

We can use the component form of the 2D cross product to find a component form for the moment $\vec{M}_{/C}$ of eqn. 2.4. Given $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ acting at P, where $\vec{r}_{P/C} = r_x\hat{\imath} + r_y\hat{\jmath}$, the moment of the force about C is

$$\vec{M}_{/C} = (r_x F_y - r_y F_x))\hat{k}$$

or the moment of $\vec{F}$ about the axis at C is

$$M_C = r_x F_y - r_y F_x \tag{2.8}$$

We can derive this component formula with the sequence of vector manipulations shown graphically in fig. 2.23.

3D moment about an axis

The concept of moment about an axis is historically, theoretically, and practically important. Moment about an axis describes the principle of the lever, which far precedes Newton's laws. The net moment of a force system about enough different axes determines everything needed in mechanics about a force system. And one can sometimes quickly solve a statics problem by considering moment about a judiciously chosen axis.

Lets start by thinking about a teeter totter again. Looking from the side we thought of a teeter totter as a 2D system. But the teeter totter really lives in the 3D world. We now re-interpret the 2D moment M as the moment of the 2D forces about the $\hat{\boldsymbol{k}}$ axis of rotation at the hinge. It is plain that a force pushing a teeter totter parallel to the axle causes no tendency to rotate. So we see that the moment a force causes about an axis is the distance of the force from the axis times the part of the force that is neither parallel to the axis nor directed towards the axis.

Now look at this in the more 3-dimensional context of fig. 2.24. Here an imagined axis of rotation is defined as the line through C that is in the $\hat{\boldsymbol{\lambda}}$ direction. A force $\vec{\boldsymbol{F}}$ is applied at P. We can break $\vec{\boldsymbol{F}}$ into a sum of three vectors

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}^{\parallel} + \vec{\boldsymbol{F}}^{r} + \vec{\boldsymbol{F}}^{\perp}$$

where $\vec{\boldsymbol{F}}^{\parallel}$ is parallel to the axis, $\vec{\boldsymbol{F}}^{r}$ is directed along the shortest connection between the axis and P (and is thus perpendicular to the axis) and $\vec{\boldsymbol{F}}^{\perp}$ is out of the plane defined by C, P and $\hat{\boldsymbol{\lambda}}$. By analogy with the teeter totter we see that $\vec{\boldsymbol{F}}^{r}$ and $\vec{\boldsymbol{F}}^{\parallel}$ cause no tendency to rotate about the axis. So only the $\vec{\boldsymbol{F}}^{\perp}$ contributes.

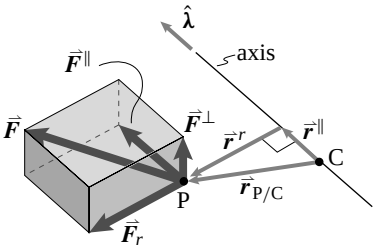


Figure 2.24: Moment about an axis

(Filename:tfigure2.mom.axis)

> **Example:** Try this. Stand facing a partially open door with the front of your body parallel to the plane of the door (a door with no springs is best). Hold the outer edge of the door with one hand. Press down and note that the door is not opened or closed. Push towards the hinge and note that the door is not opened or closed. Push and pull away and towards your body and note how easily you cause the door to rotate. Thus the only force component that tends to rotate the door is perpendicular to the plane of the door (which is the plane of the hinge and line from the hinge to your hand). Now move your hand to the middle of the door (half the distance from the hinge) and note that about twice as much pushing force is needed to rotate the door with the same authority. □

We can similarly decompose $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{\mathrm{P/C}}$ into two parts as

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}^{\parallel} + \vec{\boldsymbol{r}}^{r}.$$

Clearly $\vec{\boldsymbol{r}}^{\parallel}$ has no affect on how much rotation $\vec{\boldsymbol{F}}$ causes about the axis. If for example the point of force application was moved parallel to the axis a few centimeters, the tendency to rotate would not be changed. Altogether, we have that the moment of the force $\vec{\boldsymbol{F}}$ about the axis $\hat{\boldsymbol{\lambda}}$ through C is given by

$$M_{\lambda \mathrm{C}} = r^{r}\ F^{\perp}.$$

The perpendicular distance from the axis to the point of force application is $|\vec{\boldsymbol{r}}^{r}|$ and $\vec{\boldsymbol{F}}^{\perp}$ is the part of the force that causes right-handed rotation about the axis. A moment about an axis is defined as positive if curling the fingers of your right hand give the sense of rotation when your outstretched thumb is pointing along the axis (as in fig. 2.24). The force of the left person on the teeter totter causes a positive moment about the $\hat{\boldsymbol{k}}$ axis through the hinge.

So long as you interpret the quantities correctly, the freshman physics line

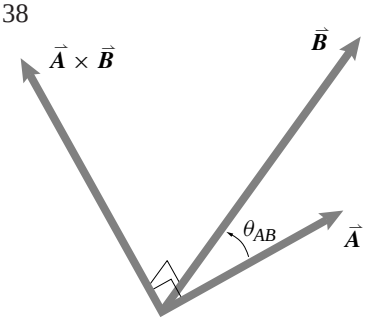


Figure 2.25: The cross product of $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$ in the direction given by the right hand rule. The magnitude of $\vec{A} \times \vec{B}$ is $AB \sin\theta_{AB}$.

(Filename:tfigure1.12)

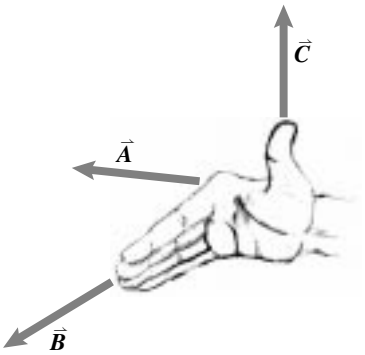


Figure 2.26: The right hand rule for determining the direction of the cross product of two vectors. $\vec{C} = \vec{A} \times \vec{B}$.

(Filename:tfigure.rhr)

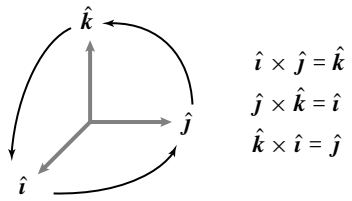


Figure 2.27: Mnemonic device to remember the cross product of the standard base unit vectors.

(Filename:tfigure1.e)

"Moment is distance times force"

perfectly defines moment about an axis.

Three dimensional geometry is difficult, so a formula for moment about an axis in terms of components would be most useful. The needed formula depends on the 3D moment vector defined by the 3D cross product.

The 3D cross product (or vector product)

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is written $\vec{A} \times \vec{B}$ and pronounced 'A cross B.' In contrast to the dot product, which gives a scalar and measures how much two vectors are parallel, the cross product is a vector and measures how much they are perpendicular. The cross product is also called *the vector product*.

The cross product is defined by:

$$\begin{aligned} \vec{A} \times \vec{B} \quad &\stackrel{def}{=} \quad |\vec{A}||\vec{B}| \sin\theta_{AB}\, \hat{n} \\ &\text{where} \quad |\hat{n}| = 1, \\ &\qquad \hat{n} \perp \vec{A}, \\ &\qquad \hat{n} \perp \vec{B}, \\ &\qquad 0 \le \theta_{AB} \le \pi, \text{ and} \\ &\qquad \hat{n} \text{ is in the direction given by the right hand rule, that is, in the direction of the right thumb when the fingers of the right hand are pointed in the direction of } \vec{A} \text{ and then wrapped towards the direction of } \vec{B}. \end{aligned}$$

If $\vec{A}$ and $\vec{B}$ are perpendicular then θ_{AB} is $\pi/2$, $\sin\theta_{AB} = 1$, and the magnitude of the cross product is AB. If $\vec{A}$ and $\vec{B}$ are parallel then θ_{AB} is 0, $\sin\theta_{AB} = 0$ and the cross product is $\vec{0}$ (the zero vector). Using this definition you should be able to verify to your own satisfaction that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. Applying the definition to the standard base unit vectors you can see that $\hat{i} \times \hat{j} = \hat{k}$, *etc.*(figure 2.27).

The geometric definition above and the geometric (tip to tale) definition of vector addition imply that the cross product follows the distributive rule (see box 2.5 on page 39).

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

Applying the distributive rule to the cross products of $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ leads to the algebraic formula for the Cartesian components of the cross product.

$$\begin{aligned} \vec{A} \times \vec{B} \quad = \quad & [A_y B_z - A_z B_y]\hat{i} \\ & +[A_z B_x - A_x B_z]\hat{j} \\ & +[A_x B_y - A_y B_x]\hat{k} \end{aligned}$$

There are various mnemonics for remembering the component formula for cross products. The most common is to calculate a 'determinant' of the 3×3 matrix with one row given by $\hat{i}$, $\hat{j}$, $\hat{k}$ and the other two rows the components of $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The following identities and special cases of cross products are worth knowing well:

- $(a\vec{A}) \times \vec{B} = \vec{A} \times (a\vec{B}) = a(\vec{A} \times \vec{B})$ (a distributive law)
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (the cross product is not commutative!)
- $\vec{A} \times \vec{B} = \vec{0}$ if $\vec{A} \parallel \vec{B}$ (parallel vectors have zero cross product)
- $|\vec{A} \times \vec{B}| = AB$ if $\vec{A} \perp \vec{B}$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$ (assuming the x, y, z coordinate system is right handed — if you use your right hand and point your fingers along the positive x axis, then curl them towards the positive y axis, your thumb will point in the same direction as the positive z axis.)
- $\hat{\imath}' \times \hat{\jmath}' = \hat{k}'$, $\hat{\jmath}' \times \hat{k}' = \hat{\imath}'$, $\hat{k}' \times \hat{\imath}' = \hat{\jmath}'$ (assuming the $x'y'z'$ coordinate system is also right handed.)
- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$, $\quad \hat{\imath}' \times \hat{\imath}' = \hat{\jmath}' \times \hat{\jmath}' = \hat{k}' \times \hat{k}' = \vec{0}$

The moment vector

We now define the moment of a force $\vec{F}$ applied at P, relative to point C as

$$\vec{M}_{/\mathrm{C}} = \vec{r}_{\mathrm{P/C}} \times \vec{F}$$

2.5 *THEORY*

The 3D cross product is distributive over sums

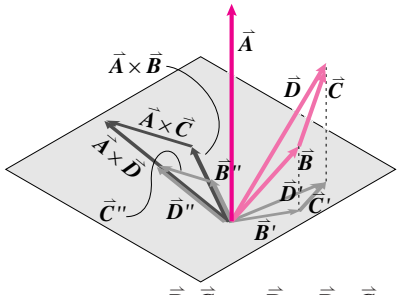
Calculating cross products using vector components depends on the cross product obeying the distributive rule

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

Here is a 3D construction demonstrating this fact.

First we present another geometric definition of the cross product of $\vec{A}$ and any vector $\vec{V}$. Consider a plane P that is perpendicular to $\vec{A}$. Now look at $\vec{V}'$, the projection of $\vec{V}$ on to P. The (right hand rule) normal of $\vec{A}$ and $\vec{V}$ is the same as the normal of $\vec{A}$ and $\vec{V}'$. Also, $|\vec{V}'| = |\vec{V}| \sin\theta_{AV}$. So $\vec{A} \times \vec{V} = \vec{A} \times \vec{V}'$. Now consider $\vec{V}''$ which is the rotation of $\vec{V}'$ by 90° around $\vec{A}$. Note that $\vec{V}''$ is still in P. Finally stretch $\vec{V}''$ by $|\vec{A}|$. The result is a vector in the P plane that is $\vec{A} \times \vec{V}$ since it has the correct magnitude and direction.

Thus $\vec{A} \times \vec{V}$ is given by projecting $\vec{V}$ onto P, rotating 90° about $\vec{A}$, and stretching by $|\vec{A}|$.



Now consider $\vec{B}$, $\vec{C}$, and $\vec{D} \equiv \vec{B} + \vec{C}$. All three cross products

$$\vec{A} \times \vec{B}, \quad \vec{A} \times \vec{C}, \quad \text{and} \quad \vec{A} \times \vec{D},$$

can be calculated by this projection, rotation, and stretch. But each of these three operations is distributive since

- the projection of a sum is the sum of the projections ($\vec{D}' = \vec{B}' + \vec{C}'$);
- the sum of two 90° rotated vectors is the rotation of the sum ($\vec{D}'' = \vec{B}'' + \vec{C}''$); and
- stretched $\vec{D}''$ is stretched $\vec{B}''$ + stretched $\vec{C}''$.

Thus the act of taking the cross product of $\vec{A}$ with $\vec{B}$ and adding that to the cross product of $\vec{A}$ with $\vec{C}$ gives the same result as taking the cross product of $\vec{A}$ with $\vec{D}$ ($\equiv \vec{B} + \vec{C}$). This demonstrates the distributive law.

Application of the distributive rule to vectors expressed in terms of the standard unit base vectors yields the oft-used component expression for the cross product as follows

$$\begin{aligned}
\vec{A} \times \vec{B} &= [A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}] \times [B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}] \\
&= A_xB_x\hat{\imath} \times \hat{\imath} + A_xB_y\hat{\imath} \times \hat{\jmath} + A_xB_z\hat{\imath} \times \hat{k} \\
&\quad + A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_y\hat{\jmath} \times \hat{\jmath} + A_yB_z\hat{\jmath} \times \hat{k} \\
&\quad + A_zB_x\hat{k} \times \hat{\imath} + A_zB_y\hat{k} \times \hat{\jmath} + A_zB_z\hat{k} \times \hat{k} \\
&= A_xB_x\vec{0} + A_xB_y\hat{k} - A_xB_z\hat{\jmath} \\
&\quad - A_yB_x\hat{k} + A_yB_y\vec{0} + A_yB_z\hat{\imath} \\
&\quad + A_zB_x\hat{\jmath} - A_zB_y\hat{\imath} + A_zB_z\vec{0} \\
&= [A_yB_z - A_zB_y]\hat{\imath} + [A_zB_x - A_xB_z]\hat{\jmath} \\
&\quad + [A_xB_y - A_yB_x]\hat{k}
\end{aligned}$$

which we read in short as 'M is r cross F.' The moment vector is admittedly a difficult idea to intuit. A look at its components is helpful.

$$\vec{M}_{/\mathrm{C}} = (r_y F_z - r_z F_y)\hat{\imath} + (r_z F_x - r_x F_z)\hat{\jmath} + (r_x F_y - r_y F_x)\hat{k}$$

You can recognize the z component of the moment vector is the moment of the force about the $\hat{k}$ axis through C (eqn. 2.8). Similarly the x and y components of $\vec{M}_{\mathrm{C}}$ are the moments about the $\hat{\imath}$ and $\hat{\jmath}$ axis through C. So at least the components of of $\vec{M}_{\mathrm{C}}$ have intuitive meaning.

The component form for each of the three components can be deduced graphically by the moves shown in fig. 2.29. The force is first broken into components. The components are then moved along their lines of action to the coordinate planes. From the resulting picture it is clear that, say, the moment about the $+y$ axis gets a positive contribution from F_x with lever arm r_z and a negative contribution from F_z with lever arm r_x. Thus the y component of $\vec{M}$ is $r_z F_x - r_x F_z$.

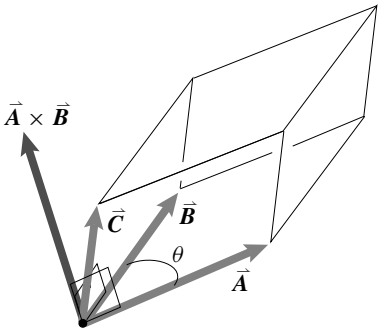


Figure 2.28: One interpretation of the mixed triple product of $\vec{A} \times \vec{B} \cdot \vec{C}$ is as the volume (a scalar) of a parallelepiped with $\vec{A}$, $\vec{B}$, and $\vec{C}$ as the three edges emanating from one corner. This interpretation only works if $\vec{A}$, $\vec{B}$, and $\vec{C}$ are taken in the appropriate order, otherwise $\vec{A} \times \vec{B} \cdot \vec{C}$ is minus the volume which is calculated.

(Filename:tfigure1.13)

The mixed triple product

The 'mixed triple product' of $\vec{A}$, $\vec{B}$, and $\vec{C}$ is so called because it mixes both the dot product and cross product in a single expression. The mixed triple product is also sometimes called the scalar triple product because its value is a scalar. The mixed triple product is useful for calculating the moment of a force about an axis and for related dynamics quantities. The *mixed triple product* of $\vec{A}$, $\vec{B}$, and $\vec{C}$ is defined by and written as

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$$

and pronounced 'A dot B cross C.' The parentheses () are sometimes omitted (*i.e.*, ., $\vec{A} \cdot \vec{B} \times \vec{C}$) because the wrong grouping $(\vec{A} \cdot \vec{B}) \times \vec{C}$ is nonsense (you can't take the cross product of a scalar with a vector) . It is apparent that one way of calculating the mixed triple product is to calculate the cross product of $\vec{B}$ and $\vec{C}$ and then to take the dot product of that result with $\vec{A}$. Some people use the notation $\left(\vec{A}, \vec{B}, \vec{C}\right)$ for the mixed triple product but it will not occur again in this book.

The mixed triple product has the same value if one takes the cross product of $\vec{A}$ and $\vec{B}$ and then the dot product of the result with $\vec{C}$. That is $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$. This identity can be verified geometrically or by looking at the (complicated) expression for the mixed triple product of three general vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in terms of their components as calculated the two different ways. One thus obtains the string of results

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C} = -\vec{B} \times \vec{A} \cdot \vec{C} = -\vec{B} \cdot \vec{A} \times \vec{C} = \ldots$$

The minus signs in the above expressions follow from the cross product identity that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

The mixed triple product has various geometric interpretations, one of them is that $\vec{A} \cdot \vec{B} \times \vec{C}$ is (plus or minus) the volume of the parallelepiped, the crooked shoe box, edged by $\vec{A}$, $\vec{B}$ and $\vec{C}$ as shown in figure 2.28.

Another way of calculating the value of the mixed triple product is with the determinant of a matrix whose rows are the components of the vectors.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{array}{r} A_x(B_y C_Z - B_z C_y) \\ +A_y(B_z C_x - B_x C_z) \\ +A_z(B_x C_y - B_y C_x) \end{array}$$

The mixed triple product of three vectors is zero if①

- any two of them are parallel, or
- if all three of the vectors have one common plane.

① In the language of linear algebra, the mixed triple product of three vectors is zero if the vectors are linearly dependent.

More on moment about an axis

We defined moment about an axis geometrically using fig. 2.24 on page 37 as $M_{\hat{\boldsymbol{\lambda}}} = r^r F^\perp$. We can now verify that the mixed triple product gives the desired result by guessing the formula and seeing that it agrees with the geometric definition.

$$M_{\lambda \mathrm{C}} = \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{M}}_{/\mathrm{C}} \quad \text{(An inspired guess...)} \tag{2.9}$$

We break both $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$ into sums indicated in the figure, use the distributive law, and note that the mixed triple product gives zero if any two of the vectors are parallel. Thus,

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{M}}_{/\mathrm{C}} &= \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{F}} \\
&= \hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{r}}^r + \vec{\boldsymbol{r}}^\parallel) \times (\vec{\boldsymbol{F}}^\perp + \vec{\boldsymbol{F}}^\parallel + \vec{\boldsymbol{F}}^r) \\
&= \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}^r \times \vec{\boldsymbol{F}}^\perp + \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}^r \times \vec{\boldsymbol{F}}^\parallel + \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}^r \times \vec{\boldsymbol{F}}^r \ldots \\
&\quad + \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}^\parallel \times \vec{\boldsymbol{F}}^\perp + \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}^\parallel \times \vec{\boldsymbol{F}}^\parallel + \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}^\parallel \times \vec{\boldsymbol{F}}^r \\
&= r^r F^\perp + 0 + 0 + 0 + 0 + 0 \\
&= r^r F^\perp. \quad \text{(... and a good guess too.)}
\end{aligned}$$

We can calculate the cross and dot product any convenient way, say using vector components.

Example: **Moment about an axis**

Given a force, $\vec{\boldsymbol{F}}_1 = (5\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}})$ N acting at a point P whose position is given by $\vec{\boldsymbol{r}}_{P/O} = (3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}})$ m, what is the moment about an axis through the origin O with direction $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{k}}$?

$$\begin{aligned}
M_{\hat{\boldsymbol{\lambda}}} &= (\vec{\boldsymbol{r}}_{P/O} \times \vec{\boldsymbol{F}}_1) \cdot \hat{\boldsymbol{\lambda}} \\
&= [(3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}})\,\mathrm{m} \times (5\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}})\,\mathrm{N}] \cdot (\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{k}}) \\
&= -\frac{17}{\sqrt{2}}\,\mathrm{m\,N}.
\end{aligned}$$

□

The power of our abstract reasoning is apparent when we consider calculating the moment of a force about an axis with two different coordinate systems. Each of the vectors in eqn. 2.3 will have different components in the different systems. Yet the resulting scalar, after all the arithmetic, will be the same no matter what the coordinate system.

Finally, the moment about an axis gives us an interpretation of the moment vector. The direction of the moment vector $\vec{\boldsymbol{M}}_{\mathrm{C}}$ is the direction of the axis through C about which $\vec{\boldsymbol{F}}$ has the greatest moment. The magnitude of $\vec{\boldsymbol{M}}_{\mathrm{C}}$ is the moment of $\vec{\boldsymbol{F}}$ about that axis.

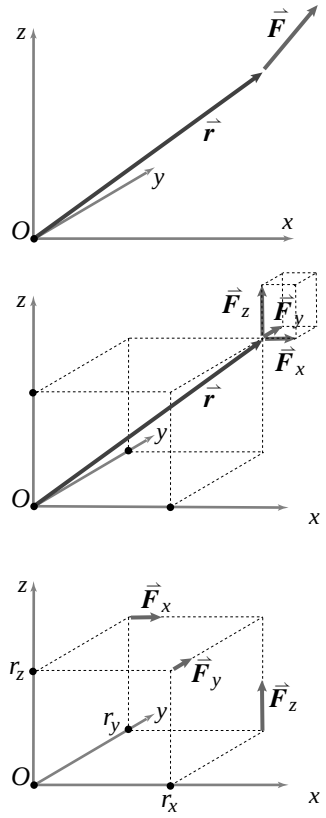


Figure 2.29: The three components of the 3D moment vector are the moments about the three axis. These can be found by sequentially breaking the force into components, sliding each component along its line of action to the coordinate planes, and noting the contribution of each component to moment about each axis.

(Filename:tfigure1.3Dcrosscomps)

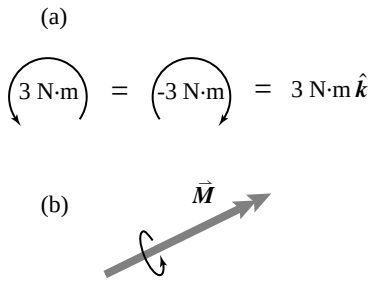


Figure 2.30: Optional drawing method for moment vectors. (a) shows an arced arrow to represent vectors having to do with rotation in 2 dimensions. Such vectors point directly out of, or into, the page so are indicated with an arc in the direction of the rotation. (b) shows a double-headed arrow for torque or rotation quantities in three dimensions.

(Filename:tfigure1rot.d)

Special optional ways to draw moment vectors

Neither of the special rotation notations below is needed because moment (and later, angular velocity, and angular momentum) is a vector like any other. None-the-less, sometimes it is nice to use a notation that suggests the rotational nature of these quantities.

Arced arrow for 2-D moment and angular velocity. In 2D problems in the xy plane, the relevant moment, angular velocity, and angular momentum point straight out of the plane in the z ($\hat{k}$) direction. A way of drawing this is to use an arced arrow. Wrap the fingers of your right hand in the direction of the arc and your thumb points in the direction of the unit vector that the scalar multiplies. The three representations in Fig. 2.30a indicate the same moment vector.

Double headed arrow for 3-D rotations and moments. Some people like to distinguish vectors for rotational motion and torque from other vectors. Two ways of making this distinction are to use double-headed arrows or to use an arrow with an arced arrow around it as shown in Fig. 2.30b.

Vector algebra

We will often be concerned with manipulating equations that involve vectors (like $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{0}$) and scalars (like a, b,, c, and 0). Without knowing anything about mechanics or the geometric meaning of vectors, one can learn to do correct vector algebra. Basically all symbols and collections of symbols can be manipulated just like in elementary algebra. Addition and all three kinds of multiplication (scalar multiplication, dot product, cross product) all follow the usual commutative, associative, and distributive laws of scalar addition and multiplication with the following exceptions:

- $a + \vec{A}$ is nonsense,
- $a/\vec{A}$ is nonsense,
- $\vec{A}/\vec{B}$ is nonsense,
- $a \cdot \vec{A}$ is nonsense,
- $a \times \vec{A}$ is nonsense,
- $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$,

and the following extra simplification rules,

- $a\vec{A}$ is a vector,
- $\vec{A}\cdot\vec{B}$ is a scalar,
- $\vec{A} \times \vec{B}$ is a vector,
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (so $\vec{A} \times \vec{A} = \vec{0}$)
- $\vec{A}\cdot(\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B})\cdot\vec{C}$.

Knowing just these rules you can do correct manipulations. Armed with insight one can direct these manipulations towards a desired end.

Example. Say you know $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ and you know that

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

but you don't know a, b, and c. How could you find a? First dot both sides with $\vec{B} \times \vec{C}$ and then blindly follow the rules:

$$\begin{aligned} \left\{a\vec{A} + b\vec{B} + c\vec{C}\right\} \cdot \left(\vec{B} \times \vec{C}\right) &= \vec{D} \cdot \left(\vec{B} \times \vec{C}\right) \\ a\vec{A}\cdot\left(\vec{B} \times \vec{C}\right) + b\underbrace{\vec{B}\cdot\left(\vec{B} \times \vec{C}\right)}_{0} + c\underbrace{\vec{C}\cdot\left(\vec{B} \times \vec{C}\right)}_{0} &= \vec{D}\cdot\left(\vec{B} \times \vec{C}\right) \\ a &= \frac{\vec{D}\cdot\left(\vec{B} \times \vec{C}\right)}{\vec{A}\cdot\left(\vec{B} \times \vec{C}\right)}. \end{aligned}$$

The two zeros followed from the general rules that $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$) and $\vec{A} \times \vec{A} = \vec{0}$. The last line of the calculation assumes that $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) \neq 0$. (The linear algebra savvy reader will recognize this thoughtless manipulation as a derivation of Kramer's rule for 3×3 matrices.) □

The point of the example above was to show the vector algebra rules at work. However, to get to the end took the first 'move' of dotting the equation with the right vector. That move could be motivated this way. We are trying to find a and not b or c. We can get rid of the terms in the equation that contain b and c if we can dot $\vec{B}$ and $\vec{C}$ with a vector perpendicular to both of them. $\vec{B} \times \vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ so can be used to kill them off with a dot product. The 0s in the example calculation were thus expected for geometric reasons.

Count equations and unknowns.

One cannot (usually) find more unknowns than one has scalar equations. Before you do lots of algebra, you should check that you have as many equations as unknowns. If not, you probably can't find all the unknowns. How do you count vector equations and vector unknowns? A two- dimensional vector is fully described by two numbers. For example, a $2D$ vector is described by its x and y components or its magnitude and the angle it makes with the positive x axis. A three-dimensional vector is described by three numbers. So a vector equation counts as 2 or 3 equations in 2 or 3 dimensional problems, respectively. And an unknown vector counts as 2 or 3 unknowns in 2 or 3 dimensions, respectively. If the direction of a vector is known but its magnitude is not, then the magnitude is the only unknown. Magnitude is a scalar, so it counts as one unknown.

Example: **Counting equations**

Say you are doing a 2-D problem where you already know the vector $\hat{\boldsymbol{\lambda}} = \sqrt{2}\hat{\boldsymbol{\imath}} + \sqrt{2}\hat{\boldsymbol{\jmath}}$ and you are given the vector equation

$$C\hat{\boldsymbol{\lambda}} = \vec{a}.$$

You then have two equations (a vector equation in 2-D) and three unknowns (the scalar C and the vector $\vec{a}$). There are more unknowns than equations so this vector equation is not sufficient for finding C and $\vec{a}$. □

Cross products and computers

The components of the cross product can be calculated with computer code that may look something like this.

```
A = [  1  2  5 ]
B = [ -2  4 19 ]
C = [ ( A(2)*B(3) - A(3)*B(2) ) ...
      ( A(3)*B(1) - A(1)*B(3) ) ...
      ( A(1)*B(2) - A(2)*B(1) ) ]
```

giving the result `C=[18  -29  8]`. Many computer languages have a shorter way to write the cross product like `cross(A,B)`. The mixed triple product might be calculated by assembling a 3×3 matrix of rows and then taking a determinant like this:

```
       A   = [  1  2  5 ]
       B   = [ -2  4 19 ]
       C   = [ 32  4  5 ]
    matrix = [A ;  B ; C]
 mixedprod = det(matrix)
```

giving the result `mixedprod = 500`. A versatile language might well allow the command `dot( A, cross(B,C) )` to calculate the mixed triple product.

SAMPLE 2.17 *Cross product in 2-D:* Two vectors $\vec{a}$ and $\vec{b}$ of length 10 ft and 6 ft, respectively, are shown in the figure. The angle between the two vectors is $\theta = 60^o$. Find the cross product of the two vectors.

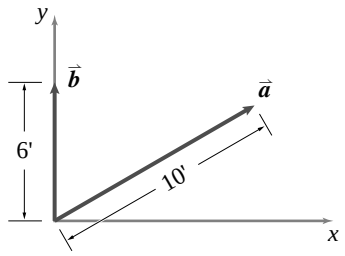


Figure 2.31: (Filename:sfig2.vec2.cross1)

Solution Both vectors $\vec{a}$ and $\vec{b}$ are in the xy plane. Therefore, their cross product is,

$$\begin{aligned}\vec{a}\times\vec{b} &= |\vec{a}||\vec{b}|\sin\theta\hat{n}\\ &= (10\,\text{ft})\cdot(6\,\text{ft})\cdot\sin 60^o\hat{k}\\ &= 60\,\text{ft}^2\cdot\frac{\sqrt{3}}{2}\hat{k}\\ &= 30\sqrt{3}\,\text{ft}^2\hat{k}.\end{aligned}$$

$$\boxed{\vec{a}\times\vec{b} = 30\sqrt{3}\,\text{ft}^2\hat{k}}$$

SAMPLE 2.18 *Computing 2-D cross product in different ways:* The two vectors shown in the figure are $\vec{a} = 2\hat{\imath} - \hat{\jmath}$ and $\vec{b} = 4\hat{\imath} + 2\hat{\jmath}$. The angle between the two vectors is $\theta = \sin^{-1}(4/5)$ (this information can be found out from the given vectors). Find the cross product of the two vectors

(a) using the angle θ, and
(b) using the components of the vectors.

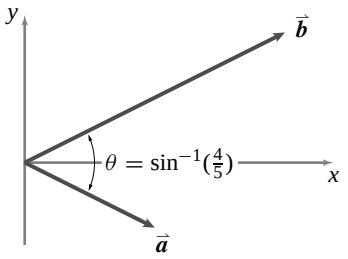


Figure 2.32: (Filename:sfig2.vec2.cross2)

Solution

(a) Cross product using the angle θ:

$$\begin{aligned}\vec{a}\times\vec{b} &= |\vec{a}||\vec{b}|\sin\theta\hat{n}\\ &= |2\hat{\imath}-\hat{\jmath}||4\hat{\imath}+2\hat{\jmath}|\cdot\sin(\sin^{-1}\frac{4}{5})\hat{k}\\ &= (\sqrt{2^2+1^2})(\sqrt{4^2+2^2})\cdot\frac{4}{5}\hat{k}\\ &= \sqrt{5}\cdot\sqrt{20}\cdot\frac{4}{5}\hat{k} = 10\cdot\frac{4}{5}\hat{k}\\ &= 8\hat{k}.\end{aligned}$$

(b) Cross product using components:

$$\begin{aligned}\vec{a}\times\vec{b} &= (2\hat{\imath}-\hat{\jmath})\times(4\hat{\imath}+2\hat{\jmath})\\ &= 2\hat{\imath}\times(4\hat{\imath}+2\hat{\jmath}) - \hat{\jmath}\times(4\hat{\imath}+2\hat{\jmath})\\ &= 8\underbrace{\hat{\imath}\times\hat{\imath}}_{\vec{0}} + 4\underbrace{\hat{\imath}\times\hat{\jmath}}_{\hat{k}} - 4\underbrace{\hat{\jmath}\times\hat{\imath}}_{-\hat{k}} - 2\underbrace{\hat{\jmath}\times\hat{\jmath}}_{\vec{0}}\\ &= 4\hat{k}+4\hat{k}\\ &= 8\hat{k}.\end{aligned}$$

The answers obtained from the two methods are, of course, the same as they must be.

$$\boxed{\vec{a}\times\vec{b} = 8\hat{k}}$$

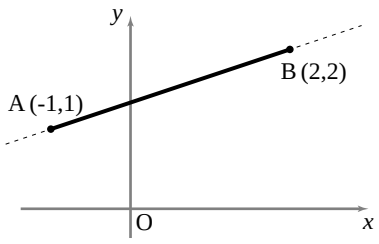


Figure 2.33: (Filename:sfig2.vec2.perp2D)

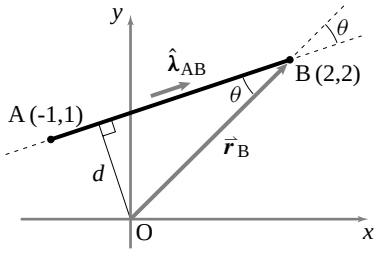


Figure 2.34: (Filename:sfig2.vec2.perp2Da)

SAMPLE 2.19 *Finding the minimum distance from a point to a line:* A straight line passes through two points, A (-1,4) and B (2,2), in the xy plane. Find the shortest distance from the origin to the line.

Solution Let $\hat{\boldsymbol{\lambda}}_{AB}$ be a unit vector along line AB. Then,

$$\hat{\boldsymbol{\lambda}}_{AB} \times \vec{r}_B = \underbrace{|\hat{\boldsymbol{\lambda}}_{AB}|}_{1} |\vec{r}_B| \sin\theta \hat{\boldsymbol{n}} = |\vec{r}_B| \sin\theta \hat{\boldsymbol{k}}.$$

Now $|\vec{r}_B| \sin\theta$ is the component of $\vec{r}_B$ that is perpendicular to $\hat{\boldsymbol{\lambda}}_{AB}$ or line AB, *i.e.*, it is the perpendicular, and hence the shortest, distance from the origin (the root of vector $\vec{r}_B$) to the line AB. Thus, the shortest distance d from the origin to the line AB is computed from,

$$\begin{aligned} d &= |\hat{\boldsymbol{\lambda}}_{AB} \times \vec{r}_B| \\ &= \left| \left(\frac{3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{3^2+1^2}}\right) \times (2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) \right| = \left| \frac{6}{\sqrt{10}}\hat{\boldsymbol{k}} - \frac{2}{\sqrt{10}}\hat{\boldsymbol{k}} \right| = \left| \frac{4}{\sqrt{10}}\hat{\boldsymbol{k}} \right| \\ &= \frac{4}{\sqrt{10}}. \end{aligned}$$

$$\boxed{d = 4/\sqrt{10}}$$

Comments: In this calculation, $\vec{r}_B$ is an arbitrary vector from the origin to some point on line AB. You can take any convenient vector. Since the shortest distance is unique, any such vector will give you the same answer. In fact, you can check your answer by selecting another vector and repeating the calculations, *e.g.*, vector $\vec{r}_A$.

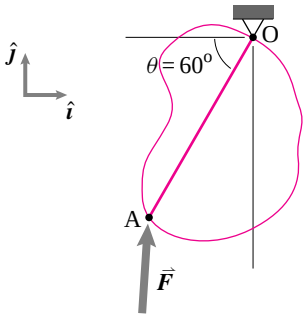


Figure 2.35: (Filename:sfig2.vec2.mom1)

SAMPLE 2.20 *Moment of a force:* Find the moment of force $\vec{F} = 1\,\text{N}\hat{\boldsymbol{\imath}} + 20\,\text{N}\hat{\boldsymbol{\jmath}}$ shown in the figure about point O.

Solution The force acts through point A on the body. Therefore, we can compute its moment about O as follows.

$$\begin{aligned} \vec{M}_O &= \vec{r}_{OA} \times \vec{F} \\ &= \underbrace{(-2\,\text{m} \cdot \cos 60^o \hat{\boldsymbol{\imath}} - 2\,\text{m} \cdot \sin 60^o \hat{\boldsymbol{\jmath}})}_{\vec{r}_{OA}} \times \underbrace{(1\,\text{N}\hat{\boldsymbol{\imath}} + 20\,\text{N}\hat{\boldsymbol{\jmath}})}_{\vec{F}} \\ &= (-1\,\text{m}\hat{\boldsymbol{\imath}} - \sqrt{3}\,\text{m}\hat{\boldsymbol{\jmath}}) \times (1\,\text{N}\hat{\boldsymbol{\imath}} + 20\,\text{N}\hat{\boldsymbol{\jmath}}) \\ &= -20\,\text{N·m}\hat{\boldsymbol{k}} + 1.732\,\text{N·m}\hat{\boldsymbol{k}} \\ &= -18.268\,\text{N·m}\hat{\boldsymbol{k}}. \end{aligned}$$

$$\boxed{\vec{M}_O = -18.268\,\text{N·m}\hat{\boldsymbol{k}}}$$

SAMPLE 2.21 A 2 m square plate hangs from one of its corners as shown in the figure. At the diagonally opposite end, a force of 50 N is applied by pulling on the string AB. Find the moment of the applied force about the center C of the plate.

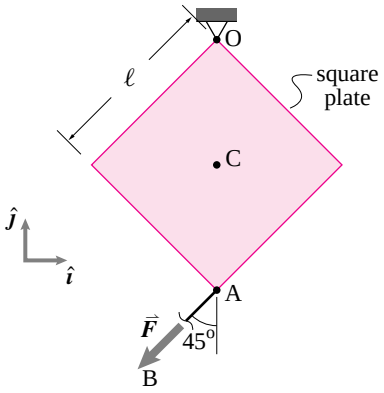


Figure 2.36: (Filename:sfig2.vec2.mom2)

Solution The moment of $\vec{F}$ about point C is

$$\vec{M}_C = \vec{r}_{A/C} \times \vec{F}.$$

So, to compute $\vec{M}_C$, we need to find the vectors $\vec{r}_{A/C}$ and $\vec{F}$.

$$\begin{aligned} \vec{r}_{A/C} &= -CA\hat{j} = -\frac{\ell}{\sqrt{2}}\hat{j} \quad (\text{since OA} = 2\ \text{CA} = \sqrt{2}\ell) \\ \vec{F} &= F(-\cos\theta\hat{i} - \sin\theta\hat{j}) = -F(\cos\theta\hat{i} + \sin\theta\hat{j}) \end{aligned}$$

Hence,

$$\begin{aligned} \vec{M}_C &= -\frac{\ell}{\sqrt{2}}\hat{j} \times [-F(\cos\theta\hat{i} + \sin\theta\hat{j})] \\ &= \frac{\ell}{\sqrt{2}}F(\cos\theta \underbrace{\hat{j} \times \hat{i}}_{-\hat{k}} + \sin\theta \underbrace{\hat{j} \times \hat{j}}_{\vec{0}}) \\ &= -\frac{\ell}{\sqrt{2}}F\cos\theta\hat{k} \\ &= -\frac{2\,\text{m}}{\sqrt{2}} \cdot 50\,\text{N} \cdot \cos 45^\circ\hat{k} = -50\,\text{N·m}\hat{k}. \end{aligned}$$

$$\boxed{\vec{M}_C = -50\,\text{N·m}\hat{k}}$$

SAMPLE 2.22 *Calculation of cross products:* Compute $\vec{a} \times \vec{b}$, where $\vec{a} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = 3\hat{\imath} + -4\hat{\jmath} + \hat{k}$.

Solution The calculation of a cross product between two 3-D vectors can be carried out by either using a determinant or the distributive rule. Usually, if the vectors involved have just one or two components, it is easier to use the distributive rule. We show you both methods here and encourage you to learn both. We are given two vectors:

$$\begin{aligned} \vec{a} &= a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = \hat{\imath} + \hat{\jmath} - 2\hat{k}, \\ \vec{b} &= b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k} = 3\hat{\imath} + -4\hat{\jmath} + \hat{k}. \end{aligned}$$

- **Calculation using the determinant formula:** In this method, we first write a 3×3 matrix whose first row has the basis vectors as its elements, the second row has the components of the first vector as its elements, and the third row has the components of the second vector as its elements. Thus,

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{\imath}(a_2b_3 - a_3b_2) + \hat{\jmath}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \hat{\imath}(1-8) + \hat{\jmath}(-6-1) + \hat{k}(-4-3) \\ &= -7(\hat{\imath} + \hat{\jmath} + \hat{k}). \end{aligned}$$

- **Calculation using the distributive rule:** In this method, we carry out the cross product by distributing the cross product properly over the three basis vectors. The steps involved are shown below.

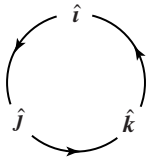


Figure 2.37: The cross product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, e.g. $\hat{\imath} \times \hat{\jmath} = \hat{k}$ but $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.
(Filename:ijkcircle)

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) \\ &= a_1\hat{\imath} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) + \\ &\quad a_2\hat{\jmath} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) + \\ &\quad a_3\hat{k} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) \\ &= a_1b_1(\overbrace{\hat{\imath} \times \hat{\imath}}^{\mathbf{0}}) + a_1b_2(\overbrace{\hat{\imath} \times \hat{\jmath}}^{\hat{k}}) + a_1b_3(\overbrace{\hat{\imath} \times \hat{k}}^{-\hat{\jmath}}) + \\ &\quad a_2b_1(\overbrace{\hat{\jmath} \times \hat{\imath}}^{-\hat{k}}) + a_2b_2(\overbrace{\hat{\jmath} \times \hat{\jmath}}^{\mathbf{0}}) + a_2b_3(\overbrace{\hat{\jmath} \times \hat{k}}^{\hat{\imath}}) + \\ &\quad a_3b_1(\overbrace{\hat{k} \times \hat{\imath}}^{\hat{\jmath}}) + a_3b_2(\overbrace{\hat{k} \times \hat{\jmath}}^{-\hat{\imath}}) + a_3b_3(\overbrace{\hat{k} \times \hat{k}}^{\mathbf{0}}) \\ &= \hat{\imath}(a_2b_3 - a_3b_2) + \hat{\jmath}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \hat{\imath}(1-8) + \hat{\jmath}(-6-1) + \hat{k}(-4-3) \\ &= -7(\hat{\imath} + \hat{\jmath} + \hat{k}) \end{aligned}$$

which, of course, is the same result as obtained above using the determinant. Making a sketch such as Fig. 2.37 is helpful while calculating cross products this way. The product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, *e.g.*, $\hat{\imath} \times \hat{\jmath} = \hat{k}$ but $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.

$$\boxed{\vec{a} \times \vec{b} = -7(\hat{\imath} + \hat{\jmath} + \hat{k})}$$

SAMPLE 2.23 *Cross product:* Find a unit vector perpendicular to the vectors $\vec{r}_A = \hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{r}_b = 3\hat{\jmath} + 2\hat{k}$.

Solution The cross product between two vectors gives a vector perpendicular to the plane formed by the two vectors. The sense of direction is determined by the right hand rule.

Let $\vec{N} = N\hat{\lambda}$ be the perpendicular vector.

$$\begin{aligned}\vec{N} &= \vec{r}_A \times \vec{r}_B \\ &= (\hat{\imath} - 2\hat{\jmath} + \hat{k}) \times (3\hat{\jmath} + 2\hat{k}) \\ &= -7\hat{\imath} - 2\hat{\jmath} + 3\hat{k}.\end{aligned}$$

This calculation can be done in any of the two ways shown in the previous sample problem.

Therefore,

$$\begin{aligned}\hat{\lambda} &= \frac{\vec{N}}{N} \\ &= \frac{-7\hat{\imath} - 2\hat{\jmath} + 3\hat{k}}{\sqrt{7^2 + 2^2 + 3^2}} \\ &= -0.89\hat{\imath} - 0.25\hat{\jmath} + 0.38\hat{k}\end{aligned}$$

$$\boxed{\hat{\lambda} = -0.89\hat{\imath} - 0.25\hat{\jmath} + 0.38\hat{k}}$$

Check:

- $|\hat{\lambda}| = (0.89)^2 + (0.25)^2 + (0.38)^2 \stackrel{\surd}{=} 1$. (it is a unit vector)
- $\hat{\lambda} \cdot \vec{r}_A = 1(-0.89) - 2(-0.25) + 1(0.38) \stackrel{\surd}{=} 0$. $(\hat{\lambda} \perp \vec{r}_A)$.
- $\hat{\lambda} \cdot \vec{r}_B = 3(-0.25) + 2(0.38) \stackrel{\surd}{=} 0$. $(\hat{\lambda} \perp \vec{r}_B)$.

Comments: If $\hat{\lambda}$ is perpendicular to $\vec{r}_{\mathrm{A}}$ and $\vec{r}_{\mathrm{B}}$, then so is $-\hat{\lambda}$. The perpendicularity does not change by changing the *sense* of direction (from positive to negative) of the vector. In fact, if $\hat{\lambda}$ is perpendicular to a vector $\vec{r}$then any scalar multiple of $\hat{\lambda}$, *i.e.*, $\alpha\hat{\lambda}$, is also perpendicular to $\vec{r}$. This follows from the fact that

$$\alpha\hat{\lambda} \cdot \vec{r} = \alpha(\hat{\lambda} \cdot \vec{r}) = \alpha(0) = 0.$$

The case of $-\hat{\lambda}$ is just a particular instance of this rule with $\alpha = -1$.

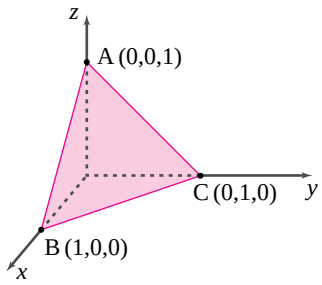


Figure 2.38: (Filename:sfig2.vec2.normal)

SAMPLE 2.24 *Finding a vector normal to a plane:* Find a unit vector normal to the plane ABC shown in the figure.

Solution A vector normal to the plane ABC would be normal to any vector in that plane. In particular, if we take any two vectors, say $\vec{r}_{AB}$ and $\vec{r}_{AC}$, the normal to the plane would be perpendicular to both $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Since the cross product of two vectors gives a vector perpendicular to both vectors, we can find the desired normal vector by taking the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Thus,

$$\begin{aligned}
\vec{N} &= \vec{r}_{AB} \times \vec{r}_{AC} \\
&= (\hat{\imath} - \hat{k}) \times (\hat{\jmath} - \hat{k}) \\
&= \underbrace{\hat{\imath} \times \hat{\jmath}}_{\hat{k}} - \underbrace{\hat{\imath} \times \hat{k}}_{-\hat{\jmath}} - \underbrace{\hat{k} \times \hat{\jmath}}_{-\hat{\imath}} + \underbrace{\hat{k} \times \hat{k}}_{\vec{0}} \\
&= \hat{\imath} + \hat{\jmath} + \hat{k} \\
\Rightarrow \quad \hat{n} &= \frac{\vec{N}}{|\vec{N}|} \\
&= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}).
\end{aligned}$$

$$\boxed{\hat{n} = \tfrac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})}$$

Check: Now let us check if $\hat{n}$ is normal to any vector in the plane ABC. It is fairly easy to show that $\hat{n} \cdot \vec{r}_{AB} = \hat{n} \cdot \vec{r}_{AC} = 0$. It is, however, not a surprise; it better be since we found $\hat{n}$ from the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Let us check if $\hat{n}$ is normal to $\vec{r}_{BC}$:

$$\begin{aligned}
\hat{n} \cdot \vec{r}_{BC} &= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot (-\hat{\imath} + \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-\hat{\imath} \cdot \hat{\imath} + \hat{\jmath} \cdot \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-1 + 1) \overset{\surd}{=} 0.
\end{aligned}$$

SAMPLE 2.25 *The shortest distance between two lines:* Two lines, AB and CD, in 3-D space are defined by four specified points, A(0,2 m,1 m), B(2 m,1 m,3 m), C(-1 m,0,0), and D(2 m,2 m,2 m) (see Fig. **??**). Find the shortest distance between the two lines.

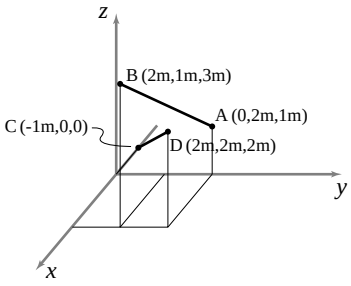


Figure 2.39: (Filename:sfig2.vec2.perp3D)

Solution The shortest distance between any pair of lines is the length of the line that is perpendicular to both the lines. We can find the shortest distance in three steps:

(a) First find a vector that is perpendicular to both the lines. This is easy. Take two vectors $\vec{r}_1$ and $\vec{r}_2$, one along each of the two given lines. Take the cross product of the two unit vectors and make the resulting vector a unit vector $\hat{n}$.

(b) Find a vector that parallel to $\hat{n}$ that connects the two lines. This is a little tricky. We don't know where to start on any of the two lines. However, we can take any vector from one line to the other and then, take its component along $\hat{n}$.

(c) Find the length (magnitude) of the vector just found (in the direction of $\hat{n}$). This is simply the component we find in step (b) devoid of its sign.

Now let us carry out these steps on the given problem.

(a) Step-1: Find a unit vector $\hat{n}$ that is perpendicular to both the lines.

$$\begin{aligned}
\vec{r}_{\rm AB} &= 2\,{\rm m}\hat{\imath} - 1\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k} \\
\vec{r}_{\rm CD} &= 3\,{\rm m}\hat{\imath} + 2\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k} \\
\Rightarrow \quad \vec{r}_{\rm AB} \times \vec{r}_{\rm CD} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} {\rm m}^2 \\
&= \hat{\imath}(-2-4)\,{\rm m}^2 + \hat{\jmath}(6-4)\,{\rm m}^2 + \hat{k}(4+3)\,{\rm m}^2 \\
&= (-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k})\,{\rm m}^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{n} &= \frac{\vec{r}_{\rm AB} \times \vec{r}_{\rm CD}}{|\vec{r}_{\rm AB} \times \vec{r}_{\rm CD}|} \\
&= \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k}).
\end{aligned}$$

(b) Step-2: Find any vector from one line to the other line and find its component along $\hat{n}$.

$$\begin{aligned}
\vec{r}_{\rm AC} &= -1\,{\rm m}\hat{\imath} - 2\,{\rm m}\hat{\jmath} - 1\,{\rm m}\hat{k} \\
\vec{r}_{\rm AC} \cdot \hat{n} &= -(\hat{\imath} + 2\hat{\jmath} + \hat{k})\,{\rm m} \cdot \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k}) \\
&= \frac{1}{\sqrt{89}}(6-4-7)\,{\rm m} = -\frac{5}{\sqrt{89}}\,{\rm m}.
\end{aligned}$$

(c) Step-3: Find the required distance d by taking the magnitude of the component along $\hat{n}$.

$$d = |-\frac{5}{\sqrt{89}}\,{\rm m}| = 0.53\,{\rm m}$$

$$\boxed{d = 0.53\,{\rm m}}$$

SAMPLE 2.26 *The mixed triple product:* Calculate the mixed triple product $\hat{\boldsymbol{\lambda}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})$ for $\hat{\boldsymbol{\lambda}}=\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}})$, $\vec{\boldsymbol{a}}=3\hat{\boldsymbol{\imath}}$, and $\vec{\boldsymbol{b}}=\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+3\hat{\boldsymbol{k}}$.

Solution We compute the given mixed triple product in two ways here:

- Method-1: Straight calculation using cross product and dot product.

$$\begin{aligned} \text{Let} \quad \vec{\boldsymbol{c}} &= \vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}} \\ &= (3\hat{\boldsymbol{\imath}})\times(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+3\hat{\boldsymbol{k}}) \\ &= 3\underbrace{\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\imath}}}_{\vec{\mathbf{0}}}+3\underbrace{\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}}}_{\hat{k}}+9\underbrace{\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{k}}}_{-\hat{\jmath}}=-9\hat{\boldsymbol{\jmath}}+3\hat{\boldsymbol{k}} \\ \text{So,} \quad \hat{\boldsymbol{\lambda}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}) &= \hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{c}} \\ &= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}})\cdot(-9\hat{\boldsymbol{\jmath}}+3\hat{\boldsymbol{k}})=-\frac{9}{\sqrt{2}}. \end{aligned}$$

- Method-2: Using the determinant formula for mixed product.

$$\begin{aligned} \hat{\boldsymbol{\lambda}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}) &= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 3 \end{vmatrix} \\ &= \frac{1}{\sqrt{2}}(0-0)+\frac{1}{\sqrt{2}}(0-9)+0=-\frac{9}{\sqrt{2}}. \end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})=-\tfrac{9}{\sqrt{2}}}$$

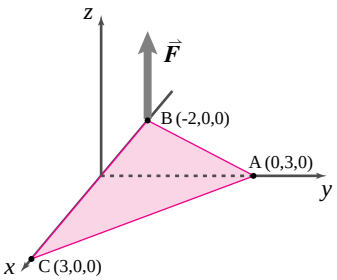


Figure 2.40: (Filename:sfig2.vec2.momxx)

SAMPLE 2.27 *Moment about an axis:* A vertical force of unknown magnitude F acts at point B of a triangular plate ABC shown in the figure. Find the moment of the force about edge CA of the plate.

Solution The moment of a force $\vec{\boldsymbol{F}}$ about an axis x-x is given by

$$M_{xx}=\hat{\boldsymbol{\lambda}}_{\text{xx}}\cdot(\vec{\boldsymbol{r}}\times\vec{\boldsymbol{F}})$$

where $\hat{\boldsymbol{\lambda}}_{\text{xx}}$ is a unit vector along the axis x-x, $\vec{\boldsymbol{r}}$ is a position vector from any point on the axis to the applied force. In this problem, the given axis is CA. Therefore, we can take $\vec{\boldsymbol{r}}$ to be $\vec{\boldsymbol{r}}_{\text{AB}}$ or $\vec{\boldsymbol{r}}_{\text{CB}}$. Here,

$$\hat{\boldsymbol{\lambda}}_{\text{CA}}=\frac{\vec{\boldsymbol{r}}_{\text{CA}}}{|\vec{\boldsymbol{r}}_{\text{CA}}|}=\frac{3(-\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}})}{\sqrt{9+9}}=-\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}}+\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}.$$

Now, moment about point A is

$$\begin{aligned} \vec{\boldsymbol{M}}_{\text{A}} &= \vec{\boldsymbol{r}}_{\text{AB}}\times\vec{\boldsymbol{F}} \\ &= (-2\hat{\boldsymbol{\imath}}-3\hat{\boldsymbol{\jmath}})\times F\hat{\boldsymbol{k}}=2F\hat{\boldsymbol{\jmath}}-3F\hat{\boldsymbol{\imath}}. \end{aligned}$$

Therefore, the moment about CA is

$$\begin{aligned} M_{CA} &= \hat{\boldsymbol{\lambda}}_{\text{CA}}\cdot(\vec{\boldsymbol{r}}_{\text{AB}}\times\vec{\boldsymbol{F}})=\hat{\boldsymbol{\lambda}}_{\text{CA}}\cdot\vec{\boldsymbol{M}}_{\text{A}} \\ &= (-\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}}+\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}})\cdot(-3F\hat{\boldsymbol{\imath}}+2F\hat{\boldsymbol{\jmath}}) \\ &= (\frac{3}{\sqrt{2}}+\frac{2}{\sqrt{2}})F=\frac{5}{\sqrt{2}}F. \end{aligned}$$

$$\boxed{M_{CA}=\tfrac{5}{\sqrt{2}}F}$$

2.4 Equivalent force systems

Most often one does not want to know the complete details of all the forces acting on a system. When you think of the force of the ground on your bare foot you do not think of the thousands of little forces at each micro-asperity or the billions and billions of molecular interactions between the wood (say) and your skin. Instead you think of some kind of equivalent force. In what way equivalent? Well, because all that the equations of mechanics know about forces is their net force and net moment, you have a criterion. You replace the actual force system with a simpler force system, possibly just a single well-placed force, that has the same total force and same total moment with respect to a reference point C.

The replacement of one system with an equivalent system is often used to help simplify or solve mechanics problems. Further, the concept of equivalent force systems allows us to define a *couple*, a concept we will use throughout the book. Here is the definition of the word *equivalent*① when applied to force systems in mechanics.

① Other phrases used to describe the same concept in other books include: *statically equivalent*, *mechanically equivalent*, and *equipollent*.

> Two force systems are said to be *equivalent* if they have the same sum (the same *resultant*) and the same net moment about some point C.

We have already discussed two important cases of equivalent force systems. On page 11 we stated the mechanics assumption that a set of forces applied at one point is equivalent to a single resultant force, their sum, applied at that point. Thus when doing a mechanics analysis you can replace a collection of forces at a point with their sum. If you think of your whole foot as a 'point' this justifies the replacement of the billions of little atomic ground contact forces with a single force.

On page 35 we discovered that a force applied at a different point is equivalent to the same force applied at a point displaced in the direction of the force. You can thus harmlessly move the point of force application along the line of the force.

More generally, we can compare two sets of forces. The first set consists of $\vec{F}_1^{(1)}$, $\vec{F}_2^{(1)}$, $\vec{F}_3^{(1)}$, *etc*. applied at positions $\vec{r}_{1/C}^{(1)}$, $\vec{r}_{2/C}^{(1)}$, $\vec{r}_{3/C}^{(1)}$, *etc*. In short hand, the first set are the forces $\vec{F}_i^{(1)}$ applied at positions $\vec{r}_{i/C}^{(1)}$, where each value of i describes a different force ($i = 7$ refers to the seventh force in the set). The second set of forces consists of $\vec{F}_j^{(2)}$ applied at positions $\vec{r}_{j/C}^{(2)}$ where each value of j describes a different force in the second set.

Now we compare the net (resultant) force and net moment of the two sets. If

$$\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)} \qquad \text{and} \qquad \vec{M}_{\text{C}}^{(1)} = \vec{M}_{\text{C}}^{(2)} \tag{2.11}$$

then the two sets are *equivalent*. Here we have defined the net forces and net moments by

$$\begin{aligned} \vec{F}_{\text{tot}}^{(1)} &= \sum_{\text{all forces } i} \vec{F}_i^{(1)}, & \vec{M}_{\text{C}}^{(1)} &= \sum_{\text{all forces } i} \vec{r}_{i/C}^{(1)} \times \vec{F}_i^{(1)}, \\ \vec{F}_{\text{tot}}^{(2)} &= \sum_{\text{all forces } j} \vec{F}_j^{(2)}, \quad \text{and} & \vec{M}_{\text{C}}^{(2)} &= \sum_{\text{all forces } j} \vec{r}_{j/C}^{(2)} \times \vec{F}_j^{(2)}. \end{aligned}$$

If you find the $\sum$ (sum) symbol intimidating see box 2.4 on page 54.

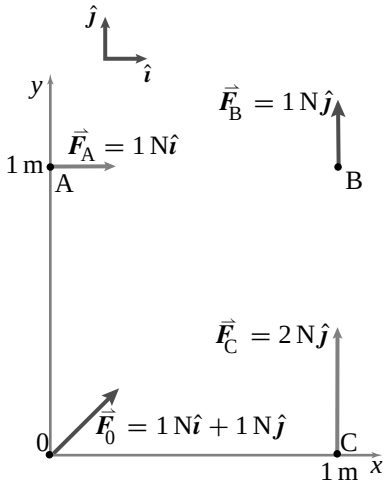


Figure 2.41: The force system $\vec{F}_A$, $\vec{F}_C$ is equivalent to the force system $\vec{F}_0$, $\vec{F}_B$.

(Filename:tfigure.equivforcepair)

Example:

Consider force system (1) with forces $\vec{F}_A$ and $\vec{F}_C$ and force system (2) with forces $\vec{F}_0$ and $\vec{F}_B$ as shown in fig. 2.41. Are the systems equivalent? First check the sum of forces.

$$\begin{aligned}
\vec{F}^{(1)}_{\text{tot}} &\stackrel{?}{=} \vec{F}^{(2)}_{\text{tot}} \\
\sum \vec{F}^{(1)}_i &\stackrel{?}{=} \sum \vec{F}^{(2)}_j \\
\vec{F}_A + \vec{F}_C &\stackrel{?}{=} \vec{F}_0 + \vec{F}_B \\
1\,\mathrm{N}\hat{i} + 2\,\mathrm{N}\hat{j} &\stackrel{\surd}{=} (1\,\mathrm{N}\hat{i} + 1\,\mathrm{N}\hat{j}) + 1\,\mathrm{N}\hat{j}
\end{aligned}$$

Then check the sum of moments about C.

$$\begin{aligned}
\vec{M}^{(1)}_C &\stackrel{?}{=} \vec{M}^{(2)}_C \\
\sum \vec{r}^{(1)}_{i/C} \times \vec{F}^{(1)}_i &\stackrel{?}{=} \sum \vec{r}^{(2)}_{j/C} \times \vec{F}^{(2)}_j \\
\vec{r}_{A/C} \times \vec{F}_A + \vec{r}_{C/C} \times \vec{F}_C &\stackrel{?}{=} \vec{r}_{0/C} \times \vec{F}_0 + \vec{r}_{B/C} \times \vec{F}_B \\
(-1\,\mathrm{m}\hat{i} + 1\,\mathrm{m}\hat{j}) \times 1\,\mathrm{N}\hat{i} + \vec{0} \times 2\,\mathrm{N}\hat{j} &\stackrel{?}{=} (-1\,\mathrm{m}\hat{i}) \times (1\,\mathrm{N}\hat{i} + 1\,\mathrm{N}\hat{j}) + 1\,\mathrm{m}\hat{j} \times 1\,\mathrm{N}\hat{j} \\
-1\,\mathrm{m\,N}\hat{k} &\stackrel{\surd}{=} -1\,\mathrm{m\,N}\hat{k}
\end{aligned}$$

So the two force systems are indeed equivalent. □

What is so special about the point C in the example above? Nothing.

2.6 $\sum$ means add

In mechanics we often need to add up lots of things: all the forces on a body, all the moments they cause, all the mass of a system, *etc*. One notation for adding up all 14 forces on some body is

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6 + \vec{F}_7 \\
&\quad + \vec{F}_8 + \vec{F}_9 + \vec{F}_{10} + \vec{F}_{11} + \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}.
\end{aligned}$$

which is a bit long, so we might abbreviate it as

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_{14}.$$

But this is definition by pattern recognition. A more explicit statement would be

$$\vec{F}_{\text{net}} = \text{The sum of all 14 forces } \vec{F}_{\text{i}} \text{ where } i = 1 \ldots 14$$

which is too space consuming. This kind of summing is so important that mathematicians use up a whole letter of the greek alphabet as a short hand for 'the sum of all'. They use the capital greek 'S' (for Sum) called sigma which looks like this:

$$\sum.$$

When you read $\sum$ aloud you don't say 'S' or'sigma' but rather 'the sum of.' The $\sum$ (sum) notation may remind you of infinite series, and convergence thereof. We will rarely be concerned with infinite sums in this book and never with convergence issues. So panic on those grounds is unjustified. We just want to easily write about adding things. For example we use the $\sum$ (sum) to write the sum of 14 forces $\vec{F}_{\text{i}}$ explicitly and concisely as

$$\sum_{i=1}^{14} \vec{F}_{\text{i}}$$

and say 'the sum of F sub i where i goes from one to fourteen'. Sometimes we don't know, say, how many forces are being added. We just want to add all of them so we write (a little informally)

$$\sum \vec{F}_{\text{i}} \quad \text{meaning} \quad \vec{F}_1 + \vec{F}_2 + etc.,$$

where the subscript i lets us know that the forces are numbered.

Rather than panic when you see something like $\sum_{i=1}^{14}$, just relax and think: oh, we want to add up a bunch of things all of which look like the next thing written. In general,

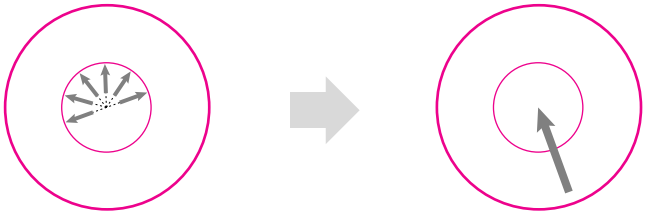
$$\sum (\text{thing})_i \quad \text{translates to} \quad (\text{thing})_1 + (\text{thing})_2 + (\text{thing})_3 + etc.$$

no matter how intimidating the 'thing' is. In time you can skip writing out the translation and will enjoy the concise notation.

See box 2.5 for a similar discussion of integration ($\int$) and addition.

If two force systems are equivalent with respect to some point C, they are equivalent with respect to *any* point.

For example, both of the force systems in the example above have the same moment of $2\,\mathrm{N\,m}\hat{\boldsymbol{k}}$ about the point A. See box 2.4 for the proof of the general case.



Example: **Frictionless wheel bearing**

If the contact of an axle with a bearing housing is perfectly frictionless then each of the contact forces has no moment about the center of the wheel. Thus the whole force system is equivalent to a single force at the center of the wheel. □

Couples

Consider a pair of equal and opposite forces that are not colinear. Such a pair is called a *couple*.① The net moment caused by a couple is the size of the force times the distance between the two lines of action and doesn't depend on the reference point. In fact, any force system that has $\vec{\boldsymbol{F}}_{\text{tot}} = \vec{\boldsymbol{0}}$ causes the same moment about all different reference points (as shown at the end of box 2.4). So, in modern usage, any force system with any number of forces and with $\vec{\boldsymbol{F}}_{\text{tot}} = \vec{\boldsymbol{0}}$ is called a couple. A couple is described by its net moment.②

A *couple* is any force system that has a total force of $\vec{\boldsymbol{0}}$. It is described by the net moment $\vec{\boldsymbol{M}}$ that it causes.

We then think of $\vec{\boldsymbol{M}}$ as representing an equivalent force system that contributes $\vec{\boldsymbol{0}}$ to the net force and $\vec{\boldsymbol{M}}$ to the net moment with respect to every reference point.

The concept of a couple (also called an applied moment or an applied torque) is especially useful for representing the net effect of a complicated collection of forces that causes some turning. The complicated set of electromagnetic forces turning a motor shaft can be replaced by a couple.

① **Caution:** Just because a collection of forces adds to zero doesn't mean the net moment they cause adds to zero.

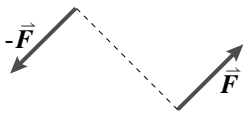


Figure 2.42: One couple. The forces add to zero. Then net moment they cause does not.

(Filename:tfigure.onecouple)

② People who have been in difficult long term relationships don't need a mechanics text to know that a couple is a pair of equal and opposite forces that push each other round and round.

Every system of forces is equivalent to a force and a couple

Given any point C, we can calculate the net moment of a system of forces relative to C. We then can replace the sum of forces with a single force at C and the net moment with a couple at C and we have an equivalent force system.

A force system is equivalent to a force $\vec{F} = \vec{F}_{\text{tot}}$ acting at C and a couple M equal to the net moment of the forces about C, *i.e.*, $\vec{M} = \vec{M}_{\text{C}}$.

If instead we want a force system at D we could recalculate the net moment about D or just use the translation formula (see box 2.4).

$$\begin{aligned} \vec{F}_{\text{tot}} &= \vec{F}_{\text{tot}}, \quad \text{and} \\ \vec{M}_{\text{D}} &= \vec{M}_{\text{C}} + \vec{r}_{\text{C/D}} \times \vec{F}_{\text{tot}}. \end{aligned}$$

stays the same and the moment at D is the moment at C plus the moment caused by $\vec{F}_{\text{net}}$ acting at position C relative to D. The net effect of the forces of the ground on a tree, for example, is of a force and a couple acting on the base of the tree.

2.7 *THEORY*

Two force systems that are equivalent for one reference point are equivalent for all reference points.

Consider two sets of forces $\vec{F}_i^{(1)}$ and $\vec{F}_j^{(2)}$ with corresponding points of application $\text{P}_i^{(1)}$ and $\text{P}_j^{(2)}$ at positions relative to the origin of $\vec{r}_i^{(1)}$ and $\vec{r}_j^{(2)}$. To simplify the discussion let's define the net forces of the two systems as

$$\vec{F}_{\text{tot}}^{(1)} \equiv \sum \vec{F}_i^{(1)} \quad \text{and} \quad \vec{F}_{\text{tot}}^{(2)} \equiv \sum \vec{F}_j^{(2)},$$

and the net moments about the origin as

$$\vec{M}_0^{(1)} \equiv \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} \quad \text{and} \quad \vec{M}_0^{(2)} \equiv \sum \vec{r}_j^{(2)} \times \vec{F}_j^{(2)}.$$

Using point 0 as a reference, the statement that the two systems are equivalent is then $\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)}$ and $\vec{M}_0^{(1)} = \vec{M}_0^{(2)}$. Now consider point C with position $\vec{r}_{\text{C}} = \vec{r}_{\text{C/0}} = -\vec{r}_{\text{0/C}}$. What is the net moment of force system (1) about point C?

$$\begin{aligned} \vec{M}_{\text{C}}^{(1)} &\equiv \sum \vec{r}_{i/\text{C}}^{(1)} \times \vec{F}_i^{(1)} \\ &= \sum \left(\vec{r}_i^{(1)} - \vec{r}_{\text{C}} \right) \times \vec{F}_i^{(1)} \\ &= \sum \left(\vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \vec{r}_{\text{C}} \times \vec{F}_i^{(1)} \right) \\ &= \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \sum \vec{r}_{\text{C}} \times \vec{F}_i^{(1)} \\ &= \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \vec{r}_{\text{C}} \times \left(\sum \vec{F}_i^{(1)} \right) \\ &= \vec{M}_0^{(1)} - \vec{r}_{\text{C}} \times \vec{F}_{\text{tot}}^{(1)}. \end{aligned}$$

$$= \vec{M}_0^{(1)} + \vec{r}_{\text{0/C}} \times \vec{F}_{\text{tot}}^{(1)}.$$

[**Aside.** The calculation above uses the 'move' of factoring a constant out of a sum. This mathematical move will be used again and again in the developement of the theory of mechanics.]

Similarly, for force system (2)

$$\vec{M}_{\text{C}}^{(2)} = \vec{M}_0^{(2)} + \vec{r}_{\text{0/C}} \times \vec{F}_{\text{tot}}^{(2)}.$$

If the two force systems are equivalent for reference point 0 then $\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)}$ and $\vec{M}_0^{(1)} = \vec{M}_0^{(2)}$ and the expressions above imply that $\vec{M}_{\text{C}}^{(1)} = \vec{M}_{\text{C}}^{(2)}$. Because we specified nothing special about the point C, the systems are equivalent for *any* reference point. Thus, to demonstrate equivalence we need to use a reference point, but once equivalence is demonstrated we need not name the point since the equivalence holds for all points.

By the same reasoning we find that once we know the net force and net moment of a force system ($\vec{F}_{\text{tot}}$) relative to some point C (call it $\vec{M}_{\text{C}}$), we know the net moment relative to point D as

$$\vec{M}_{\text{D}} = \vec{M}_{\text{C}} + \vec{r}_{\text{C/D}} \times \vec{F}_{\text{tot}}.$$

Note that if the net force is $\vec{0}$ (and the force system is then called a couple) that $\vec{M}_{\text{D}} = \vec{M}_{\text{C}}$ so the net moment is the same for all reference points.

The tidiest representation of a force system: a “wrench”

Any force system can be represented by an equivalent force and a couple at any point. But force systems can be reduced to simpler forms. That this is so is of more theoretical than practical import. We state the results here without proof.

In 2D one of these two things is true:

- The system is equivalent to a couple, or
- There is a line of points for which the system can be described by an equivalent force with no couple.

In 3D one of these two things is true:

- The system is equivalent to a couple, or
- There is a line of points for which the system can be reduced to a force and a couple where the force, couple, and line are all parallel. The representation of the system of forces as a force and a parallel moment is called a *wrench*.

Equivalent does not mean equivalent for all purposes

We have perhaps oversimplified.

Imagine you stayed up late studying and overslept. Your roommate was not so diligent; woke up on time and went to wake you by gently shaking you. Having read this chapter so far and no further, and being rather literal, your roommate gets down on the floor and presses on the linoleum underneath your bed applying a force that is *equivalent* to pressing on you. Obviously this is not equivalent in the ordinary sense of the word. It isn’t even equivalent in all of its mechanics effects. One force moves you even if you don’t wake up, and the other doesn’t.

Any two force systems that are ‘equivalent’ but different *do* have different mechanical effects. So, in what sense are two force systems that have the same net force and the same net moment really equivalent? They are equivalent in their contributions to the equations of mechanics (equations 0-II on the inside cover) for any system to which they are both applied. But full mechanical analysis of a situation requires looking at the mechanics equations of many subsystems. In the mechanics equations for each subsystem, two ‘equivalent’ force systems are equivalent if they are both applied to that subsystem.

For the analysis of the subsystem that is you sleeping, the force of your roommate’s hand on the floor isn’t applied to you, so doesn’t show up in the mechanics equations for you, and doesn’t have the same effect as a force on you.

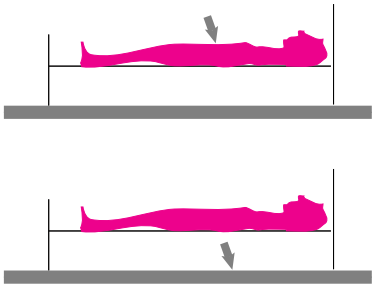


Figure 2.43: It feels different if someone presses on you or presses on the floor underneath you with an ‘equivalent’ force. The equivalence of ‘equivalent’ force systems depends on them both being applied to the same system.

(Filename:tfigure.inbed)

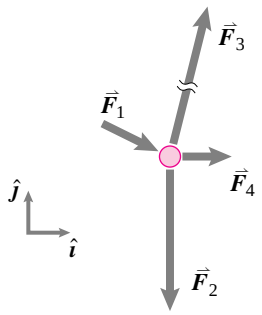


Figure 2.44: (Filename:sfig2.vec3.particle)

SAMPLE 2.28 *Equivalent force on a particle:* Four forces $\vec{F}_1 = 2\,\text{N}\hat{\imath} - 1\,\text{N}\hat{\jmath}$, $\vec{F}_2 = -5\,\text{N}\hat{\jmath}$, $\vec{F}_3 = 3\,\text{N}\hat{\imath} + 12\,\text{N}\hat{\jmath}$, and $\vec{F}_4 = 1\,\text{N}\hat{\imath}$ act on a particle. Find the equivalent force on the particle.

Solution The equivalent force on the particle is the net force, *i.e.*, the vector sum of all forces acting on the particle. Thus,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 \\ &= (2\,\text{N}\hat{\imath} - 1\,\text{N}\hat{\jmath}) + (-5\,\text{N}\hat{\jmath}) + (3\,\text{N}\hat{\imath} + 12\,\text{N}\hat{\jmath}) + (1\,\text{N}\hat{\imath}) \\ &= 6\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath}.\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = 6\,\text{N}(\hat{\imath} + \hat{\jmath})}$$

Note that there is no net couple since all the four forces act at the same point. This is always true for particles. Thus, the equivalent force-couple system for particles consists of only the net force.

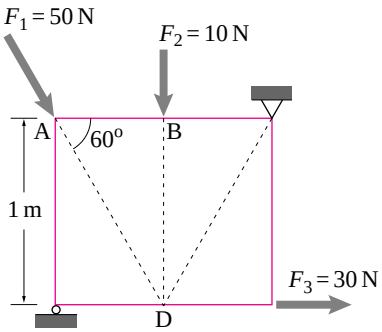


Figure 2.45: (Filename:sfig2.vec3.plate)

SAMPLE 2.29 *Equivalent force with no net moment:* In the figure shown, $F_1 = 50\,\text{N}$, $F_2 = 10\,\text{N}$, $F_3 = 30\,\text{N}$, and $\theta = 60^\text{o}$. Find the equivalent force-couple system about point D of the structure.

Solution From the given geometry, we see that the three forces $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ pass through point D. Thus they are *concurrent* forces. Since point D is on the line of action of these forces, we can simply slide the three forces to point D without altering their mechanical effect on the structure. Then the equivalent force-couple system at point D consists of only the net force, $\vec{F}_{\text{net}}$, with no couple (the three forces passing through point D produce no moment about D). This is true for all concurrent forces. Thus,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= F_1(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) - F_2\hat{\jmath} + F_3\hat{\imath} \\ &= (F_1\cos\theta + F_3)\hat{\imath} - (F_1\sin\theta + F_2)\hat{\jmath} \\ &= (50\,\text{N}\cdot\frac{1}{2} + 30\,\text{N})\hat{\imath} - (50\,\text{N}\cdot\frac{\sqrt{3}}{2} + 10\,\text{N})\hat{\jmath} \\ &= 50\,\text{N}\hat{\imath} - 53.3\,\text{N}\hat{\jmath}, \\ \text{and}\qquad \vec{M}_\text{D} &= \vec{0}.\end{aligned}$$

Graphically, the solution is shown in Fig. 2.46

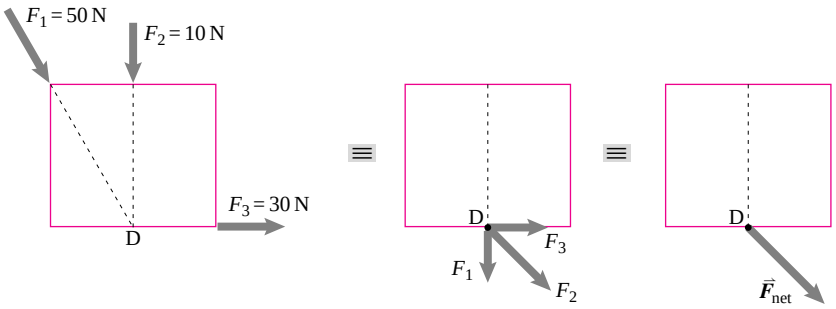


Figure 2.46: (Filename:sfig2.vec3.plate.a)

$$\boxed{\vec{F}_{\text{net}} = 50\,\text{N}\hat{\imath} - 53.3\,\text{N}\hat{\jmath},\ \vec{M}_\text{D} = \vec{0}}$$

SAMPLE 2.30 *An equivalent force-couple system:* Three forces $F_1 = 100\,\text{N}$, $F_2 = 50\,\text{N}$, and $F_3 = 30\,\text{N}$ act on a structure as shown in the figure where $\alpha = 30^\text{o}$, $\theta = 60^\text{o}$, $\ell = 1\,\text{m}$ and $h = 0.5\,\text{m}$. Find the equivalent force-couple system about point D.

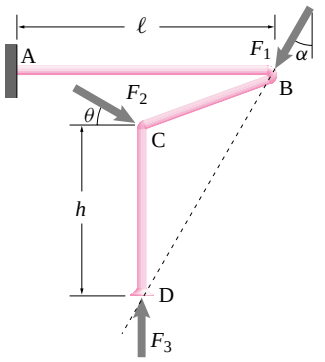


Figure 2.47: (Filename:sfig2.vec3.bar)

Solution The net force is the sum of all applied forces, *i.e.*,

$$\begin{aligned}
\vec{F}_\text{net} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\
&= F_1(-\sin\alpha\hat{\imath} - \cos\alpha\hat{\jmath}) + F_2(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) + F_3\hat{\imath} \\
&= (-F_1\sin\alpha + F_2\cos\theta)\hat{\imath} + (-F_1\cos\alpha - F_2\sin\theta + F_3)\hat{\jmath} \\
&= (-100\,\text{N}\cdot\frac{1}{2} + 50\,\text{N}\cdot\frac{1}{2})\hat{\imath} + (-100\,\text{N}\cdot\frac{\sqrt{3}}{2} - 50\,\text{N}\cdot\frac{\sqrt{3}}{2} + 30\,\text{N})\hat{\jmath} \\
&= -25\,\text{N}\hat{\imath} - 99.9\,\text{N}\hat{\jmath}.
\end{aligned}$$

Forces $\vec{F}_1$ and $\vec{F}_3$ pass through point D. Therefore, they do not produce any moment about D. So, the net moment about D is the moment caused by force $\vec{F}_2$:

$$\begin{aligned}
\vec{M}_\text{D} &= \vec{r}_{\text{C/D}} \times \vec{F}_2 \\
&= h\hat{\jmath} \times F_2(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= -F_2 h\cos\theta\hat{k} \\
&= -50\,\text{N}\cdot 0.5\,\text{m}\frac{1}{2}\hat{k} = -12.5\,\text{N·m}\hat{k}.
\end{aligned}$$

The equivalent force-couple system is shown in Fig. 2.48

$$\boxed{\vec{F}_\text{net} = -25\,\text{N}\hat{\imath} - 99.9\,\text{N}\hat{\jmath} \text{ and } \vec{M}_\text{D} = -12.5\,\text{N·m}\hat{k}}$$

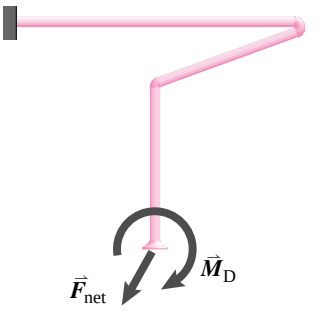


Figure 2.48: (Filename:sfig2.vec3.bar.a)

SAMPLE 2.31 *Translating a force-couple system:* The net force and couple acting about point B on the 'L' shaped bar shown in the figure are 100 N and 20 N·m, respectively. Find the net force and moment about point G.

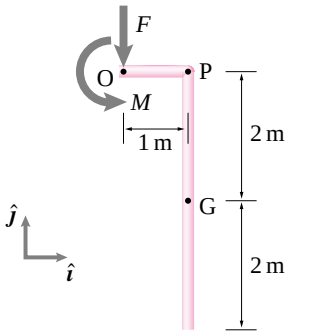


Figure 2.49: (Filename:sfig2.vec3.bentbar)

Solution The net force on a structure is the same about any point since it is just the vector sum of all the forces acting on the structure and is independent of their point of application. Therefore,

$$\vec{F}_\text{net} = \vec{F} = -100\,\text{N}\hat{\jmath}.$$

The net moment about a point, however, depends on the location of points of application of the forces with respect to that point. Thus,

$$\begin{aligned}
\vec{M}_\text{G} &= \vec{M}_\text{O} + \vec{r}_{\text{O/G}} \times \vec{F} \\
&= M\hat{k} + (-\ell\hat{\imath} + h\hat{\jmath}) \times (-F\hat{\jmath}) \\
&= (M + F\ell)\hat{k} \\
&= (20\,\text{N·m} + 100\,\text{N}\cdot 1\,\text{m})\hat{k} = 120\,\text{N·m}\hat{k}.
\end{aligned}$$

$$\boxed{\vec{F}_\text{net} = -100\,\text{N}\hat{\jmath}, \text{ and } \vec{M}_\text{G} = 120\,\text{N·m}\hat{k}}$$

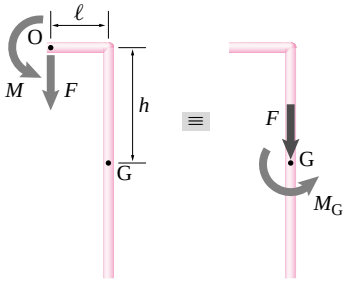


Figure 2.50: (Filename:sfig2.vec3.bentbar.a)

SAMPLE 2.32 *Checking equivalence of force-couple systems:* In the figure shown below, which of the force-couple systems shown in (b), (c), and (d) are equivalent to the force system shown in (a)?

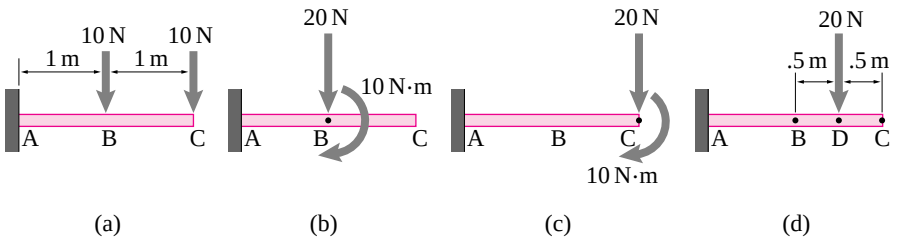


Figure 2.51: (Filename:sfig2.vec3.beam)

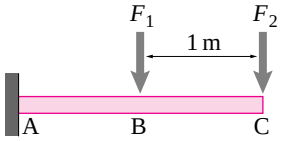


Figure 2.52: (Filename:sfig2.vec3.beam.a)

Solution The equivalence of force-couple systems require that (i) the net force be the same, and (ii) the net moment about any reference point be the same. For the given systems, let us choose point B as our reference point for comparing their equivalence. For the force system shown in Fig. 2.51(a), we have,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 = -10\,\text{N}\hat{j} - 10\,\text{N}\hat{j} = -20\,\text{N}\hat{j}\\ \vec{M}_{\text{B}_{\text{net}}} &= \vec{r}_{\text{C/B}} \times \vec{F}_2 = 1\,\text{m}\hat{i} \times (-10\,\text{N}\hat{j}) = -10\,\text{N·m}\hat{k}.\end{aligned}$$

Now, we can compare the systems shown in (b), (c), and (d) against the computed equivalent force-couple system, $\vec{F}_{\text{net}}$ and $\vec{M}_{\text{D}}$.

- Figure (b) shows exactly the system we calculated. Therefore, it represents an equivalent force-couple system.
- Figure (c): Let us calculate the net force and moment about point B for this system.

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_{\text{C}} \overset{\surd}{=} -20\,\text{N}\hat{j}\\ \vec{M}_{\text{B}} &= \vec{M}_{\text{C}} + \vec{r}_{\text{C/B}} \times \vec{F}_{\text{C}}\\ &= -10\,\text{N·m}\hat{k} + 1\,\text{m}\hat{i} \times (-20\,\text{N}\hat{j}) = -30\,\text{N·m}\hat{k} \neq \vec{M}_{\text{B}_{\text{net}}}.\end{aligned}$$

  Thus, the given force-couple system in this case is not equivalent to the force system in (a).
- Figure (d): Again, we compute the net force and the net couple about point B:

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_{\text{D}} \overset{\surd}{=} -20\,\text{N}\hat{j}\\ \vec{M}_{\text{B}} &= \vec{r}_{\text{D/B}} \times \vec{F}_{\text{D}}\\ &= 0.5\,\text{m}\hat{i} \times (-20\,\text{N}\hat{j}) = -10\,\text{N·m}\hat{k} \overset{\surd}{=} \vec{M}_{\text{B}_{\text{net}}}.\end{aligned}$$

  Thus, the given force-couple system (with zero couple) at D is equivalent to the force system in (a).

(b) and (d) are equivalent to (a); (c) is not.

SAMPLE 2.33 *Equivalent force with no couple:* For a body, an equivalent force-couple system at point A consists of a force $\vec{F} = 20\,\text{N}\hat{\imath} + 16\,\text{N}\hat{\jmath}$ and a couple $\vec{M}_\text{A} = 10\,\text{N·m}\hat{k}$. Find a point on the body such that the equivalent force-couple system at that point consists of only a force (zero couple).

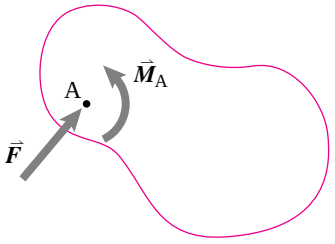


Figure 2.53: (Filename:sfig2.vec3.body)

Solution The net force in the two equivalent force-couple systems has to be the same. Therefore, for the new system, $\vec{F}_\text{net} = \vec{F} = 20\,\text{N}\hat{\imath} + 15\,\text{N}\hat{\jmath}$. Let B be the point at which the equivalent force-coupel system consists of only the net force, with zero couple. We need to find the location of point B. Let A be the origin of a xy cordinate system in which the coordinates of B are (x, y). Then, the moment about point B is,

$$\begin{aligned}\vec{M}_\text{B} &= \vec{M}_\text{A} + \vec{r}_\text{A/B} \times \vec{F} \\ &= M_A\hat{k} + (-x\hat{\imath} - y\hat{\jmath}) \times (F_x\hat{\imath} + F_y\hat{\jmath}) \\ &= M_A\hat{k} + (-F_y x + F_x y)\hat{k}.\end{aligned}$$

Since we require that $\vec{M}_\text{B}$ be zero, we must have

$$\begin{aligned}F_y x - F_x y &= M_A \\ \Rightarrow \quad y &= \frac{F_y}{F_x}x - \frac{M_A}{F_x} \\ &= \frac{15\,\text{N}}{20\,\text{N}}x - \frac{10\,\text{N·m}}{20\,\text{N}} \\ &= 0.75x - 0.5\,\text{m}.\end{aligned}$$

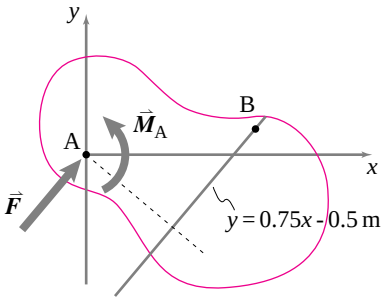


Figure 2.54: (Filename:sfig2.vec3.body.a)

This is the equation of a line. Thus, we can select any point on this line and apply the force $\vec{F} = 20\,\text{N}\hat{\imath} + 15\,\text{N}\hat{\jmath}$ with zero couple as an equivalent force-couple system.

Any point on the line $y = 0.75x - 0.5\,\text{m}$.

So, how or why does it work? The line we obtained is shown in gray in Fig. 2.54. Note that this line has the same slope as that of the given force vector (slope $= 0.75 = F_y/F_x$) and the offset is such that shifting the force $\vec{F}$ to this line counter balances the given couple at A. To see this clearly, let us select three points C, D, and E on the line as shown in Fig. 2.55. From the equation of the line, we find the coordinates of C(0,-.5m), D(.24m,.32m) and E(.67m,0). Now imagine moving the force $\vec{F}$ to C, D, or E. In each case, it must produce the same moment $\vec{M}_\text{A}$ about point A. Let us do a quick check.

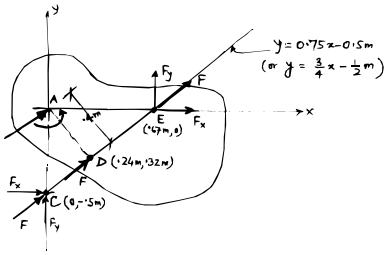


Figure 2.55: (Filename:sfig2.vec3.body.b)

- $\vec{F}$ at point C: The moment about point A is due to the horizontal component $F_x = 20\,\text{N}$, since F_y passes through point A. The moment is $F_x \cdot AC = 20\,\text{N} \cdot 0.5\,\text{m} = 10\,\text{N·m}$, same as M_A. The direction is counterclockwise as required.
- $\vec{F}$ at point D: The moment about point A is $|\vec{F}| \cdot AD = 25\,\text{N} \cdot 0.4\,\text{m} = 10\,\text{N·m}$, same as M_A. The direction is counterclockwise as required.
- $\vec{F}$ at point E: The moment about point A is due to the vertical component F_y, since F_x passes through point A. The moment is $F_y \cdot AE = 15\,\text{N} \cdot 0.67\,\text{m} = 10\,\text{N·m}$, same as M_A. The direction here too is counterclockwise as required.

Once we check the calculation for one point on the line, we should not have to do any more checks since we know that sliding the force along its line of action (line CB) produces no couple and thus preserves the equivalence.

2.5 Center of mass and gravity

For every system and at every instant in time, there is a unique location in space that is the average position of the system's mass. This place is called the *center of mass*, commonly designated by cm, c.o.m., G, c.g., or (◓).

One of the routine but important tasks of many real engineers is to find the center of mass of a complex machine. Just knowing the location of the center of mass of a car, for example, is enough to estimate whether it can be tipped over by maneuvers on level ground. The center of mass of a boat must be low enough for the boat to be stable. Any propulsive force on a space craft must be directed towards the center of mass in order to not induce rotations. Tracking the trajectory of the center of mass of an exploding plane can determine whether or not it was hit by a massive object. Any rotating piece of machinery must have its center of mass on the axis of rotation if it is not to cause much vibration.

Also, many calculations in mechanics are greatly simplified by making use of a system's center of mass. In particular, the whole complicated distribution of gravity forces on a body is equivalent to a single force at the center of mass. Many of the important quantities in dynamics are similarly simplified using the center of mass.

The center of mass of a system is the point at the position $\vec{r}_{\text{cm}}$ defined by

$$\begin{aligned} \vec{r}_{\text{cm}} &= \frac{\sum \vec{r}_{\text{i}} m_i}{m_{\text{tot}}} \quad \text{for discrete systems} \\ &= \frac{\int \vec{r}\, dm}{m_{\text{tot}}} \quad \text{for continuous systems} \end{aligned} \tag{2.12}$$

where $m_{\text{tot}} = \sum m_i$ for discrete systems and $m_{\text{tot}} = \int dm$ for continuous systems. See boxes 2.4 and 2.5 for a discussion of the $\sum$ and $\int$ sum notations.

Sometimes it is convenient to remember the rearranged definition of center of mass as

$$m_{\text{tot}} \vec{r}_{\text{cm}} = \sum m_i \vec{r}_{\text{i}} \quad \text{or} \quad m_{\text{tot}} \vec{r}_{\text{cm}} = \int \vec{r}\, dm.$$

For theoretical purposes we rarely need to evaluate these sums and integrals, and for simple problems there are sometimes shortcuts that reduce the calculation to a matter of observation. For complex machines one or both of the formulas 2.12 must be evaluated in detail.

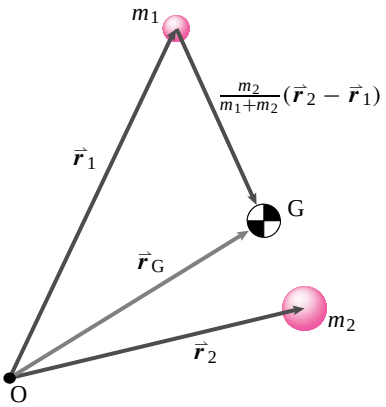


Figure 2.56: Center of mass of a system consisting of two points.

(Filename:tfigure3.com.twomass)

Example: **System of two point masses**

Intuitively, the center of mass of the two masses shown in figure 2.56 is between the two masses and closer to the larger one. Referring to equation 2.12,

$$\begin{aligned} \vec{r}_{\text{cm}} &= \frac{\sum \vec{r}_{\text{i}} m_i}{m_{tot}} \\ &= \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2} \\ &= \frac{\vec{r}_1 (m_1 + m_2) - \vec{r}_1 m_2 + \vec{r}_2 m_2}{m_1 + m_2} \end{aligned}$$

$$= \vec{r}_1 + \underbrace{\left(\frac{m_2}{m_1+m_2}\right)}_{\text{the fraction of the distance that the cm is from } \vec{r}_1 \text{ to } \vec{r}_2} \underbrace{(\vec{r}_2 - \vec{r}_1)}_{\text{the vector from } \vec{r}_1 \text{ to } \vec{r}_2}.$$

so that the math agrees with common sense — the center of mass is on the line connecting the masses. If $m_2 \gg m_1$, then the center of mass is near m_2. If $m_1 \gg m_2$, then the center of mass is near m_1. If $m_1 = m_2$ the center of mass is right in the middle at $(\vec{r}_1 + \vec{r}_2)/2$. □

① Note: writing $\int_{m_1}^{m_2}(\text{something})\,dm$ is nonsense because m is not a scalar parameter which labels points in a material (there is no point at $m = 3\,\text{kg}$).

Continuous systems

How do we evaluate integrals like $\int (\text{something})\,dm$? In center of mass calculations, (something) is position, but we will evaluate similar integrals where (something) is some other scalar or vector function of position. Most often we label the material by its spatial position, and evaluate dm in terms of increments of position. For 3D solids $dm = \rho dV$ where ρ is density (mass per unit volume). So $\int(\text{something})dm$ turns into a standard volume integral $\int_V(\text{something})\rho\,dV$①. For thin flat things

2.8 $\int$ means add

As discussed in box 2.4 on page 54 we often add things up in mechanics. For example, the total mass of some particles is

$$m_{\text{tot}} = m_1 + m_2 + m_3 + \ldots = \sum m_i$$

or more specifically the mass of 137 particles is, say, $m_{\text{tot}} = \sum_{i=1}^{137} m_i$.

And the total mass of a bicycle is:

$$m_{\text{bike}} = \sum_{i=1}^{100,000,000,000,000,000,000,000} m_i$$

where m_i are the masses of each of the 10^{23} (or so) atoms of metal, rubber, plastic, cotton, and paint. But atoms are so small and there are so many of them. Instead we often think of a bike as built of macroscopic parts. The total mass of the bike is then the sum of the masses of the tires, the tubes, the wheel rims, the spokes and nipples, the ball bearings, the chain pins, and so on. And we would write:

$$m_{\text{bike}} = \sum_{i=1}^{2,000} m_i$$

where now the m_i are the masses of the 2,000 or so bike parts. This sum is more manageable but still too detailed in concept for some purposes.

An approach that avoids attending to atoms or ball bearings, is to think of sending the bike to a big shredding machine that cuts it up into very small random small bits. Now we write

$$m_{\text{bike}} = \sum m_i$$

where the m_i are the masses of the very small bits. We don't fuss over whether one bit is a piece of ball bearing or fragment of cotton from the tire walls. We just chop the bike into bits and add up the contribution of each bit. If you take the letter S, as in SUM, and distort it ($\mathsf{S}\,\mathcal{S}\,\int\int$) and you get a big old fashioned German 'S' as in $\int \mathcal{UM}$ (sum). So we write

$$m_{\text{bike}} = \int dm$$

to mean the $\int$um of all the teeny bits of mass. More formally we mean the value of that sum in the limit that all the bits are infinitesimal (not minding the technical fine point that its hard to chop atoms into infinitesimal pieces).

The mass is one of many things we would like to add up, though many of the others also involve mass. In center of mass calculations, for example, we add up the positions 'weighted' by mass.

$$\int \vec{r}\,dm \quad \text{which means} \sum_{\lim m_i \to 0} \vec{r}_i m_i.$$

That is, you take your object of interest and chop it into a billion pieces and then re-assemble it. For each piece you make the vector which is the position vector of the piece multiplied by ('weighted by') its mass and then add up the billion vectors. Well really you chop the thing into a trillion trillion . . . pieces, but a billion gives the idea.

like metal sheets we often take ρ to mean mass per unit area A so then $dm = \rho dA$ and $\int(\text{something})dm = \int_A(\text{something})\rho\, dA$. For mass distributed along a line or curve we take ρ to be the mass per unit length or arc length s and so $dm = \rho ds$ and $\int(\text{something})dm = \int_{curve}(\text{something})\rho\, ds$.

Example. The center of mass of a uniform rod is naturally in the middle, as the calculations here show (see fig. 2.57a). Assume the rod has length $L = 3\,\text{m}$ and mass $m = 7\,\text{kg}$.

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r}\, dm}{m_{\text{tot}}} = \frac{\int_0^L x\hat{\imath}\, \overbrace{\rho dx}^{dm}}{\int_0^L \rho\, dx} = \frac{\rho(x^2/2)|_0^L}{\rho(1)|_1^L}\hat{\imath} = \frac{\rho(L^2/2)}{\rho L}\hat{\imath} = (L/2)\hat{\imath}$$

So $\vec{r}_{\text{cm}} = (L/2)\hat{\imath}$, or by dotting with $\hat{\imath}$ (taking the x component) we get that the center of mass is on the rod a distance $d = L/2 = 1.5\,\text{m}$ from the end. □

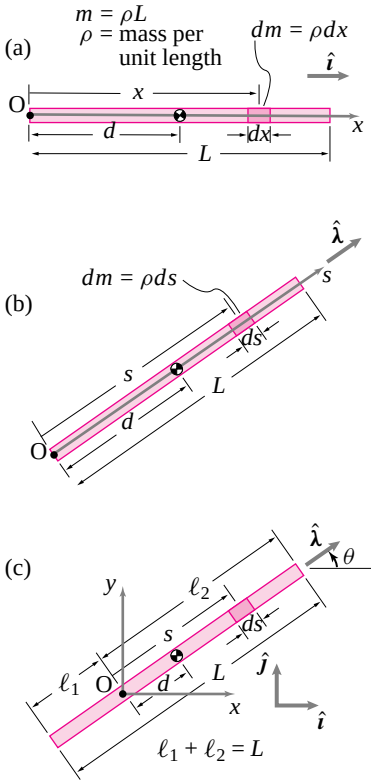


Figure 2.57: Where is the center of mass of a uniform rod? In the middle, as you can find calculating a few ways or by symmetry.
(Filename:tfigure1.rodcm)

The center of mass calculation is *objective*. It describes something about the object that does not depend on the coordinate system. In different coordinate systems the center of mass for the rod above will have different coordinates, but it will always be at the middle of the rod.

Example. Find the center of mass using the coordinate system with s & $\hat{\boldsymbol{\lambda}}$ in fig. 2.57b:

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r}\, dm}{m_{\text{tot}}} = \frac{\int_0^L s\hat{\boldsymbol{\lambda}}\, \rho ds}{\int_0^L \rho\, ds}\hat{\boldsymbol{\lambda}} = \frac{\rho(s^2/2)|_0^L}{\rho(1)|_0^L}\hat{\boldsymbol{\lambda}} = \frac{\rho(L^2/2)}{\rho L}\hat{\boldsymbol{\lambda}} = (L/2)\hat{\boldsymbol{\lambda}},$$

again showing that the center of mass is in the middle. □

Note, one can treat the center of mass vector calculations as separate scalar equations, one for each component. For example:

$$\hat{\imath}\cdot\left\{\vec{r}_{\text{cm}} = \frac{\int \vec{r}\, dm}{m_{\text{tot}}}\right\} \quad\Rightarrow\quad r_{x\text{cm}} = x_{\text{cm}} = \frac{\int x\, dm}{m_{\text{tot}}}.$$

Finally, there is no law that says you have to use the best coordinate system. One is free to make trouble for oneself and use an inconvenient coordinate system.

Example. Use the xy coordinates of fig. 2.57c to find the center of mass of the rod.

$$x_{\text{cm}} = \frac{\int x dm}{m_{\text{tot}}} = \frac{\int_{-\ell_1}^{\ell_2} \overbrace{s\cos\theta}^{x}\, \rho ds}{\int_0^L \rho\, ds} = \frac{\rho\cos\theta\frac{s^2}{2}|_{-\ell_1}^{\ell_2}}{\rho(1)|_{-\ell_1}^{\ell_2}} = \frac{\rho\cos\theta\frac{(\ell_2^2-\ell_1^2)}{2}}{\rho(\ell_1+\ell_2)} = \frac{\cos\theta(\ell_2-\ell_1)}{2}$$

Similarly $y_{\text{cm}} = \sin\theta(\ell_2-\ell_1)/2$ so

$$\vec{r}_{\text{cm}} = \frac{\ell_2-\ell_1}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\boldsymbol{\jmath}})$$

which still describes the point at the middle of the rod. □

The most commonly needed center of mass that can be found analytically but not directly from symmetry is that of a triangle (see box 2.5 on page 70). You can find more examples using integration to find the center of mass (or centroid) in your calculus text.

Center of mass and centroid

For objects with uniform material density we have

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r}\, dm}{m_{\text{tot}}} = \frac{\int_V \vec{r}\, \rho dV}{\int_V \rho dV} = \frac{\rho \int_V \vec{r}\, dV}{\rho \int_V dV} = \frac{\int_V \vec{r}\, dV}{V}$$

where the last expression is just the formula for geometric centroid. Analogous calculations hold for 2D and 1D geometric objects. Thus for objects with density that does not vary from point to point, the geometric centroid and the center of mass coincide.

Center of mass and symmetry

The center of mass respects any symmetry in the mass distribution of a system. If the word 'middle' has unambiguous meaning in English then that is the location of the center of mass, as for the rod of fig. 2.57 and the other examples in fig. 2.58.

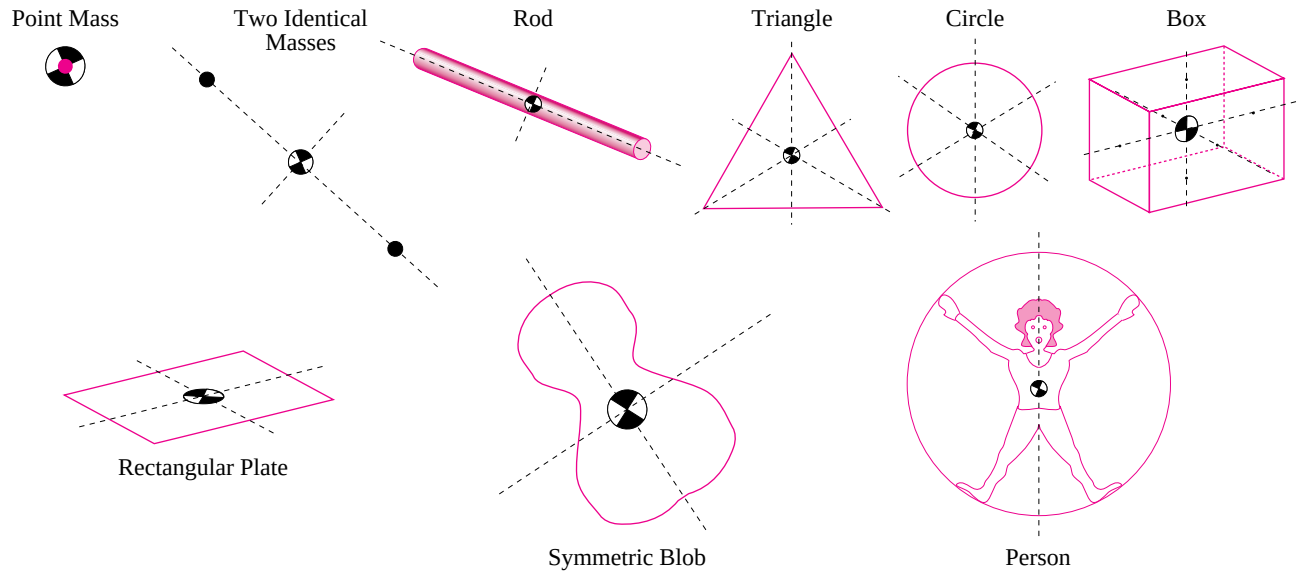


Figure 2.58: The center of mass and the geometric centroid share the symmetries of the object. (Filename:tfigure3.com.symm)

Systems of systems and composite objects

Another way of interpreting the formula

$$\vec{r}_{\text{cm}} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots}{m_1 + m_2 + \cdots}$$

is that the m's are the masses of subsystems, not just points, and that the $\vec{r}_i$ are the positions of the centers of mass of these systems. This subdivision is justified in box 2.9 on page 67. The center of mass of a single complex shaped object can be found by treating it as an assembly of simpler objects.

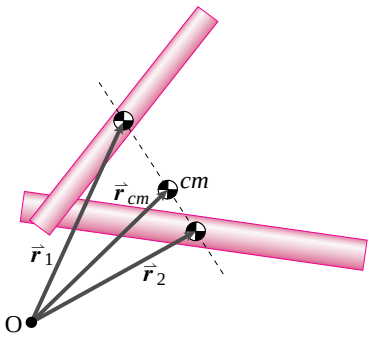


Figure 2.59: Center of mass of two rods

(Filename:tfigure3.com.tworods)

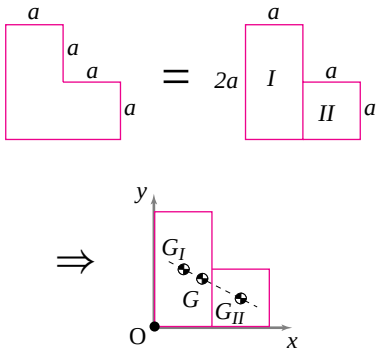


Figure 2.60: The center of mass of the 'L' shaped object can be found by thinking of it as a rectangle plus a square.

(Filename:tfigure3.1.Lshaped)

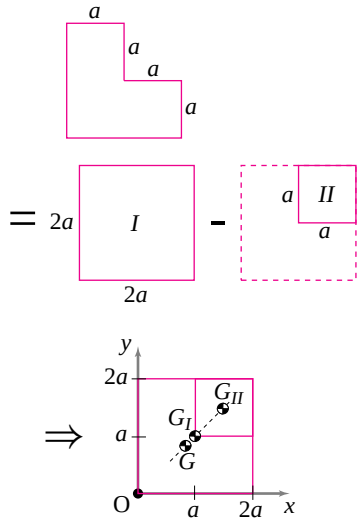


Figure 2.61: Another way of looking at the 'L' shaped object is as a square minus a smaller square in its upper right-hand corner.

(Filename:tfigure3.1.Lshaped.a)

Example: **Two rods**

The center of mass of two rods shown in figure 2.59 can be found as

$$\vec{r}_{\rm cm} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of the centers of mass of each rod and m_1 and m_2 are the masses. □

Example: **'L' shaped plate**

Consider the plate with uniform mass per unit area ρ.

$$\begin{aligned}\vec{r}_{\rm G} &= \frac{\vec{r}_{\rm I} m_I + \vec{r}_{\rm II} m_{II}}{m_I + m_{II}} \\ &= \frac{(\frac{a}{2}\hat{\imath} + a\hat{\jmath})(2\rho a^2) + (\frac{3}{2}a\hat{\imath} + \frac{a}{2}\hat{\jmath})(\rho a^2)}{(2\rho a^2) + (\rho a^2)} \\ &= \frac{5}{6}a(\hat{\imath} + \hat{\jmath}).\end{aligned}$$

□

Composite objects using subtraction

It is sometimes useful to think of an object as composed of pieces, some of which have negative mass.

Example: **'L' shaped plate, again**

Reconsider the plate from the previous example.

$$\begin{aligned}\vec{r}_{\rm G} &= \frac{\vec{r}_{\rm I} m_I + \vec{r}_{\rm II} m_{II}}{m_I + m_{II}} \\ &= \frac{(a\hat{\imath} + a\hat{\jmath})(\rho(2a)^2) + (\frac{3}{2}a\hat{\imath} + \frac{3}{2}a\hat{\jmath})\overbrace{(-\rho a^2)}^{m_{II}}}{(\rho(2a)^2) + \underbrace{(-\rho a^2)}_{m_{II}}} \\ &= \frac{5}{6}a(\hat{\imath} + \hat{\jmath}).\end{aligned}$$

□

Center of gravity

The force of gravity on each little bit of an object is gm_i where g is the local gravitational 'constant' and m_i is the mass of the bit. For objects that are small compared to the radius of the earth (a reasonable assumption for all but a few special engineering calculations) the gravity constant is indeed constant from one point on the object to another (see box **??** on page **??** for a discussion of the meaning and history of g.)

Not only that, all the gravity forces point in the same direction, down. (For engineering purposes, the two intersecting lines that go from your two hands to the center of the earth are parallel.). Lets call this the $-\hat{\boldsymbol{k}}$ direction. So the net force of gravity on an object is:

$$\begin{aligned}\vec{\boldsymbol{F}}_{\text{net}} &= \sum \vec{\boldsymbol{F}}_{\text{i}} = \sum m_i g(-\hat{\boldsymbol{k}}) = -mg\hat{\boldsymbol{k}} \quad \text{for discrete systems, and}\\ &= \int d\vec{\boldsymbol{F}} = \int \underbrace{-g\hat{\boldsymbol{k}}\, dm}_{d\vec{\boldsymbol{F}}} = -mg\hat{\boldsymbol{k}} \quad \text{for continuous systems.}\end{aligned}$$

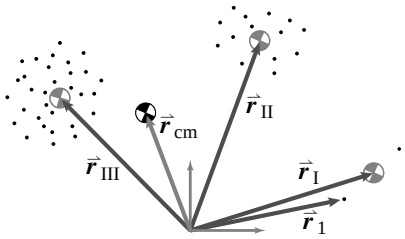
That's easy, the billions of gravity forces on an objects microscopic constituents add up to mg pointed down. What about the net moment of the gravity forces? The answer turns out to be simple. The top line of the calculation below poses the question, the last line gives the lucky answer.①

① We do the calculation here using the $\int$ notation for sums. But it could be done just as well using $\sum$.

$$\begin{aligned}\vec{\boldsymbol{M}}_{\text{C}} &= \int \vec{\boldsymbol{r}} \times d\vec{\boldsymbol{F}} && \text{The net moment with respect to C.}\\ &= \int \vec{\boldsymbol{r}}_{/\text{C}} \times \left(-g\hat{\boldsymbol{k}}\, dm\right) && \text{A force bit is gravity acting on a mass bit.}\\ &= \left(\int \vec{\boldsymbol{r}}_{/\text{C}} dm\right) \times \left(-g\hat{\boldsymbol{k}}\right) && \text{Cross product distributive law } (g,\ \hat{\boldsymbol{k}} \text{ are constants}).\\ &= (\vec{\boldsymbol{r}}_{\text{cm/C}} m) \times \left(-g\hat{\boldsymbol{k}}\right) && \text{Definition of center of mass.}\\ &= \vec{\boldsymbol{r}}_{\text{cm/C}} \times \left(-mg\hat{\boldsymbol{k}}\right) && \text{Re-arranging terms.}\end{aligned}$$

2.9 *THEORY*

Why can subsystems be treated like particles when finding the center of mass?



Lets look at the collection of 47 particles above and then think of it as a set of three subsystems: I, II, and III with 2, 14, and 31 particles respectively. We treat masses 1 and 2 as subsystem I with center of mass $\vec{\boldsymbol{r}}_{\text{I}}$ and total mass m_I. Similarly, we call subsystem II masses m_3 to m_{16}, and subsystem III, masses m_{17} to m_{47}. We can calculate the center of mass of the system by treating it as 47 particles, or we can re-arrange the sum as follows:

$$\begin{aligned}\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2 + \cdots + \vec{\boldsymbol{r}}_{46} m_{46} + \vec{\boldsymbol{r}}_{47} m_{47}}{m_1 + m_2 + \cdots + m_{47}}\\ &= \frac{\frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2}{m_1+m_2}(m_1+m_2)}{m_1 + m_2 + \cdots + m_{47}}\\ &\quad + \frac{\frac{\vec{\boldsymbol{r}}_3 m_3 + \cdots + \vec{\boldsymbol{r}}_{16} m_{16}}{m_3+\cdots+m_{16}}(m_3 + \cdots + m_{16})}{m_1 + m_2 + \cdots + m_{47}}\\ &\quad + \frac{\frac{\vec{\boldsymbol{r}}_{17} m_{17} + \cdots \vec{\boldsymbol{r}}_{47} m_{47}}{m_{17}+\cdots+m_{47}}(m_{17} + \cdots + m_{47})}{m_1 + m_2 + \cdots + m_{47}}\\ &= \frac{\vec{\boldsymbol{r}}_{\text{I}} m_I + \vec{\boldsymbol{r}}_{\text{II}} m_{II} + \vec{\boldsymbol{r}}_{\text{III}} m_{III}}{m_I + m_{II} + m_{III}}, \text{ where}\end{aligned}$$

$$\begin{aligned}\vec{\boldsymbol{r}}_{\text{I}} &= \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2}{m_1 + m_2},\\ m_I &= m_1 + m_2\\ \vec{\boldsymbol{r}}_{\text{II}} &\quad etc.\end{aligned}$$

The formula for the center of mass of the whole system reduces to one that looks like a sum over three (aggregate) particles.

This idea is easily generalized to the integral formulae as well like this.

$$\begin{aligned}\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{\int \vec{\boldsymbol{r}}\, dm}{\int dm}\\ &= \frac{\int_{\text{region 1}} \vec{\boldsymbol{r}}\, dm + \int_{\text{region 2}} \vec{\boldsymbol{r}}\, dm + \int_{\text{region 3}} \vec{\boldsymbol{r}}\, dm + \cdots}{\int_{\text{region 1}} dm + \int_{\text{region 2}} dm + \int_{\text{region 1}} dm + \cdots}\\ &= \frac{\vec{\boldsymbol{r}}_{\text{I}} m_I + \vec{\boldsymbol{r}}_{\text{II}} m_{II} + \vec{\boldsymbol{r}}_{\text{III}} m_{III} + \cdots}{m_I + m_{II} + m_{III} + \cdots}.\end{aligned}$$

The general idea of the calculations above is that center of mass calculations are basically big sums (addition), and addition is 'associative.'

$$= \quad \vec{r}_{\mathrm{cm/C}} \times \vec{F}_{\mathrm{net}} \qquad \text{Express in terms of net gravity force.}$$

Thus the net moment is the same as for the total gravity force acting at the center of mass.

> The near-earth gravity forces acting on a system are *equivalent* to a single force, mg, acting at the system's center of mass.

For the purposes of calculating the net force and moment from near-earth (constant g) gravity forces, a system can be replaced by a point mass at the center of gravity. The words 'center of mass' and 'center of gravity' both describe the same point in space.

Although the result we have just found seems plain enough, here are two things to ponder about gravity when viewed as an inverse square law (and thus not constant like we have assumed) that may make the result above seem less obvious.

- The net gravity force on a sphere is indeed equivalent to the force of a point mass at the center of the sphere. It took the genius Isaac Newton 3 years to deduce this result and the reasoning involved is too advanced for this book.
- The net gravity force on systems that are not spheres is generally *not* equivalent to a force acting at the center of mass (this is important for the understanding of tides as well as the orientational stability of satellites).

A recipe for finding the center of mass of a complex system

You find the center of mass of a complex system by knowing the masses and mass centers of its components. You find each of these centers of mass by

- Treating it as a point mass, or
- Treating it as a symmetric body and locating the center of mass in the middle, or
- Using integration, or
- Using the result of an experiment (which we will discuss in statics), or
- Treating the component as a complex system itself and applying this very recipe.

The recipe is just an application of the basic definition of center of mass (eqn. 2.12) but with our accumulated wisdom that the locations and masses in that sum can be the centers of mass and total masses of complex subsystems.

One way to arrange one's data is in a table or spreadsheet, like below. The first four columns are the basic data. They are the x, y, and z coordinates of the subsystem center of mass locations (relative to some clear reference point), and the masses of the subsystems, one row for each of the N subsystems.

Subsys#	1	2	3	4	5	6	7
Subsys 1	x_1	y_1	z_1	m_1	m_1x_1	m_1y_1	m_1z_1
Subsys 2	x_2	y_2	z_2	m_2	m_2x_2	m_2y_2	m_2z_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Subsys N	x_N	y_N	z_N	m_N	m_Nx_N	m_Ny_N	m_Nz_N
Row N+1 sums				$m_{\mathrm{tot}} = \sum m_i$	$\sum m_i x_i$	$\sum m_i y_i$	$\sum m_i z_i$

					x_{cm}	y_{cm}	z_{cm}
Result					$\frac{\sum m_i x_i}{m_{\mathrm{tot}}}$	$\frac{\sum m_i y_i}{m_{\mathrm{tot}}}$	$\frac{\sum m_i z_i}{m_{\mathrm{tot}}}$

One next calculates three new columns (5,6, and 7) which come from each coordinate multiplied by its mass. For example the entry in the 6th row and 7th column is the z component of the 6th subsystem's center of mass multiplied by the mass of the 6th subsystem. Then one sums columns 4 through 7. The sum of column 4 is the total mass, the sums of columns 5 through 7 are the total mass-weighted positions. Finally the result, the system center of mass coordinates, are found by dividing columns 5-7 of row N+1 by column 4 of row N+1.

Of course, there are multiple ways of systematically representing the data. The spreadsheet-like calculation above is just one way to organize the calculation.

Summary of center of mass

All discussions in mechanics make frequent reference to the concept of center of mass because

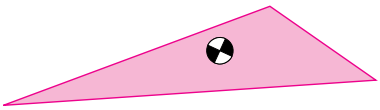
> *For systems with distributed mass, the expressions for gravitational moment, linear momentum, angular momentum, and energy are all simplified by using the center of mass.*

Simple center of mass calculations also can serve as a check of a more complicated analysis. For example, after a computer simulation of a system with many moving parts is complete, one way of checking the calculation is to see if the whole system's center of mass moves as would be expected by applying the net external force to the system. These formulas tell the whole story if you know how to use them:

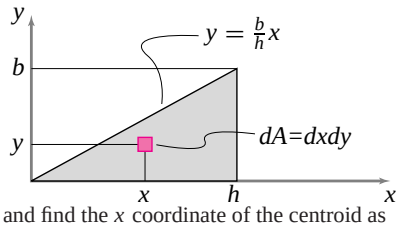
$$\begin{aligned}
\vec{r}_{\rm cm} &= \frac{\sum \vec{r}_{\rm i} m_i}{m_{tot}} && \text{for discrete systems or systems of systems} \\
&= \frac{\int \vec{r}\, dm}{m_{tot}} && \text{for continuous systems} \\
m_{\rm tot} &= \sum {\rm m}_i && \text{for discrete systems or systems of systems} \\
&= \int dm && \text{for continuous systems.}
\end{aligned}$$

2.10 The center of mass of a uniform triangle is a third of the way up from the base

The center of mass of a 2D uniform triangular region is the centroid of the area.



First we consider a right triangle with perpendicular sides b and h



and find the x coordinate of the centroid as

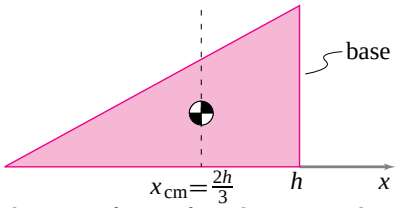
$$x_{\rm cm} A = \int x\, dA$$

$$= \int_0^h \left[\int_0^{\frac{b}{h}x} x\, dy \right] dx = \int_0^h [xy] \Big|_{y=0}^{y=\frac{b}{h}x} dx$$

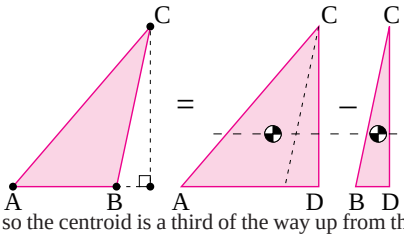
$$x_{\rm cm} \left(\frac{bh}{2}\right) = \int_0^h x \left(\frac{b}{h}x\right) dx = \frac{b}{h}\frac{x^3}{3}\Big|_0^h = \frac{bh^2}{3}$$

$$\Rightarrow \qquad x_{\rm cm} = \frac{2h}{3},$$

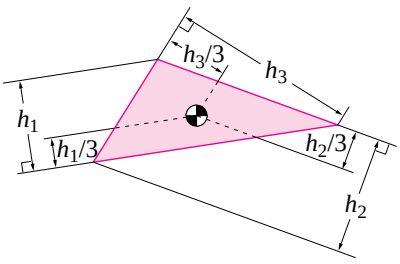
a third of the way to the left of the vertical base on the right. By similar reasoning, but in the y direction, the centroid is a third of the way up from the base.



The center of mass of an arbitrary triangle can be found by treating it as the sum of two right triangles

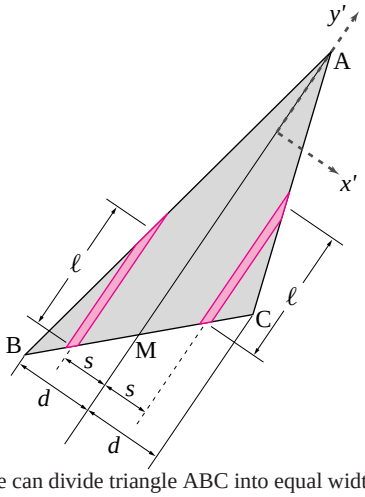


so the centroid is a third of the way up from the base of any triangle. Finally, the result holds for all three bases. Summarizing, the centroid of a triangle is at the point one third up from each of the bases.



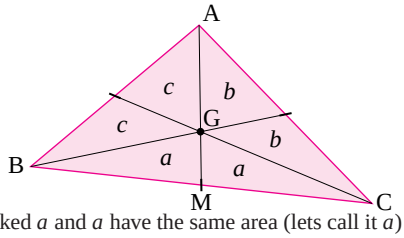
Non-calculus approach

Consider the line segment from A to the midpoint M of side BC.



We can divide triangle ABC into equal width strips that are parallel to AM. We can group these strips into pairs, each a distance s from AM. Because M is the midpoint of BC, by proportions each of these strips has the same length ℓ. Now in trying to find the distance of the center of mass from the line AM we notice that all contributions to the sum come in canceling pairs because the strips are of equal area and equal distance from AM but on opposite sides. Thus the centroid is on AM. Likewise for all three sides. Thus the centroid is at the point of intersection of the three side bisectors.

That the three side bisectors intersect a third of the way up from the three bases can be reasoned by looking at the 6 triangles formed by the side bisectors.



The two triangles marked a and a have the same area (lets call it a) because they have the same height and bases of equal length (BM and CM). Similar reasoning with the other side bisectors shows that the pairs marked b have equal area and so have the pairs marked c. But the triangle ABM has the same base and height and thus the same area as the triangle ACM. So $a + b + b = a + c + c$. Thus $b = c$ and by similar reasoning $a = b$ and all six little triangles have the same area. Thus the area of big triangle ABC is 3 times the area of GBC. Because ABC and GBC share the base BC, ABC must have 3 times the height as GBC, and point G is thus a third of the way up from the base.

Where is the middle of a triangle?

We have shown that the centroid of a triangle is at the point that is at the intersection of: the three side bisectors; the three area bisectors (which are the side bisectors); and the three lines one third of the way up from the three bases.

If the triangle only had three equal point masses on its vertices the center of mass lands on the same place. Thus the 'middle' of a triangle seems pretty well defined. *But*, there is some ambiguity. If the triangle were made of bars along each edge, each with equal cross sections, the center of mass would be in a different location for all but equilateral triangles. Also, the three angle bisectors of a triangle do not intersect at the centroid. Unless we define middle to mean centroid, the "middle" of a triangle is not well defined.

SAMPLE 2.34 *Center of mass in 1-D:* Three particles (point masses) of mass 2 kg, 3 kg, and 3 kg, are welded to a straight massless rod as shown in the figure. Find the location of the center of mass of the assembly.

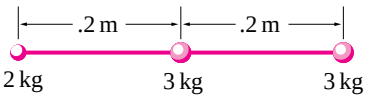


Figure 2.62: (Filename:sfig2.cm.1D)

Solution Let us select the first mass, $m_1 = 2\,\text{kg}$, to be at the origin of our co-ordinate system with the x-axis along the rod. Since all the three masses lie on the x-axis, the center of mass will also lie on this axis. Let the center of mass be located at x_{cm} on the x-axis. Then,

$$\begin{aligned} m_{\text{tot}} x_{cm} &= \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3 \\ &= m_1(0) + m_2(\ell) + m_3(2\ell) \\ \Rightarrow \quad x_{cm} &= \frac{m_2\ell + m_3 2\ell}{m_1 + m_2 + m_3} \\ &= \frac{3\,\text{kg} \cdot 0.2\,\text{m} + 3\,\text{kg} \cdot 0.4\,\text{m}}{(2+3+3)\,\text{kg}} \\ &= \frac{1.8\,\text{m}}{8} = 0.225\,\text{m}. \end{aligned}$$

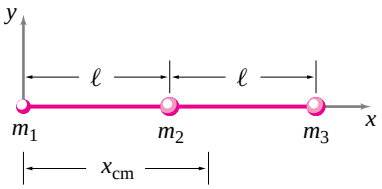


Figure 2.63: (Filename:sfig2.cm.1Da)

$$\boxed{x_{cm} = 0.225\,\text{m}}$$

Alternatively, we could find the center of mass by first replacing the two 3 kg masses with a single 6 kg mass located in the middle of the two masses (the center of mass of the two equal masses) and then calculate the value of x_{cm} for a two particle system consisting of the 2 kg mass and the 6 kg mass (see Fig. 2.64):

$$x_{cm} = \frac{6\,\text{kg} \cdot 0.3\,\text{m}}{8\,\text{kg}} = \frac{1.8\,\text{m}}{8} = 0.225\,\text{m}.$$

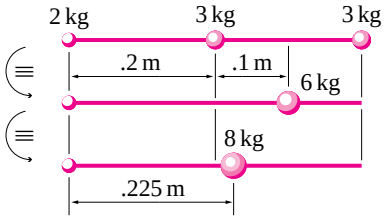


Figure 2.64: (Filename:sfig2.cm.1Db)

SAMPLE 2.35 *Center of mass in 2-D:* Two particles of mass $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$ are located at coordinates (1m, 2m) and (-2m, 5m), respectively, in the xy-plane. Find the location of their center of mass.

Solution Let $\vec{r}_{\text{cm}}$ be the position vector of the center of mass. Then,

$$\begin{aligned} m_{\text{tot}} \vec{r}_{\text{cm}} &= m_1 \vec{r}_1 + m_2 \vec{r}_2 \\ \Rightarrow \quad \vec{r}_{\text{cm}} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_{\text{tot}}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \\ &= \frac{1\,\text{kg}(1\,\text{m}\,\hat{\imath} + 2\,\text{m}\,\hat{\jmath}) + 2\,\text{kg}(-2\,\text{m}\,\hat{\imath} + 5\,\text{m}\,\hat{\jmath})}{3\,\text{kg}} \\ &= \frac{(1\,\text{m} - 4\,\text{m})\hat{\imath} + (2\,\text{m} + 10\,\text{m})\hat{\jmath}}{3} = -1\,\text{m}\,\hat{\imath} + 4\,\text{m}\,\hat{\jmath}. \end{aligned}$$

Thus the center of mass is located at the coordinates(-1m, 4m).

$$\boxed{(x_{cm}, y_{cm}) = (-1\,\text{m}, 4\,\text{m})}$$

Geometrically, this is just a 1-D problem like the previous sample. The center of mass has to be located on the straight line joining the two masses. Since the center of mass is a point about which the distribution of mass is *balanced*, it is easy to see (see Fig. 2.65) that the center of mass must lie one-third way from m_2 on the line joining the two masses so that $2\,\text{kg} \cdot (d/3) = 1\,\text{kg} \cdot (2d/3)$.

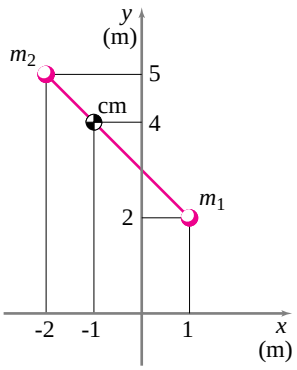


Figure 2.65: (Filename:sfig2.cm.2Da)

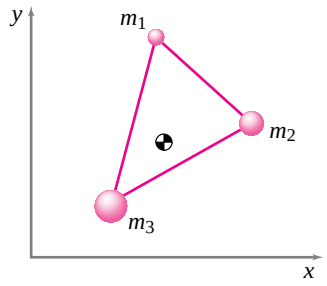


Figure 2.66: (Filename:sfig2.4.2)

SAMPLE 2.36 *Location of the center of mass.* A structure is made up of three point masses, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$, connected rigidly by massless rods. At the moment of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\text{m/s}\hat{\boldsymbol{\imath}}$, $2\,\text{m/s}(\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}})$ and $1\,\text{m/s}\hat{\boldsymbol{\jmath}}$, respectively. Find the coordinates of the center of mass of the structure.

Solution Just for fun, let us do this problem two ways — first using scalar equations for the coordinates of the center of mass, and second, using vector equations for the position of the center of mass.

(a) **Scalar calculations:** Let (x_{cm}, y_{cm}) be the coordinates of the mass-center. Then from the definition of mass-center,

$$\begin{aligned} x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 1.25\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.75\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{7.5\,\cancel{\text{kg}}\cdot\,\text{m}}{6\,\cancel{\text{kg}}} = 1.25\,\text{m}. \end{aligned}$$

Similarly,

$$\begin{aligned} y_{cm} &= \frac{\sum m_i y_i}{\sum m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 3\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.5\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{8.5\,\cancel{\text{kg}}\cdot\,\text{m}}{6\,\cancel{\text{kg}}} = 1.42\,\text{m}. \end{aligned}$$

Thus the center of mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\text{m},\ 1.42\,\text{m})}$$

(b) **Vector calculations:** Let $\vec{r}_{\text{cm}}$ be the position vector of the mass-center. Then,

$$\begin{aligned} m_{\text{tot}}\vec{r}_{\text{cm}} &= \sum_{i=1}^{3} m_i \vec{r}_i = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 \\ \Rightarrow \quad \vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} \end{aligned}$$

Substituting the values of m_1, m_2, and m_3, and $\vec{r}_1 = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$, $\vec{r}_2 = 2\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{\jmath}}$, and $\vec{r}_3 = 0.75\,\text{m}\hat{\boldsymbol{\imath}} + 0.5\,\text{m}\hat{\boldsymbol{\jmath}}$, we get,

$$\begin{aligned} \vec{r}_{\text{cm}} &= \frac{1\,\text{kg}\cdot(1.25\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})\,\text{m} + 2\,\text{kg}\cdot(2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{m} + 3\,\text{kg}\cdot(0.75\hat{\boldsymbol{\imath}} + 0.5\hat{\boldsymbol{\jmath}})\,\text{m}}{(1+2+3)\,\text{kg}} \\ &= \frac{(7.5\hat{\boldsymbol{\imath}} + 8.5\hat{\boldsymbol{\jmath}})\,\cancel{\text{kg}}\cdot\,\text{m}}{6\,\cancel{\text{kg}}} \\ &= 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}} \end{aligned}$$

which, of course, gives the same location of the mass-center as above.

$$\boxed{\vec{r}_{\text{cm}} = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}}}$$

SAMPLE 2.37 *Center of mass of a bent bar:* A uniform bar of mass 4 kg is bent in the shape of an asymmetric 'Z' as shown in the figure. Locate the center of mass of the bar.

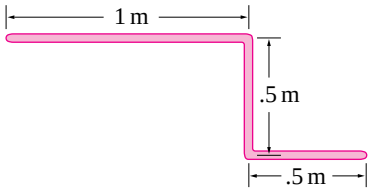


Figure 2.67: (Filename:sfig2.cm.wire)

Solution Since the bar is uniform along its length, we can divide it into three straight segments and use their individual mass-centers (located at the geometric centers of each segment) to locate the center of mass of the entire bar. The mass of each segment is proportional to its length. Therefore, if we let $m_2 = m_3 = m$, then $m_1 = 2m$; and $m_1 + m_2 + m_3 = 4m = 4\,\text{kg}$ which gives $m = 1\,\text{kg}$. Now, from Fig. 2.68,

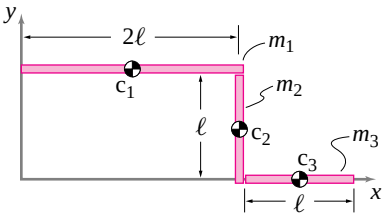


Figure 2.68: (Filename:sfig2.cm.wire.a)

$$\begin{aligned}
\vec{r}_1 &= \ell\hat{\imath} + \ell\hat{\jmath} \\
\vec{r}_2 &= 2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath} \\
\vec{r}_3 &= (2\ell + \frac{\ell}{2})\hat{\imath} = \frac{5\ell}{2}\hat{\imath}
\end{aligned}$$

So,

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_{\text{tot}}} \\
&= \frac{2m(\ell\hat{\imath} + \ell\hat{\jmath}) + m(2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath}) + m(\frac{5\ell}{2}\hat{\imath})}{4m} \\
&= \frac{\cancel{m}\ell(2\hat{\imath} + 2\hat{\jmath} + 2\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{5}{2}\hat{\imath})}{4\cancel{m}} \\
&= \frac{\ell}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= \frac{0.5\,\text{m}}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}}$$

Geometrically, we could find the center of mass by considering two masses at a time, connecting them by a line and locating their mass-center on that line, and then repeating the process as shown in Fig. 2.69. The center of mass of m_2 and m_3 (each

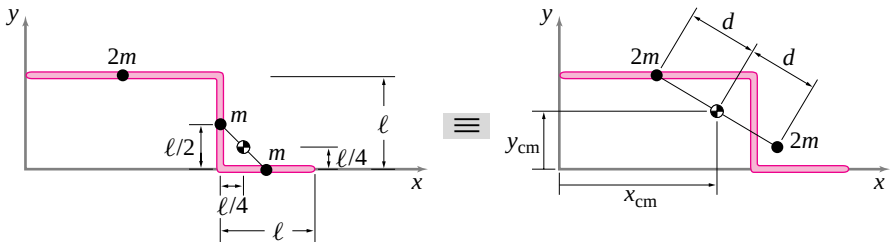


Figure 2.69: (Filename:sfig2.cm.wire.b)

of mass m) is at the mid-point of the line connecting the two masses. Now, we replace these two masses with a single mass $2m$ at their mass-center. Next, we connect this mass-center and m_1 with a line and find their combined mass-center at the mid-point of this line. The mass-center just found is the center of mass of the entire bar.

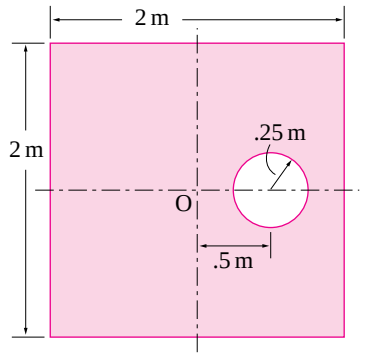


Figure 2.70: (Filename:sfig2.cm.plate)

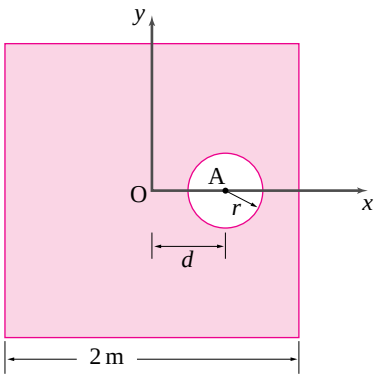


Figure 2.71: (Filename:sfig2.cm.plate.a)

SAMPLE 2.38 *Shift of mass-center due to cut-outs:* A 2 m × 2 m uniform square plate has mass $m = 4$ kg. A circular section of radius 250 mm is cut out from the plate as shown in the figure. Find the center of mass of the plate.

Solution Let us use an xy-coordinate system with its origin at the geometric center of the plate and the x-axis passing through the center of the cut-out. Since the plate and the cut-out are symmetric about the x-axis, the new center of mass must lie somewhere on the x-axis. Thus, we only need to find x_{cm} (since $y_{cm} = 0$). Let m_1 be the mass of the plate with the hole, and m_2 be the mass of the circular cut-out. Clearly, $m_1 + m_2 = m = 4$ kg. The center of mass of the circular cut-out is at A, the center of the circle. The center of mass of the intact square plate (without the cut-out) must be at O, the middle of the square. Then,

$$\begin{aligned} m_1 x_{cm} + m_2 x_A &= m x_O = 0 \\ \Rightarrow \quad x_{cm} &= -\frac{m_2}{m_1} x_A. \end{aligned}$$

Now, since the plate is uniform, the masses m_1 and m_2 are proportional to the surface areas of the geometric objects they represent, *i.e.*,

$$\frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi}.$$

Therefore,

$$\begin{aligned} x_{cm} &= -\frac{m_2}{m_1} d = -\frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} d \qquad (2.13) \\ &= -\frac{\pi}{\left(\frac{2\,\text{m}}{.25\,\text{m}}\right)^2 - \pi} \cdot 0.5\,\text{m} \\ &= -25.81 \times 10^{-3}\,\text{m} = -25.81\,\text{mm} \end{aligned}$$

Thus the center of mass shifts to the left by about 26 mm because of the circular cut-out of the given size.

$$\boxed{x_{cm} = -25.81\,\text{mm}}$$

Comments: The advantage of finding the expression for x_{cm} in terms of r and ℓ as in eqn. (2.13) is that you can easily find the center of mass of any size circular cut-out located at any distance d on the x-axis. This is useful in design where you like to select the size or location of the cut-out to have the center of mass at a particular location.

SAMPLE 2.39 *Center of mass of two objects:* A square block of side 0.1 m and mass 2 kg sits on the side of a triangular wedge of mass 6 kg as shown in the figure. Locate the center of mass of the combined system.

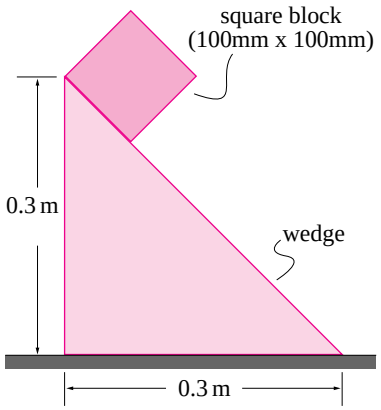


Figure 2.72: (Filename:sfig2.cm.2blocks)

Solution The center of mass of the triangular wedge is located at $h/3$ above the base and $\ell/3$ to the right of the vertical side. Let m_1 be the mass of the wedge and $\vec{r}_1$ be the position vector of its mass-center. Then, referring to Fig. 2.73,

$$\vec{r}_1 = \frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}.$$

The center of mass of the square block is located at its geometric center C_2. From geometry, we can see that the line AE that passes through C_2 is horizontal since $\angle OAB = 45^o$ ($h = \ell = 0.3\,\text{m}$) and $\angle DAE = 45^o$. Therefore, the coordinates of C_2 are ($d/\sqrt{2}, h$). Let m_2 and $\vec{r}_2$ be the mass and the position vector of the mass-center of the block, respectively. Then,

$$\vec{r}_2 = \frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath}.$$

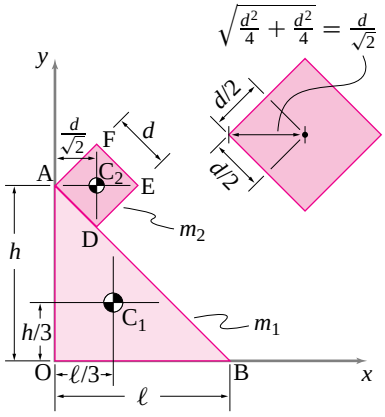


Figure 2.73: (Filename:sfig2.cm.2blocks.a)

Now, noting that $m_1 = 3m_2$ or $m_1 = 3m$, and $m_2 = m$ where $m = 2\,\text{kg}$, we find the center of mass of the combined system:

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{(m_1 + m_2)} \\
&= \frac{3m(\frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}) + m(\frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath})}{3m + m} \\
&= \frac{\cancel{m}[(\ell + \frac{d}{\sqrt{2}})\hat{\imath} + 2h\hat{\jmath}]}{4\cancel{m}} \\
&= \frac{1}{4}(\frac{d}{\sqrt{2}} + \ell)\hat{\imath} + \frac{h}{2}\hat{\jmath} \\
&= \frac{1}{4}(\frac{0.1\,\text{m}}{\sqrt{2}} + 0.3\,\text{m})\hat{\imath} + \frac{0.3\,\text{m}}{2}\hat{\jmath} \\
&= 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}}$$

Thus, the center of mass of the wedge and the block together is slightly closer to the side OA and higher up from the bottom OB than $C_1(0.1\,\text{m}, 0.1\,\text{m})$. This is what we should expect from the placement of the square block.

Note that we could have, again, used a 1-D calculation by placing a point mass $3m$ at C_1 and m at C_2, conneced the two points by a straight line, and located the center of mass C on that line such that $CC_2 = 3CC_1$. You can verify that the distance from $C_1(0.1\,\text{m}, 0.1\,\text{m})$ to $C(.093\,\text{m}, 0.15\,\text{m})$ is one third the distance from C to $C_2(.071\,\text{m}, 0.3\,\text{m})$.

3 Free body diagrams

The zeroth laws of mechanics

One way to understand something is to isolate it, see how it behaves on its own, and see how it responds to various stimuli. Then, when the thing is not isolated, you still think of it as isolated, but think of the effects of all its surroundings as stimuli. We can also see its behavior as causing stimulus to other things around it, which themselves can be thought of as isolated and stimulating back, and so on.

This reductionist approach is used throughout the physical and social sciences. A tobacco plant is understood in terms of its response to light, heat flow, the chemical environment, insects, and viruses. The economy of Singapore is understood in terms of the flow of money and goods in and out of the country. And social behavior is regarded as being a result of individuals reacting to the sights, sounds, smells, and touch of other individuals and thus causing sights, sounds, smell and touch that the others react to in turn, etc.

The isolated system approach to understanding is made most clear in thermodynamics courses. A system, usually a fluid, is isolated with rigid walls that allow no heat, motion or material to pass. Then, bit by bit, as the subject is developed, the response of the system to certain interactions across the boundaries is allowed. Eventually, enough interactions are understood that the system can be viewed as isolated even when in a useful context. The gas expanding in a refrigerator follows the same rules of heat-flow and work as when it was expanded in its 'isolated' container.

The subject of mechanics is also firmly rooted in the idea of an isolated system. As in elementary thermodynamics we will be solely concerned with *closed* systems. A (closed) system, in mechanics, is a fixed collection of material. You can draw an imaginary boundary around a system, then in your mind paint all the atoms inside the

boundary red, and then define the system as being the red atoms, no matter whether they cross the original boundary markers or not. Thus mechanics depends on bits of matter as being durable and non-ephemeral. A given bit of matter in a system exists forever, has the same mass forever, and is always in that system.①

Mechanics is based on the notion that any part of a system is itself a system and that all interactions between systems or subsystems have certain simple rules, most basically:

The measure of mechanical interaction is force,

and

What one system does to another, the other does back to the first.

Thus a person can be moved by forces, but not by the sight of a tree falling towards them or the attractive smell of a flower (these things may cause, by rules that fall outside of mechanics, forces that move a person). And when a person is moved by the force of the ground on her feet, the ground is pushed back just as hard. The two simple rules above, which we call the zeroth① laws of mechanics, imply that all the mechanical effects of interaction on a system can be represented by a sketch of the system with arrows showing the forces of interaction. If we want to know how the system in turn effects its surroundings we draw the opposite arrows on a sketch of the surroundings.

In mechanics a system is often called a *body* and when it is isolated it is *free* (as in *free* from its surroundings). In mechanics a sketch of an isolated system and the forces which act on it is called a *free body diagram*.

3.1 Free body diagrams

A *free body diagram* is a sketch of the system of interest and the forces that act on the system. A free body diagram precisely defines the system to which you are applying mechanics equations and the forces to be considered. Any reader of your calculations needs to see your free body diagrams. To put it directly, if you want to be right and be seen as right, then ②

Draw a Free Body Diagram!

The concept of the free body diagram is simple. In practice, however, drawing useful free body diagrams takes some thought, even for those practiced at the art. Here are some free body diagram properties and features:

- A free body diagram is a picture of the system for which you would like to apply linear or angular momentum balance or power balance. It shows the system isolated ('free') from its environment. That is, the free body diagram does *not* show things that are near or touching the system of interest. See figure 3.1.

① The mechanics of *open systems*, where material crosses the system boundaries, is important in fluid mechanics and even in some elementary dynamics problems (like rockets), where material is allowed to cross the system boundaries. The equations governing *open* systems are deduced from careful application of the more fundamental governing mechanics equations of *closed* systems.

① Why do we awkwardly number the first law as zero? Because it is really more of an underlying assumption, a background concept, than a law. As a law it is a little imprecise since force has not yet been defined. You could take the zeroth law as an implicit and partial definition of force. The phrase "zeroth law" means "important implicit assumption". The second part of the zeroth law is usually called "Newton's third law."

②**Free-body Perkins.** At Cornell University, in the 1950's, a professor Harold C. Perkins earned the nick name 'Free-body Perkins' by stopping random mechanics students in the hall and saying "You! Come in my office! Draw a free body diagram!" Students learned that they should draw free body diagrams, at least to please Free-body Perkins. Perkins was right. If you want to get the right answers to mechanics problems, and want to convince someone else that you have done so, you must draw good free body diagrams.

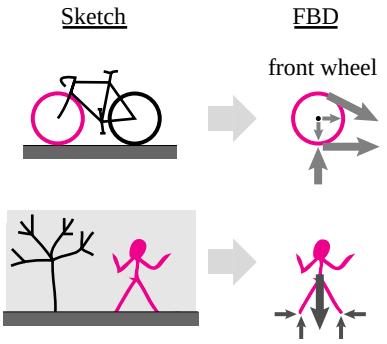


Figure 3.1: A sketch of a bicycle and a free body diagram of the braked front wheel. A sketch of a person and a free body diagram of a person.

(Filename:tfigure2.1)

- A free body diagram may show one or more particles, rigid bodies, deformable bodies, or parts thereof such as a machine, a component of a machine, or a part of a component of a machine. You can draw a free body diagram of any collection of material that you can identify. The word 'body' connotes a standard object in some people's minds. In the context of free body diagrams, 'body' means system. The body in a free body diagram may be a subsystem of the overall system of interest.
- The free body diagram of a system shows the forces and moments that the surroundings impose on the system. That is, since the only method of mechanical interaction that God has invented is force (and moment), the free body diagram shows what it would take to mechanically fool the system if it was literally cut free. That is, the motion of the system would be totally unchanged if it were cut free and the forces shown on the free body diagram were applied as a replacement for all external interactions.
- The forces and moments are shown on the free body diagram at the points where they are applied. These places are where you made 'cuts' to *free* the *body*.
- At places where the outside environment causes or restricts translation of the isolated system, a contact force is drawn on the free body diagram. Draw the contact force outside the sketch of the system for viewing clarity. A block supported by a hinge with friction in figure 3.2 illustrates how the reaction force on the block due to the hinge is best shown outside the block.
- At connections to the outside world that cause or restrict rotation of the system a contact torque (or couple or moment) is drawn. Draw this moment outside the system for viewing clarity. Refer again to figure 3.2 to see how the moment on the block due to the friction of the hinge is best shown outside the block.
- The free body diagram shows the system cut free from the source of any *body forces* applied to the system. Body forces are forces that act on the inside of a body from objects outside the body. It is best to draw the body forces on the interior of the body, at the center of mass if that correctly represents the net effect of the body forces. Figure 3.2 shows the cleanest way to represent the gravity force on the uniform block acting at the center of mass. ①.
- The free body diagram shows all external forces acting on the system but *no* internal forces — forces between objects within the body are not shown.
- The free body diagram shows nothing about the motion ②. It shows: no "centrifugal force", no "acceleration force", and no "inertial force". For statics this is a non-issue because inertial terms are neglected for all purposes.
  Velocities, inertial forces, and acceleration forces do not show on a free body diagram.
  The prescription that you *not show inertial forces* is a practical lie. In the D'Alembert approach to dynamics you can show inertial forces on the free body diagram. The D'Alembert approach is discussed in box **??** on page **??**. This legitimate and intuitive approach to dynamics is not followed in this book because of the frequent sign errors amongst beginners who use it.

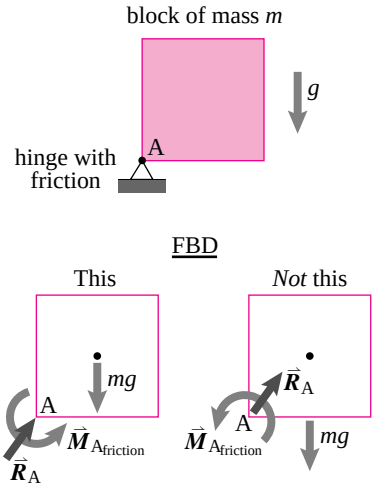


Figure 3.2: A uniform block of mass *m* supported by a hinge with friction in the presence of gravity. The free body diagram on the right is correct, just less clear than the one on the left.

(Filename:tfigure2.outside.loads)

① **Body Forces.** In this book, the only body force we consider is gravity. For near-earth gravity, gravity forces show on the free body diagram as a single force at the center of gravity, or as a collection of forces at the center of gravity of each of the system parts. For parts of electric motors and generators, not covered here in detail, electrostatic or electro-dynamic body forces also need to be considered.

②**Caution:**A common *error* made by beginning dynamics students is to put velocity and/or acceleration arrows on the free body diagram.

How to draw a free body diagram

We suggest the following procedure for drawing a free body diagram, as shown schematically in fig. 3.3

(a) Define in your own mind what system or what collection of material, you would like to write momentum balance equations for. This subsystem may be part of your overall system of interest.

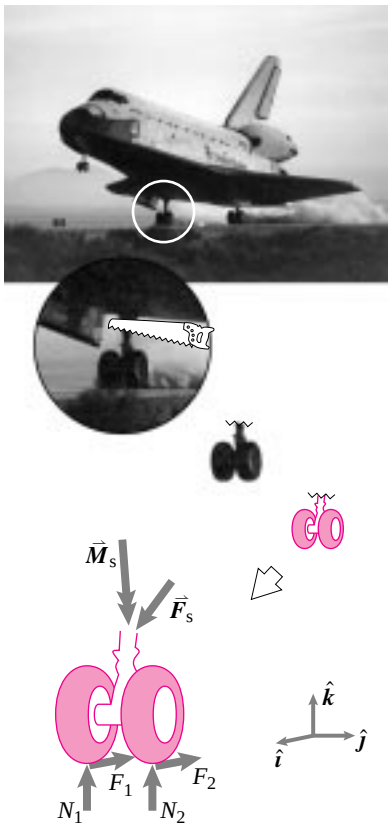


Figure 3.3: The process of drawing a FBD is illustrated by the sequence shown.

(Filename:tfigure2.howtoFBD)

(b) Draw a sketch of this system. Your sketch may include various cut marks to show how it is isolated from its environment. At each place the system has been cut free from its environment you imagine that you have cut the system free with a sharp scalpel or with a chain saw.

(c) Look systematically at the picture at the places that the system interacts with material *not* shown in the picture, places where you made 'cuts'.

(d) Use forces and torques to fool the system into thinking it has not been cut. For example, if the system is being pushed in a given direction at a given contact point, then show a force in that direction at that point. If a system is being prevented from rotating by a (cut) rod, then show a torque at that cut.

(e) To show that you have cut the system from the earth's gravity force show the force of gravity on the system's center of mass or on the centers of mass of its parts.

How to draw forces on free body diagrams

How you draw a force on a free body diagram depends on

- How much you know about the force when you draw the free body diagram. Do you know its direction? its magnitude?; and
- Your choice of notation (which may vary from vector to vector within one free body diagram). See page 13 for a description of the 'symbolic' and 'graphical' vector notations.

Some of the possibilities are shown in fig. 3.4 for three common notations for a 2D force in the cases when (a) any $\vec{F}$ possible, (b) the direction of $\vec{F}$ is fixed, and (c) everything about $\vec{F}$ is fixed.

	(a) Nothing is known about $\vec{F}$	(b) Direction of $\vec{F}$ is known	(c) $\vec{F}$ is known
Symbolic	$\vec{F}$	$\vec{F} = F\frac{\hat{i}+\hat{j}}{\sqrt{2}}$	$10\,\text{N}\hat{i} + 10\,\text{N}\hat{j}$
Graphical	F, θ	F, 45^o	14.1 N, 45^o
Components	F_x, F_y	F_1, F_1	10 N, 10 N

Figure 3.4: The various ways of notating a force on a free body diagram. (a) nothing is known or everything is variable (b) the direction is known, (c) Everything is known. In one free body diagram different notations can be used for different forces, as needed or convenient.

(Filename:tfigure.fbdvectnot)

Equivalent force systems

The concept of 'fooling' a system with forces is somewhat subtle. If the free body diagram involves 'cutting' a rope what force should one show? A rope is made of many fibers so cutting the rope means cutting all of the rope fibers. Should one show hundreds of force vectors, one for each fiber that is cut? The answer is: yes and no. You would be correct to draw all of these hundreds of forces at the fiber cuts. But, since the equations that are used with any free body diagram involve only the total force and total moment, you are also allowed to replace these forces with an equivalent force system (see section 2.4).

Any force system acting on a given free body diagram can be replaced by an equivalent force and couple.

In the case of a rope, a single force directed nearly parallel to the rope and acting at about the center of the rope's cross section is equivalent to the force system consisting of all the fiber forces. In the case of an ideal rope, the force is exactly parallel to the rope and acts exactly at its center.

Action and reaction

For some systems you will want to draw free body diagrams of subsystems. For example, to study a machine, you may need to draw free body diagrams of its parts; for a building, you may draw free body diagrams of various structural components; and, for a biomechanics analysis, you may 'cut up' a human body. When separating a system into parts, you must take account of how the subsystems interact. Say these subsystems, e.g. two touching parts of a machine, are called $\mathcal{A}$ and $\mathcal{B}$. We then have that

If	$\mathcal{A}$	feels force	$\vec{F}$	and couple	$\vec{M}$	from $\mathcal{B}$,
then	$\mathcal{B}$	feels force	$-\vec{F}$	and couple	$-\vec{M}$	from $\mathcal{A}$.

To be precise we must make clear that $\vec{F}$ and $-\vec{F}$ have the same line of action.①

The principle of action and reaction doesn't say anything about what force or moment acts on one object. It only says that the actor of a force and moment gets back the opposite force and moment.

It is easy to make mistakes when drawing free body diagrams involving action and reaction. Box 3.3 on page 94 shows some correct and incorrect partial FBD's of interacting bodies $\mathcal{A}$ and $\mathcal{B}$. Use notation consistent with box **??** on page **??** for the action and reaction vectors.

① The principle of action and reaction can be derived from the momentum balance laws by drawing free body diagrams of little slivers of material. Nonetheless, in practice you can think of the principle of action and reaction as a basic law of mechanics. Newton did. The principal of action and reaction is "Newton's third law".

Interactions

The way objects interact mechanically is by the transmission of a force or a set of forces. If you want to show the effect of body $\mathcal{B}$ on $\mathcal{A}$, in the most general case you can expect a force and a moment which are equivalent to the whole force system, however complex.

That is, the most general interaction of two bodies requires knowing

- six numbers in three dimensions (three force components and three moment components)
- and three numbers in two dimensions (two force components and one moment).

Many things often do not interact in this most general way so often fewer numbers are required. You will use what you know about the interaction of particular bodies to reduce the number of unknown quantities in your free body diagrams.

Some of the common ways in which mechanical things interact, or are assumed to interact, are described in the following sections. You can use these simplifications in your work.

Constrained motion and free motion

One general principle of interaction forces and moments concerns constraints. Wherever a *motion* of $\mathcal{A}$ is either caused *or* prevented by $\mathcal{B}$ there is a corresponding *force* shown at the interaction point on the free body diagram of $\mathcal{A}$. Similarly if $\mathcal{B}$ causes or prevents *rotation* there is a *moment* (or torque or couple) shown on the free body diagram of $\mathcal{A}$ at the place of interaction.

The converse is also true. Many kinds of mechanical attachment gadgets are specifically designed to allow motion. If an attachment allows free motion in some direction the free body diagram shows no force in that direction. If the attachment allows free rotation about an axis then the free body diagram shows no moment (couple or torque) about that axis.

You can think of each attachment point as having a variety of jobs to do. For every possible direction of translation and rotation, the attachment has to either allow free motion or restrict the motion. In every way that motion is restricted (or caused) by the connection a force or moment is required. In every way that motion is free there is no force or couple. Motion of body $\mathcal{A}$ is caused and restricted by forces and couples which act on $\mathcal{A}$. Motion is freely allowed by the absence of such forces and couples.

Here are some of the common connections and the free body diagrams with which they are associated.

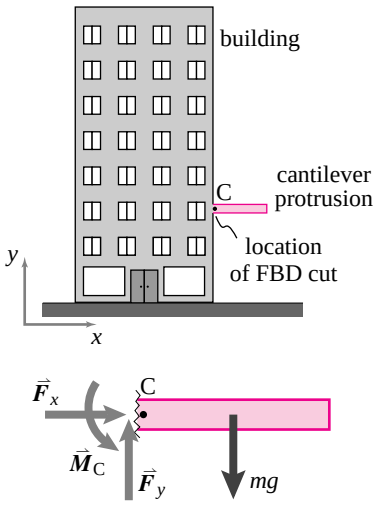


Figure 3.5: **A rigid connection: a cantilever structure on a building.** At the point C where the cantilever structure is connected to the building all motions are restricted so every possible force needs to be shown on the free body diagram cut at C.

(Filename:tfigure2.rigid)

Cuts at rigid connections

Sometimes the body you draw in a free body diagram is firmly attached to another.

Figure 3.5 shows a cantilever structure on a building. The free body diagram of the cantilever has to show all possible force and load components. Since we have used vector notation for the force $\vec{F}$ and the moment $\vec{M}_C$ we can be ambiguous about whether we are doing a two or three dimensional analysis.

A common question by new mechanics students seeing a free body diagram like in figure 3.5 is: 'gravity is pointing down, so why do we have to show a horizontal reaction force at C?' Well, for a stationary building and cantilever a quick statics analysis reveals that $\vec{F}_C$ must be vertical, so the question is reasonable. But one must remember: this book is about statics *and* dynamics and in dynamics *the forces on a body do not add to zero*. In fact, the building shown in figure 3.5 might be accelerating rapidly to the right due to the motions of a violent earthquake occurring at the instant pictured in the figure. Sometimes you know a force is going to turn out to be zero, as for the sideways force in this example if treated as a statics problem. In these cases it is a matter of taste whether or not you show the sideways force on the free body diagram (see box 3.1 on page 84).

The attachment of the cantilever to the building at C in figure 3.5 is surely intended to be rigid and prevent the cantilever from moving up or down (falling), from moving

sideways (and drifting into another building) or from rotating about point C. In most of the building's life, the horizontal reaction at C is small. But since the connection at C clearly prevents relative horizontal motion, a horizontal reaction force is drawn on the free body diagram. During an earthquake, this horizontal component will turn out to be not zero.

The situation with rigid connections is shown more abstractly in figure 3.6.

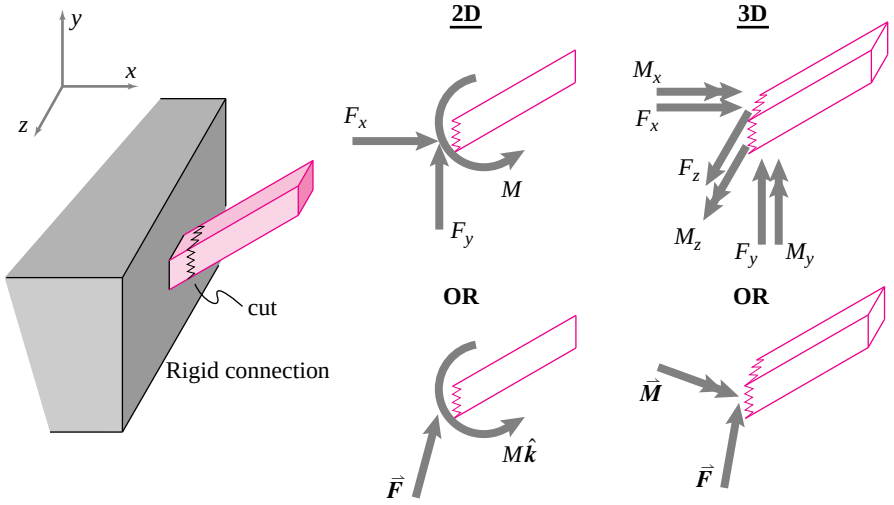


Figure 3.6: A rigid connection shown with partial free body diagrams in two and three dimensions. One has a choice between showing the separate force components (top) or using the vector notation for forces and moments (bottom). The double head on the moment vector is optional.

(Filename:tfigure2.rigidb)

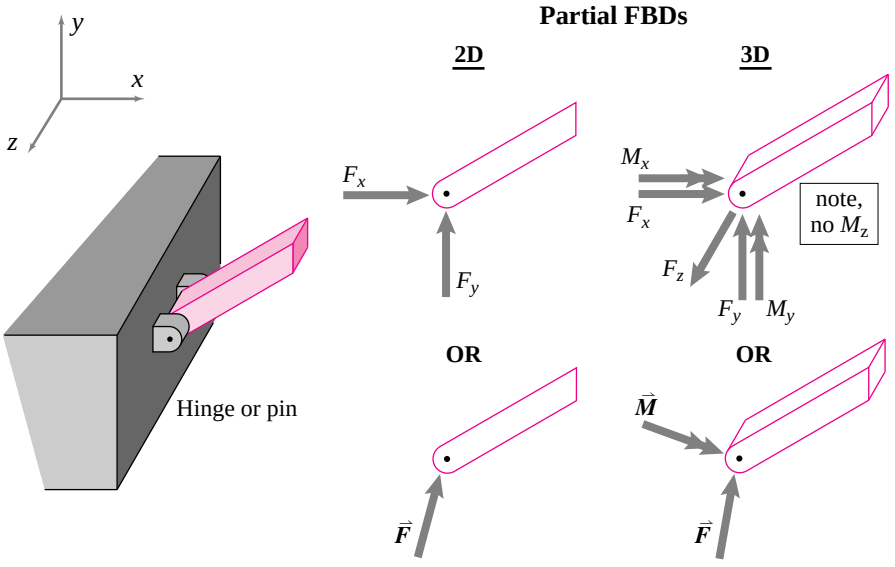


Figure 3.7: **A hinge with partial free body diagrams in 2D and 3-D.** A hinge joint is also called a pin joint because it is sometimes built by drilling a hole and inserting a pin.

(Filename:tfigure2.hinge)

Cuts at hinges

A hinge, shown in figure 3.7, allows rotation and prevents translation. Thus, the free body diagram of an object cut at a hinge shows *no* torque about the hinge axis but does show the force or its components which prevent translation.

There is some ambiguity about how to model pin joints in three dimensions. The ambiguity is shown with reference to a hinged door (figure 3.8). Clearly, one hinge, if the sole attachment, prevents rotation of the door about the x and y axes shown. So, it is natural to show a couple (torque or moment) in the x direction, M_x, and in the y direction, M_y. But, the hinge does not provide very stiff resistance to rotations in these directions compared to the resistance of the other hinge. That is, even if both hinges are modeled as ball and socket joints (see the next sub-section), offering no resistance to rotation, the door still cannot rotate about the x and y axes.

If a connection between objects prevents relative translation or rotation that is already prevented by another stiffer connection, then the more compliant connection reaction is often neglected. Even without rotational constraints, the translational constraints at the hinges A and B restrict rotation of the door shown in figure 3.8. The hinges are probably well modeled — that is, they will lead to reasonably accurate

3.1 *THEORY*

How much mechanics reasoning should you use when you draw a free body diagram?

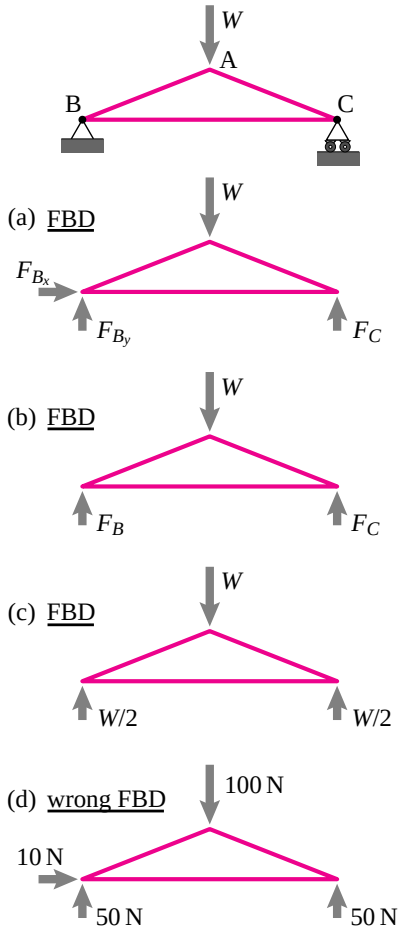
The simple rules for drawing free body diagrams prescribe an unknown force every place a motion is prevented and an unknown torque where rotation is prevented. Consider the simple symmetric truss with a load W in the middle. By this prescription the free body diagram to draw is shown as (a). There is an unknown force restricting both horizontal and vertical motion at the hinge at B.

However, a person who knows some statics will quickly deduce that the horizontal force at B is zero and thus draw the free body diagram in figure (b). Or if they really think ahead they will draw the free body diagram in (c). All three free body diagrams are correct. In particular diagram (a) is correct even though F_{Bx} turns out to be zero and (b) is correct even though F_B turns out to be equal to F_C.

Some people, thinking ahead, sometimes say that the free body diagram in (a) is wrong. But it should be pointed out that free body diagram (a) is correct because the force F_{Bx} is not specified and therefore could be zero. Free body diagram (d), on the other hand, explicitly and incorrectly assigns a non-zero value to F_{Bx}, so it is wrong.

A reasonable approach is to follow the naive rules, and then later use the force and momentum equations to find out more about the forces. That is use free body diagram (a) and discover (c) using the laws of mechanics. If you are confident about the anticipated results, it is sometimes a time saver to use diagrams analogous to (b) or (c) but beware of

- making assumptions that are not reasonable, and
- wasting time trying to think ahead when the force and momentum balance equations will tell all in the end anyway.



calculations of forces and motions — by ball and socket joints at A and B. In 2-D , a ball and socket joint is equivalent to a hinge or pin joint.

Ball and socket joint

Sometimes one wishes to attach two objects in a way that allows no relative translation but for which all rotation is free. The device that is used for this purpose is called a 'ball and socket' joint. It is constructed by rigidly attaching a sphere (the ball) to one of the objects and rigidly attaching a partial spherical cavity (the socket) to the other object.

The human hip joint is a ball and socket joint. At the upper end of the femur bone is the femoral head, a sphere to within a few thousandths of an inch. The hip bone has a spherical cup that accurately fits the femoral head. Car suspensions are constructed from a three-dimensional truss-like mechanism. Some of the parts need free relative rotation in three dimensions and thus use a joint called a 'ball joint' or 'rod end' that is a ball and socket joint.

Since the ball and socket joint allows all rotations, no moment is shown at a cut ball and socket joint. Since a ball and socket joint prevents relative translation in all directions, the possibility of force in any direction is shown.

String, rope, wires, and light chain

One way to keep a radio tower from falling over is with wire, as shown in figure 3.10. If the mass and weight of the wires seems small it is common to assume they can only transmit forces along their length. Moments are not shown because ropes, strings, and wires are generally assumed to be so compliant in bending that the bending moments are negligible. We define *tension* to be the force pulling away from a free body diagram cut. ①

① **Caution:** Sometimes string like things should not be treated as idealized strings. *Short* wires can be stiff so bending moments may not be negligible. The mass of chains can be significant so that the mass and weight may not be negligible, the direction of the tension force in a sagging chain is not in the direction connecting the two chain endpoints.

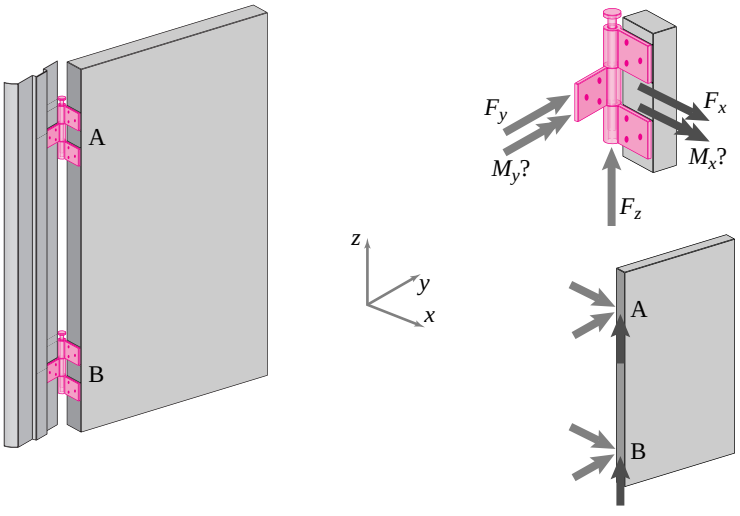


Figure 3.8: **A door held by hinges.** One must decide whether to model hinges as proper hinges or as ball and socket joints. The partial free body diagram of the door at the lower right neglects the couples at the hinges, effectively idealizing the hinges as ball and socket joints. This idealization is generally quite accurate since the rotations that each hinge might resist are already resisted by their being two connection points.

(Filename:tfigure2.door)

Springs and dashpots

Springs are used in many machines to absorb and return small amounts of energy. Dashpots are used to absorb energy. They are shown schematically in fig. 3.12. Often springs and dashpots are light in comparison to the machinery to which they are attached so their mass and weight are neglected. Often they are attached with pin joints, ball and socket joints, or other kinds of flexible connections so only forces are transmitted. Since they only have forces at their ends they are 'two-force' bodies and, by the reasoning of coming section 4.1, the forces at their ends are equal, opposite,

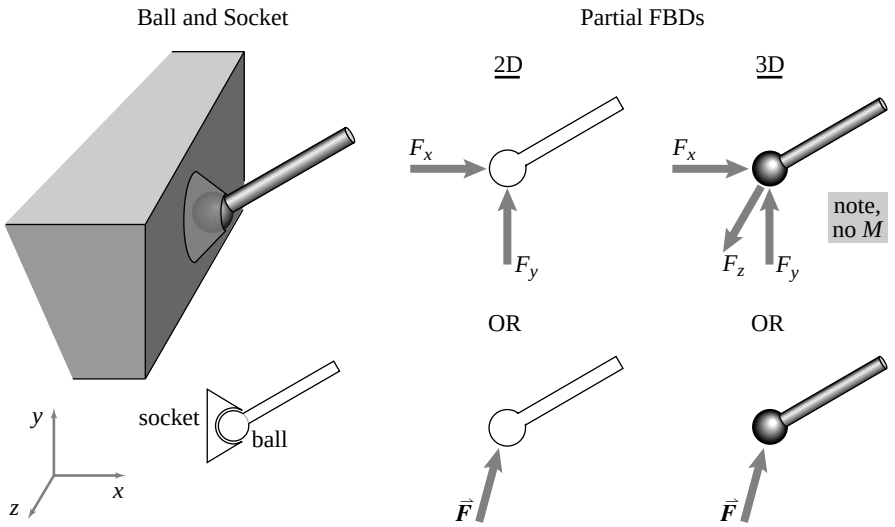


Figure 3.9: **A ball and socket joint** allows all relative rotations and no relative translations so reaction forces, but not moments, are shown on the partial free body diagrams. In two dimensions a ball and socket joint is just like a pin joint. The top partial free body diagrams show the reaction in component form. The bottom illustrations show the reaction in vector form.

(Filename:tfigure2.ballands)

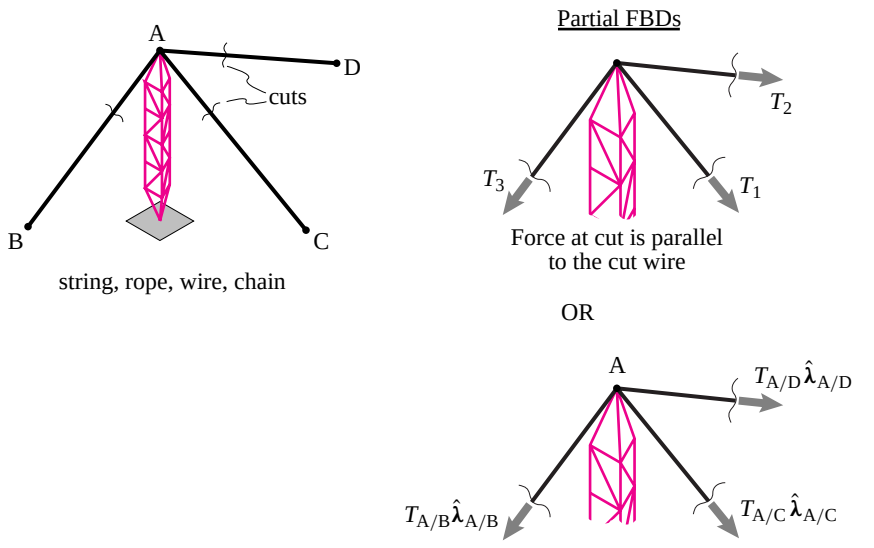


Figure 3.10: **A radio tower kept from falling with three wires.** A partial free body diagram of the tower is drawn two different ways. The upper figure shows three tensions that are parallel to the three wires. The lower partial free body diagram is more explicit, showing the forces to be in the directions of the $\hat{\boldsymbol{\lambda}}$s, unit vectors parallel to the wires.

(Filename:tfigure2.string)

and along the line of connection.

Springs

Springs often look like the standard spring drawing in figure 3.11.

If the tension in a spring is a function of its length alone, independent of its rate of lengthening, the spring is said to be 'elastic.' If the tension in the spring is proportional to its stretch the spring is said to be 'linear.' The assumption of linear elastic behavior is accurate for many physical springs. So, most often if one says one is using a spring, the linear and elastic properties are assumed.

The stretch of a spring is the amount by which the spring is longer than when it is relaxed. This relaxed length is also called the 'unstretched' length, the 'rest' length, or the 'reference' length. If we call the unstretched length, the length of the spring when its tension is zero, ℓ_0, and the present length ℓ, then the stretch of the spring is $\Delta\ell = \ell - \ell_0$. The tension in the spring is proportional to this stretch. Most often people use the letter k for the proportionality constant and say 'the spring has constant k.' So the basic equation defining a spring is

$$T = k\Delta\ell.$$

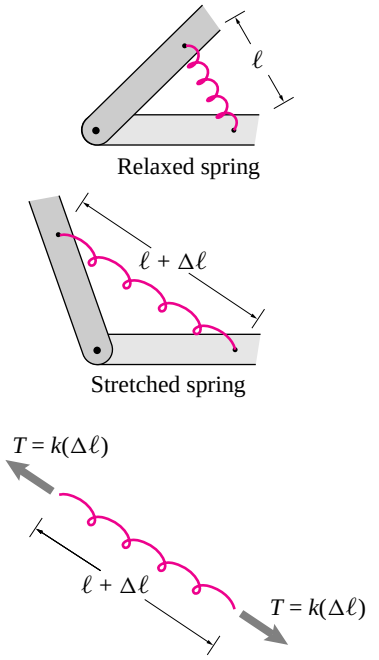


Figure 3.11: **Spring connection.** The tension in a spring is usually assumed to be proportional to its change in length, with proportionality constant k: $T = k\Delta\ell$.

(Filename:tfigure2.spring)

Dashpots

Dashpots are used to absorb, or dampen, energy. The most familiar example is in the shock absorbers of a car. The symbol for a dashpot shown in figure 3.12 is meant to suggest the mechanism. A fluid in a cylinder leaks around a plunger as the dashpot gets longer and shorter. The dashpot resists motion in both directions.

The tension in the dashpot is usually assumed to be proportional to the rate at which it lengthens, although this approximation is not especially accurate for most dampers one can buy. The relation is assumed to hold for negative lengthening as well. So the compression (negative tension) is proportional to the rate at which the dashpot shortens (negative lengthens). The defining equation for a linear dashpot is:

$$T = C\,\dot{\ell}$$

where C is the dashpot constant.

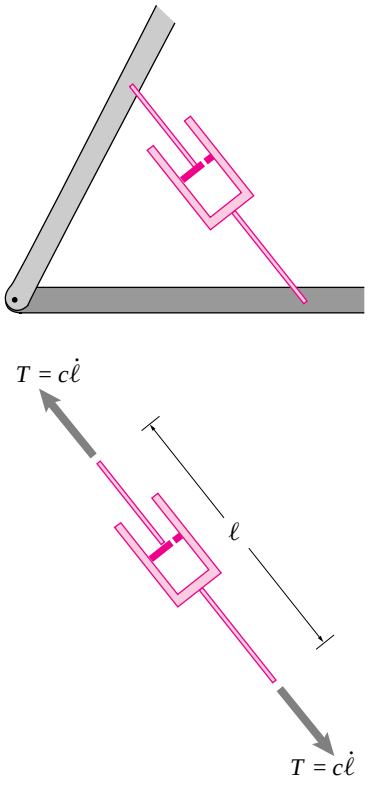


Figure 3.12: **A dashpot.** A dashpot is shown here connecting two parts of a mechanism. The tension in the dashpot is proportional to the rate at which it lengthens.

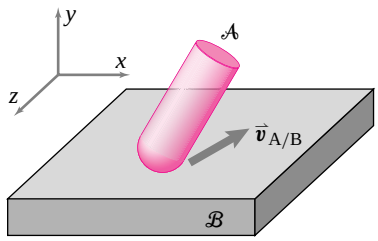
(Filename:tfigure2.dashpot)

Collisions

Two objects are said to *collide* when some interaction force or moment between them becomes very large, so large that other forces acting on the bodies become negligible. For example, in a car collision the force of interaction at the bumpers may be many times the weight of the car or the reaction forces acting on the wheels.

The analysis of collisions is a little different than the analysis of smooth motions, as will be discussed later in the text. But this analysis still depends on free body diagrams showing the non-negligible collision forces. See figure 3.13. Knowing which forces to include and which to ignore in a collision problem is an issue which can have great subtlety. Some rules of thumb:

- ignore forces from gravity, springs, and at places where contact is broken in the collision, and
- include forces at places where new contact is made, or where contact is maintained.



$\mathcal{A}$ slides frictionally on $\mathcal{B}$

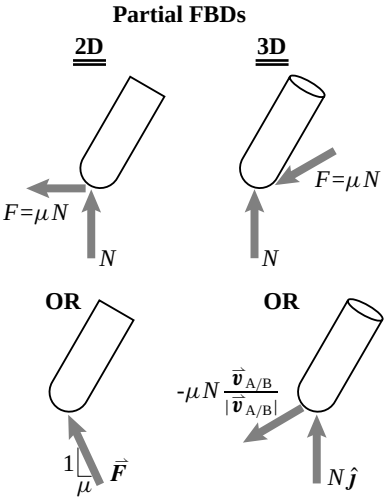
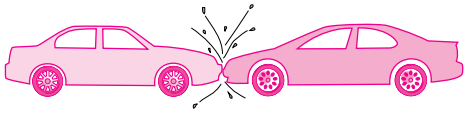
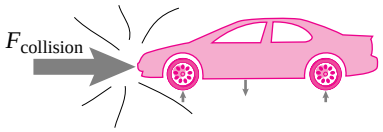


Figure 3.14: Object $\mathcal{A}$ slides on the plane $\mathcal{B}$. The friction force on $\mathcal{A}$ is in the direction that opposes the relative motion.

(Filename:tfigure2.friction)



Collision!



The collision forces are assumed to be much bigger than all other forces on the FBD during the collision.

Figure 3.13: Here cars are shown colliding. A free body diagram of the right car shows the collision force and should not show other forces which are negligibly small. Here they are shown as negligibly small forces to give the idea that they may be much smaller than the collision force. The wheel reaction forces are neglected because of the spring compliance of the suspension and tires.

(Filename:tfigure2.collisions)

Friction

When two independent solids are in contact relative slipping motion is resisted by *friction*. Friction can prevent slip and resists any slip which does occur.

The force on body $\mathcal{A}$ from body $\mathcal{B}$ is decomposed into a part which is tangent to the surface of contact $\vec{F}$, with $|\vec{F}| = F$, and a part which is normal to the surface N. The relation between these forces depends on the relative slip of the bodies $\vec{v}_{A/B}$. The magnitude of the frictional force is usually assumed to be proportional to the normal force with proportionality constant μ. So the deceptively simple defining equation for the friction force F during slip is

$$F = \mu N$$

where N is the component of the interaction force in the inwards normal direction. The problem with this simple equation is that it assumes you have drawn the friction force in the direction opposing the slip of $\mathcal{A}$ relative to $\mathcal{B}$. If the direction of the friction force has been drawn incorrectly then the formula gives the wrong answer.

If two bodies are in contact but are *not* sliding then the friction force can still keep the objects from sliding. The strength of the friction bond is often assumed to be proportional to the normal force with proportionality constant μ. Thus if there is no slip we have that the force is something less than or equal to the strength,

$$|F| \leq \mu N.$$

Partial FBD's for the cases of slip and no slip are shown in figures 3.14 and 3.15, respectively. See the appendix for a further discussion of friction. To make things a little more precise, for those more formally inclined, we can write the friction equations as follows:

$$\begin{aligned} \vec{F}(\mathcal{B} \text{ acts on } \mathcal{A}) &= -\mu N \frac{\vec{v}_{A/B}}{|\vec{v}_{A/B}|} \quad , \text{ if } \vec{v}_{A/B} \neq \vec{0}, \\ |\vec{F}(\mathcal{B} \text{ acts on } \mathcal{A})| &\leq \mu N \qquad\qquad , \text{ if } \vec{v}_{A/B} = \vec{0}. \end{aligned}$$

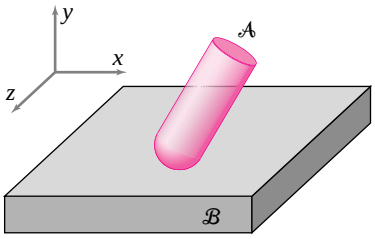
The unit vector $\frac{\vec{v}_{A/B}}{|\vec{v}_{A/B}|}$ is in the direction of relative slip. The principle of action and reaction, discussed previously, determines the force that $\mathcal{A}$ acts on $\mathcal{B}$.

The simplest friction law, the one we use in this book, uses a single constant coefficient of friction μ. Usually $.05 \leq \mu \leq 1.2$. We do not distinguish the static coefficient μ_s from the dynamic coefficient μ_d or μ_k. That is $\mu = \mu_s = \mu_k = \mu_d$ for our purposes. We promote the use of this simplest law for a few reasons.

- All friction laws used are quite approximate, no matter how complex. Unless the distinction between static and dynamic coefficients of friction is essential to the engineering calculation, using $\mu_s \neq \mu_k$ doesn't add to the calculation's usefulness.
- The concept of a static coefficient of friction that is larger than a dynamic coefficient is, it turns out, not well defined if bodies have more than one point of contact, which they often do have.
- Students learning to do dynamics are often confused about how to handle problems with friction. Since the more complex friction laws are of questionable usefulness and correctness, it seems time is better spent understanding the simplest relations.

In summary, the simple model of friction we use is:

> Friction resists relative slipping motion. *During slip the friction force opposes relative motion and has magnitude* $F = \mu N$. When there is no slip the magnitude of the friction force F cannot be determined from the friction law but it cannot exceed μN, $F \leq \mu N$.



$\mathcal{A}$ does not slide relative to $\mathcal{B}$

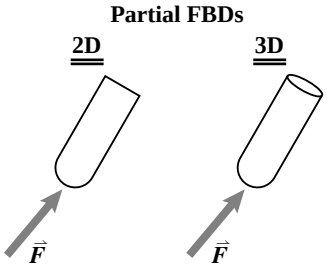


Figure 3.15: Object $\mathcal{A}$ does not slide relative to the plane $\mathcal{B}$.

(Filename:tfigure2.noslip)

Sometimes people describe the friction coefficient with a *friction angle* ϕ rather than the coefficient of friction (see fig. 3.16). The friction angle is the angle between the net interaction force (normal force plus friction force) and the normal to the sliding surface when slip is occurring. The relation between the friction coefficient μ and the friction angle ϕ is

$$\tan\phi \equiv \mu.$$

The use of ϕ or μ to describe friction are equivalent. Which you use is a matter of taste and convenience.

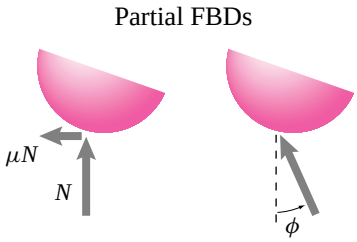


Figure 3.16: Two ways of characterizing friction: the friction coefficient μ and friction angle ϕ.

(Filename:tfigure2.friction.angle)

"Smooth" and "rough" surfaces

As a modeling simplification for situations where we would like to neglect friction forces we sometimes assume frictionless contact and thus set $\mu = \phi = 0$. In many books, but never in this one, the phrase "perfectly smooth" means frictionless. It is true that when separated by a little fluid (say water between your feet and the bathroom tile, or oil between pieces of a bearing) that smooth surfaces slide easily by each other. And even without a lubricant sometimes slipping can be reduced by roughening a surface. But making a surface progressively smoother does not diminish the friction to zero. In fact, extremely smooth surfaces sometimes have anomalously high friction. In general, there is no reliable relation correlation smoothness and low friction.

Similarly many books, but not this one, use the phrase "perfectly rough" to mean perfectly high friction ($\mu \to \infty$ and $\phi \to 90^o$) and hence that no slip is allowed. This is misleading twice over. First, as just stated, rougher surfaces do not reliably have more friction than smooth ones. Second, even when $\mu \to \infty$ slip can proceed in some situations (see, for example, box 4.1 on page 120).

We use the phrase *frictionless* to mean that there is no tangential force component and not the misleading words "perfectly smooth". We use the phrase *no slip* to mean that no tangential motion is allowed and not the misleading words "perfectly rough".

Rolling contact

An idealization for the non-skidding contact of balls, wheels, and the like is *pure rolling*.

> *Objects $\mathcal{A}$ and $\mathcal{B}$ are in pure rolling contact when their (relatively convex) contacting points have equal velocity. They are not slipping, separating, or interpenetrating.*

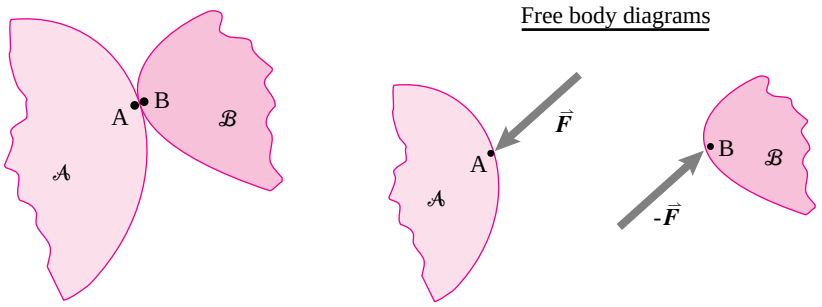


Figure 3.17: Rolling contact: Points of contact on adjoining bodies have the same velocity, $\vec{v}_A = \vec{v}_B$. But, $\vec{\omega}_{\mathcal{A}}$ is not necessarily equal to $\vec{\omega}_{\mathcal{B}}$.

(Filename:tfigure2.rolling.contact)

Most often, we are interested in cases where the contacting bodies have some non-zero relative angular velocity — a ball sitting *still* on level ground may be technically in rolling contact, but not interestingly so.

The simplest common example is the rolling of a round wheel on a flat surface in two dimensions. See figure 3.18. In practice, there is often confusion about the

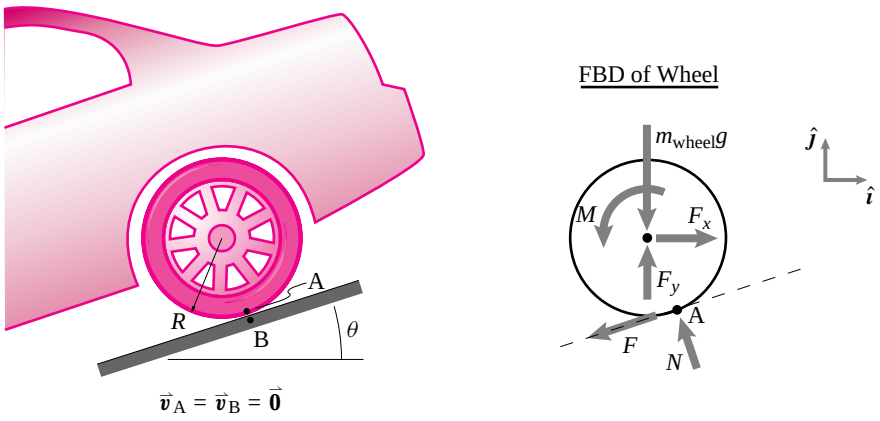


Figure 3.18: Pure rolling of a round wheel on a flat slope in two dimensions.

(Filename:tfigure2.pure.rolling.wheel)

direction and magnitude of the force F shown in the free body diagram in figure 3.18. Here is a recipe:

1.) Draw F as shown in any direction, tangent to the surface.

2.) Solve the statics or dynamics problem and find F. (It may turn out to be a negative force, which is fine.)
3.) Check that rolling is really possible; that is, that slip would not occur. If the force is greater than the frictional strength, $|F| > \mu N$, the assumption of rolling contact is not appropriate. In this case, you must assume that $F = \mu N$ or $F = -\mu N$ and that slip occurs; then, re-solve the problem.

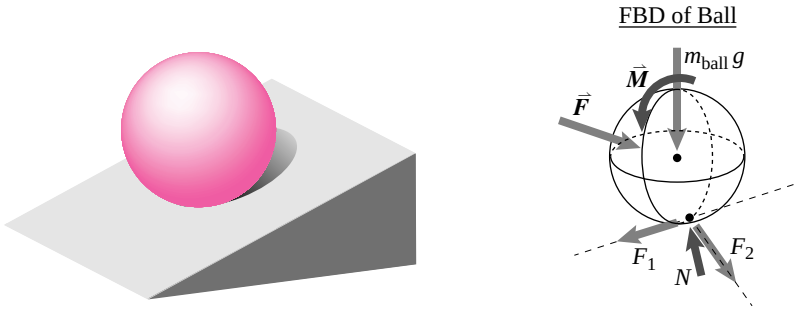


Figure 3.19: **Rolling ball in 3-D.** The force $\vec{F}$ and moment $\vec{M}$ are applied loads from, say, wind, gravity, and any attachments. N is the normal reaction and F_1 and F_2 are the in plane components of the frictional reaction. One must check the no-slip condition, $\mu^2 N^2 \geq F_1^2 + F_2^2$.
(Filename:tfigure2.3D.rolling)

In three-dimensional rolling contact, we have a free body diagram that again looks like a free body diagram for non-slipping frictional contact. Consider, for example, the ball shown in figure 3.19. For the friction force to be less than the friction coefficient times the normal force, we have

$$\sqrt{F_1^2 + F_2^2} \;\leq\; \mu N \qquad \text{or} \qquad F_1^2 + F_2^2 \;\leq\; \mu^2 N^2 \qquad \text{no slip condition}$$

Rolling is just a special case of frictional contact. It is the case where bodies contact at a single point (or on a line, as with cylinders) and have relative rotation yet have no relative velocity at their contacting points. The tricky part about rolling is the kinematic analysis. This kinematics, we take up in section 9.3 on page 513 after you have learned more about angular velocity $\vec{\omega}$.

Rolling resistance

Non-ideal rolling contact includes provision for *rolling resistance*. This resistance is simply represented by either moving the location of the point of contact force or by a contact couple. Rolling resistance leads to subtle questions which we would like to finesse here. A brief introduction is given in chapter 10.

Ideal wheels

An ideal wheel is an approximation of a real wheel. It is a sensible approximation if the mass of the wheel is negligible, bearing friction is negligible, and rolling resistance is negligible. Free body diagrams of undriven ideal wheels in two and three dimensions are shown in figure 3.20. This idealization is rationalized in chapter 4 in box 4.1 on page 120. Note that if the wheel is not massless, the 2-D free body diagram looks more like the one in figure 3.20b with $F_{\text{friction}} \leq \mu N$.

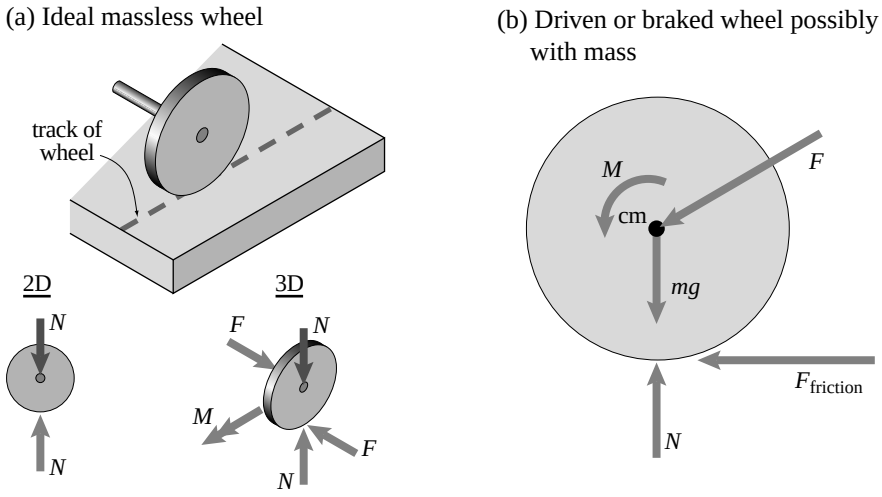


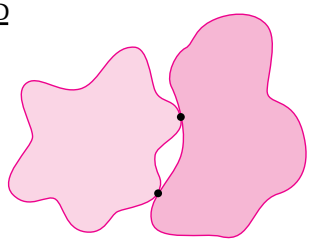
Figure 3.20: An ideal wheel is massless, rigid, undriven, round and rolls on flat rigid ground with no rolling resistance. Free body diagrams of ideal undriven wheels are shown in two and three dimensions. The force F shown in the three-dimensional picture is perpendicular to the path of the wheel. (b) 2D free body diagram of a wheel with mass, possibly driven or braked. If the wheel has mass but is not driven or braked the figure is unchanged but for the moment M being zero.

(Filename:tfigure2.idealwheel)

3.2 *THEORY*
Conformal contact of rigid bodies: a near impossibility

If you take two arbitrary shaped rigid objects and make them touch without overlapping you will most often only be able to make contact at a few points (typically 1 to 3 points in 2D, and 1 to 6 points in 3D). Cut out two pieces of cardboard (leaving no straight edges) and slide them around on a table and you will see this.

2D



But machines are not made of random parts. Many parts are made to conform, like an axle and a bearing, and many parts are machined with flat surfaces and thus seem to conform with each other whether or not by explicit intent. Many machined objects nominally (in name) conform. But do they really conform?

Let us consider the case of two rigid objects pressed together at their flat surfaces. We can think of a rigid object as a limiting case of stiffer and stiffer objects; and we can think of flat surfaces as the limiting case of less and less rough surfaces. Now imagine pressing two objects together that are not quite flat and are also not quite rigid.

On the one hand, no matter how stiff the objects so long as they have a little compliance, if you made them flatter and flatter, eventually the little bit of deformation from your pressing would make them conform and they would make contact along the contact surfaces (where the details of the pressure distribution still would depend on the shape of the bodies away from the contact area).

On the other hand, no matter how flat the contact surfaces (so long as they weren't perfectly flat), if you made them stiffer and stiffer, the deformation would be extinguished and eventually they would only make contact at a few points (as in the figure above).

To get the idea considering two springs in parallel that have almost equal length. Consider the limits as the lengths become matched and as the stiffnesses go to infinity (see problem. **??** on page **??**).

That is, the meaning of the phrase 'flat and rigid' depends on whether you first think of the objects as flat and then rigid, or first rigid and then flat. In math language this dependence on the order of limits is called a *distinguished* limit. Here it means that the idea of rigid objects touching on flat surfaces is ill-defined.

This distinguished limit is not a mathematical fine point. It corresponds to the physical reality that things which look flat and hard touch each other with a pressure distribution that is highly dependent on fine details of construction and loading.

In many mechanics problems one can, by means of the equations of elementary mechanics taught here, find an equivalent force system to that of the micro-contact force distribution. Using more advanced mechanics reasoning (the theory of elasticity) and computers (finite element programs) one can estimate certain features of the details of the contact pressure distribution if one knows the surface shapes accurately. But in many mechanics calculations the details of the contact force distribution are left unknown.

Extended contact

When things touch each other over an extended region, like the block on the plane of fig. 3.21a, it is not clear what forces to put where on the free body diagram. On the one hand one imagines reality to be somewhat reflected by millions of small forces as in fig. 3.21b which may or may not be divided into normal (n_i) and frictional (f_i) components. But one generally is not interested in such detail, and even if interested one cannot find it easily (see box 3.1 on page 92).

A simple approach is to replace the detailed force distribution with a single equivalent force, as shown in fig. 3.21c broken into components. The location of this force is not relevant for some problems.①

If one wants to make clear that the contact forces serve to keep the block from rotating, one may replace the contact force distribution with a pair of contacts at the corners as in fig. 3.21d.

Summary of free body diagrams.

- Draw one or more clear free body diagrams!
- Forces and moments on the free body diagram show *all* mechanical interactions.
- Every point on the boundary of a body has a force in every direction that motion is either being caused *or* prevented. Similarly with torques.
- If you do not know the direction of a force, use vector notation to show that the direction is yet to be determined.
- Leave off the free body diagram forces that you think are negligible such as, possibly:
  - The force of air on small slowly moving bodies;
  - Forces that prevent motion that is already prevented by a much stiffer means (as for the torques at each of a pair of hinges);
  - Non-collisional forces, such as gravity, during a collision.

① In 3D, contact force distributions cannot always be replaced with an equivalent force at an appropriate location (see section 2.4). A couple may be required. Nonetheless, many people often make the approximation that a contact force distribution can be replaced by a force at an appropriate location. This approximation neglects any frictional resistance to twisting about the normal to the contact plane.

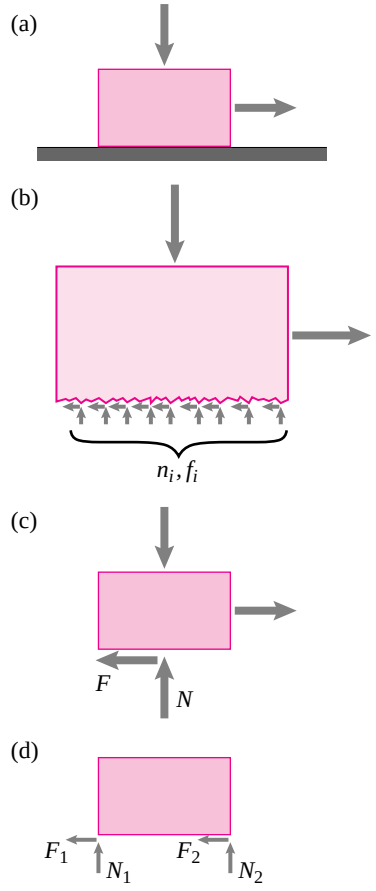
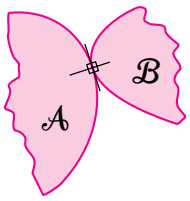


Figure 3.21: The contact forces of a block on a plane can be sensibly modeled in various ways.

(Filename:tfigure.conformalblock)

3.3 Action and reaction on partial FBD's of interacting bodies

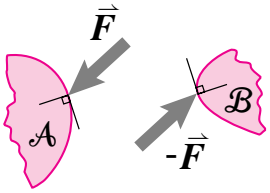
Imagine bodies $\mathcal{A}$ and $\mathcal{B}$ are interacting and that you want to draw separate free body diagrams (FBD's) of each.



Part of the FBD of each shows the interaction force. The FBD of $\mathcal{A}$ shows the force of $\mathcal{B}$ on $\mathcal{A}$ and the FBD of $\mathcal{B}$ shows the force of $\mathcal{A}$ on $\mathcal{B}$. To illustrate the concept, we show partial FBD's of both $\mathcal{A}$ and $\mathcal{B}$ using the principle of action and reaction. Items (a - d) are correct and items (e - g) are wrong.

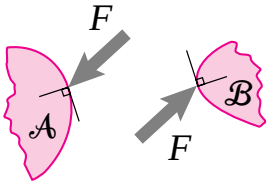
Correct partial FBD's

(a)



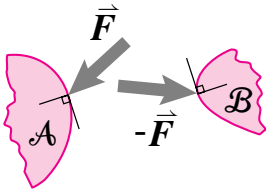
(a) Here are some good partial FBD's: the arrows are equal and opposite and the the vector notations are opposite in sign.

(b)



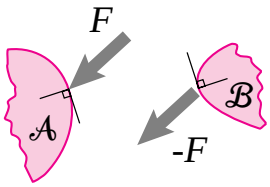
(b) These FBD's are also good since the opposite arrows multiplied by equal magnitude F produce net vectors that are equal and opposite.

(c)



(c) The FBD's may *look* wrong, and they are impractically misleading and not advised. But technically they are okay because we take the vector notation to have precedence over the drawing inaccuracy.

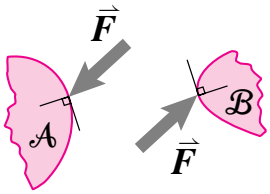
(d)



(d) The FBD's may look wrong but since no vector notation is used, the forces should be interpreted as in the direction of the drawn arrows and multiplied by the shown scalars. Since the same arrow is multiplied by F and $-F$, the net vectors are actually equal and opposite.

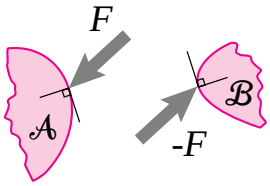
Wrong partial FBD's

(e)



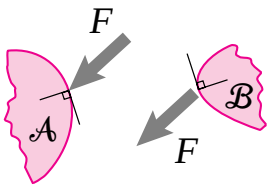
(e) The FBD's are wrong because the vector notation $\vec{F}$ takes precedence over the drawn arrows. So the drawing shows the *same* force $\vec{F}$ acting on both $\mathcal{A}$ and $\mathcal{B}$, rather than the opposite force.

(f)



(f) Since the opposite arrow is multiplied by the negative scalars, the FBD's here show the *same* force acting on both $\mathcal{A}$ and $\mathcal{B}$. Treating a double-negative as a negative is a common mistake.

(g)



(g) The FBD's are obviously wrong since they again show the same force acting on $\mathcal{A}$ and $\mathcal{B}$. These FBD's would represent *the principle of double action* which applies to laundry detergents but not to mechanics.

SAMPLE 3.1 *A mass and a pulley.* A block of mass m is held up by applying a force F through a massless pulley as shown in the figure. Assume the string to be massless. Draw free body diagrams of the mass and the pulley separately and as one system.

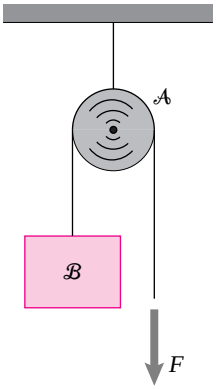


Figure 3.22: (Filename:sfig2.1.02)

Solution The free body diagrams of the block and the pulley are shown in Fig. 3.23. Since the string is massless and we assume an ideal massless pulley, the tension in the string is the same on both sides of the pulley. Therefore, the force applied by the string on the block is simply F. When the mass and the pulley are considered as one system, the force in the string on the left side of the pulley doesn't show because it is internal to the system.

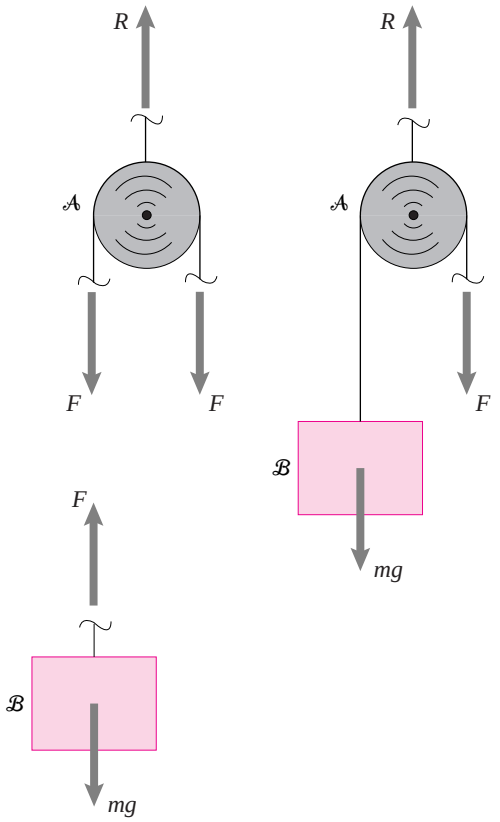


Figure 3.23: The free body diagrams of the mass, the pulley, and the mass-pulley system. Note that for the purpose of drawing the free body diagram we need not show that we know that $R = 2F$. Similarly, we could have chosen to show two different rope tensions on the sides of the pulley and reasoned that they are equal as is done in the text.

(Filename:sfig2.1.02a)

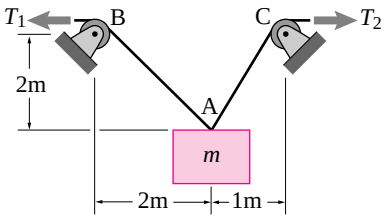


Figure 3.24: (Filename:sfig2.1.2a)

SAMPLE 3.2 *Forces in strings.* A block of mass m is held in position by strings AB and AC as shown in Fig. 3.24. Draw a free body diagram of the block and write the vector sum of all the forces shown on the diagram. Use a suitable coordinate system.

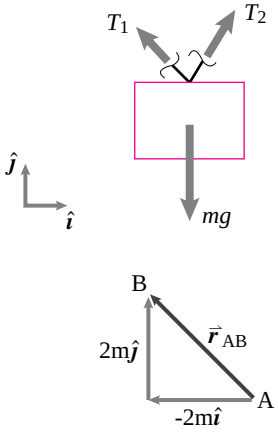


Figure 3.25: Free body diagram of the block and a diagram of the vector $\vec{r}_{AB}$.
(Filename:sfig2.1.2b)

Solution To draw a free body diagram of the block, we first *free* the block. We cut strings AB and AC very close to point A and show the forces applied by the cut strings on the block. We also isolate the block from the earth and show the force due to gravity. (See Fig. 3.25.)

To write the vector sum of all the forces, we need to write the forces as vectors. To write these vectors, we first choose an xy coordinate system with basis vectors $\hat{\imath}$ and $\hat{\jmath}$ as shown in Fig. 3.25. Then, we express each force as a product of its magnitude and a unit vector in the direction of the force. So,

$$\vec{T}_1 = T_1\hat{\lambda}_{AB} = T_1\frac{\vec{r}_{AB}}{|\vec{r}_{AB}|},$$

where $\vec{r}_{AB}$ is a vector from A to B and $|\vec{r}_{AB}|$ is its magnitude. From the given geometry,

$$\begin{aligned} \vec{r}_{AB} &= -2\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath} \\ \Rightarrow \quad \hat{\lambda}_{AB} &= \frac{2\,\cancel{\text{m}}(-\hat{\imath}+\hat{\jmath})}{\sqrt{2^2+2^2}\,\cancel{\text{m}}} = \frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}). \end{aligned}$$

Thus,

$$\vec{T}_1 = T_1\frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}).$$

Similarly,

$$\begin{aligned} \vec{T}_2 &= T_2\frac{1}{\sqrt{5}}(\hat{\imath}+2\hat{\jmath}) \\ m\vec{g} &= -mg\hat{\jmath}. \end{aligned}$$

Now, we write the sum of all the forces:

$$\begin{aligned} \sum\vec{F} &= \vec{T}_1 + \vec{T}_2 + m\vec{g} \\ &= \left(-\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}}\right)\hat{\imath} + \left(\frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg\right)\hat{\jmath}. \end{aligned}$$

$$\boxed{\sum\vec{F} = \left(-\tfrac{T_1}{\sqrt{2}} + \tfrac{T_2}{\sqrt{5}}\right)\hat{\imath} + \left(\tfrac{T_1}{\sqrt{2}} + \tfrac{2T_2}{\sqrt{5}} - mg\right)\hat{\jmath}}$$

SAMPLE 3.3 *Two bodies connected by a massless spring.* Two carts A and B are connected by a massless spring. The carts are pulled to the left with a force F and to the right with a force T as shown in Fig. 3.26. Assume the wheels of the carts to be massless and frictionless. Draw free body diagrams of

- cart A,
- cart B, and
- carts A and B together.

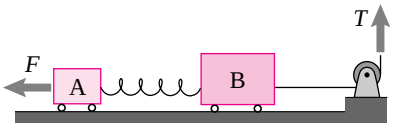


Figure 3.26: Two carts connected by a massless spring

(Filename:sfig2.1.3a)

Solution The three free body diagrams are shown in Fig. 3.27 (a) and (b). In Fig. 3.27 (a) the force F_s is applied by the spring on the two carts. Why is this force the same on both carts? In Fig. 3.27(b) the spring is a part of the system. Therefore, the forces applied by the spring on the carts and the forces applied by the carts on the spring are internal to the system. Therefore these forces do not show on the free body diagram.

Note that the normal reaction of the ground can be shown either as separate forces on the two wheels of each cart or as a resultant reaction.

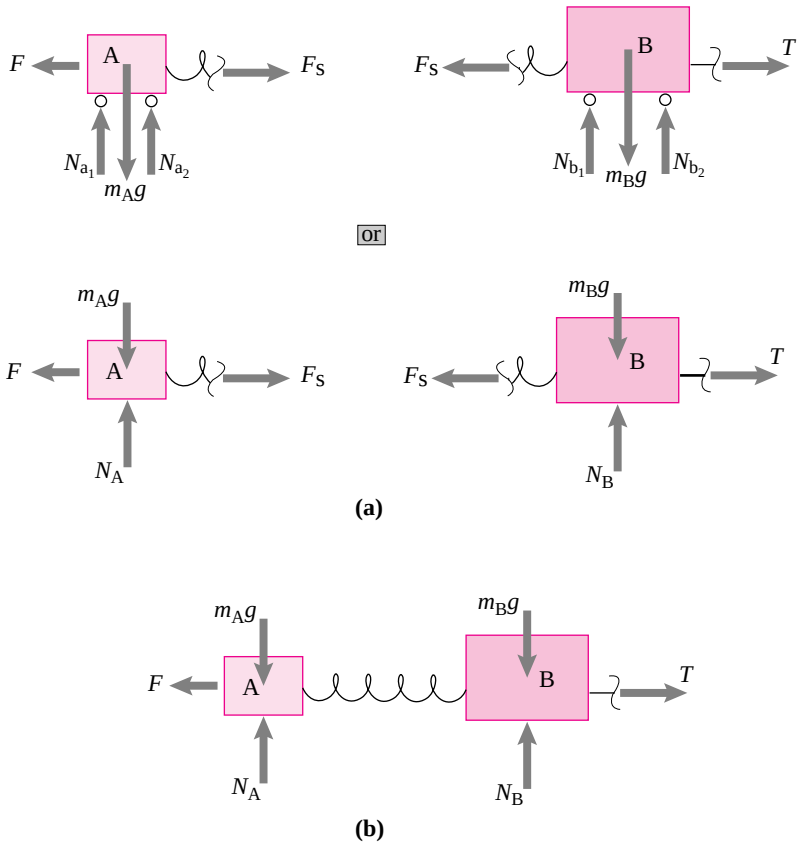


Figure 3.27: Free body diagrams of (a) cart A and cart B separately and (b) cart A and B together

(Filename:sfig2.1.3b)

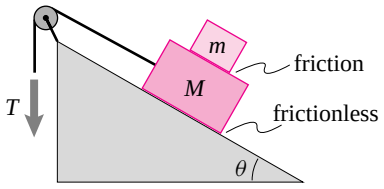


Figure 3.28: Two blocks held in place on a frictionless inclined surface

(Filename:sfig2.1.4a)

SAMPLE 3.4 *Stacked blocks at rest on an inclined plane.* Blocks A and B with masses m and M, respectively, rest on a frictionless inclined surface with the help of force T as shown in Fig. 3.28. There is friction between the two blocks. Draw free body diagrams of each of the the two blocks separately and a free body diagram of the two blocks as one system.

Solution The three free body diagrams are shown in Fig. 3.29 (a) and (b). Note the action and reaction pairs between the two blocks; the normal force N_A and the friction force F_f between the two bodies A and B. If we consider the two blocks together as a system, then the forces N_A and F_f do not show on the free body diagram of the system (See Fig. 3.29(b)), because now they are internal to the system.

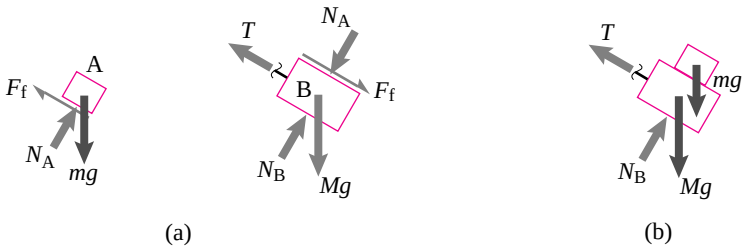


Figure 3.29: Free body diagrams of (a) block A and block B separately and (b) blocks A and B together.

(Filename:sfig2.1.4b)

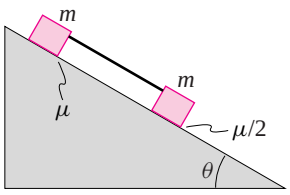


Figure 3.30: Two blocks slide down a frictional inclined plane. The blocks are connected by a light rigid rod.

(Filename:sfig2.1.15)

SAMPLE 3.5 *Two blocks slide down a frictional inclined plane.* Two blocks of identical mass but different material properties are connected by a massless rigid rod. The system slides down an inclined plane which provides different friction to the two blocks. Draw free body diagrams of the two blocks separately and of the system (two blocks with the rod).

Solution The Free body diagrams are shown in Fig. 3.31. Note that the friction forces on the two blocks are different because the coefficients of friction are different for the two blocks. The normal reaction of the plane, however, is the same for each block (why?).

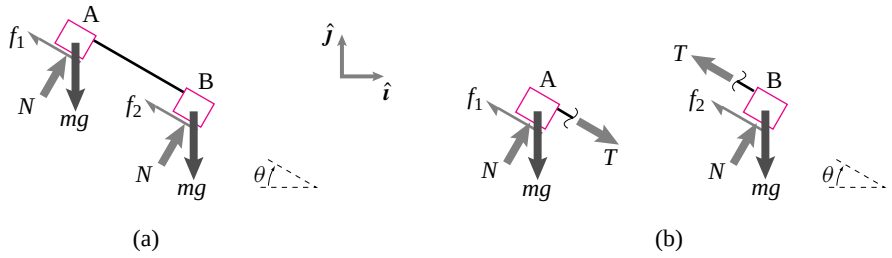


Figure 3.31: Free body diagrams of (a) the two blocks and the rod as a system and (b) the two blocks separately.

(Filename:sfig2.1.15a)

SAMPLE 3.6 *Massless pulleys.* A force F is applied to the pulley arrangement connected to the cart of mass m shown in Fig. 3.32. All the pulleys are massless and frictionless. The wheels of the cart are also massless but there is friction between the wheels and the horizontal surface. Draw a free body diagram of the cart, its wheels, and the two pulleys attached to the cart, all as one system.

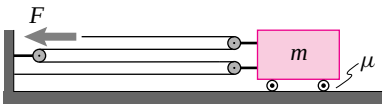


Figure 3.32: A cart with pulleys

(Filename:sfig2.1.7a)

Solution The free body diagram of the cart system is shown in Fig. 3.33. The force in each part of the string is the same because it is the same string that passes over all the pulleys.

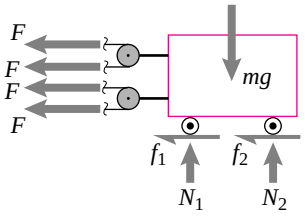


Figure 3.33: Free body diagram of the cart. (Filename:sfig2.1.7b)

SAMPLE 3.7 *Two carts connected by pulleys.* The two masses shown in Fig. 3.34 have frictionless bases and round frictionless pulleys. The inextensible massless cord connecting them is always taut. Mass A is pulled to the left by force F and mass B is pulled to the right by force P as shown in the figure. Draw free body diagrams of each mass.

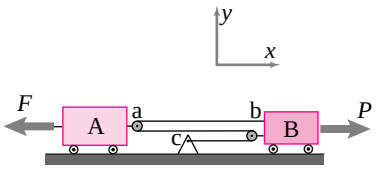


Figure 3.34: Two carts connected by massless pulleys.

(Filename:sfig2.1.12)

Solution Let the tension in the cord be T. Since the pulleys and the cord are massless, the tension is the same in each section of the cord. This equality is clearly shown in the Free body diagrams of the two masses below.

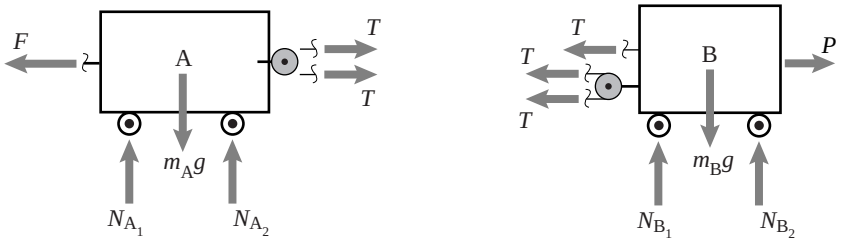


Figure 3.35: Free body diagrams of the two masses. (Filename:sfig2.1.12a)

Comments: We have shown unequal normal reactions on the wheels of mass B. In fact, the two reactions would be equal only if the forces applied by the cord on mass B satisfy a particular condition. Can you see what condition they must satisfy for, say, $N_{A_1} = N_{A_2}$. [Hint: think about the moment of forces about the center of mass A.]

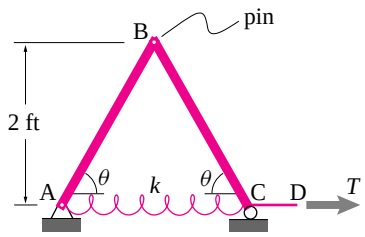


Figure 3.36: (Filename:sfig2.1.5)

SAMPLE 3.8 *Structures with pin connections.* A horizontal force T is applied on the structure shown in the figure. The structure has pin connections at A and B and a roller support at C. Bars AB and BC are rigid. Draw free body diagrams of each bar and the structure including the spring.

Solution The free body diagrams are shown in figure 3.37. Note that there are both vertical and horizontal forces at the pin connections because pins restrict translation in any direction. At the roller support at point C there is only vertical force from the support (T is an externally applied force).

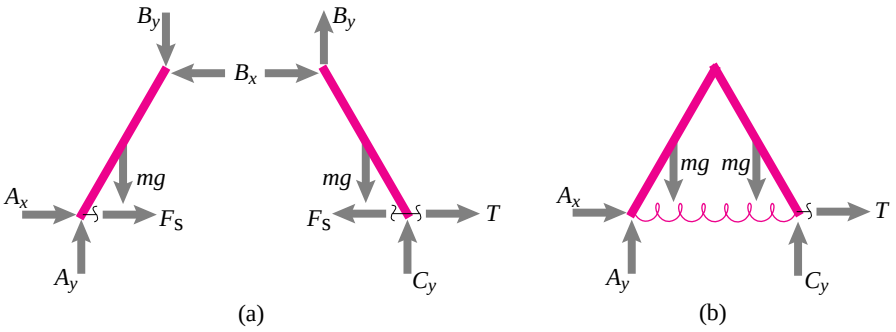


Figure 3.37: Free body diagrams of (a) the individual bars and (b) the structure as a whole.
(Filename:sfig2.1.5a)

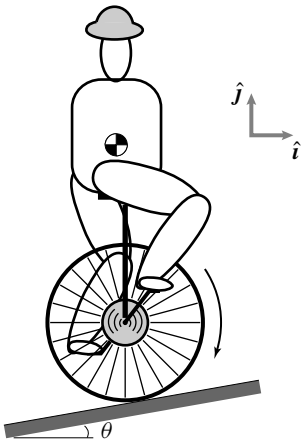


Figure 3.38: The unicyclist
(Filename:sfig2.1.8)

SAMPLE 3.9 *A unicyclist in action.* A unicyclist weighing 160 lbs exerts a force on the front pedal with a vertical component of 30 lbf at the instant shown in figure 3.38. The rear pedal barely touches the other foot. Assume the wheel and the frame are massless. Draw free body diagrams of the cyclist and the cycle. Make other reasonable assumptions if required.

Solution Let us assume, there is friction between the seat and the cyclist and between the pedal and the cyclist's foot. Let's also assume a 2-D analysis. The free body diagrams of the cyclist and the cycle are shown in Fig. 3.39. We assume no couple interaction at the seat.

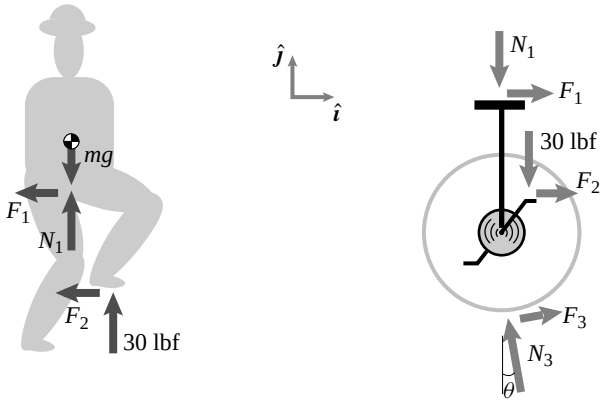


Figure 3.39: Free body diagram of the cyclist and the cycle. (Filename:sfig2.1.8a)

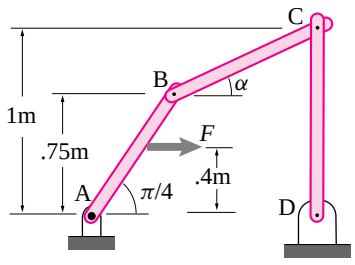


Figure 3.40: A four bar linkage.

(Filename:sfig2.2.1)

SAMPLE 3.10 The four bar linkage shown in the figure is pushed to the right with a force F as shown in the figure. Pins A, C & D are frictionless but joint B is rusty and has friction. Neglect gravity; and assume that bar AB is massless. Draw free body diagrams of each of the bars separately and of the whole structure. Use consistent notation for the interaction forces and moments. Clearly mark the action-reaction pairs.

Solution A 'good' pin resists any translation of the pinned body, but allows free rotation of the body about an axis through the pin. The body reacts with an equal and opposite force on the pin. When two bodies are connected by a pin, the pin exerts separate forces on the two bodies. Ideally, in the free body diagram , we should show the pin, the first body, and the second body separately and draw the interaction forces. This process, however, results in too many free body diagrams. Therefore, usually, we let the pin be a part of one of the objects and draw the free body diagrams of the two objects.

Note that the pin at joint B is rusty, which means, it will resist a relative rotation of the two bars. Therefore, we show a moment, in addition to a force, at point B of each of the two rods AB and BC.

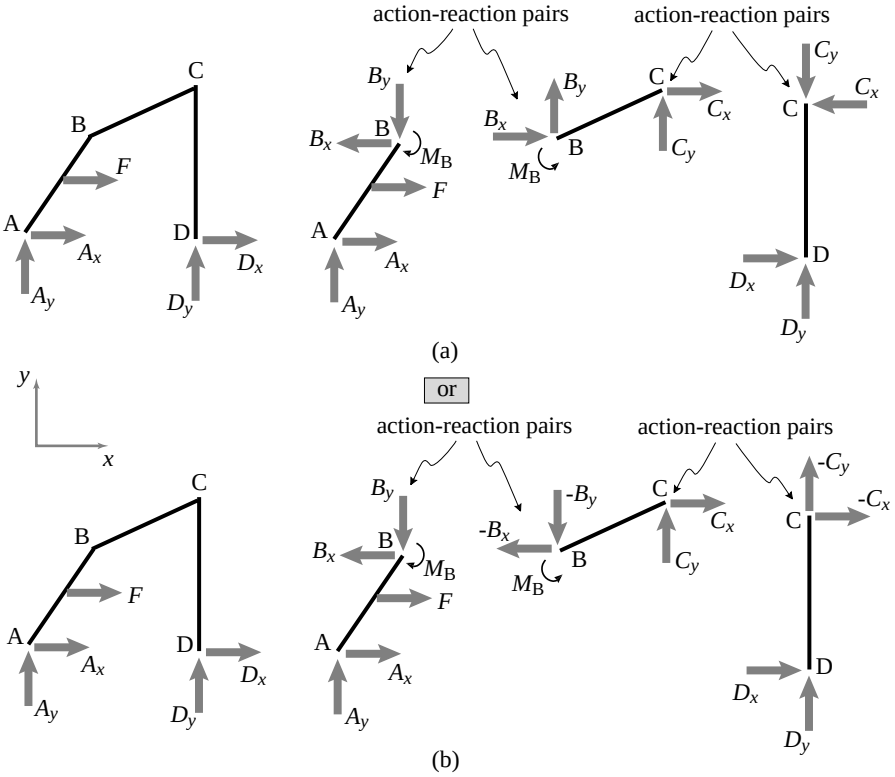


Figure 3.41: Style 1: Free body diagrams of the structure and the individual bars. The forces shown in (a) and (b) are the same.

(Filename:sfig2.2.1b)

Figure 3.41 shows the free body diagrams of the structure and the individual rods. In this figure, we show the forces in terms of their x- and y-components. The directions of the forces are shown by the arrows and the magnitude is labeled as A_x, A_y, etc. Therefore, a force, shown as an arrow in the positive x-direction with 'magnitude' A_x, is the same as that shown as an arrow in the negative x-direction with magnitude $-A_x$. Thus, the free body diagrams in Fig. 3.41(a) show exactly the

same forces as in Fig. 3.41(b).

In Fig. 3.42, we show the forces by an arrow in an arbitrary direction. The corresponding labels represent their magnitudes. The angles represent the unknown directions of the forces.

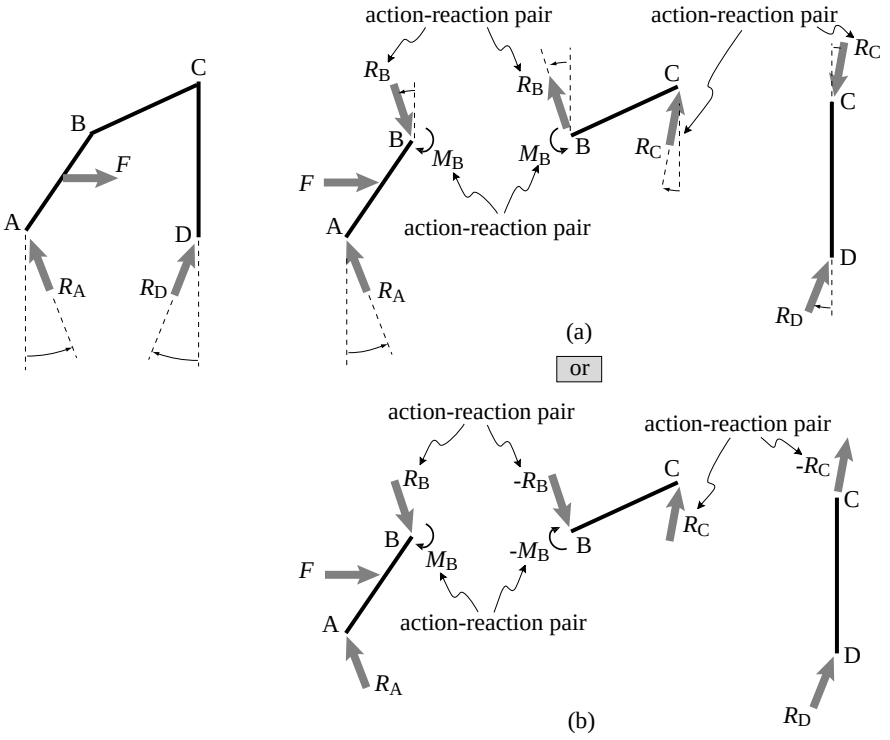


Figure 3.42: Style 2: Free body diagrams of the structure and the individual bars. The forces shown in (a) and (b) are the same.

(Filename:sfig2.2.1c)

In Fig. 3.43, we show yet another way of drawing and labeling the free body diagrams, where the forces are labeled as vectors.

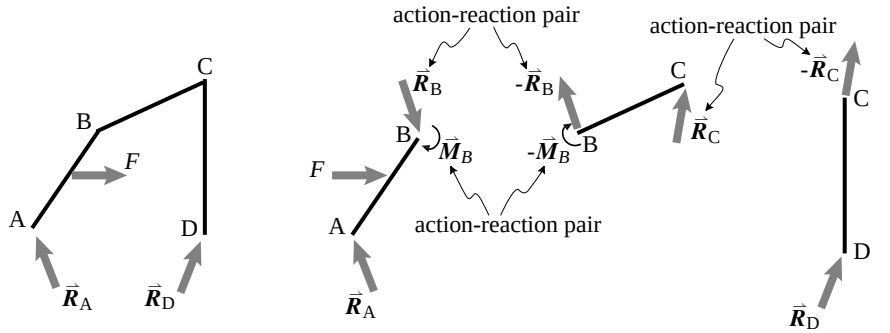


Figure 3.43: Style 3: Free body diagrams of the structure and the individual bars. The label of a force indicates both its magnitude and direction. The arrows are arbitrary and merely indicate that a force or a moment acts on those locations.

(Filename:sfig2.2.1d)

Note: There are *no* two-force bodies in this problem. Bar AB is massless but is not a two-force member because it has a couple at its end.

4 Statics

Statics is the mechanics of things that don't move. But everything does move, at least a little. So statics doesn't exactly apply to anything. The statics equations are, however, a very good approximation of the more general dynamics equations for many practical problems. The statics equations are also easier to manage than the dynamics equations. So with little loss of accuracy, sometimes very little loss, and a great saving of effort, sometimes a very great saving, many calculations can be performed using a statics model instead of a more general dynamics model. Thus it is not surprising that typical engineers perform many more statics calculations than dynamics calculations. Statics is the core of structural and strength analysis. And even for a moving system, say an accelerating car, statics calculations are appropriate for many of the parts. Simply put, and perhaps painful to remember when you complete this chapter and begin the chapters on dynamics, statics is more useful to most engineers than dynamics.

One possible motivation for studying statics is that the statics skills all carry over to dynamics which is a more general subject. But the opposite is maybe closer to truth. Statics is indeed a special case of dynamics. But for many engineers the benefit of going on from statics to dynamics is the sharpening of the more-useful statics skills that ensue.

How does general mechanics simplify to statics?

The mechanics equations in the front cover are applicable to everything most engineers will ever encounter. The statics equations are a special case that apply only approximately to many things. In statics we set the right hand sides of equations I and II to zero. The neglected terms involve mass times acceleration and are called the *inertial terms*. For statics we set the inertial terms $\dot{\vec{L}}$ and $\dot{\vec{H}}_C$ to zero. Thus we replace

the linear and angular momentum balance equations with their simplified forms

$$\sum_{\substack{\text{All external}\\ \text{forces}}} \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}} \qquad \text{and} \qquad \sum_{\substack{\text{All external}\\ \text{torques}}} \vec{\boldsymbol{M}}_{\mathrm{C}} = \vec{\boldsymbol{0}} \qquad \text{(Ic,IIc)}$$

which are called the *force balance* and *moment balance* equations and together are called the *equilibrium* equations. The forces to be summed are those that show on a free body diagram of the system. The torques that are summed are those due to the same forces (by means of $\vec{\boldsymbol{r}}_{\mathrm{i/C}} \times \vec{\boldsymbol{F}}_{\mathrm{i}}$) plus those due to any force systems that have been replaced with equivalent couples. If the forces on a system satisfy eqs. Ic and IIc the system is said to be in *static equilibrium* or just in equilibrium.

A system is in static equilibrium if the applied forces and moments add to zero.

Which can also be stated as

The forces on a system in static equilibrium, considered as a system, are equivalent to a zero force and a zero couple.

The approximating assumption that an object is in static equilibrium is that the forces mediated by an object are much larger than the forces needed to accelerate it. The statics equations are generally reasonably accurate for

- Things that a normal person would call “still” such as a building or bridge on a calm day, and a sleeping person; for
- Things that move slowly or with little acceleration, such as a tractor plowing a field or the arm of a person holding up a book while seated in a smooth-flying airplane; and for
- Parts that mediate the forces needed to accelerate more massive parts, such as gears in a transmission, the rear wheel of an accelerating bicycle, the strut in the landing gear of an airplane, and the individual structural members of a building swaying in an earthquake.

Quantitative estimation of the goodness of the statics approximation is not a statics problem, so we defer it until the chapters on dynamics.

How is statics done?

The practice of statics involves:

- Drawing free body diagrams of the system of interest and of appropriate subsystems;
- writing equations Ic and IIc for each free body diagram; and
- using vector manipulation skills to solve for unknown features of the applied loads or geometry.

The organization of this chapter

This whole chapter involves drawing free body diagrams and apply the force and moment balance equations. The chapter development is, roughly, the application of this procedure to more and more complex systems. We start with single bodies in the next key section. We then go on to the most useful examples of composite bodies, trusses. The relation between statics and the prediction of structural failure is next explained to be based on the concept of "internal" forces. Springs are ubiquitous in mechanics, so we devote a section to them. More difficult statics problems with composite bodies, mechanisms and frames, come next. Hydrostatics, useful for understanding the forces of water on a structure, is next. The final section serves as a cover for harder and three dimensional problems associated with all of the statics topics but has little new content.

Further statics skills will be developed later in the dynamics portion of the book. In particular, statics methods that depend on kinematics (work methods) are deferred.

Two dimensional and three dimensional mechanics

The world we live in is three dimensional. So two dimensional models and equations are necessarily approximations. The theory of mechanics is a three dimensional theory that is simplified in two dimensional models. To appreciate the simplification one needs to understand 3D mechanics. But to understand 3D mechanics it is best to start practicing with 2D mechanics. Thus, until the last section of this chapter, we emphasize use of the two dimensional approximation and are intentionally casual about its precise meaning. We will think of a cylinders and spheres as circles, of boxes as squares, and of cars as things with two wheels (one in front, one in back). In the last section on three dimensional statics we will look more closely at the meaning of the 2 dimensional approximation.

4.1 Static equilibrium of one body

A body is in static equilibrium if and only if the force balance and moment balance equations are hold:

$$\underbrace{\sum_{\substack{\text{All external}\\ \text{forces}}} \vec{F} = \vec{0}}_{\text{force balance}} \qquad \text{and} \qquad \underbrace{\sum_{\substack{\text{All external}\\ \text{torques}}} \vec{M}_{\mathrm{C}} = \vec{0}}_{\text{moment balance}} \qquad \text{(Ic,IIc)}$$

for some point C. Is C a special point? No. Why? Because the statics equations say that the net force system is equivalent to a zero force and zero couple at C. We know from our study of equivalent force systems that this implies that the force system is equivalent to a zero force and zero couple at any and every point. So you can use any convenient point for the reference point in the moment balance equation.

> **Example.** As you sit still reading, gravity is pulling you down and forces from the floor on your feet, the chair on your seat, and the table on your elbows hold you up. All of these forces add to zero. The net moment of these forces about the front-left corner of your desk adds to zero. □

In two dimensions the equilibrium equations make up 3 independent scalar equations (2 components of force, 1 of moment). In 3 dimensions the equilibrium equations make up 6 independent scalar equations (3 components of force and 3 components of moment).

We now proceed to consider a sequence of special loading situations. In principle you don't need to know any of them, force balance and moment balance spell out the whole statics story.

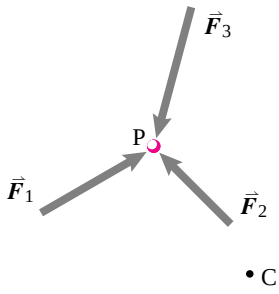
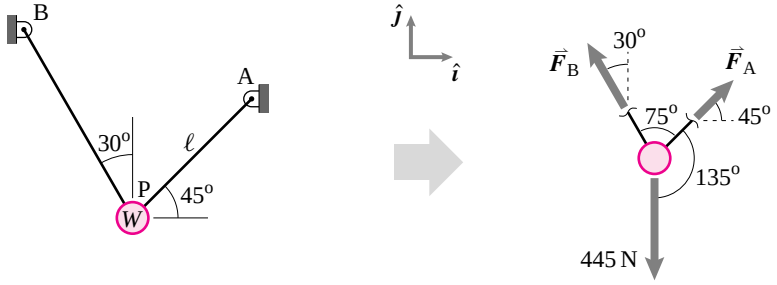


Figure 4.1: A set of forces acting concurrently on a particle.

(Filename:tfigure.particleequilib)

Concurrent forces, equilibrium of a particle

The word particle usually means something small. In mechanics a *particle* is something whose spatial extent is ignored for one reason or another. If the 'body' in a free body diagram is a particle then all forces on it act at the same point, namely at the particle, and are said to be *concurrent* (see fig. 4.1). Force balance says that the forces add to zero. The moment balance equation adds no information because it is automatically satisfied (concurrent forces adding to zero have no moment about any point).



Example. A 100 pound weight hangs from 2 lines. So

$$\sum \vec{F}_i = \vec{0} \qquad \Rightarrow \qquad 445\,\mathrm{N}(-\hat{j}) + F_A \frac{(\hat{\imath}+\hat{j})}{\sqrt{2}} + F_B\left(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{j}\right) = \vec{0}.$$

This can be solved any number of ways to get $F_A = 230.3\,\mathrm{N}$ and $F_B = 325.8\,\mathrm{N}$. □

Although the moment balance equation has nothing to add in the case of concurrent forces, it can be used instead of force balance.

Example. Consider the same weight hanging from 2 strings. Moment balance about point A gives

$$\sum \vec{M}_\mathrm{A} = \vec{0} \qquad \Rightarrow \qquad \vec{r}_{\mathrm{P/A}} \times 445\,\mathrm{N}(-\hat{j}) + \vec{r}_{\mathrm{P/A}} \times F_B\left(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{j}\right) + \vec{0} = \vec{0}.$$

Evaluating the cross products one way or another one again gets $F_B = 325.8\,\mathrm{N}$. Similarly moment balance about B could be used to find $F_A = 230.3\,\mathrm{N}$. □

If we thought of moment balance first we could have solved this problem using moments and said the force balance had nothing to add. In either case, we only have two useful scalar equilibrium equations in 2D and 3 in 3D for concurrent force systems. The other equations are satisfied automatically because of the force concurrence.

One-force body

Lets first dispose of the case of a "one-force" body. Consider a finite body with only one force acting on it. Assume it is in equilibrium. Force balance says that the sum of forces must be zero. So that force must be zero.

> If only one force is acting on a body in equilibrium that force is zero.

That was too easy, but a count to 3 wouldn't feel complete if it didn't start at 1.

Two-force body

When only two forces act on a system the situation is also simplified, though not so drastically as the case with one force. To determine the simplification, we apply the equilibrium equations of statics (Ic and IIc) to the body. Consider the free body diagram of a body $\mathcal{B}$ in figure 4.2a. Forces $\vec{F}_P$ and $\vec{F}_Q$ are acting on $\mathcal{B}$ at points P and Q. First, we have that the sum of all forces on the body are zero,

$$\begin{aligned} \sum_{\substack{\text{All external}\\ \text{forces}}} \vec{F} &= \vec{0} \\ \vec{F}_P + \vec{F}_Q &= \vec{0} \qquad \Rightarrow \qquad \vec{F}_P = -\vec{F}_Q. \end{aligned}$$

Thus, the two forces must be equal in magnitude and opposite in direction. So, thus far, we can conclude that the forces must be parallel as shown in figure 4.2b. But the forces still seem to have a net turning effect, thus still violating the concept of static equilibrium. The sum of all external torques on the body about any point are zero. So, summing moments about point P, we get,

$$\begin{aligned} \sum_{\substack{\text{All external}\\ \text{torques}}} \vec{M}_{/P} &= \vec{0} \\ \vec{r}_{Q/P} \times \vec{F}_Q &= \vec{0} \qquad (\vec{F}_P \text{ produces no torque about } P.) \\ |\vec{r}_{Q/P}| \left(\hat{\lambda}_{Q/P} \times \vec{F}_Q\right) &= \vec{0} \qquad (\hat{\lambda}_{Q/P} = \frac{\vec{r}_{Q/P}}{|\vec{r}_{Q/P}|} = -\frac{\vec{r}_{P/Q}}{|\vec{r}_{P/Q}|}) \end{aligned}$$

So $\vec{F}_Q$ has to be parallel to the line connecting P and Q. Similarly, taking the sum of moments about point Q, we get

$$-\hat{\lambda}_{Q/P} \times \vec{F}_Q = \vec{0}$$

and $\vec{F}_P$ also must be parallel to the line connecting P and Q. So, not only are $\vec{F}_P$ and $\vec{F}_Q$ equal and opposite, they are collinear as well since they are parallel to the axis passing through their points of action (see fig. 4.2c). Summarizing,

> *If a body in static equilibrium is acted on by two forces, then those forces are equal, opposite, and have a common line of action.*

A body with with only two forces acting on it is called a *two-force bodies* or *two-force member*. If you recognize a two-force body you can draw it in a free body diagram as in fig. 4.2c and the equations of force and moment balance applied to this body will provide no new information. This shortcut is sometimes useful for systems with several parts some of which are two-force members. Most often springs, dashpots, struts, and strings are idealized as two-force bodies as for bar BC in the example below.

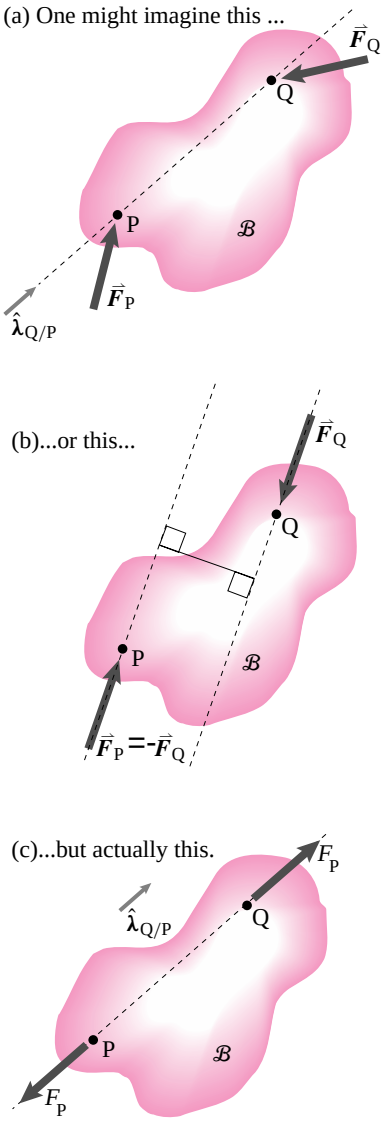
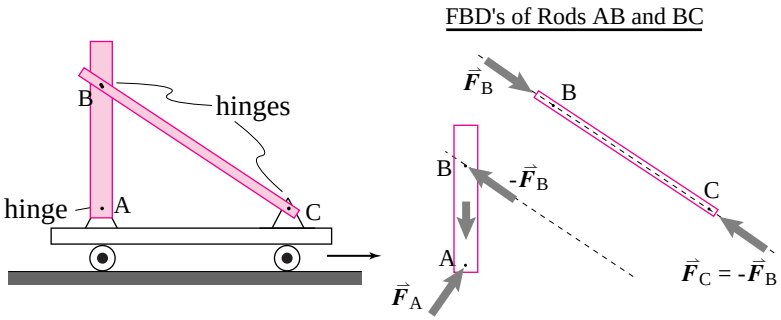


Figure 4.2: (a) Two forces acting on a body $\mathcal{B}$. (b) force balance implies that the forces are equal in magnitude and opposite in direction. (c) moment balance implies that the forces are colinear. Body $\mathcal{B}$ is a two-force member; the two forces are equal, opposite, and collinear.

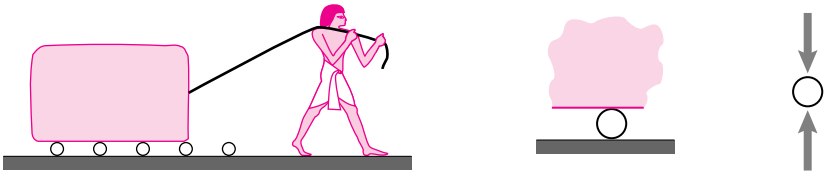
(Filename:tfigure2.two.force)

Example: **Tower and strut**



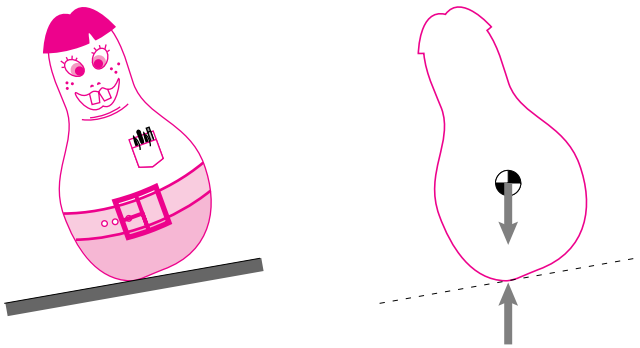
Consider an accelerating cart holding up massive tower AB which is pinned at A and braced by the light strut BC. The rod BC qualifies as a two-force member. The rod AB does not because it has three forces and is also not in static equilibrium (non-negligible accelerating mass). Thus, the free body diagram of rod BC shows the two equal and opposite colinear forces at each end parallel to the rod and the tower AB does not. □

Example: **Logs as bearings**



Consider the ancient egyptian dragging a big stone. If the stone and ground are flat and rigid, and the log is round, rigid and much lighter than the stone we are led to the free body diagram of the log shown. *With these assumptions there can't be any resistance to rolling.* Note that this effectively frictionless rolling occurs no matter how big the friction coefficient between the contacting surfaces. That the egyptian got tired comes from logs, ground, stone,not being perfectly flat (or round) and rigid. (Also, it is tiring to keep replacing the logs in the front.) □

Example: **One point of support**



If an object with weight is supported at just one point, that point must be directly above or below the center of mass. Why? The gravity forces are

equivalent to a single force at the center of mass. The body is then a two force body. Since the direction of the gravity force is down, the support point and center of mass must be above one another. Similarly if a body is suspended from one point, the center of gravity must be directly above or below that point. □

Three-force body

If a body in equilibrium has only three forces on it, again there is a general simplification that one can deduce from the general equations of statics

$$\sum_{\substack{\text{All external}\\ \text{forces}}} \vec{F} = \vec{0} \qquad \text{and} \qquad \sum_{\substack{\text{All external}\\ \text{torques}}} \vec{M}_C = \vec{0}.$$

The simplification is not as great as for two-force bodies but is remarkably useful for more difficult statics problems. In box 4.1 on page 111 moment balance about various axes is used to prove that

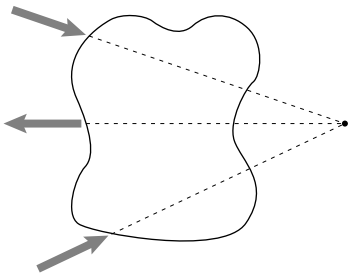
for a three-force body to be in equilibrium, the forces

(a) must be coplanar, and

(b) must either have lines of action which intersect at a single point, or the three forces are parallel.

That is, one could imagine three random forces acting on a body. But, for equilibrium they must be coplanar and concurrent.

2D



3D

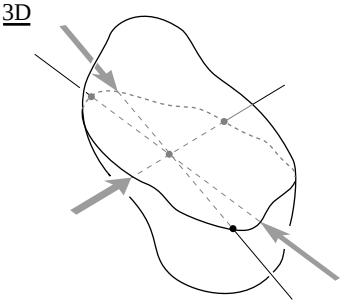
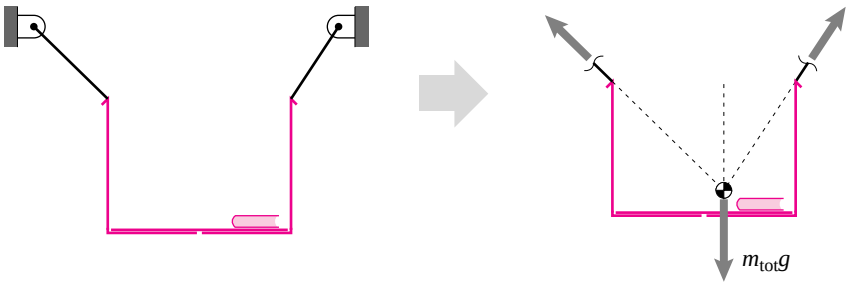


Figure 4.3: In a three-force body, the lines of action of the forces intersect at a single point and are coplanar. The point of intersection does not have to lie within the body.

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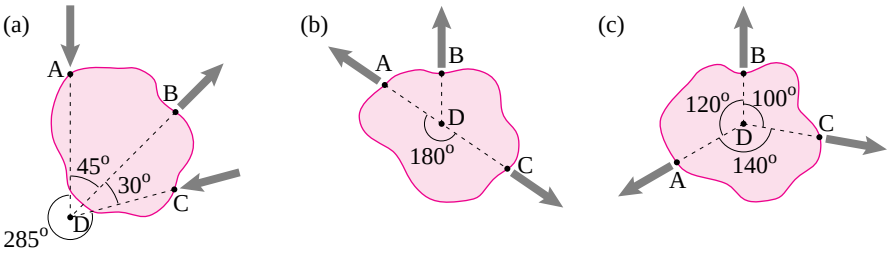
Example: **Hanging book box**



4.1 *THEORY*
Three-force bodies

Consider a body in static equilibrium with just three forces on it; $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ acting at $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$. Taking moment balance about the axis through points at $\vec{r}_2$ and $\vec{r}_3$ implies that the line of action of $\vec{F}_1$ must pass through that axis. Similarly, for equilibrium to hold, the line of action of $\vec{F}_2$ must intersect the axis through points at $\vec{r}_1$ and $\vec{r}_3$ and the line of action of $\vec{F}_3$ must intersect the axis through $\vec{r}_1$ and $\vec{r}_2$. So, the lines of action of all three forces are in the plane defined by the three points of action and the lines of action of $\vec{F}_2$ and $\vec{F}_3$ must intersect. Taking moment balance about this point of intersection implies that $\vec{F}_1$ has line of action passing through the same point. (The exceptional case is when $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ are parallel and have a common plane of action.)

A box with a book inside is hung by two strings so that it is in equilibrium on when level. The lines of action of the two strings must intersect directly above the center of mass of the box/book system. □



Example: **Which way do the forces go?**

The maximum angle between pairs of forces can be (a) greater than, (b) equal to, or (c) less than 180°. In case (b) force balance in the direction perpendicular to line ADC shows that the odd force must be zero. In case (a) force balance perpendicular to the middle force implies that the outer two forces are both directed from D or both directed away from D. Force balance in the direction of the middle force shows that it has to have the opposite sense than the outer forces. If the others are pushing in then it is pulling away. If the outer forces are pulling away than it is pushing in. In case (c) application of force balance perpendicular to the force at C shows that the other two forces must both pull away towards D or both push in. Then force balance along C shows that all three forces must have the same sense. All three forces are pulling away from D or all three are pushing in. □

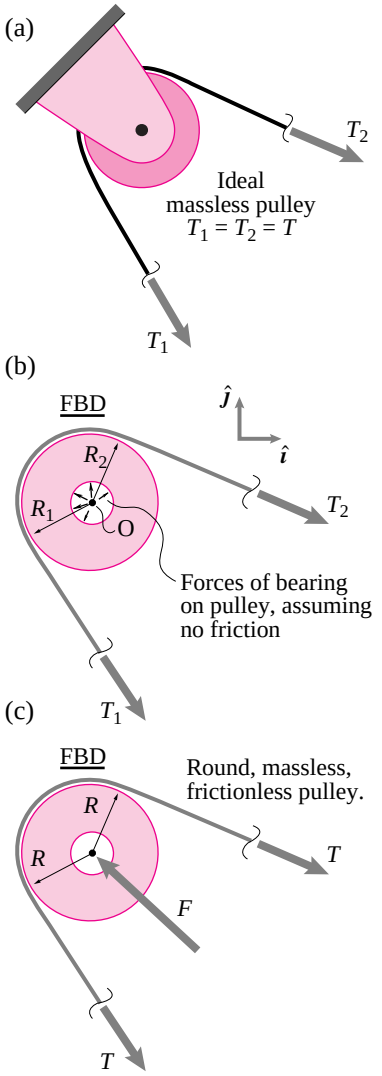


Figure 4.4: (a) An ideal massless pulley, (b) FBD of idealized massless pulley, detailing the frictionless bearing forces and showing forces at the cut strings, (c) final FBD after analysis.

(Filename:tfigure3.pulleytheory1)

The idealized massless pulley

Both real machines and mechanical models are built of various building blocks. One of the standards is a pulley. We often draw pulleys schematically something like in figure 4.4a which shows that we believe that the tension in a string, line, cable, or rope that goes around an ideal pulley is the same on both sides, $T_1 = T_2 = T$. An ideal pulley is

(i.) Round,
(ii.) Has frictionless bearings,
(iii.) Has negligible inertia, and
(iv.) Is wrapped with a line which only carries forces along its length.

We now show that these assumptions lead to the result that $T_1 = T_2 = T$. First, look at a free body diagram of the pulley with a little bit of string at both ends. Since we assume the bearing has no friction, the interaction between the pulley bearing shaft and the the pulley has no component tangent to the bearing.

To find the relation between tensions, we apply angular momentum balance (equation II) about point O

$$\left\{\sum \vec{M}_O = \dot{\vec{H}}_O\right\} \cdot \hat{k}. \tag{4.1}$$

Evaluating the left hand side of eqn. 4.1

$$\begin{aligned} \sum \vec{M}_O \cdot \hat{k} &= R_2 T_2 - R_1 T_1 + \underbrace{\text{bearing friction}}_{0} \\ &= R(T_2 - T_1), \text{ since } R_1 = R_2 = R. \end{aligned}$$

Because there is no friction, the bearing forces acting perpendicular to the round bearing shaft have no moment about point O (see also the short example on page 55). Because the pulley is round, $R_1 = R_2 = R$.

When mass is negligible, dynamics reduces to statics because, for example, all the terms in the definition of angular momentum are multiplied by the mass of the system parts. So the right hand side of eqn. 4.1 reduces to $\dot{\vec{H}}_O \cdot \hat{k} = 0$.

Putting these assumptions and results together gives

$$\begin{aligned} \left\{ \sum \vec{M}_O = \dot{\vec{H}}_O \right\} \cdot \hat{k} \\ \Rightarrow R(T_2 - T_1) = 0 \\ \Rightarrow T_1 = T_2 \end{aligned}$$

Thus, the tensions on the two lines of an ideal massless pulley are equal.

Lopsided pulleys are not often encountered, so it is usually satisfactory to assume round pulleys. But, in engineering practice, the assumption of frictionless bearings is often suspect. In dynamics, you may not want to neglect pulley mass.

Lack of equilibrium as a sign of dynamics

Surprisingly, statics calculations often give useful information about dynamics. If, in a given problem, you find that forces cannot be balanced this is a sign that the related physical system will accelerate in the direction of imbalance. If you find that moments cannot be balanced, this is a sign of rotational acceleration in the physical system. The first example ('block on ramp') in the next subsection illustrates the point.

Conditional contact, consistency, and contradictions

There is a natural hope that a subject will reduce to the solution of some well defined equations. For statics problems one would like to specify the object(s) the forces on the them, the nature of the interactions and then just write the force balance and moment balance equations and be sure that the solution follows by solving the equations.

For better and worse, things are not always this simple. For better because it means that the recipes are still not so well defined that computers can easily steal the subject of mechanics from people. For worse because it means people have to think hard to do mechanics problems.

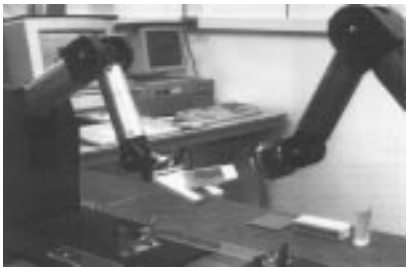
Many mechanics problems do have a solution, just one, that follows from the governing equations. But some reasonable looking problems have no solutions. And some problems have multiple solutions. When these mathematical anomalies arise, they usually have some physical importance. Even for problems with one solution, the route to finding that solution can involve more than simple equation manipulation.

One source of these difficulties is the conditional nature of the equations that govern contact. For example:

- The ground pushes up on something to prevent interpenetration *if* the pushing is positive, *otherwise* the ground does not push up.
- The force of friction opposes motion and has magnitude μN *if* there is slip, *otherwise* the force of friction is something less than μN in magnitude.
- The distance between two points is kept from increasing by the tension in the string between them *if* the tension is positive, *otherwise* the tension is zero.

These conditions are, implicitly or explicitly, in the equations that govern these interactions. One does not always know which of the contact conditions, if either, apply when one starts a problem. Sometimes multiple possibilities need to be checked.

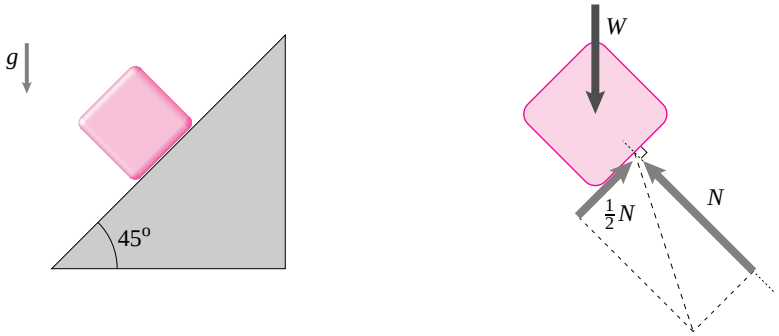
Example: **Robot hand**



Roboticist Michael Erdmann has designed a palm manipulator that manipulates objects without squeezing them. The flat robot palms just move around and the object consequently slides. Determining whether the object slides on one the other or possibly on both hands in a given movement is a matter of case study. The computer checks to see if the equilibrium equations can be solved with the assumption of sticking or slipping at one or the other contact. □

Sometimes there is no statics solution as the following simple example shows.

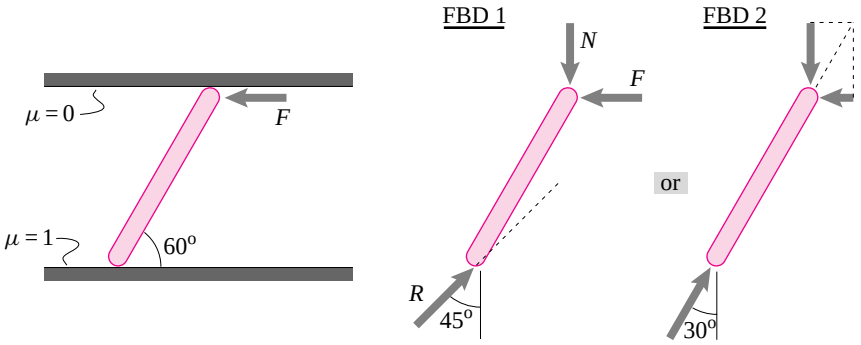
Example: **Block on ramp.**



A block with coefficient of friction $\mu = .5$ is in static equilibrium sliding steadily down a 45^o ramp. Not! The two forces in the free body diagram cannot add to zero (since they are not parallel). The assumptions are not consistent. They lead to a contradiction. Given the geometry and friction coefficient one could say that the assumption of equilibrium was inconsistent (and actually the block accelerates down the ramp). If equilibrium is demanded, say you saw the block just sitting there, then you can pin the contradiction on a mis-measured slope or a mis-estimated coefficient of friction. □

The following problem shows a case where a statics problem has multiple solutions.

Example: **Rod pushed in a channel.**



A light rod is just long enough to make a 60^o angle with the walls of a channel. One channel wall is frictionless and the other has $\mu = 1$. What is the force needed to keep it in equilibrium in the position shown? If we assume it is sliding we get the first free body diagram. The forces shown can be in equilibrium if all the forces are zero. Thus we have the solution that the rod slides in equilibrium with no force. If we assume that the block is not sliding the friction force on the lower wall can be at any angle between $\pm 45^o$. Thus we have equilibrium with the second FBD for arbitrary positive F. This is a second set of solutions. A rod like this is said to be *self locking* in that it can hold arbitrary force F without slipping. That we have found freely slipping solutions with no force and jammed solutions with arbitrary force corresponds physically to one being able to easily slide a rod like this down a slot and then have it totally jamb. Some rock-climbing equipment depends on such self-locking and easy release. □

One might not at first thing of string connections as being a form of contact, but the whether a string is taught or not is the same as whether contact is made with a frictionless spherical wall or not.

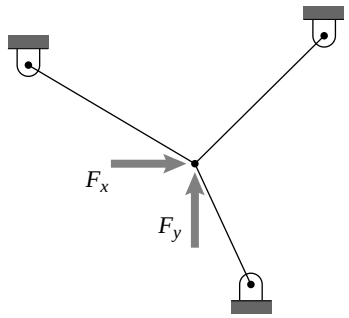
Example: **Particle held by two strings.**



Two inextensible strings are slightly slack when no load is applied to the knot in the middle. When a load is applied what is the tension in the strings? Force balance along the strings gives us one equation for the two unknown tensions. There are many solutions. There are even solutions where both tensions are positive. But geometry does not allow both of the strings to be at full length simultaneously. Thus we have to assume one of the strings has no tension when applying force balance. If we pick the wrong string we will get the contradiction that its tension is negative. □

The triviality of this example perhaps hides the problem, so here it is again with three strings.

Example: **Particle held by three strings.**



Three inextensible strings are just slightly slack when no load is applied to the knot in the middle. When a load is applied what is the tension in the strings? Planar force balance gives us two equations for the 3 unknown tensions. These equations have many solutions, even some with positive tension in all three strings. But geometry does not allow all three strings to be at their extended lengths simultaneously. So at least one string has to be slack and have no tension. If you guess the right one you will find positive tension in the other two strings. If you guess the wrong one you will get the contradiction that one of the strings has negative tension. □

If this example still seems too easy to demonstrate that sometimes you have to think about which of two or more conditionals needs to be enforced, try a case with four strings in three dimensions.

These examples, and one could construct many more, show that you have to look out for static equilibrium being not consistent with other information given. This contradiction could arise in an il-posed problem, a problem that is really a dynamics problem, or as you eliminate possibilities that a given well-posed statics problem superficially allows.

The general case

For one body, whether in 1D,2D or 3D the equations of equilibrium are:

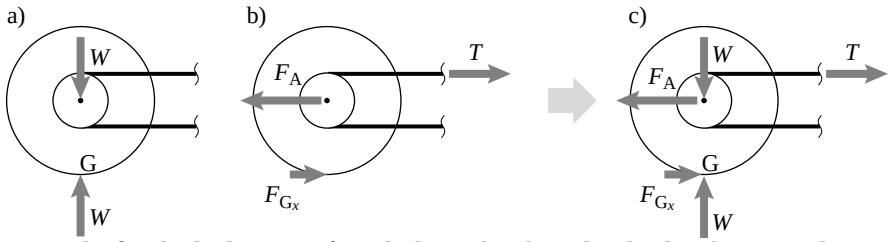
$$\underbrace{\sum_{\substack{\text{All external}\\ \text{forces}}} \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}}_{\text{force balance}} \qquad \text{and} \qquad \underbrace{\sum_{\substack{\text{All external}\\ \text{torques}}} \vec{\boldsymbol{M}}_{\mathrm{C}} = \vec{\boldsymbol{0}}}_{\text{moment balance}} \qquad \text{(Ic,IIc)}$$

Solving a statics problem means using these equations, along with any available information about the forces involved, to find various unknowns. For some problems, the various tricks involving one-force, two-force, and three-force bodies can serve as a time saver for solving these equations and can help build your intuition. For some contact problems you may have to try various cases. But ultimately, always, statics means applying the force balance and moment balance equations.

Linearity and superposition

For a given geometry the equilibrium equations are *linear*. That is: If for a given object you know a set of forces that is in equilibrium and you also know a second set of forces that is in equilibrium, then the sum of the two sets is also in equilibrium.

Example: **A bicycle wheel**



The free body diagram of an ideal massless bicycle wheel with a vertical load is shown in (a) above. The same wheel driven by a chain tension but with no weight is shown in equilibrium in (b) above. The sum of these two load sets (c) is therefor in equilibrium. □

The idea that you can add two solutions to a set of equations is called the *principle of superposition*, sometimes called the principle of superimposition①. The principle of superposition provides a useful shortcut for some mechanics problems.

① Here's a pun to help you remember the idea. When talkative Sam comes over you get bored. When hungry Sally comes over you reluctantly go get a snack for her. When Sam and Sally come over together you get bored *and* reluctantly go get a snack. Each one of them is imposing. By the principle of superimposition their effects add when they are together and they are super imposing.

Vectors, matrices, and linear algebraic equations

Once has drawn a free body diagram and written the force and moment balance equations one is left with vector equations to solve for various unknowns. The vector equations of mechanics can be reduced to scalar equations by using dot products. The simplest dot product to use is with the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. This use of dot products is equivalent to taking the x, y, and z components of the vector equation. The two vector equations

$$\begin{aligned} a\hat{\imath} + b\hat{\jmath} &= (c-5)\hat{\imath} + (d+7)\hat{\jmath} \\ (a-c)\hat{\imath} + (a+b)\hat{\jmath} &= (c+b)\hat{\imath} + (2a+c)\hat{\jmath} \end{aligned}$$

with four scalar unknowns a, b, c, and d, can be rewritten as four scalar equations, two from each two-dimensional vector equation. Taking the dot product of the first equation with $\hat{\imath}$ gives $a = c-5$. Similarly dotting with $\hat{\jmath}$ gives $b = d+7$. Repeating the procedure with the second equation gives 4 scalar equations:

$$\begin{aligned} a &= c-5 \\ b &= d+7 \\ a-c &= c+b \\ a+b &= 2a+c. \end{aligned}$$

These equations can be re-arranged putting unknowns on the left side and knowns on the right side:

$$\begin{aligned} 1a + 0b + -1c + 0d &= -5 \\ 0a + 1b + 0c + -1d &= 7 \\ 1a + -1b + -2c + 0d &= 0 \\ -1a + 1b + -1c + 0d &= 0 \end{aligned}$$

These equations can in turn be written in standard matrix form. The standard matrix form is a short hand notation for writing (linear) equations, such as the equations above:

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}}_{[A]} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[x]} = \underbrace{\begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}}_{[y]}$$

$$\Rightarrow \quad [A] \cdot [x] = [y].$$

The matrix equation $[\boldsymbol{A}] \cdot [\boldsymbol{x}] = [\boldsymbol{y}]$ is in a form that is easy to input to any of several programs that solve linear equations. The computer (or a do-able but probably untrustworthy hand calculation) should return the following solution for $[x]$ (a, b, c, and d).

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix}.$$

That is, $a = -5$, $b = -5$, $c = 0$, and $d = -12$. If you doubt the solution, check it. To check the answer, plug it back into the original equation and note the equality (or lack thereof!). In this case, we have done our calculations correctly and

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}.$$

Going back to the original vector equations we can also check that

$$\begin{aligned} -5\hat{\boldsymbol{\imath}} + -5\hat{\boldsymbol{\jmath}} &= (0-5)\hat{\boldsymbol{\imath}} + (-12+7)\hat{\boldsymbol{\jmath}} \\ (-5-0)\hat{\boldsymbol{\imath}} + (-5+-5)\hat{\boldsymbol{\jmath}} &= (0+-5)\hat{\boldsymbol{\imath}} + (2\cdot -5+0)\hat{\boldsymbol{\jmath}}. \end{aligned}$$

‘Physical’ vectors and row or column vectors

The word ‘vector’ has two related but subtly different meanings. One is a physical vector like $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$, a quantity with magnitude and direction. The other meaning is a list of numbers like the row vector

$$[\boldsymbol{x}] = [x_1, x_2, x_3]$$

or the column vector

$$[\boldsymbol{y}] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Once you have picked a basis, like $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, you can represent a physical vector $\vec{\boldsymbol{F}}$ as a row vector $\begin{bmatrix} F_x, F_y, F_z \end{bmatrix}$ or a column vector $\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$. But the components of a given vector depend on the base coordinate system (or base vectors) that are used. For clarity it is best to distinguish a physical vector from a list of components using a notation like the following:

$$[\vec{\boldsymbol{F}}]_{XYZ} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

The square brackets around $\vec{\boldsymbol{F}}$ indicate that we are looking at its components. The subscript XYZ identifies what coordinate system or base vectors are being used. The right side is a list of three numbers (in this case arranged as a column, the default arrangement in linear algebra).

Matrices and tensors

In chapter 5, we will introduce the 3 by 3 moment of inertia matrix $[\boldsymbol{I}]$. We will find it in expressions having to do with angular momentum sitting next to either a vector $\vec{\boldsymbol{\omega}}$ or a vector $\vec{\boldsymbol{\alpha}}$: $[\boldsymbol{I}]\cdot\vec{\boldsymbol{\omega}}$ or $[\boldsymbol{I}]\cdot\vec{\boldsymbol{\alpha}}$. What we mean by this expression is the three element column vector that comes from matrix multiplication of the matrix $[\boldsymbol{I}]$ and the column vector for $\vec{\boldsymbol{\omega}}$, $\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$, an expression that only makes sense if everyone knows what bases are being used.

More formally, and usually only in more advanced treatments, people like to define a coordinate-free quantity called the tensor $\underline{\underline{\boldsymbol{I}}}$. Then we would have

$$\underline{\underline{\boldsymbol{I}}}\cdot\vec{\boldsymbol{\omega}}$$

by which we would mean the vector whose components would be found by $[\boldsymbol{I}]\cdot\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$.

Eigenvectors and Eigenvalues

A square matrix $[\boldsymbol{A}]$ when multiplied by a column vector $[\boldsymbol{v}]$ yields a new vector $[\boldsymbol{w}]$. A given matrix has a few special vectors, somehow characteristic of that matrix, called *eigenvectors*. The vector $\vec{\boldsymbol{v}}$ is an eigenvector of $[\boldsymbol{A}]$ if

$$[\boldsymbol{A}]\cdot[\boldsymbol{v}] \text{ is parallel to } [\boldsymbol{v}].$$

In other words, if

$$[\boldsymbol{A}]\cdot[\boldsymbol{v}] = \lambda\,[\boldsymbol{v}]$$

for some λ.

The scalar λ is called the *eigenvalue* associated with the eigenvector $[\boldsymbol{v}]$ of the matrix $[\boldsymbol{A}]$. The eigen-values and eigen-vectors of a matrix are found with a single command in many computer math programs. In statics you will have little or no use for eigen-values and eigen-vectors. In dynamics, eigenvectors and eigenvalues are useful for understanding dynamic balance, 3-D rigid body rotations, and normal mode vibrations.

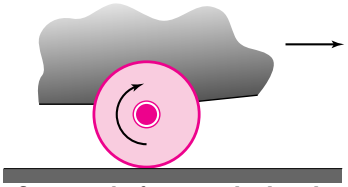
4.2 Wheels and two force bodies

One often hears whimsical reverence for the "invention of the wheel." Now, using elementary mechanics, we can gain some appreciation for this revolutionary way of sliding things.

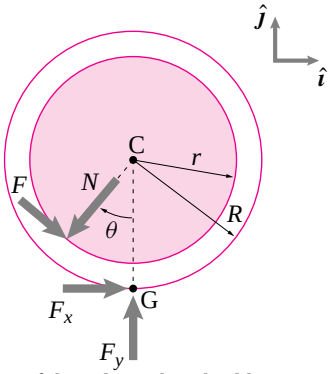
Without a wheel the force it takes to drag something is about μW. Since μ ranges between about .1 for teflon, to about .6 for stone on ground, to about 1 for rubber on pavement, you need to pull with a force that is on the order of a half of the full weight of the thing you are dragging.

You have seen how rolling on round logs cleverly take advantage of the properties of two-force bodies (page 110). But that good idea has the major deficiency of requiring that logs be repeatedly picked up from behind and placed in front again.

The simplest wheel design uses a dry "journal" bearing consisting of a non-rotating shaft protruding through a near close fitting hole in the wheel. Here is shown part of a cart rolling to the right with a wheel rotating steadily clockwise.



To figure out the forces involved we draw a free body diagram of the wheel. We neglect the wheels weight because it is generally much smaller than the forces it mediates. To make the situation clear the picture shows too-large a bearing hole r.



The force of the axle on the wheel has a normal component N and a frictional component F. The force of the ground on the wheel has a part holding the cart up F_y and a part along the ground F_x which will surely turn out to be negative for a cart moving to the right. If we take the wheel dimensions to be known and also the vertical part of the ground reaction force F_y we have as unknowns N, F, θ and F_x. To find these we could use the friction equation for the sliding bearing contact

$$F = \mu N;$$

force balance

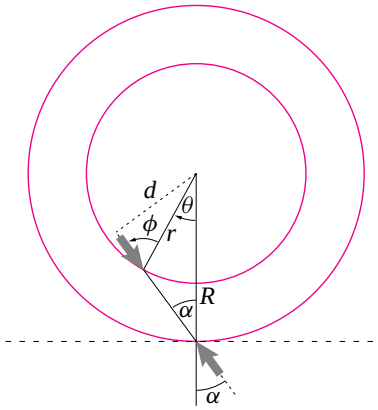
$$F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + N(-\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) + F(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}) = \vec{\mathbf{0}},$$

which could be reduced to 2 scalar equations by taking components or dot products; and moment balance which is easiest to see in terms of forces and perpendicular distances as

$$Fr + F_x R = 0.$$

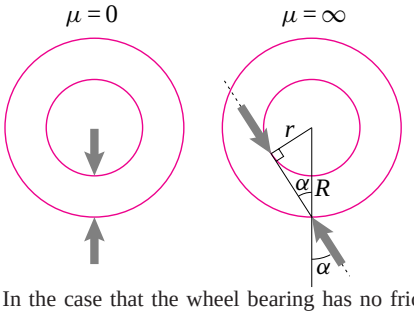
Of key interest is finding the force resisting motion F_x. With a little mathematical manipulation we could solve the 4 scalar equations above for any of F_x, N, F, and θ in terms of r, R, F_y, and μ. We follow a more intuitive approach instead.

As modeled, the wheel is a two-force body so the free body diagram shows equal and opposite colinear forces at the two contact points.



The friction angle ϕ describes the friction between the axle and wheel (with $\tan\phi = \mu$). The angle α describes the *effective* friction of the wheel. This is not the friction angle for sliding between the wheel and ground which is assumed to be larger (if not, the wheel would skid and not roll), probably much larger. The *specific resistance* or the *coefficient of rolling resistance* or the *specific cost of transport* is $\mu_{\text{eff}} = \tan\alpha$. (If there was no wheel, and the cart or whatever was just dragged, the specific resistance would be the friction between the cart and ground $\mu_{\text{eff}} = \mu$.)

Lets consider two extreme cases: one is a frictionless bearing and the other is a bearing with infinite friction coefficient $\mu \to \infty$ and $\phi \to 90^{\text{o}}$.



In the case that the wheel bearing has no friction we satisfyingly see clearly that there is no ground resistance to motion. The case of infinite friction is perhaps surprising. Even with infinite friction we have that

$$\sin\alpha = \frac{r}{R}.$$

Thus if the axle has a diameter of 10 cm and the wheel of 1 m then $\sin\alpha$ is less than .1 no matter how bad the bearing material. For such small values we can make the approximation $\mu_{\text{eff}} = \tan\alpha \approx \sin\alpha$ so that the effective coefficient of friction is .1 or less no matter what the bearing friction.

The genius of the wheel design is that it makes the effective friction less than r/R no matter how bad the bearing friction.

Going back to the two-force body free body diagram we can see that

$$\begin{aligned} & d & = & \quad d \\ \Rightarrow \quad & r\sin\phi & = & \quad R\sin\alpha \\ \Rightarrow \quad & \sin\alpha & = & \quad \frac{r}{R}\sin\phi. \qquad (*) \end{aligned}$$

From this formula we can extract the limiting cases discussed previously ($\phi = 0$ and $\phi = 90^o$). We can also plug in the small angle approximations ($\sin\alpha \approx \tan\alpha$ and $\sin\phi \approx \tan\phi$) if the friction coefficient is low to get

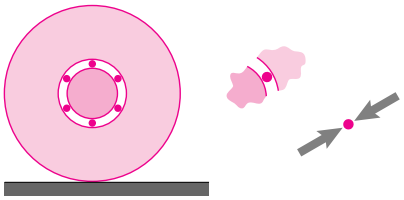
$$\mu_{\text{eff}} \approx \mu\frac{r}{R}.$$

The effective friction is the bearing friction attenuated by the radius ratio. Or, we can use the trig identity $\sin = \sqrt{1+\tan^{-2}}^{-1}$ to solve the exact equation (*) for

$$\mu_{\text{eff}} = \mu\frac{r}{R}\left(\frac{1}{\sqrt{1+\mu^2(1-r^2/R^2)}}\right),$$

where the term in parenthesis is always less than one and close to one if the sliding coefficient in the bearing is low.

Finally we combine the genius of the wheel with the genius of the rolling log and invent a wheel with rolling logs inside, a ball bearing wheel.



Each ball is a two force body and thus only transmits radial loads. Its as if there were no friction on the bearing and we get a specific resistance of zero, $\mu_{\text{eff}} = 0$. Of course real ball bearings are not perfectly smooth or perfectly rigid, so its good to keep r/R small as a back up plan even with ball bearings.

By this means some wheels have effective friction coefficients as low as about .003. The force it takes to drag something on wheels can be as little as one three hundredth the weight.

SAMPLE 4.1 *Concurrent forces:* A block of mass $m = 10\,\text{kg}$ hangs from strings AB and AC in the vertical plane as shown in the figure. Find the tension in the strings.

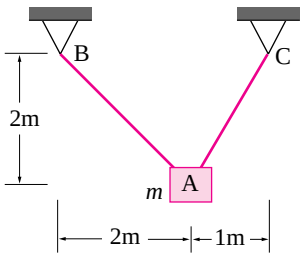


Figure 4.5: (Filename:sfig2.4new.1)

Solution The free body diagram of the block is shown in figure 4.6. Since the block is at rest, the equation of force balance is

$$\sum \vec{F} = \vec{0}$$
$$\text{or} \qquad T_1\hat{\boldsymbol{\lambda}}_{\text{AB}} + T_2\hat{\boldsymbol{\lambda}}_{\text{AC}} - mg\hat{\boldsymbol{\jmath}} = \vec{0}, \tag{4.2}$$

where $\hat{\boldsymbol{\lambda}}_{\text{AB}}$ and $\hat{\boldsymbol{\lambda}}_{BC}$ are unit vectors in the AB and AC directions, respectively. From geometry,

$$\hat{\boldsymbol{\lambda}}_{AB} = \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = \frac{-2\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{\jmath}}}{2\sqrt{2}\,\text{m}} = \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$$
$$\hat{\boldsymbol{\lambda}}_{\text{AC}} = \frac{\vec{r}_{AC}}{|\vec{r}_{AC}|} = \frac{1\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{\jmath}}}{\sqrt{5}\,\text{m}} = \frac{1}{\sqrt{5}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})$$

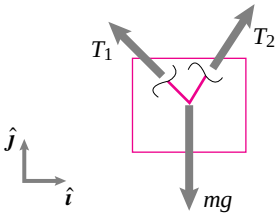


Figure 4.6: (Filename:sfig2.4new.1a)

Dotting eqn. (4.2) with $\hat{\boldsymbol{\imath}}$ we get

$$T_1(\underbrace{\hat{\boldsymbol{\lambda}}_{\text{AB}}\cdot\hat{\boldsymbol{\imath}}}_{-1/\sqrt{2}}) + T_2(\underbrace{\hat{\boldsymbol{\lambda}}_{\text{AC}}\cdot\hat{\boldsymbol{\imath}}}_{1/\sqrt{5}}) = 0 \quad \Rightarrow \qquad T_1 = \sqrt{\frac{2}{5}}\,T_2$$

Dotting eqn. (4.2) with $\hat{\boldsymbol{\jmath}}$ and substituting $T_1 = \sqrt{2/5}\,T_2$, we get

$$\underbrace{\sqrt{\frac{2}{5}}\,T_2}_{T_1}(\underbrace{\hat{\boldsymbol{\lambda}}_{\text{AB}}\cdot\hat{\boldsymbol{\jmath}}}_{1/\sqrt{2}}) + T_2(\underbrace{\hat{\boldsymbol{\lambda}}_{\text{AC}}\cdot\hat{\boldsymbol{\jmath}}}_{2/\sqrt{5}}) - mg = 0$$
$$\Rightarrow \qquad \frac{3}{\sqrt{5}}T_2 - mg = 0 \quad \Rightarrow \qquad T_2 = \frac{\sqrt{5}}{3}mg = 73.12\,\text{N}$$

Substituting in $T_1 = \sqrt{\frac{2}{5}}\,T_2$, we have $T_1 = \sqrt{\frac{2}{5}}\cdot(73.12\,\text{N}) = 46.24\,\text{N}$

$$\boxed{T_1 = 46.24\,\text{N},\ T_2 = 73.12\,\text{N}}$$

• • •

Note: We could also write eqn. (4.2) in matrix form and solve the matrix equation to find T_1 and T_2. Substituting $\hat{\boldsymbol{\lambda}}_{\text{AB}}$ and $\hat{\boldsymbol{\lambda}}_{\text{AC}}$ in terms of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ in eqn. (4.2) and dotting the resulting equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, we can write eqn. (4.2) as

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ mg \end{pmatrix} \Rightarrow \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1}\begin{pmatrix} 0 \\ mg \end{pmatrix}$$

Using Cramer's rule for the inverse of a matrix, we get

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = -\frac{\sqrt{10}}{3}\begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{pmatrix} 0 \\ mg \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{3}mg \\ \frac{\sqrt{5}}{3}mg \end{pmatrix}$$

which is, of course, the same result as we got above.

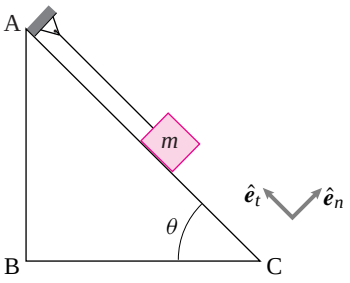


Figure 4.7: A mass-particle on an inclined plane.

(Filename:sfig2.1.11)

SAMPLE 4.2 A small block of mass m rests on a frictionless inclined plane with the help of a string that connects the mass to a fixed support at A. Find the force in the string.

Solution The free body diagram of the mass is shown in Fig. 4.8. The string force F_s and the normal reaction of the plane N are unknown forces. To determine the

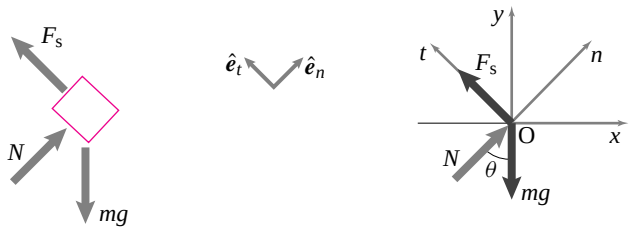


Figure 4.8: Free body diagram of the mass and the geometry of force vectors. (Filename:sfig2.1.11a)

unknown forces, we write the force balance equation, $\sum \vec{F} = \vec{0}$,

$$\vec{F}_s + \vec{N} + m\vec{g} = \vec{0}$$

We can express the forces in terms of their components in various ways and then dot the vector equation with appropriate unit vectors to get two independent scalar equations. For example, let us draw two unit vectors $\hat{e}_t$ and $\hat{e}_n$ along and perpendicular to the plane. Now we write the force balance equation using mixed basis vectors $\hat{e}_t$ and $\hat{e}_n$, and $\hat{\imath}$ and $\hat{\jmath}$:

$$F_s\hat{e}_t + N\hat{e}_n - mg\hat{\jmath} = \vec{0} \tag{4.3}$$

We can now find F_s directly by taking the dot product of the above equation with $\hat{e}_t$ since the other unknown N is in the $\hat{e}_n$ direction and $\hat{e}_n \cdot \hat{e}_t = 0$:

$$[\text{eqn. (4.3)}] \cdot \hat{e}_t \quad \Rightarrow \quad F_s - mg\overbrace{(\hat{\jmath} \cdot \hat{e}_t)}^{\sin\theta} = 0 \quad \Rightarrow \quad F_s = mg\sin\theta$$

$$\boxed{F_s = mg\sin\theta}$$

• • •

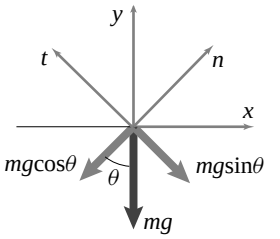
Note that we did not have to separate out two scalar equations and solve for F_s and N simultaneously. If we needed to find N, we could do that too from a single equation by taking the dot product of eqn. (4.3) with $\hat{n}$:

$$[\text{eqn. (4.3)}] \cdot \hat{e}_n \quad \Rightarrow \quad N - mg\overbrace{(\hat{\jmath} \cdot \hat{e}_n)}^{\cos\theta} = 0 \quad \Rightarrow \quad N = mg\cos\theta$$

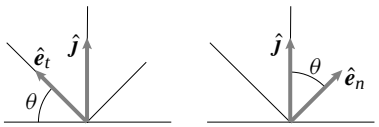
Writing direct scalar equations: You are familiar with this method from your elementary physics courses. We resolve all forces into their components along the desired directions and then sum the forces. Here, F_s is along the plane and therefore, has no component perpendicular to the plane. Force N is perpendicular to the plane and therefore, has no component along the plane. We resolve the weight mg into two components: (1) $mg\cos\theta$ perpendicular to the plane (n direction) and (2) $mg\sin\theta$ along the plane (t direction). Now we can sum the forces:

$$\sum F_t = 0 \quad \Rightarrow \quad F_s - mg\sin\theta = 0; \quad \text{and} \quad \sum F_n = 0 \quad \Rightarrow \quad N - mg\cos\theta = 0$$

which, of course, is essentially the same as the equations obtained above.



(a)



(b)

Figure 4.9: (a) Components of mg along t and n directions. (b) The mixed basis dot products: $\hat{\jmath} \cdot \hat{e}_t = \sin\theta$ and $\hat{\jmath} \cdot \hat{e}_n = \cos\theta$

(Filename:sfig2.1.11b)

SAMPLE 4.3 *A bar as a 2-force body:* A 4 ft long horizontal bar supports a load of 60 lbf at one of its ends. The other end is pinned to a wall. The bar is also supported by a string connected to the load-end of the bar and tied to the wall. Find the force in the bar and the tension in the string.

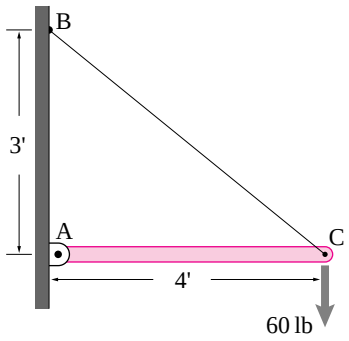


Figure 4.10: (Filename:sfig4.single.bar)

Solution Let us do this problem two ways — using equilibrium equations without much thought, and using those equations with some insight.

(a) The free body diagram of the bar is shown in Fig. 4.11. The moment balance about point A, $\sum \vec{M}_{\rm A} = 0$, gives

$$\begin{aligned} \vec{r}_{\rm C/A} \times T\hat{\boldsymbol{\lambda}} + \vec{r}_{\rm C/A} \times (-P\hat{\boldsymbol{\jmath}}) &= \vec{\mathbf{0}} \\ \underbrace{\ell\hat{\boldsymbol{\imath}} \times T(-\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})}_{\ell T\sin\theta\hat{\boldsymbol{k}}} + \underbrace{\ell\hat{\boldsymbol{\imath}} \times (-P\hat{\boldsymbol{\jmath}})}_{-\ell P\hat{\boldsymbol{k}}} &= \vec{\mathbf{0}} \\ (T\ell\sin\theta - P\ell)\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \qquad (4.4) \\ [\text{eqn. (4.4)}]\cdot\hat{\boldsymbol{k}} \quad\Rightarrow\quad T = \frac{P}{\sin\theta} = \frac{60\,\text{lbf}}{\frac{3}{5}} &= 100\,\text{lbf}. \end{aligned}$$

The force equilibrium, $\sum \vec{F} = 0$, gives

$$\begin{aligned} (A_x - T\cos\theta)\hat{\boldsymbol{\imath}} + (A_y + T\sin\theta - P)\hat{\boldsymbol{\jmath}} &= \vec{\mathbf{0}} \qquad (4.5) \\ [\text{eqn. (4.5)}]\cdot\hat{\boldsymbol{\imath}} \quad\Rightarrow\quad A_x = T\cos\theta = (100\,\text{lbf})\cdot\frac{4}{5} &= 80\,\text{lbf} \\ [\text{eqn. (4.5)}]\cdot\hat{\boldsymbol{\jmath}} \quad\Rightarrow\quad A_y = P - T\sin\theta &= 0 \end{aligned}$$

where the last equation, $A_y = P - T\sin\theta = 0$ follows from eqn. (4.4). Thus, the force in the rod is $\vec{A} = 80\,\text{lbf}\hat{\boldsymbol{\imath}}$, *i.e.*, a purely compressive force, and the tension in the string is 100 lbf.

$$\boxed{\vec{A} = 80\,\text{lbf}\hat{\boldsymbol{\imath}}, \quad T = 100\,\text{lbf}}$$

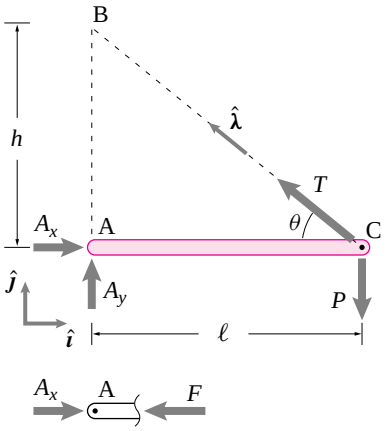


Figure 4.11: (Filename:sfig4.single.bar.a)

(b) From the free body diagram of the rod, we realize that the rod is a *two-force body*, since the forces act at only two points of the body, A and C. The reaction force at A is a single force $\vec{A}$, and the forces at end C, the tension $\vec{T}$ and the load $\vec{P}$, sum up to a single net force, say $\vec{F}$. So, now using the fact that the rod is a two-force body, the equilibrium equation requires that $\vec{F}$ and $\vec{A}$ be equal, opposite, and colinear (along the longitudinal axis of the bar). Thus,

$$\vec{A} = -\vec{F} = -F\hat{\boldsymbol{\imath}}.$$

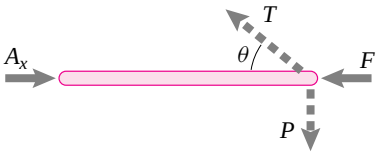


Figure 4.12: (Filename:sfig4.single.bar.b)

Now,

$$\begin{aligned} \vec{F} &= \vec{P} + \vec{T} \\ -F\hat{\boldsymbol{\imath}} &= -P\hat{\boldsymbol{\jmath}} + T\sin\theta\hat{\boldsymbol{\jmath}} - T\cos\theta\hat{\boldsymbol{\imath}} \qquad (4.6) \\ [\text{eqn. (4.6)}]\cdot\hat{\boldsymbol{\jmath}} \quad\Rightarrow\quad P &= T\sin\theta \\ \Rightarrow\quad T &= \frac{P}{\sin\theta} = \frac{60\,\text{lbf}}{\frac{3}{5}} = 100\,\text{lbf} \\ [\text{eqn. (4.6)}]\cdot\hat{\boldsymbol{\imath}} \quad\Rightarrow\quad F &= T\cos\theta = (100\,\text{lbf})\cdot\frac{4}{5} = 80\,\text{lbf}. \end{aligned}$$

The answers, of course, are the same.

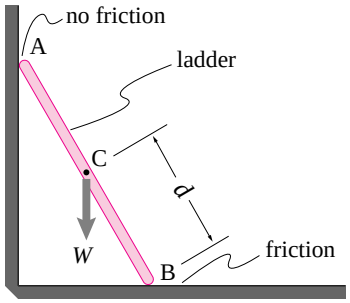


Figure 4.13: (Filename:sfig4.single.ladder)

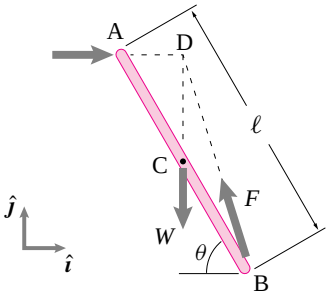


Figure 4.14: The free body diagram of the ladder indicates that it is a three force body. Since the direction of the forces acting at points B and C are known (the normal, horizontal reaction at B and the vertical gravity force at C), it is easy to find the direction of the net ground reaction at A — it must pass through point D. The ground reaction F at A can be decomposed into a normal reaction and a horizontal reaction (the force of friction, F_s) at A.

(Filename:sfig4.single.ladder.a)

SAMPLE 4.4 *Will the ladder slip?* A ladder of length $\ell = 4\,\text{m}$ rests against a wall at $\theta = 60^\text{o}$. Assume that there is no friction between the ladder and the vertical wall but there is friction between the ground and the ladder with $\mu = 0.5$. A person weighing 700 N starts to climb up the ladder.

(a) Can the person make it to the top safely (without the ladder slipping)? If not, then find the distance d along the ladder that the person can climb safely. Ignore the weight of the ladder in comparision to the weight of the person.

(b) Does the "no slip" distance d depend on θ? If yes, then find the angle θ which makes it safe for the person to reach the top.

Solution

(a) The free body diagram of the ladder is shown in Fig. 4.14. There is only a normal reaction $\vec{\boldsymbol{R}} = \text{R}\hat{\boldsymbol{\imath}}$ at A since there is no friction between the wall and the ladder. The force of friction at B is $\vec{\boldsymbol{F}}_\text{s} = -F_s\hat{\boldsymbol{\imath}}$ where $F_\text{s} \le \mu N$. To determine how far the person can climb the ladder without the ladder slipping, we take the critical case of impending slip. In this case, $F_s = \mu N$. Let the person be at point C, a distance d along the ladder from point B.
From moment balance about point B, $\sum \vec{\boldsymbol{M}}_\text{B} = \vec{\boldsymbol{0}}$, we find

$$\begin{aligned} \vec{\boldsymbol{r}}_{\text{A/B}} \times \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_{\text{C/B}} \times \vec{\boldsymbol{W}} &= \vec{\boldsymbol{0}} \\ -R\ell\sin\theta\hat{\boldsymbol{k}} + Wd\cos\theta\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad R &= W\frac{d\cos\theta}{\ell\sin\theta} \end{aligned}$$

From force equilibrium, we get

$$(R - \mu N)\hat{\boldsymbol{\imath}} + (N - W)\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}} \tag{4.7}$$

Dotting eqn. (4.7) with $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\imath}}$, respectively, we get

$$\begin{aligned} N &= W \\ R &= \mu N = \mu W \end{aligned}$$

Substituting this value of R in eqn. (4.7) we get

$$\begin{aligned} \mu W &= W\frac{d\cos\theta}{\ell\sin\theta} \\ \Rightarrow \quad d &= \mu\ell\tan\theta \qquad (4.8) \\ &= 0.5\cdot(4\,\text{m})\cdot\tan 60^\text{o} \qquad (4.9) \\ &= 3.46\,\text{m} \end{aligned}$$

Thus, the person cannot make it to the top safely.

$$\boxed{d = 3.46\,\text{m}}$$

(b) The "no slip" distance d depends on the angle θ via the relationship in eqn. (4.8). The person can climb the ladder safely up to the top (*i.e.*, $d = \ell$), if

$$\tan\theta = \frac{1}{\mu} \quad \Rightarrow \quad \theta = \tan^{-1}(\mu^{-1}) = 63.43^\text{o}$$

Thus, any angle $\theta \ge 64^\text{o}$ will allow the person to climb up to the top safely.

$$\boxed{\theta \ge 64^\text{o}}$$

SAMPLE 4.5 *How much friction does the ball need?* A ball of mass m sits between an incline and a vertical wall as shown in the figure. There is no friction between the wall and the ball but there is friction between the incline and the ball. Take the coefficient of friction to be μ and the angle of incline with the horizontal to be θ. Find the force of friction on the ball from the incline.

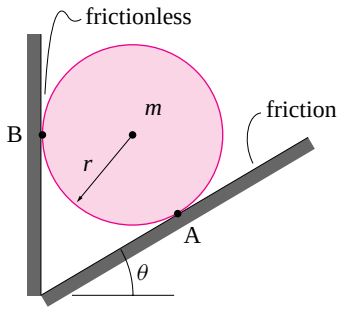


Figure 4.15: (Filename:sfig4.single.ball)

Solution The free body diagram of the ball is shown in Fig. 4.16. Note that the normal reaction of the vertical wall, N, the force of gravity, mg, and the normal reaction of the incline, R, all pass through the center C of the ball. Therefore, the moment balance about point C, $\sum \vec{M}_{\rm C} = \vec{0}$, gives

$$\begin{aligned} \vec{r}_{\rm A/C} \times F_s\hat{\boldsymbol{\lambda}} &= \vec{0} \\ \Rightarrow \quad F_s &= 0 \end{aligned}$$

Thus the force of friction on the ball is zero! Note that F_s is independent of θ, the angle of incline. Thus, irrespective of what the angle of incline is, in the static equilibrium condition, there is no force of friction on the ball.

$$\boxed{F_s = 0}$$

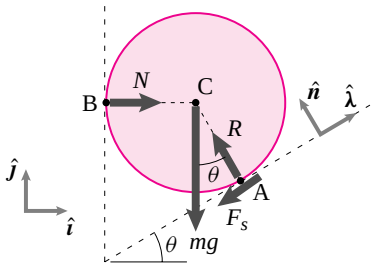


Figure 4.16: (Filename:sfig4.single.ball.a)

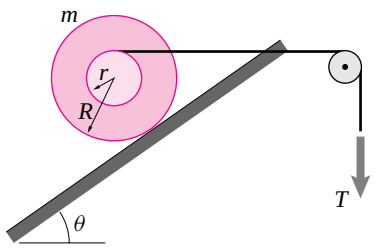


Figure 4.17: (Filename:sfig4.single.spool)

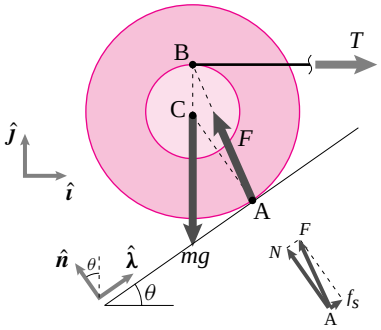


Figure 4.18: (Filename:sfig4.single.spool.a)

SAMPLE 4.6 *Can you balance this?* A spool of mass $m = 2\,\text{kg}$ rests on an incline as shown in the figure. The inner radius of the spool is $r = 200\,\text{mm}$ and the outer radius is $R = 500\,\text{mm}$. The coefficient of friction between the spool and the incline is $\mu = 0.4$, and the angle of incline $\theta = 60^\text{o}$.

(a) Which way does the force of friction act, up or down the incline?
(b) What is the required horizontal pull T to balance the spool on the incline?
(c) Is the spool about to slip?

Solution

(a) The free body diagram of the spool is shown in Fig. 4.18. Note that the spool is a 3-force body. Therefore, in static equilibrium all the three forces — the force of gravity mg, the horizontal pull T, and the incline reaction F — must intersect at a point. Since T and mg intesect at the top of the inner drum (point B), the incline reaction force $\vec{F}$ must be along the direction AB. Now the incline reaction $\vec{F}$ is the vector sum of two forces — the normal (to the incline) reaction N and the friction force F_s (along the incline). The normal reaction force N passes though the center C of the spool. Therefore, the force of friction F_s must point up along the incline to make the resultant $\vec{F}$ point along AB.

(b) From the moment equilibrium about point A, $\sum \vec{M}_\text{A} = \vec{0}$, we get

$$\vec{r}_{\text{C/A}} \times (-mg\hat{j}) + \vec{r}_{\text{B/A}} \times (T\hat{i}) = \vec{0}$$

Substituting the cross products

$$\vec{r}_{\text{C/A}} \times (-mg\hat{j}) = mgR\sin\theta\hat{k} \quad \text{and} \quad \vec{r}_{\text{B/A}} \times (T\hat{i}) = -T(R\cos\theta + r)\hat{k}$$

and dotting the entire equation with $\hat{k}$, we get

$$\begin{aligned} mgR\sin\theta &= T(R\cos\theta + r) \\ \Rightarrow \quad T &= mg\,\frac{\sin\theta}{\cos\theta + r/R} \\ &= 2\,\text{kg}\cdot 9.81\,\text{m/s}^2 \cdot \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2} + \frac{.2\,\text{m}}{.5\,\text{m}}} = 18.88\,\text{N} \end{aligned}$$

$$\boxed{T = 18.88\,\text{N}}$$

(c) To find if the spool is about to slip, we need to find the force of friction F_s and see if $F_s = \mu N$. The force balance on the spool, $\sum \vec{F} = \vec{0}$ gives

$$T\hat{i} - mg\hat{j} + F_s\hat{\lambda} + N\hat{n} = \vec{0} \tag{4.10}$$

where $\hat{\lambda}$ and $\hat{n}$ are unit vectors along the incline and normal to the incline, respectively. Dotting eqn. (4.10) with $\hat{\lambda}$ we get

$$\begin{aligned} F_s &= -T(\underbrace{\hat{i}\cdot\hat{\lambda}}_{\cos\theta}) + mg(\underbrace{\hat{j}\cdot\hat{\lambda}}_{\sin\theta}) = -T\cos\theta + mg\sin\theta \\ &= -18.88\,\text{N}(1/2) + 19.62\,\text{N}(\sqrt{3}/2) = 7.55\,\text{N} \end{aligned}$$

Similarly, we compute the normal force N by dotting eqn. (4.10) with $\hat{n}$:

$$\begin{aligned} N &= -T(\hat{i}\cdot\hat{n}) + mg(\hat{j}\cdot\hat{n}) = T\sin\theta + mg\cos\theta \\ &= 18.88\,\text{N}(\sqrt{3}/2) + 19.62\,\text{N}(1/2) = 26.16\,\text{N} \end{aligned}$$

Now we find that $\mu N = 0.4(26.16\,\text{N}) = 10.46\,\text{N}$ which is greater than $F_s = 7.55\,\text{N}$. Thus $F_s < \mu N$, and therefore, the spool is not in the condition of impending slip.

4.2 Elementary truss analysis

Join two pencils (or pens, chopsticks, or popsicle sticks) tightly together with a rubber band as in fig. 4.19a. You can feel that the pencils rotate relative to each other relatively easily. But it is hard to slide one against the other. Add a third pencil to complete the triangle (fig. 4.19b). The relative rotation of the first two pencils is now almost totally prohibited. Now tightly strap strap four pencils (or whatever) into a square with rubber bands as in fig. 4.19c, making 4 rubber band joints at the corners. Put the square down on a table. The pencils don't stretch or bend visibly, nor do they slide much along each-other's lengths, but the connections allow the pencils to rotate relative to each other so the square easily distorts into a parallelogram. Because a triangle is fully determined by the lengths of its sides and a quadrilateral is not, the triangle is *much* harder to distort than the square. A triangle is sturdy even without restraint against rotation at the joints and a square is not.

Now add two more pencils to your triangle to make two triangles (fig. 4.19d). So long as you keep this structure flat on the table, it is also sturdy. You have just observed the essential inspiration of a truss: triangles make sturdy structures.

A different way to imagine discovering a truss is by means of swiss cheese. Imagine your first initial design for a bridge is to make it from one huge piece of solid steel. This would be heavy and expensive. So you could cut holes out of the chunk here and there, greatly diminishing the weight and amount of material used, but not much reducing the strength. Between these holes you would see other heavy regions of metal from which you might cut more holes leading to a more savings of weight at not much cost in strength. In fact, the reduced weight in the middle decreases the load on the outer parts of the structure possibly making the whole structure stronger. Eventually you would find yourself with a structure that looks much like a collection of bars attached from end to end in vaguely triangular patterns. As opposed to a solid block, a truss

- Uses less material;
- Puts less gravity load on other parts of the structure;
- Leaves space for other things of interest (*e.g.*, cars, cables, wires, people).

Real trusses are usually not made by removing material from a solid but by joining bars of steel, wood, or bamboo with with welds, bolts, rivets, nails, screws, glue, or lashings. Now that you are aware you will probably notice trusses in bridges, radio towers, and large-scale construction equipment. Early airplanes were flying trusses.① Trusses have been used as scaffoldings for millennia. Birds have had bones whose internal structure is truss-like since they were dinosaurs. Trusses are worth study on their own, since they are a practical way to design sturdy light structures. But trusses also are useful

- As a first example of a complex mechanical system that a student can analyze;
- As an example showing the issues involved in structural analysis;
- As an intuition builder for understanding structures that are not really trusses (The engineering mind often sees an underlying conceptual truss where no physical truss is externally visible).

What is a truss?

> A *truss* is a structure made from connecting long narrow elements at their ends.

The sturdiness of most trusses comes from the inextensibility of the bars, not the resistance to rotation at the joints. To make the analysis simpler the (generally small) resistance to rotation in the joints is totally neglected in truss analysis. Thus

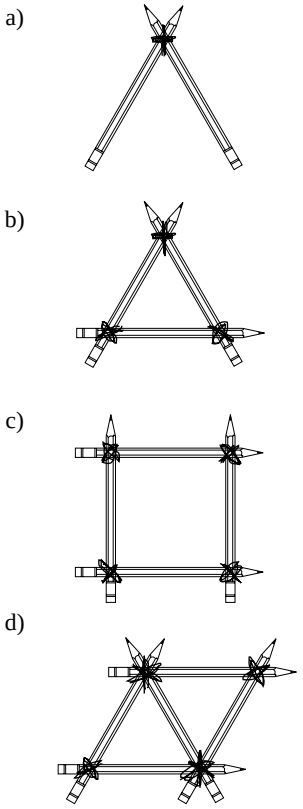


Figure 4.19: a) Two pencils strapped together with a rubber band are not sturdy. b) A triangle made of pencils feels sturdy. c) A square made of 4 pencils easily distorts into a parallelogram. d) A structure made of two triangles feels sturdy (if held on a table).

(Filename:tfigure.pencil)

① The Wright brothers first planes were near copies of the planes built a few years earlier by Octave Chanute, a retired bridge designer. With regard to structural design, these early biplanes were essentially flying bridges. Take away the outer skin from many small modern planes and you will also find trusses.

An *ideal truss* is an assembly of two force members.

Or, if you like, an ideal truss is a collection of bars connected at their ends with frictionless pins. Loads are only applied at the pins. In engineering analysis, the word 'truss' refers to an ideal truss even though the object of interest might have, say, welded joint connections. Had we assumed the presence of welding equipment in your room, the opening paragraph of this section would have described the welding of metal bars instead of the attachment of pencils with rubber bands. Even welded, you would have found that a triangle is more rigid than a square.

Bars, joints, loads, and supports

An ideal truss is a collection of *bars* connected at frictionless *joints* at which are applied *loads* as shown in fig. 4.20a (the load at a joint can be $\vec{\mathbf{0}}$ and thus not show on either the sketch of the truss or the free body diagram of the truss). Each bar is a two-force body so has a free body diagram like that shown in fig. 4.20b, with the same tension force pulling away from each end. A joint can be cut free with a conceptual chain saw, fooling each bar stub with the bar tension, as in the free body diagram 4.20c. A truss is held in place with supports which are idealized in 2D as either being fixed pins (as for joint E in fig. 4.20a) or as a pin on a roller (as for joint G in fig. 4.20a). The forces of the outside world on the truss at the supports are called the *reaction forces*.

The bar tensions can be negative. A bar with a tension of, say, $T = -5000\,\text{N}$ is said to to be in *compression*.

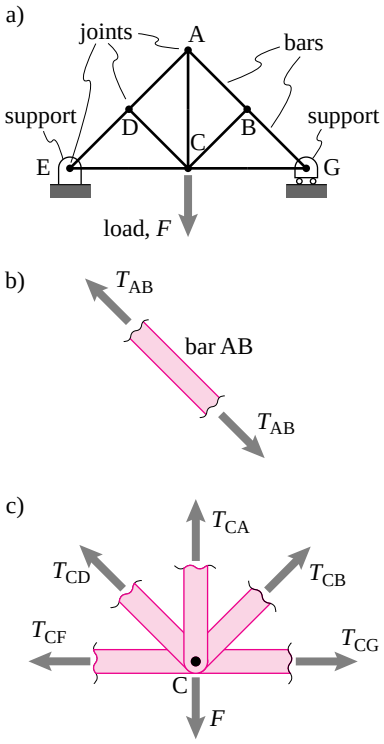


Figure 4.20: a) a truss, b) each bar is a two-force body, c) A joint is acted on by bar tensions and from applied loads.

(Filename:tfigure.trussdef)

Elementary truss analysis

In elementary truss analysis you are given a truss design to which given loads are applied. Your goal is to 'solve the truss' which means you are to find the reaction forces and the tensions in the bars (sometimes called the 'bar forces'). As an engineer, this allows you to determine the needed strengths for the bars.

The elementary truss analysis you are about to learn is straightforward and fun. You will learn it without difficulty. However, the analysis of trusses at a more advanced level is mysteriously deep and has occupied great minds from the mid-nineteenth century (*e.g.*, Maxwell and Cauchy) to the present.

The method of free body diagrams

Trusses are always analyzed by the method of free body diagrams. Free body diagrams are drawn of the whole truss and of various parts of the truss, the equilibrium equations are applied to each free body diagram, and the resulting equations are solved for the unknown bar forces and reactions.

The method of free body diagrams is sometimes subdivided into two sub-methods.

- In the *method of joints* you draw free body diagrams of every joint and apply the force balance equations to each free body diagram. The method of joints is systematic and complete; if a truss can be solved, it can be solved with the method of joints.

- In the *method of sections* you draw a free body diagrams of one or more sections of the structure each of which includes 2 or more joints and apply force and moment balance to the section. The method of sections is powerful tool but is generally not applied systematically. Rather, the method of sections is a mostly used for determining 1-3 bar forces in trusses that have a simple aspect to them. The method of sections can add to your intuitive understanding of how a structure carries a load.

For either of these methods, it is often useful to first draw a free body diagram of the whole structure and use the equilibrium equations to determine what you can about the reaction forces.

Consider this planar approximation to the arm of a derrick used in construction where F and d are known (see fig.4.21). This truss has joints A-S (skipping 'F' to avoid confusion with the load). As is common in truss analysis, we totally neglect the force of gravity on the truss elements①. From the free body diagram of the whole

① To include the force of gravity on the truss elements replace the single gravity force at the center of each bar with a pair of equivalent forces at the ends. The gravity loads then all apply at the joints and the truss can still be analyzed as a collection of two-force members.

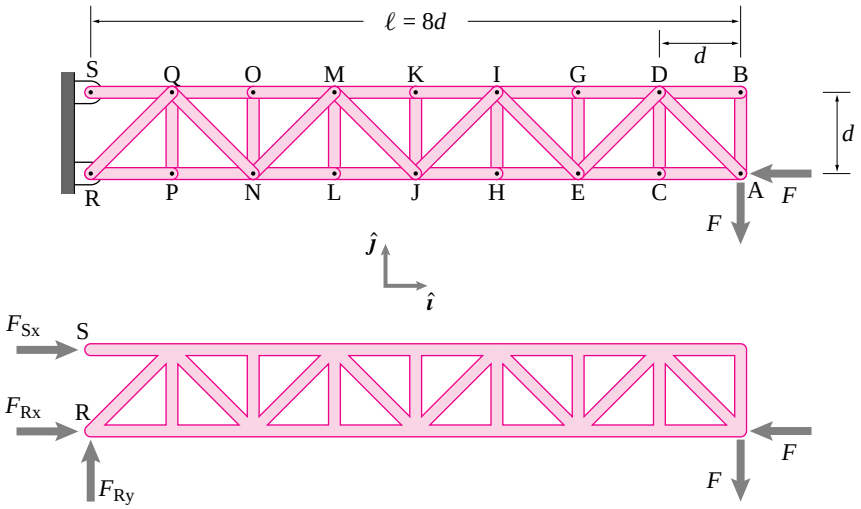


Figure 4.21: A truss. (Filename:tfigure.derrick)

structure we find that

$$\begin{aligned} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} \quad &\Rightarrow \quad F_{Ry} = F \\ \left\{\sum \vec{M}_S = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad F_{Rx} = 9F \\ \left\{\sum \vec{M}_R = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad F_{Sx} = -8F. \end{aligned}$$

The method of joints

The sure-fire approach to solve a truss is the brute force method of joints. For the truss above you draw 18 free body diagrams, one for each joint. For each joint free body diagram you write the force balance equations, each of which can be broken down into 2 scalar equations. You then solve these 36 equations for the 33 unknown bar tensions and the 3 reactions (which we found already, but need not have). In general solving 36 simultaneous equations is really only feasible with a computer, which is one way to go about things.

For simple triangulated structures, like the one in fig. 4.21, you can find a sequence of joints for which there are at most two unknown bar forces at each joint. So hand solution of the joint force balance equations is actually feasible. For this truss we could start at joint B (see fig. 4.22) where force balance tells us at a glance that

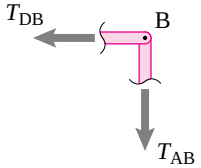


Figure 4.22: Free body diagram of joint B.

(Filename:tfigure.derrickB)

$$T_{AB} = 0 \quad \text{and} \quad T_{DB} = 0.$$

Just by looking at the joint and thinking about the free body diagram you could probably pick out these *zero force members*. Now you can draw a free body diagram of joint A where there are only two unknown tensions (since we just found T_{AB}), namely T_{AD} and T_{AC}. Force balance will give two scalar equations which you can solve to find these. Now you can move on to joint C. Here, without drawing the free body diagram on paper, you might see that bar CD is also a zero force member (its the only thing pulling up on joint C and the net up force has to be zero). In any case force balance for joint C will tell you T_{CD} and T_{CE}. You can then work your way through the alphabet of joints and find all the bar tensions, using the bar tensions you have already found as you go on to new joints.

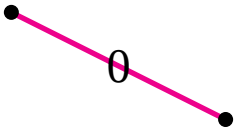


Figure 4.23: A zero force member is sometimes indicated by writing a zero on top of the bar.

(Filename:tfigure.zeroforce)

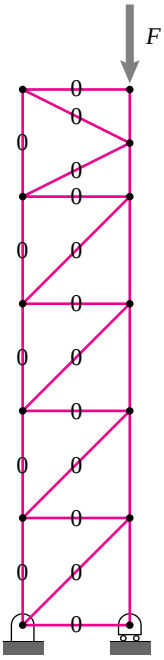


Figure 4.24: A tower with many zero force members. Although they carry no load they prevent structural collapse.

(Filename:tfigure.zeroforcetower)

Zero force members

The unnecessary but useful trick of recognizing zero-force members, like we just did for bars AB,BD and CD in the truss of fig. 4.21, can be systematized. The basic idea is this: if there is any direction for which only one bar contributes a force, that bar tension must be zero. In particular:

- At any joint where there are no loads, where there are only two unknown non-parallel bar forces, and where all known bar-tensions are zero, the two new bar tensions are both zero (joint B in the example above).
- At any joint where all bars but one are in the same direction as the applied load (if any), the one bar is a zero-force member (see joints C, G, H, K, L, O, and P in the example above).

In the truss of fig. 4.21 bars AB, BD, CD, EG, IH, JK, ML, NO, and PQ are all zero force members. Sometimes it is useful to keep track of the zero force members by marking them with a zero (see fig. 4.23). Although zero-force members seem to do nothing, they are generally needed. For this or that reason there are small loads, imperfections, or load induced asymmetries in a structure that give the 'zero-force' bars a small job to do, a job not noticed by the equilibrium equations in elementary truss analysis, but one that can prevent total structural collapse. Imagine, for example, the tower of fig. 4.24 if all of the zero-force members were removed.

The method of sections

Say you are interested in the truss of fig. 4.21, but only in the tension of bar KM. You already know how to find T_{KM} using the brute-force method of joints or by working through the joints one at time. The method of sections provides a shortcut.

You look for a way to isolate a section of the structure using a *section cut* that cuts the bar of interest and at most two other bars as in free body diagram 4.25. For the method of sections to bear easy fruit, the truss must be simple in that it has a place where it can be divided with only three bar cuts.

Because 2D statics of finite bodies gives three scalar equations we can find all three unknown tensions. In particular:

$$\left\{\sum \vec{M}_J = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad T_{KM} = 4\ F.$$

Using this same section cut we can also find:

$$\begin{aligned} \left\{\sum \vec{M}_M = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad T_{JL} = -5\ F, \text{ and} \\ \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} \quad &\Rightarrow \quad F_{JM} = \sqrt{2}\ F. \end{aligned}$$

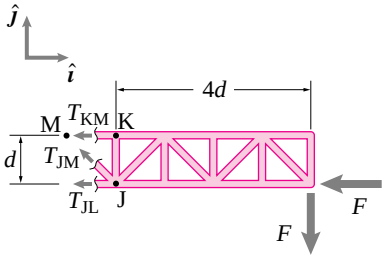


Figure 4.25: Free body diagram of a 'section' of the structure.

(Filename:tfigure.derricksection)

A traditional part of the shortcut in the method of sections is to avoid the solution of even two or three simultaneous equations by judicious choice of equilibrium equations following this general rule.

> Use equilibrium equations that don't contain terms that you don't know and don't care about.

The two common implementations of this rule are:

- Use moment balance about points where the lines of action of two unknown forces meet. In the free body diagram of fig. 4.25 moment balance about point J eliminates T_{JM} and T_{JL} and gives one equation for T_{KM}.
- Use force balance perpendicular to the direction of a pair of parallel unknown forces. In the free body diagram of fig. 4.25 force balance in the $\hat{j}$ direction eliminates T_{KM} and T_{JL} and gives one equation for T_{JM}.

In the method of joints, as you worked your way along the structure fig. 4.21 from right to left you would have found the tensions getting bigger and bigger on the top bars and the compressions (negative tensions) getting bigger and bigger on the bottom bars. With the method of sections you can see that this comes from the lever arm of the load F being bigger and bigger for longer and longer sections of truss. The moment caused by the vertical load F is carried by the tension in the top bars and compression in the bottom bars.

Why aren't trusses everywhere?

Trusses can carry big loads with little use of material and can look nice (See fig. 4.26)., so why don't engineers use them for all structural designs? Here are some reasons to consider other designs:

- Trusses are relatively difficult to build and thus possibly expensive.
- They are sensitive to damage when loads are not applied at the anticipated joints. They are especially sensitive to loads on the middle of the bars.
- Trusses inevitably depend on the tension strength in some bars. Some common building materials (*e.g.*, concrete, stone, and clay) crack easily when pulled.
- Trusses usually have little or no redundancy, so failure in one part can lead to total structural failure.
- The triangulation that trusses require can use space that is needed for other purposes (*e.g.*, doorways or rooms)
- Trusses tend to be stiff, and sometimes more flexibility is desirable (*e.g.*, diving boards, car suspensions).
- In some places some people consider trusses unaesthetic.

None-the-less, for situations where you want a stiff, light structure that can carry known loads at pre-defined points, a truss is often a great design choice.



Figure 4.26: Sometimes trusses are used only because they look nice. The tensegrity structure 'Needle Tower' was designed by artist Kenneth Snelson and is on display in the Hirshhorn Museum in Washington, DC. Here you are looking straight up the middle. Photograph by Christopher Rywalt.

(Filename:tfigure.Needle)

Summary

Using free body diagrams of the whole structure, sections of the structure, or the joints, you can find the tensions in the bars and the reaction forces for some elementary trusses. There are trusses that do not yield to this analysis, however, which are discussed in the next section.

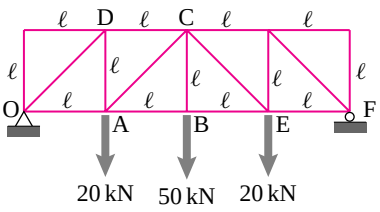


Figure 4.27: (Filename:sfig4.truss.simple)

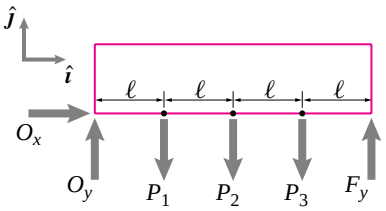
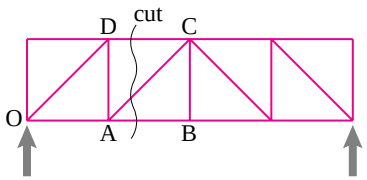


Figure 4.28: (Filename:sfig4.truss.simple.a)



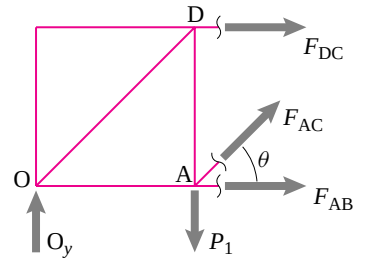


Figure 4.29: (Filename:sfig4.truss.simple.b)

SAMPLE 4.7 *A 2-D truss:* The box truss shown in the figure is loaded by three vertical forces acting at joints A, B, and E. All horizontal and vertical bars in the truss are of length 2 m. Find the forces in members AB, AC, and DC.

Solution First, we need to find the support reactions at points O and F. We do this by drawing the free body diagram of the whole truss and writing the equilibrium equations for it. Referring to Fig. 4.28, the force equilibrium, $\sum \vec{F} = \vec{0}$ implies,

$$O_x\hat{\imath} + (O_y + F_y - P_1 - P_2 - P_3)\hat{\jmath} = \vec{0} \tag{4.11}$$

Dotting eqn. (4.11) with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$\begin{aligned} O_x &= 0 \\ O_y + F_y &= P_1 + P_2 + P_3 \end{aligned} \tag{4.12}$$

The moment equilibrium about point O, $\vec{M}_O = \vec{0}$, gives

$$(-P_1\ell - P_2\,2\ell - P_3\,3\ell + F_y\,4\ell)\hat{k} = \vec{0} \tag{4.13}$$

$$\text{or} \qquad F_y = \frac{1}{4}(P_1 + 2P_2 + 3P_3) \tag{4.14}$$

Solving eqns. (4.12) and (4.14), we get

$$F_y = 45\,\text{kN}, \text{ and } O_y = 45kN.$$

In fact, from the symmetry of the structure and the loads, we could have guessed that the two vertical reactions must be equal, *i.e.*, $O_y = F_y$. Then, from eqn. (4.12) it follows that $O_y = F_y = (P_1 + P_2 + P_3)/2 = 45\,\text{kN}$.

Now, we proceed to find the forces in the members AB, AC, and DC. For this purpose, we make a cut in the truss such that it cuts members AD, AC, and DC, just to the right of joints A and D. Next, we draw the free body diagram of the left (or right) portion of the truss and use the equilibrium equations to find the required forces. Referring to Fig. 4.29, the force equilibrium requires that

$$(F_{AB} + F_{DC} + F_{AC}\cos\theta)\hat{\imath} + (O_y - P_1 + F_{AC}\sin\theta)\hat{\jmath} = \vec{0} \tag{4.15}$$

Dotting eqn. (4.15) with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$F_{AB} + F_{DC} + F_{AC}\cos\theta = 0 \tag{4.16}$$

$$O_y - P_1 + F_{AC}\sin\theta = 0 \tag{4.17}$$

So far, we have two equations in three unknowns (F_{AB}, F_{DC}, F_{AC}). We need one more independent equation to be able to solve for the unknown forces. We now write moment equilibrium equation about point A, *i.e.*, $\sum \vec{M}_A = \vec{0}$,

$$\begin{aligned} (-O_y\ell - F_{DC}\ell)\hat{k} &= \vec{0} \\ \Rightarrow \qquad O_y + F_{DC} &= 0. \end{aligned} \tag{4.18}$$

We can now solve eqns. (4.16–4.18) any way we like, *e.g.*, using elimination or a computer. The solution we get (see next page for details) is:

$$F_{AC} = -25\sqrt{2}\,\text{kN}, \quad F_{DC} = -45\,\text{kN}, \quad \text{and} \quad F_{AB} = 70\,\text{kN}.$$

$$\boxed{F_{AC} = -25\sqrt{2}\,\text{kN}, \quad F_{DC} = -45\,\text{kN}, \quad F_{AB} = 70\,\text{kN}.}$$

Comments:

- Note that the values of F_{AC} and F_{DC} are negative which means that bars AC and DC are in compression, not tension, as we initially assumed. Thus the solution takes care of our incorrect assumptions about the directionality of the forces.
- **Short-cuts:** In the solution above, we have not used any tricks or any special points for moment equilibrium. However, with just a little bit of mechanics intution we can solve for the required forces in five short steps as shown below.
  (i) No external force in $\hat{\boldsymbol{\imath}}$ direction implies $O_x = 0$.
  (ii) Symmetry about the middle point B implies $O_y = F_y$. But,

$$O_y + F_y = \sum P_i = 90\,\text{kN} \quad \Rightarrow \quad O_y = F_y = 45\,\text{kN}.$$

  (iii) $(\sum \vec{\boldsymbol{M}}_{\text{A}} = \vec{\boldsymbol{0}}) \cdot \hat{\boldsymbol{k}}$ gives

$$O_y \ell + F_{DC} \ell = 0 \quad \Rightarrow \quad F_{DC} = -O_y = -45\,\text{kN}.$$

  (iv) $(\sum \vec{\boldsymbol{M}}_{\text{C}} = \vec{\boldsymbol{0}}) \cdot \hat{\boldsymbol{k}}$ gives

$$-O_y\, 2\ell + P_1 \ell + F_{AB} \ell = 0 \quad \Rightarrow \quad F_{AB} = 2O_y - P_1 = 70\,\text{kN}.$$

  (v) $(\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}) \cdot \hat{\boldsymbol{\jmath}}$ gives

$$O_y - P_1 + F_{AC} \sin\theta = 0 \quad \Rightarrow \quad F_{AC} = (P_1 - O_y)/\sin\theta = -25\sqrt{2}\,\text{kN}.$$

- **Solving equations:** On the previous page, we found F_{AB}, F_{DC}, and F_{AC} by solving eqns. (4.15–4.17) simultaneously. Here, we show you two ways to solve those equations.
  (a) *By elimination:* From eqn. (4.17), we have

$$F_{AC} = \frac{O_y - P_1}{\sin\theta} = \frac{20\,\text{kN} - 45\,\text{kN}}{1/\sqrt{2}} = -25\sqrt{2}\,\text{kN}.$$

  From eqn. (4.18), we get

$$F_{DC} = -O_y = -45\,\text{kN},$$

  and finally, substituting the values found in eqn. (4.15), we get

$$F_{AB} = -F_{DC} - F_{AC}\cos\theta = 45\,\text{kN} + 25\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 70\,\text{kN}.$$

  (b) *On a computer:* We can write the three equations in the matrix form:

$$\underbrace{\begin{bmatrix} 1 & 1 & \cos\theta \\ 0 & 0 & \sin\theta \\ 0 & 1 & 0 \end{bmatrix}}_{\mathsf{A}} \underbrace{\begin{Bmatrix} F_{AB} \\ F_{DC} \\ F_{AC} \end{Bmatrix}}_{\mathsf{x}} = \begin{Bmatrix} 0 \\ P_1 - O_y \\ -O_y \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ -25 \\ -45 \end{Bmatrix}}_{\mathsf{b}} \text{kN}$$

  We can now solve this matrix equation on a computer by keying in matrix A (with θ specified as $\pi/4$) and vector b as input and solving for x.①

① Pseudocode:

```
A = [1 1 cos(pi/4)
     0 0 sin(pi/4)
     0 1 0]
b = [0 -25 -45]
solve A*x = b for x
```

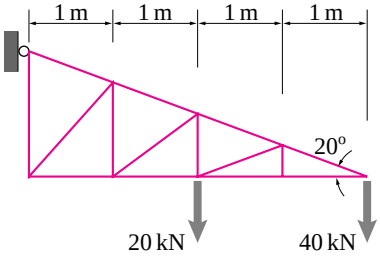


Figure 4.30: (Filename:sfig4.truss.comp)

SAMPLE 4.8 The truss shown in the figure has four horizontal bays, each of length 1 m. The top bars make 20° angle with the horizontal. The truss carries two loads of 40 kN and 20 kN as shown. Find the forces in each bar. In particular, find the bars that carry the maximum tensile and compressive forces.

Solution Since we need to find the forces in all the 15 bars, we need to find enough equations to solve for these 15 forces in addition to 3 unknown reactions A_x, A_y, and I_x. Thus we have a total of 18 unknowns. Note that there are 9 joints and therefore, we can generate 18 scalar equations by writing force equilibrium equations (one vector equation per joint) for each joint.

Number of unknowns	15 + 3 = 18
Number of joints	9
Number of equations	9 × 2 = 18

So, we go joint by joint, draw the free body diagram of each joint and write the equilibrium equations. After we get all the equations, we can solve them on a computer. All joint equations are just force equilibrium equations, *i.e.*, $\sum \vec{F} = \vec{0}$.

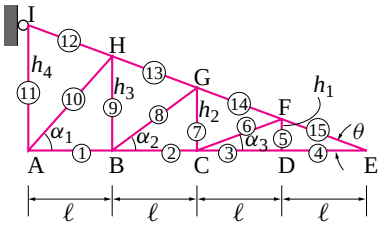


Figure 4.31: (Filename:sfig4.truss.comp.a)

- Joint **A**:

$$(A_x + T_1 + T_{10}\cos\alpha_1)\hat{\boldsymbol{\imath}} + (A_y + T_{11} + T_{10}\sin\alpha_1)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{4.19}$$

- Joint **B**:

$$(-T_1 + T_2 + T_8\cos\alpha_2)\hat{\boldsymbol{\imath}} + (T_9 + T_8\sin\alpha_2)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{4.20}$$

- Joint **C**:

$$(-T_2 + T_3 + T_6\cos\alpha_3)\hat{\boldsymbol{\imath}} + (T_7 + T_6\sin\alpha_3)\hat{\boldsymbol{\jmath}} = P\,\hat{\boldsymbol{\jmath}} \tag{4.21}$$

- Joint **D**:

$$(T_4 - T_3)\hat{\boldsymbol{\imath}} + T_5\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{4.22}$$

- Joint **E**:

$$(-T_4 - T_{15}\cos\theta)\hat{\boldsymbol{\imath}} + T_{15}\sin\theta\,\hat{\boldsymbol{\jmath}} = 2P\,\hat{\boldsymbol{\jmath}} \tag{4.23}$$

- Joint **F**:

$$(-T_6\cos\alpha_3 + (T_{15} - T_{14})\cos\theta)\hat{\boldsymbol{\imath}} + (-T_6\sin\alpha_3 + (T_{14} - T_{15})\sin\theta - T_5)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{4.24}$$

- Joint **G**:

$$(-T_8\cos\alpha_2 + (T_{14} - T_{13})\cos\theta)\hat{\boldsymbol{\imath}} + ((T_{13} - T_{14})\sin\theta - T_8\sin\alpha_2 - T_7)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{4.25}$$

- Joint **H**:

$$(-T_{10}\cos\alpha_1 + (T_{13} - T_{12})\cos\theta)\hat{\boldsymbol{\imath}} + ((T_{12} - T_{13})\sin\theta - T_{10}\sin\alpha_1 - T_9)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{4.26}$$

- Joint **I**:

$$(-I_x + T_{12}\cos\theta)\hat{\boldsymbol{\imath}} + (-T_{11} - T_{12}\sin\theta)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{4.27}$$

Dotting each equation from (4.19) to (4.27) with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, we get the required 18 equations. We need to define all the angles that appear in these equations ($\alpha_1, \alpha_2, \alpha_3,$ and θ) before we are ready to solve the equations on a computer.

Let ℓ be the length of each horizontal bar and let DF $= h_1$, CG $= h_2$, and BH $= h_3$. Then, $h_1/\ell = h_2/2\ell = h_3/3\ell = \tan\theta$. Therefore,

$$\begin{aligned}
\tan\alpha_1 = \frac{h_1}{\ell} = \tan\theta \quad &\Rightarrow \quad \alpha_1 = \tan^{-1}(\tan\theta) = \theta \\
\tan\alpha_2 = \frac{h_2}{\ell} = 2\tan\theta \quad &\Rightarrow \quad \alpha_2 = \tan^{-1}(2\tan\theta) \\
\tan\alpha_3 = \frac{h_3}{\ell} = 3\tan\theta \quad &\Rightarrow \quad \alpha_3 = \tan^{-1}(3\tan\theta)
\end{aligned}$$

Now, we are ready for a computer solution. You can enter the 18 equations in matrix form or as your favourite software package requires and get the solution by solving for the unknowns. Here are two examples of pseudocodes. Let us order the unknown forces in the form

$$x = [T_1 \quad T_2 \quad \ldots \quad T_{15} \quad A_x \quad A_y \quad I_x]^T$$

so that x_1–x_{15} = T_1–T_{15}, $x_{16} = A_x$, $x_{17} = A_y$, and $x_{18} = I_x$

(a) Entering full matrix equation:

```
theta = pi/9                        % specify theta in radians
alpha1 = theta                      % calculate alpha1
alpha2 = atan(2*tan(theta))         % calculate alpha2 from arctan
alpha3 = atan(3*tan(theta)          % calculate alpha3 from arctan

C = cos(theta), S = sin(theta)      % compute all sines and cosines
C1 = cos(alpha1), S1 = sin(alpha1)
C2 = .. ..

A = [1 0 0 0 0 0 0 0 0 C1 0 0 0 0 0 1 0 0     % enter matrix A row-wise
     0 0 0 0 0 0 0 0 0 S1 1 0 0 0 0 0 1 0
     .
     .
     0 0 0 0 0 0 0 0 0 0 -1 -S 0 0 0 0 0 0]
b = [0 0 0 0 0 20 0 0 0 40 0 0 0 0 0 0 0 0]'  % enter column vector b
solve A*x = b for x
```

(b) Entering each equation as part of matrix A and vector b:

```
A(1,[1 10 16]) = [1 C1 1]
A(2,[10 11 17]) = [S1 1 1]
.
.
A(18,[11 12]) = [-1 -S]
b(6,1) = 20
b(10,1) = 40
form A and b setting all other entries to zero
solve A*x = b for x
```

The solution obtained from the computer is

$$\begin{gathered}
T_1 = -128.22\,\text{kN}, \quad T_2 = T_3 = T_4 = -109.9\,\text{kN}, \quad T_5 = T_6 = 0, \\
T_7 = 20\,\text{kN}, \quad T_8 = -22.66\,\text{kN}, \quad T_9 = -T_{10} = 13.33\,\text{kN}, \quad T_{11} = -50\,\text{kN}, \\
T_{12} = 146.19\,\text{kN}, \quad T_{13} = 136.44\,\text{kN}, \quad T_{14} = T_{15} = 116.95\,\text{kN}, \\
A_x = 137.37\,\text{kN}, \quad A_y = 60\,\text{kN}, \quad I_x = -137.37\,\text{kN}.
\end{gathered}$$

4.3 Advanced truss analysis: determinacy, rigidity, and redundancy

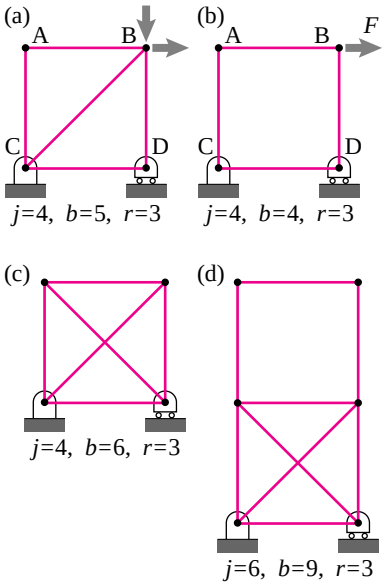


Figure 4.32: a) a statically determinate truss, b) a non-rigid truss, c) a redundant truss, and d) a non-rigid and redundant truss.

(Filename:tfigure.4cases)

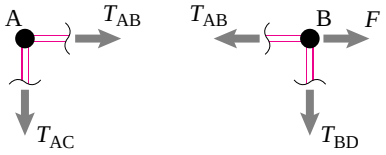


Figure 4.33: Free body diagrams of joints A and B from 4.32b

(Filename:tfigure.squarejoints)

After you have mastered the elementary truss analysis of the previous section, namely the method of free body diagrams in its two incarnations (the method of joints and the method of sections) you might wonder if at least one of these methods always work. The answer is yes, if you just look at the homework problems for elementary truss analysis, but 'no' if you look at the variety of real (good and bad) structures in the world. In this section we discuss the classification of trusses into types. In the previous section all of the examples were from one of these types.

Determinate, rigid, and redundant trusses

Your first concern when studying trusses is to develop the ability to solve a truss using free body diagrams and equilibrium equations. A truss that yields a solution, and only one solution, to such an analysis for all possible loadings is called *statically determinate* or just *determinate*. The braced box supported with one pin joint and one pin on rollers (see fig. 4.32a) is a classic statically determinate truss. A statically determinate truss is *rigid* and does not have *redundant* bars.

You should beware, however, that there are a few other possibilities.

Some trusses are *non-rigid*, like the one shown in fig. 4.32b, and can not carry arbitrary loads at the joints.

Example: **Joint equations and non-rigid structures**

Free body diagrams of joints A and B of fig. 4.32b are shown in fig. 4.33.

$$\begin{array}{lrcl} \text{jointB}: & \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{\imath} & \Rightarrow & T_{AB} = F \\ \text{jointA}: & \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{\imath} & \Rightarrow & T_{AB} = 0 \end{array}$$

The contradiction that T_{AB} is both F and 0 implies that the equations of statics have no solution for a horizontal load at joint B. □

A non-rigid truss can carry some loads, and you can find the bar tensions using the joint equilibrium equations when these loads are applied. For example, the structure of fig. 4.32b can carry a vertical load at joint B. Engineers sometimes choose to design trusses that are not rigid, the simplest example being a single piece of cable hanging a weight. A more elaborate example is a suspension bridge which, when analyzed as a truss, is not rigid.

A *redundant* truss has more bars than needed for rigidity. As you can tell from inspection or analysis, the braced square of fig. 4.32a is rigid. None the less engineers will often choose to add extra redundant bracing as in fig. 4.32c for a variety of reasons.

- Redundancy is a safety feature. If one member brakes the whole structure holds up.
- Redundancy can increase a structure's strength.
- Redundancy can allow tensile bracing. In the structure of Fig. 4.32a a top load to the left puts bar BC in compression. Thus bar BC can't be, say, a cable. But in structure fig. 4.32c both diagonals can be cables and neither need carry compression for any load①.

① As a curiosity notice that you could make the diagonals in fig. 4.32c both sticks and all of the outside square from cables and the truss would still carry all loads. This is the simplest 'tensegrity' structure. In a tensegrity structure no more than one bar in compression is connected to any one joint. (See fig. 4.26 for a more elegant example.). The label 'Tensegrity structure' was coined by the truss-pre-occupied designer Buckminster Fuller. Fuller is also responsible for re-inventing the "geodesic dome" a type of structure studied previously by Cauchy.

A property of redundant structures is that you can find more than one set of bar forces that satisfy the equilibrium equations. Even when the loads are all zero these structures can have non-zero *locked in* forces (sometimes called ('locked in stress', or 'self stress'). In the structure of fig. 4.32c, for example, if one of the diagonals got hot and stretched both it and the opposite diagonal would be put in compression while the outside was in tension. For structures whose parts are likely to expand or contract, or for which the foundation may shift, this locked in stress can be a contributor to structural failure. So redundancy is not all good.

Finally, a structure can be both non-rigid and redundant as shown in fig. 4.32d. This structure can't carry all loads, but the loads it can carry it can carry with various locked in bar forces.

More examples of statically determinate, non-rigid, and redundant truss are given on pages 143 and 144.

Note, one of the basic assumptions in elementary truss analysis which we have thus far used without comment is that motions and deformations of the structure are not taken into account when applying the equilibrium equations. If a bar is vertical in the drawing then it is taken as vertical for all joint equilibrium equations.

Example: **Hanging rope**

For elementary truss analysis, a hanging rope would be taken as hanging vertically even if side loads are applied to its end. This obviously ridiculous assumption manifests itself in truss analysis by the discovery that a hanging rope cannot carry any sideways loads (if it must stay vertical this is true). □

Determining determinacy: counting equations and unknowns

How can you tell if a truss is statically determinate? The only sure test is to write all the joint force balance equations and see if they have a unique solution for all possible joint loads. Because this is an involved linear algebra calculation (which we skip in this book), it is nice to have shortcuts, even if not totally reliable. Here are three:

- See, using your intuition, if the structure can deform without any of the bars changing length. You can see that the structures of fig. 4.32b and d can distort. If a structure can distort it is not rigid and thus is not statically determinate.
- See, using your intuition, if there are any redundant bars. A redundant bar is one that prevents a structural deformation that already is prevented. It is easy to see that the second diagonal in structures of fig. 4.32c and d is clearly redundant so these structures are not statically determinate.
- Count the total number of joint equations, two for each joint. See if this is equal to the number of unknown bar forces and reactions. If not, the structure is not statically determinate.

The counting formula in the third criterion above is:

$$2j = b + r \tag{4.28}$$

where j is the number of joints, including joints at reaction points, b is the number of bars, and r is the number of reaction components that shows on a free body diagram of the whole structure (2 from pin joints, 1 from a pin on a roller).

If $2j > b + r$ the structure is necessarily not rigid because then there are more equations than unknowns①. For such a structure there are some loads for which there is no set of bar forces and reactions that can satisfy the joint equilibrium equations. A structure that is non-redundant and non-rigid always has $2j > b + r$ (see fig. 4.32b).

If $2j < b + r$ the structure is redundant because there are not as many equations as unknowns; if the equations can be solved there is more than one combination of forces that solve them. A structure that is rigid and redundant always has $2j < b + r$ (see fig. 4.32b).

But the possibility of structures that are both non-rigid and redundant makes the counting formulas an imperfect way to classify structures②. Non-rigid redundant structures can have $2j < b + r$, $2j = b + r$, or $2j > b + r$. The redundant non-rigid structure in fig. 4.32d has $2j = b + r$.

The discussion above can be roughly summarized by this table (refer to fig. 4.32 for a simple example of each entry and to pages 143 and 144 for several more examples).

Truss Type	Rigid	Non-rigid
Non-redundant	a) $2j = b + r$ (Statically determinate)	b) $2j > b + r$
Redundant	c) $2j < b + r$	d) $2j < b + r$, $2j = b + r$, or $2j > b + r$

A basic summary is this:

If

- $2j = b + r$ *and*
- you cannot see any ways the structure can distort, *and*
- you cannot see any redundant bars

then the truss is likely statically determinate. But the only way you can know for sure is through either a detailed study of the joint equilibrium equations, or familiarity with similar structures.

On the other hand if

- $2j > b + r$, *or*
- $2j < b + r$, *or*
- you can see a way the structure can distort, *or*
- you can see one or more redundant bars,

then the truss is *not* statically determinate.

Example: **The classic statically determinate structure**

A *triangulated truss* can be drawn as follows:

(a) draw one triangle,

(b) then another by adding two bars to an edge,

① A non-rigid truss is sometimes called 'over-determinate' because there are more equations than unknowns. However, the term 'over-determinate' may incorrectly conjure up the image of there being too many bars (which we call *redundant*) rather than too many joints. So we avoid use of this phrase.

② In the language of mathematics we would say that satisfaction of the counting equation $2j = b+r$ is a *necessary* condition for static determinacy but it is not *sufficient*.

(c) then another by adding two bars to an existent edge
(d) and so on, but never adding a triangle by adding just one bar, and
(e) you hold this structure in place with a pin at one joint and one pin on roller at another joint

then the structure is statically determinate. Many elementary trusses are of exactly this type. (*Note*: if you violate the 'but' in rule (d) you can make a truss that looks 'triangulated' but is redundant and therefore not statically determinate.) □

Floating trusses

Sometimes one wants to know if a structure is rigid and non-redundant when it is floating unconnected to the ground (but still in 2D, say). For example, a triangle is rigid when floating and a square is not. The truss of fig. 4.34a is rigid as connected but not when floating (fig. 4.34b). A way to find out if a floating structure is rigid is to connect one bar of the truss to the ground by connecting one end of the bar with a pin and the other with a pin on a roller, as in fig. 4.34c. All determinations of rigidity for the floating truss are the same as for a truss grounded this way. The counting formula eqn. 4.28, is reduced to

$$2j = b + 3$$

because this minimal way of holding the structure down uses $r = 3$ reaction force components.

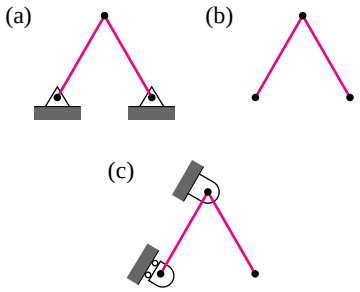


Figure 4.34: a) a determinate two bar truss connected to the ground, b) the same truss is not rigid when floating, which you can tell by seeing that c) it is not rigid when one bar is fixed to the ground.

(Filename:tfigure.rigidonground)

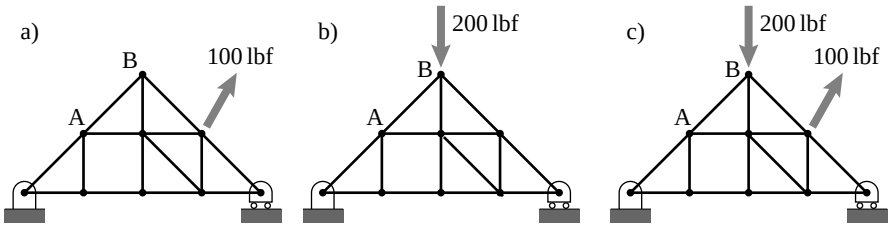
The principle of superposition for trusses

Say you have solved a truss with a certain load and have also solved it with a different load. Then if both loads were applied the reactions would be the sums of the previously found reactions and the bar forces would be the sums of the previously found bar forces.

This useful fact follows from the linearity of the equilibrium equations①.

① A careful derivation would also show that the linearity depends on the nature of the foundation. Linearity holds for pins and pins on rollers, but not for frictional contact.

Example: **Superposition and a truss**



If for the loading (a) you found $T_{AB} = 50\,\text{lbf}$ and for loading (b) you found $T_{AB} = -140\,\text{lbf}$ then for loading (c) $T_{AB} = 50\,\text{lbf} - 140\,\text{lbf} = -90\,\text{lbf}$ □

4.3 Theory: Rigidity, redundancy, linear algebra and maps

This mathematical aside is only for people who have had a course in linear algebra. For definiteness this discussion is limited to 2D trusses, but the ideas also apply to 3D trusses.

For beginners trusses fall into two types, those that are uniquely solvable (statically determinate) and those that are not. Statically determinate trusses are rigid and non-redundant. However, a truss could be non-rigid and non-redundant, rigid and redundant, or non-rigid and redundant. These four possibilities are shown with a simple example each in figure 4.32 on page 138, as a simple table on page 140, and as a big table of examples on pages 143 and 144. The table below, which we now proceed to discuss in detail, is a more abstract mathematical representation of this same set of possibilities.

• • •

We can number the bars of the truss followed by the reaction components $1, 2, \ldots, n$, where $n = b + r$. The bar tensions and support reaction forces can be put in a vertical list $[F_1, F_2, \ldots, F_n]'$. The set of lists of all conceivable tensions and reaction forces we call the "vector space" V (it is also R^n).

We can also make a list of all possible applied loads. In a 2D truss there can be a horizontal or vertical load at each joint. So, we can write a list of $m = 2j$ numbers to represent the load. If there is only an applied load at a few joints most of the elements of this load vector will be zero. The set of all possible loads we call the vector space W.

If we use the method of joints we can write two scalar equilibrium equations for each joint. These are linear algebraic equations. Thus we can write them in matrix form as:

$$[A][v] = [w] \tag{4.29}$$

where $[v]$ is the list of bar tensions and reaction forces, and $[w]$ is the list of applied loads to the joints. The matrix $[A]$ is determined by the geometry of the truss. The classification of trusses is really a statement about the solutions of eqn. 4.29. This classification follows, in turn, from the properties of the matrix $[A]$.

Another point of view is to think of eqn. 4.29 as a function that maps one vector space onto another. For any $[v]$ eqn. 4.29 maps that $[v]$ to some $[w]$. That is, if one were given all the bar tensions and reactions one could uniquely determine the applied loads from eqn. 4.29. This map, from V to W we call T.

• • •

We can now discuss each of the truss categorizations in turn, with reference to the table at the end of this box.

The first column of the table corresponds to rigid trusses. These trusses have at least one set of bar forces that can equilibrate any particular load. This means that for every $[w]$ there is some $[v]$ that maps to (whose image is) $[w]$. In these cases the map T is onto. And the columns space of $[A]$ is W. Thus $[A]$ needs to have at least as many columns as the dimension of W which is the number of rows of $[A]$.

On the other hand if the structure is not rigid there are some loads that cannot be equilibrated by any bar forces. This is the second column of the table. There is at least some $[w]$ with no preimage $[v]$. Thus the map T is not onto and the column space of $[A]$ is less than all of W.

The first row of the table describes trusses which are not-redundant. Thus, any loads which can be equilibrated can be equilibrated with a unique set of bar tensions and reactions. Thus the columns of $[A]$ are linearly independent and the map T is one to one. The matrix $[A]$ must have at least as many rows as columns.

If a truss is redundant, as in the second row of the table, then there are various ways to equilibrate loads which can be carried. Points in W in the image of one, and the columns of A are linearly dependent.

• • •

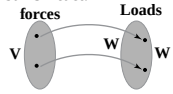
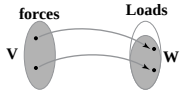
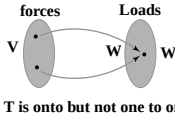
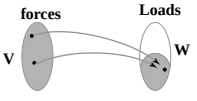
We can now look at the four entries in the table. The top left case is the statically determinate case where the structure is rigid and non-redundant. The map T is one to one and onto, $V = W$, and the matrix $[A]$ is square and non-singular.

The bottom left case corresponds to a truss that is rigid and redundant. The map to is onto but not one to one. The columns of $[A]$ are linearly dependent and it has more columns than rows (it is wide).

The top right case is not rigid and not redundant. Some loads cannot be equilibrated and those that can be are equilibrated uniquely. T is one to one but not onto. The columns of $[A]$ are linearly independent but they do not span W. The matrix $[A]$ has more rows than columns and is thus tall.

The bottom right case is the most perverse. The structure is not rigid but is redundant. Not all loads can be equilibrated but those that can be equilibrated are equilibrated non-uniquely. The matrix $[A]$ could have any shape but its columns are linearly dependent and do not span W. The map T is neither one to one nor onto.

• • •

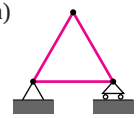
	Rigid • T is onto • col(A) = W	**Not rigid** • T is not onto • col(A) ≠ W
Not redundant • T is one to one • columns of A are linearly independent	**A is square and invertible** bar & react. forces  **T is one to one and onto**	**A is tall** bar & react. forces  **T is one to one but not onto**
Redundant • T is not one to one • columns of A are linearly dependent	**A is wide** bar & react. forces  **T is onto but not one to one**	**A can be wide, square, or tall** bar & react. forces  **T is neither one to one nor onto**

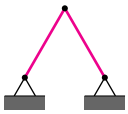
2D TRUSS CLASSIFICATION
(page 1)

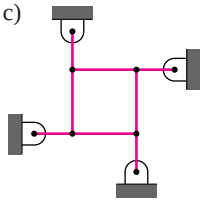
Rigid
- Not overdeterminate
- loads can be equilibriated with bar forces

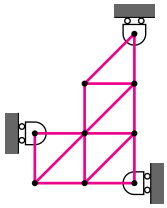
Not redundant
- Not indeterminate
- If there are bar forces that can equilibriate the loads they are unique
- No locked in stresses

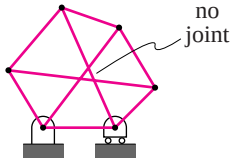
Statically determinate, rigid and not redundant,
$b + r = 2j$,
One and only one set of bar forces can equilibriate any given load.

a)

j=3, b=3, r=3

b)

j=3, b=2, r=4

c)

j=8, b=8, r=8

d)

j=9, b=15, r=3

e)


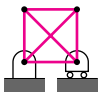
j=6, b=9, r=3

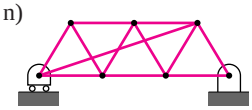
Redundant
- indeterminate
- locked in stress possible
- solutions not unique if they exist

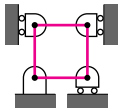
$b + r > 2j$, "too few equations", rigid and redundant,
Every possible load can be equilibrated
but the bar forces are not unique.

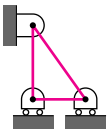
l)

j=2, b=1, r=4

m)

j=4, b=6, r=3

n)

j=7, b=12, r=3

o)

j=4, b=4, r=5

p)

j=3, b=3, r=4

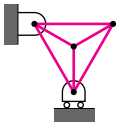
q)

j=4, b=6, r=3

Figure 4.35: Examples of 2D trusses. These two pages concern the 2-fold system for identifying trusses. Trusses can be rigid or not rigid (the two columns) and they can be redundant or not redundant (the two rows). Elementary truss analysis is only concerned with rigid and not redundant trusses (*statically determinate* trusses). Note that the only difference between trusses (b) and (s) is a change of shape (likewise for the far more subtle examples (e) and (u)). Truss (e) is interesting as a rare example of a determinate truss with no triangles.

(Filename:tfigure.trussclass1)

2D TRUSS CLASSIFICATION
(page 2)

Not rigid
- 'overdeterminate'

$b + r < 2j$, not rigid and not redundant, "too many equations"
Unique bar forces for some loads,
no solution for other loads.

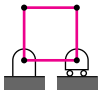
Not redundant
- Not indeterminate
- If there are bar forces that can equilibriate the loads they are unique
- No locked in stresses

f)



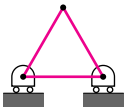
j=2
b=1
r=2

g)



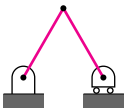
j=4, b=4, r=3

h)



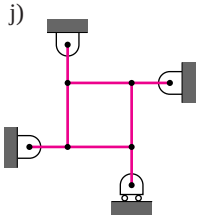
j=3, b=3, r=2

i)



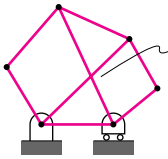
j=3, b=2, r=3

j)



j=8, b=8, r=7

k)



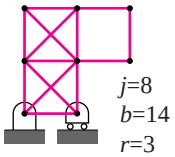
j=6, b=8, r=3

Not rigid and redundant

Redundant
- indeterminate
- locked in stress possible
- solutions not unique if they exist

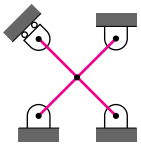
$b + r > 2j$

r)



j=8
b=14
r=3

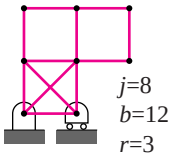
x)



j=5
b=4
r=7

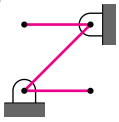
$b + r < 2j$

w)



j=8
b=12
r=3

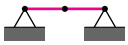
z)



j=4
b=3
r=4

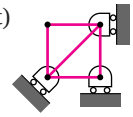
$b + r = 2j$

s)



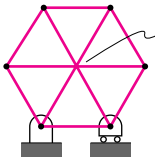
j=3, b=2, r=4

t)



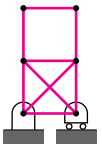
j=4
b=5
r=3

u) regular hexagon



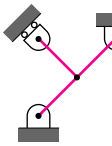
j=6, b=9, r=3

v)



j=6
b=9
r=3

y)



j=4
b=3
r=5

Figure 4.36: (Second page of a two page table.) (Filename:tfigure.trussclass2)

SAMPLE 4.9 *An indeterminate truss:* For the truss shown in the figure, find all support reactions.

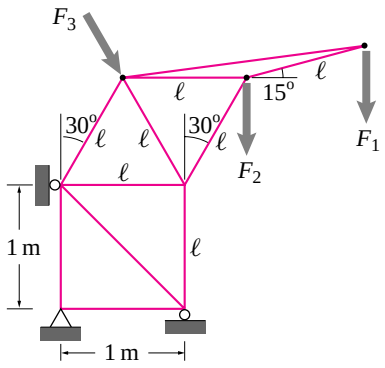


Figure 4.37: (Filename:sfig4.truss.over)

Solution The free body diagram of the truss is shown in Fig. 4.38. We need to find the support reactions A_x, A_y, B_y, and D_x.

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$(A_x + D_x + F_3\cos\theta_1)\hat{\imath} + (A_y + B_y - F_3\sin\theta_1 - F_2 - F_1)\hat{\jmath} = \vec{0} \quad (4.30)$$

$$[\text{eqn. (4.30)}]\cdot\hat{\imath} \quad \Rightarrow \quad A_x + D_x = -F_3\cos\theta_1 \quad (4.31)$$

$$[\text{eqn. (4.30)}]\cdot\hat{\jmath} \quad \Rightarrow \quad A_y + B_y = F_1 + F_2 + F_3\sin\theta_1 \quad (4.32)$$

Now we apply moment balance about point A, $\sum \vec{M}_{\rm A} = \vec{0}$. Let A be the origin of our xy-coordinate system (so that we can write $\vec{r}_{\rm D/A} = \vec{r}_{\rm D}$, etc.).

$$\vec{r}_{\rm D}\times\vec{D}_x + \vec{r}_{\rm F}\times\vec{F}_3 + \vec{r}_{\rm G}\times\vec{F}_1 + \vec{r}_{\rm E}\times\vec{F}_2 + \vec{r}_{\rm B}\times\vec{B}_y = 0$$

where,

$$\begin{aligned}
\vec{r}_{\rm D}\times\vec{D}_x &= \ell\hat{\jmath}\times D_x\hat{\imath} = -D_x\ell\hat{k}\\
\vec{r}_{\rm F}\times\vec{F}_3 &= (\vec{r}_{\rm D} + \vec{r}_{\rm F/D})\times\vec{F}_3 = [\ell\hat{\jmath} + \ell(\sin\theta_1\hat{\imath} + \cos\theta_1\hat{\jmath})]\times F_3(\cos\theta_1\hat{\imath} - \sin\theta_1\hat{\jmath})\\
&= F_3\ell\cos\theta_1\hat{k} - F_3\ell\hat{k} = -F_3\ell(1+\cos\theta_1)\hat{k}\\
\vec{r}_{\rm G}\times\vec{F}_1 &= (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath})\times(-F_1\hat{\jmath}) = -r_{G_x}F_1\hat{k}\\
&= -F_1\ell(1+\sin\theta_1+\cos\theta_2)\hat{k}\\
\vec{r}_{\rm E}\times\vec{F}_2 &= -F_2(\ell + \ell\sin\theta_1\hat{k} = -F_2\ell(1+\sin\theta_1)\hat{k}\\
\vec{r}_{\rm B}\times\vec{B}_y &= \ell\hat{\imath}\times B_y\hat{\jmath} = B_y\ell\hat{k}
\end{aligned}$$

Adding them together and dotting with $\hat{k}$ we get

$$-D_x\ell - F_3\ell(1+\cos\theta_1) - F_1\ell(1+\sin\theta_1+\cos\theta_2) - F_2\ell(1+\sin\theta_1) + B_y\ell = 0$$

$$\Rightarrow \quad B_y - D_x = F_1(1+\sin\theta_1+\cos\theta_2) + F_2(1+\sin\theta_1) + F_3(1+\cos\theta_1). \quad (4.33)$$

We have three equations (4.31–4.33) containing four unknowns A_x, A_y, B_y, and D_x. So, we cannot solve for the unknowns uniquely. This was expected as the truss is indetrminate. However, if we assume a value for one of the unknowns, we can solve for the rest in terms of the assumed one. For example, let $D_x = \alpha$. For simplicity let the right hand sides of eqns. (4.31, 4.32, and 4.33) be C_1, C_2, and C_3 (computed values), respectively. Then, we get $A_x = C_1 - \alpha$, $A_y = C_2 - C_3 - \alpha$, and $B_y = C_3 + \alpha$. The equilibrium is satisfied for any value of α. Thus there are infinite number of solutions! This is true for all indeterminate systems. However, when deformations of structures are taken into account (extra constraint equations), then solutions do turn out to be unique. You will learn about such things in courses dealing with strength of materials.

□

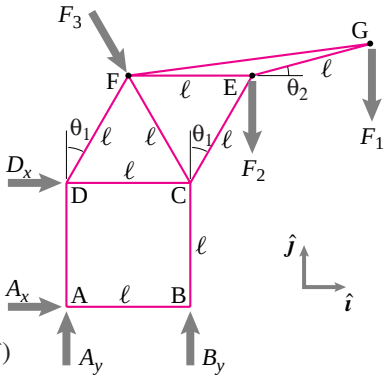


Figure 4.38: (Filename:sfig4.truss.over.a)

4.4 Internal forces

"Take one."

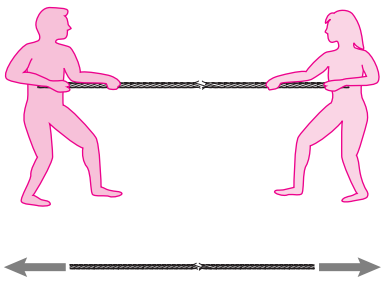


Figure 4.39: a) Two people pulling on a rope that is likely to break in the middle, b) A free body diagram of the rope.

(Filename:tfigure.ropeexternal)

Consider two people pulling on the frayed rope of fig. 4.39a. A free body diagram of the rope is shown in fig. 4.39b. The laws of mechanics use the *external* forces on an isolated system. These are the forces that show on a free body diagram. For the rope these are the forces at the ends. The free body diagram does not include internal forces. Thus nothing about the 'internal forces' at the fraying part of the rope shows up in the mechanics equations describing the rope.

Mechanics has nothing to say about so called 'internal forces' and thus nothing to say about the rope breaking in the middle. 'Internal forces' are meaningless in mechanics. End of section.

"Cut! There's got to be more to it than that. Let's try again."

4.4 Internal forces

"Take two."

On page 1 we advertised mechanics as being useful for predicting when things will break. And our intuitions strongly tell us that there is something about the forces *in* the rope that make it break. Yet mechanics equations are based on the forces that show on free body diagrams. And free body diagrams only show external forces. How can we use mechanics to describe the 'forces' inside a body? We use an idea whose simplicity hides its utility and depth:

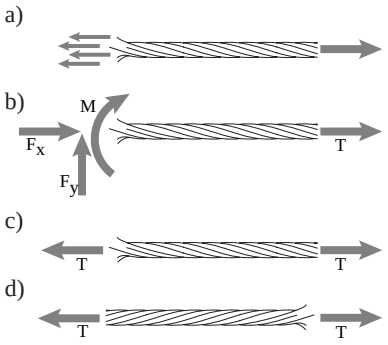


Figure 4.40: a) free body diagram of the right part of the rope, b)the same free body diagram, with the force distribution at the cut replaced with an equivalent force couple system, c) further simplified by using the laws of mechanics, and d) a free body diagram of the left portion of the rope.

(Filename:tfigure.ropeinternal)

> You cut the body, and what was inside it is now the outside of a smaller body.

In the case of the rope, we cut it in the middle. Then we fool the rope into thinking it wasn't cut using forces (remember, 'forces are *the* measure of mechanical interaction'), one force, say, at each fiber that is cut. Then we get the free body diagram of fig. 4.40a. We can simplify this to the free body diagram of fig. 4.40b because we know that every force system is equivalent to a force and couple at any point, in this case the middle of the rope. If we apply the equilibrium conditions to this cut rope we see that

$$\begin{array}{rcrcr} \text{Sum of vertical forces is zero} & \Rightarrow & F_y & = & 0 \\ \text{Sum of horizontal forces is zero} & \Rightarrow & F_x & = & -T \\ \text{Sum of moments about the cut is zero} & \Rightarrow & M & = & 0. \end{array}$$

Thus we get the simpler free body diagram of fig. 4.40c as you probably already knew without using the equilibrium equations explicitly.

Tension

We have just derived the concept of 'tension in a rope' also sometimes called the 'axial force'. The tension is the pulling force on a free body diagram of the cut rope. If we had used the same cut for a free body diagram of the left half of the rope we would see the free body diagram of fig. 4.40d. Either by the principle of action and reaction, or by the equilibrium equations for the left half of the rope, you see also a tension T. The force vector is the opposite of the force vector on the right half of the rope. So it doesn't make sense to talk about the tension force vector *in* the rope since different (opposite) force vectors manifest themselves on the two sides of the cut ($-T\hat{\boldsymbol{\imath}}$ on the left end of the right half and $T\hat{\boldsymbol{\imath}}$ on the right end of the left half). Instead we talk about the scalar tension T which expresses the force vector at the cut as

$$\vec{\boldsymbol{F}} = T\hat{\boldsymbol{\lambda}}$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector pointing out from the free body diagram cut. Because $\hat{\boldsymbol{\lambda}}$ switches direction depending on which half rope you are looking at, the same scalar T works for both pieces.

> The tension in a rope, cable, or bar is the amount of force pulling out on a free body diagram of the cut rope, cable, or bar. Tension is a scalar.

Note our abuse of language: force is a vector, tension is an 'internal force' and tension is a scalar. What we call 'internal forces' are not really forces. We can't talk about the internal force vector at a point in the string because there are two different vectors for each cut, one for each string half. An 'internal force' isn't a force unless it is made external by a free body diagram cut, in which case it is not internal. We use this confusing language because of its strong place in engineering practice and its constant reinforcement by our intuitions which sense 'internal forces'. Whenever you see the phrase 'internal force' you should substitute in your mind 'a scalar with dimensions of force from which you can find the force on a free body diagram cut' ①.

① Calling tension a scalar is one of the practical lies we tell you for relative simplicity. The clearest representation of 'internal forces' is with tensors. But that idea is too advanced for this book.

For a two force body the tension is a constant along the length (because we found T without ever using information about the location of the free body diagram cut). We used this idea without comment in trusses when we included a small stub of each bar in the free body diagrams of the joints and showed a tension force along the stub.

Getting back to the question of whether or not the rope will break, we can now characterize the rope by the tension it can carry. A $10kN$ cable can carry a tension of $10,000$ N all along its length. This means a free body diagram of the rope, cut anywhere along its length, could show forces up to but not bigger than $10,000$ N. If the rope is frayed it make break at, say, a tension of $2,000$ N, meaning a free body diagram with a cut at the fray can only show forces up to $2,000$ N.

As noted in the context of trusses, tension is not always positive. A negative tension (negative pulling out from the ends) is also called a positive compression (positive pushing in at the ends).

Shear force and bending moment

To characterize the strength of more than just 2-force bodies. we need to generalize the concept of tension. The main idea, which was emphasized in chapter 2, is this:

> You can make a free body diagram cut anywhere on any body no matter how it is loaded.

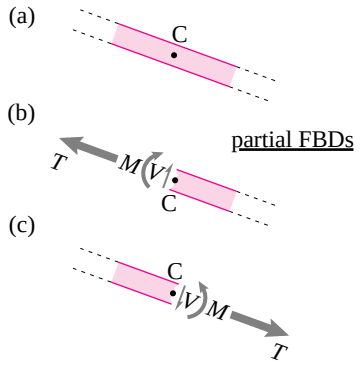


Figure 4.41: a) A piece of a structure, loads not shown; b) a partial free body diagram of the right part of the bar; c) a partial free body diagram of the left part of the bar.

(Filename:tfigure.signs)

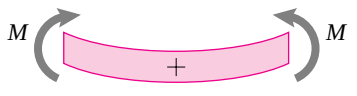


Figure 4.42: The smiling beam sign convention for bending moment. For a horizontal beam, moments which tend to make the beam smile (curve up) are called positive.

(Filename:tfigure.smilingbeam)

As for tension, we define internal forces in terms of the forces (and moments) that show up on a free body diagram cut. Again we consider things (bars) that are rather longer than they are wide or thick because

- Long narrow pieces are commonly used in construction of buildings, machines, plants and animals (not just in trusses).
- Internal forces in long narrow things are easier to understand than in bulkier objects, and so are studied first.

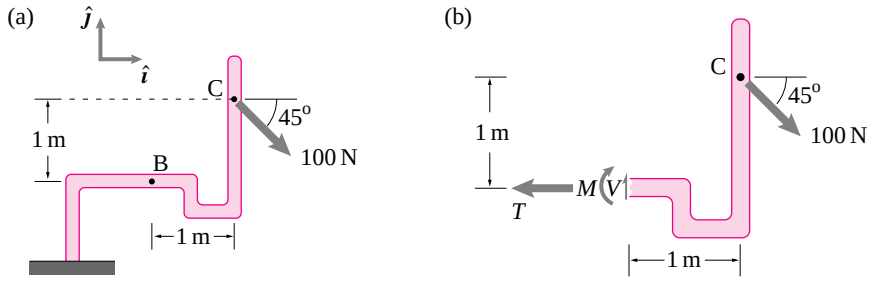
For now we limit ourselves to 2D statics. At an arbitrary cut we break the force into two components (see fig. 4.41).

- The *tension* T is the scalar part of the force directed along the bar assumed positive when pulling away from the free body diagram cut.
- The *shear force* V is the force perpendicular to the bar (tangent to the free body diagram cut. Our sign convention is that shear is positive if it tends to rotate the cut object clockwise. An equivalent statement of the sign convention is that shear is positive if down on cuts at the right of a bar and positive if up on a cut on the left of bar (and to the right on top and to the left on the bottom).

Since we are just doing 2D problems now, the moment is always in the out of plane (typically $\hat{\boldsymbol{k}}$) direction.

- *The bending moment M* is the scalar part of the bending moment. The sign convention is that for a smiling beam (Fig. 4.42): A clockwise ($-\hat{\boldsymbol{k}}$) couple is positive on a left cut and a counterclockwise ($\hat{\boldsymbol{k}}$) couple is positive on a right cut①.

The tension T, shear V, and bending moment M on fig. 4.41 follows these sign conventions.



① Note that neither V nor T changes if you rotate your paper until the picture is upside down. However, this definition for the sign convention for M has the disadvantage that the bending moment does changes sign if you turn your paper upside down. Here is a more precise definition which gets rid of this flaw. Choose the x and $\hat{\boldsymbol{\imath}}$ direction to be along the bar. Bending moment is positive for a cut with normal in the $-\hat{\boldsymbol{\imath}}$ direction if clockwise. Bending moment is positive for a cut with a normal in the $\hat{\boldsymbol{\imath}}$ direction if counterclockwise. More concisely, if $\hat{\boldsymbol{n}}$ is the normal to the cut, bending moment is positive in the $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\jmath}}$ direction.

Example: **Internal forces in a bent rod**

The internal forces at B can be found by making a free body diagram of a portion of the structure with a cut at B.

$$\begin{array}{rcrcl} \text{Sum of vertical forces is zero} & \Rightarrow & V & = & (100/\sqrt{2})\,\text{N} \\ \text{Sum of horizontal forces is zero} & \Rightarrow & T & = & (100/\sqrt{2})\,\text{N} \\ \text{Sum of moments about the cut at B is zero} & \Rightarrow & M & = & -100\sqrt{2}\,\text{N m}. \end{array}$$

□

Tension, shear force, and bending moment diagrams

Engineers often want to know how the internal forces vary from point to point in a structure. If you want to know the internal forces at a variety of points you can draw a variety of free body diagrams with cuts at those points of interest. Another approach, which we present now, is to leave the position of the free body diagram cut a variable, and then calculate the internal forces in terms of that variable.

Example: **Tension in a rod from its own weight.**

The uniform $1\,\text{cm}^2$ steel square rod with density $\rho = 7.7\,\text{gm/cm}^3$ and length $\ell = 100\,\text{m}$ has total weight $W = mg = \rho\ell Ag$ (see fig. 4.43). What is the tension a distance x_D from the top? Using the free body diagram with cut at x_D we get:

$$\begin{aligned} &\left\{\sum \vec{\boldsymbol{F}}_\text{i} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{\imath}} \\ \Rightarrow \quad T &= \rho Ag(\ell - x_\text{D}) \\ &= (7.7\,\text{gm/cm}^3)(1\,\text{cm}^2)(9.8\,\text{N/kg})(100\,\text{m} - x_\text{D}) \\ &= 7.7 \cdot 9.8 \frac{\text{gm N m}}{\text{cm kg}} \left(100 - \tfrac{x_\text{D}}{\text{m}}\right) \underbrace{\left(\frac{1\,\text{kg}}{1000\,\text{gm}}\right)}_{1} \underbrace{\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)}_{1} \\ &= 7.5\left(100 - \tfrac{x_\text{D}}{\text{m}}\right)\,\text{N}. \end{aligned}$$

So, at the bottom end at $x_\text{D} = 100\,\text{m}$ we get $T = 0$ and at the top end where $x_\text{D} = 0\,\text{m}$ we get $T = 750\,\text{N}$ and in the middle at $x_\text{D} = 50\,\text{m}$ we get $T = 375\,\text{N}$. □

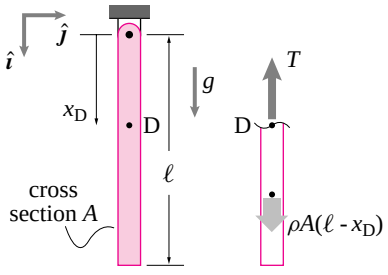


Figure 4.43: a) Rod hanging with gravity. b) free body diagram with cut at x_D.

(Filename:tfigure.tensioncut)

Because the free body diagram cut location is variable, we can plot the internal forces as a function of position. This is most useful in civil engineering where an engineer wants to know the internal forces in a horizontal beam carrying vertical loads. Common examples include bridge platforms and floor joists.

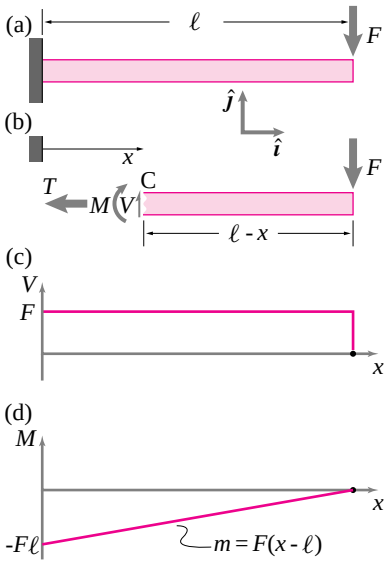


Figure 4.44: a) Cantilever beam, b) free body diagram, c) Shear force diagram, d) Bending moment diagram

(Filename:tfigure.bendandsheardiag)

Example: **Cantilever M and V diagram**

A *cantilever* beam is mounted firmly at one end and has various loads orthogonal to its length, in this case a downwards load F at the end (fig. 4.44a). By drawing a free body diagram with a cut at the arbitrary point C (fig. 4.44b) we can find the internal forces as a function of the position of C.

$$\begin{array}{lcccl} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} & \Rightarrow & V & = & F \\ \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{i} & \Rightarrow & T & = & 0 \\ \left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} & \Rightarrow & M(x) & = & F(x-\ell). \end{array}$$

That the tension is zero in these problems is so well known that the tension is often not drawn on the free body diagram and not calculated. We can now plot $V(x)$ and $M(x)$ as in figs. 4.44c and 4.44d. In this case the shear force is a constant and the bending moment varies from its maximum magnitude at the wall ($M = -F\ell$) to 0 at the end. It is the big value of $|M|$ at the fixed support that makes cantilever beams typically break there. □

Often one is interested in distributed loads from gravity on the structure itself or from a distribution (say of people on a floor). The method is the same.

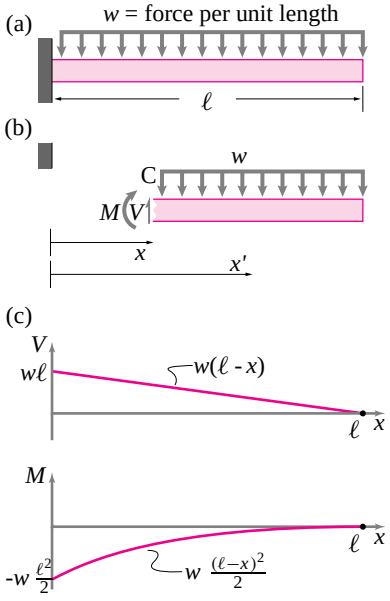


Figure 4.45: a) Cantilever beam, b) free body diagram, c) Shear force diagram, d) Bending moment diagram

(Filename:tfigure.uniformcant)

Example: **Distributed load**

A cantilever beam has a downwards uniformly distributed load of w per unit length (fig. 4.45a). Using the free body diagram shown (fig. 4.45b) we can find:

$$\begin{array}{lcccl} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} & \Rightarrow & \left\{V(x)\hat{j} + \int d\vec{F}\right\} \cdot \hat{j} & = & 0 \\ & \Rightarrow & V(x) & = & \int_x^\ell w\, dx' \\ & & & = & w \cdot (\ell - x) \end{array}$$

$$\begin{array}{lcccl} \left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} & \Rightarrow & \left\{M(x)(-\hat{k}) + \int \vec{r}_{/C} \times d\vec{F}\right\} \cdot \hat{k} & = & 0 \\ & \Rightarrow & M(x) & = & \int_x^\ell (x'-x) w dx' \\ & & & = & w \cdot (x'^2/2 - x'x)\Big|_x^\ell \\ & & & = & (\ell^2/2 - \ell x) - (x^2/2 - x^2) \\ & & & = & -w \cdot (\ell - x)^2/2. \end{array}$$

The integrals were used because of their general applicability for distributed loads. For this problem we could have avoided the integrals by using an equivalent downwards force $w \cdot (\ell - x)$ applied a distance $(\ell - x)/2$ to the right of the cut. Shear and bending moment diagrams are shown in figs. 4.45a and 4.45b. □

As for all problems based on the equilibrium equations and a given geometry, the principle of superposition applies.

Example: **Superposition**

Consider a cantilever beam that simultaneously has both of the loads from the previous two examples. By the principle of superposition:

$$\begin{array}{rcrcl} V & = & F & + & w(\ell - x) \\ M(x) & = & F(x-\ell) & + & -w(\ell-x)^2/2. \end{array}$$

The shear force at every point is the sum of the shear forces from the previous examples. The bending moment at every point is the sum of the bending moments. □

If there are concentrated loads in the middle of the region of interest the calculation gets more elaborate; the concentrated force may or may not show up on the free body diagram of the cut bar, depending on the location of the cut.

Example: **Simply supported beam with point load in the middle**

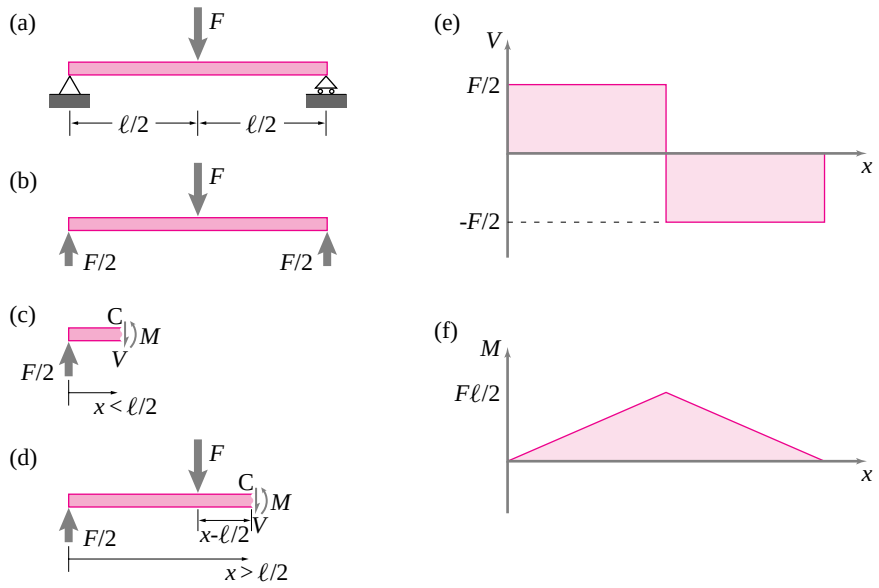


Figure 4.46: a) Simply supported beam, b) free body diagram of whole beam, c) free body diagram with cut to the left of the applied force, d) free body diagram with cut to the right of the applied force e) Shear force diagram, f) Bending moment diagram

(Filename:tfigure.simplesup)

A *simply supported* beam is mounted with pivots at both ends (fig. 4.46a). First we draw a free body diagram of the whole beam (fig. 4.46a) and then two more, one with a cut to the left of the applied force and one with a cut to the right of the applied force (figs. 4.46c and 4.46d). With the free body diagram 4.46c we can find $V(x)$ and $M(x)$ for $x < \ell/2$ and with the free body diagram 4.46d we can find $V(x)$ and $M(x)$ for $x > \ell/2$.

$$\begin{array}{lcrcll} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} & \Rightarrow & V & = & F/2 & \text{for } x < \ell/2 \\ & & & = & -F/2 & \text{for } x > \ell/2 \\ \left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} & \Rightarrow & M(x) & = & Fx/2 & \text{for } x < \ell/2 \\ & & & = & F(\ell - x)/2 & \text{for } x > \ell/2 \end{array}$$

These relations can be plotted as in figs. 4.46e and 4.46f. Some observations: For this beam the biggest bending moment is in the middle, the

place where simply supported beams often break. Instead of the free body diagram shown in (c) and (d) we could have drawn a free body diagrams of the bar to the right of the cut and would have got the same $V(x)$ and $M(x)$. We avoided drawing a free body diagram cut *at* the applied load where $V(x)$ has a discontinuity. □

How to find T, V, and M

Here are some guidelines for finding internal forces and drawing shear and bending moment diagrams.

- Draw a free body diagram of the whole bar.
- Using the free body diagram above find the reaction forces .
- Draw a free body diagram(s) of the cut bar of interest.
  - For each region between concentrated loads draw one free body diagram.
  - Show the piece from the cut to one or the other end (So that all but the internal forces are known).
  - Don't make cuts *at* intermediate points of connection or load application.
- Use the equilibrium equations to find T, V, or M (Moment balance about a point at the cut is a good way to find M.)
- Use the results above to plot $V(x)$ and $M(x)$ ($T(x)$ is rarely plotted).
  - Use the same x scale for this plot as for the free body diagram of the whole bar.
  - Put the plots directly under the free body diagram of the bar (so you can most easily relate features of the loads to features of the V and M diagrams).

Stress is force per unit area

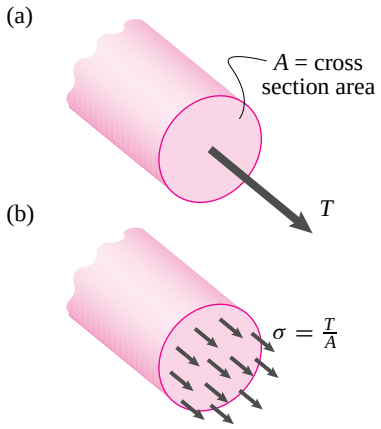


Figure 4.47: a) Tension on a free body diagram cut is equivalent to b) uniform tension stress.

(Filename:tfigure.tension)

For a given load, if you replace one bar in tension with two bars side by side you would imagine the tension in each bar would go down by a factor of 2. Thus the pair of bars should be twice as strong as a single bar. If you glued these side by side bars together you would again have one bar but it would be twice as strong as the original bar. Why? Because it has twice the cross sectional area.

What makes a solid break is the force per unit area carried by the material. For an applied tension load T, the force per unit area on an interior free body diagram cut is T/A. Force per unit area normal to an internal free body diagram cut is called *tension stress* and denoted σ (lower case 'sigma', the Greek letter s).

$$\sigma = T/A$$

Example: **Stress in a hanging bar**

Look at the hanging bar in the example on page 149. We can find the tension stress in this bar as a function of position along the bar as:

$$\sigma = \frac{T}{A} = \frac{\rho g A(\ell - x)}{A} = \rho g(\ell - x).$$

Note that the stress for this bar doesn't depend on the cross sectional area. The bigger the area the bigger the volume and hence the load. But also, the bigger the area on which to carry it. □

For reasons that are beyond this book, the tension stress tends to be uniform in homogeneous (all one material) bars, no matter what their cross sectional shape, so that the average tension stress $\frac{T}{A}$ is actually the tension stress all across the cross section.

We can similarly define the average *shear stress* τ_{ave} ('tau') on a free body diagram cut as the average force per unit area tangent to the cut,

$$\tau_{\text{ave}} = \frac{V}{A}.$$

For reasons you may learn in a strength of materials class, shear stress is not so uniformly distributed across the cross section. But the average shear stress τ_{ave} does give an indication of the actual shear stress in the bar (*e.g.*, for a rectangular elastic bar the peak shear stress is 50% larger than τ_{ave}).

The biggest stresses typically come from bending moment. Motivating formulas for these stresses here is too big a digression. The formulas for the stresses due to bending moment are a key part of elementary strength of materials. But just knowing that these stresses tend to be big, gives you the important notion that bending moment is a common cause of structural failure.

Internal force summary

'Internal forces' are the scalars which describe the force and moment on potential internal free body diagram cuts. They are found by applying the equilibrium equations to free body diagrams that have cuts at the points of interest. The internal forces are intimately associated with the internal stresses (force per unit area) and thus are important for determining the strength of structures.

"Cut. OK it's a take. Lets quit for the day."

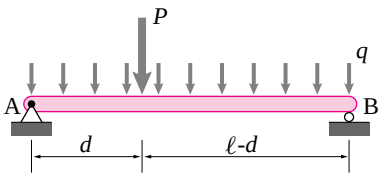


Figure 4.48: (Filename:sfig4.intern.ssup)

SAMPLE 4.10 *Support reactions on a simply supported beam:* A uniform beam of length 3 m is simply supported at A and B as shown in the figure. A uniformly distributed vertical load $q = 100\,\text{N/m}$ acts over the entire length of the beam. In addition, a concentrated load $P = 150\,\text{N}$ acts at a distance $d = 1\,\text{m}$ from the left end. Find the support reactions.

Solution Since the beam is supported at A on a pin joint and at B on a roller, the unknown reactions are

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}, \qquad \vec{B} = B_y\hat{\jmath}$$

The uniformly distributed load q can be replaced by an equivalent concentrated load $W = q\ell$ acting at the center of the beam span. The free body diagram of the beam, with the concentrated load replaced by the equivalent concentrated load is shown in Fig. 4.49. The moment equilibrium about point A, $\sum \vec{M}_\text{A} = \vec{0}$, gives

$$\begin{aligned} (-Pd - W\frac{\ell}{2} + B_y\ell)\hat{k} &= \vec{0} \\ \Rightarrow \quad B_y &= P\frac{d}{\ell} + \frac{1}{2}W \\ &= 150\,\text{N}\cdot\frac{1}{3} + \frac{1}{2}\cdot 300\,\text{N} = 200\,\text{N} \end{aligned}$$

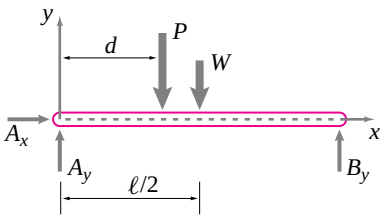


Figure 4.49: (Filename:sfig4.intern.ssup.a)

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned} \vec{A} + B_y\hat{\jmath} - P\hat{\jmath} - W\hat{\jmath} &= \vec{0} \\ \Rightarrow \quad \vec{A} &= (-B_y + P + W)\hat{\jmath} \\ &= (-200\,\text{N} + 150\,\text{N} + 300\,\text{N})\hat{\jmath} = 250\,\text{N}\hat{\jmath} \end{aligned}$$

$$\boxed{\vec{A} = 250\,\text{N}\hat{\jmath}, \quad \vec{B} = 200\,\text{N}\hat{\jmath}}$$

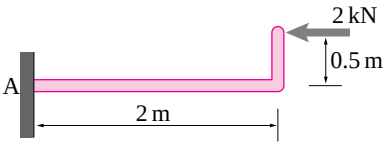


Figure 4.50: (Filename:sfig4.intern.cant)

SAMPLE 4.11 *Support reactions on a cantilever beam:* A 2 kN horizontal force acts at the tip of an 'L' shaped cantilever beam as shown in the figure. Find the support reactions at A.

Solution The free body diagram of the beam is shown in Fig. 4.51. The reaction force at A is $\vec{A}$ and the reaction moment is $\vec{M} = M\hat{k}$. Writing moment balance equation about point A, $\sum \vec{M}_\text{A} = \vec{0}$, we get

$$\begin{aligned} \vec{M} + \vec{r}_{\text{C/A}} \times \vec{F} &= \vec{0} \\ \vec{M} + (\ell\hat{\imath} + h\hat{\jmath}) \times (-F\hat{\imath}) &= \vec{0} \\ \Rightarrow \quad \vec{M} &= -Fh\,\hat{k} \\ &= -2\,\text{kN}\cdot 0.5\,\text{m}\,\hat{k} \\ &= -1\,\text{kN}\cdot\,\text{m}\,\hat{k} \end{aligned}$$

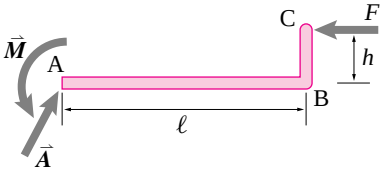


Figure 4.51: (Filename:sfig4.intern.cant.a)

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned} \vec{A} + \vec{F} &= \vec{0} \\ \Rightarrow \quad \vec{A} &= -\vec{F} = -(-2\,\text{kN}\,\hat{\imath}) = 2\,\text{kN}\,\hat{\imath} \end{aligned}$$

$$\boxed{\vec{A} = 2\,\text{kN}\,\hat{\imath}, \quad \vec{M} = -1\,\text{kN}\cdot\,\text{m}\,\hat{k}}$$

SAMPLE 4.12 *Net force of a uniformly distributed system:* A uniformly distributed vertical load of intensity 100 N/macts on a beam of length $\ell = 2\,\text{m}$ as shown in the figure.

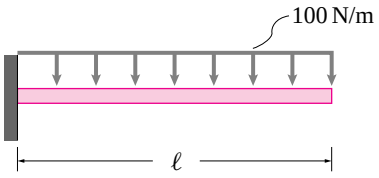


Figure 4.52: (Filename:sfig2.vec3.uniform)

(a) Find the net force acting on the beam.
(b) Find an equivalent force-couple system at the mid-point of the beam.
(c) Find an equivalent force-couple system at the right end of the beam.

Solution

(a) **The net force:** Since the load is uniformly distributed along the length, we can find the total or the net load by calculating the load on an infinitesimal segment of length dx of the beam and then integrating over the entire length of the beam. Let the load intensity (load per unit length) be q ($q = 100\,\text{N/m}$, as given). Then the vertical load on segment dx is (see Fig. 4.53),

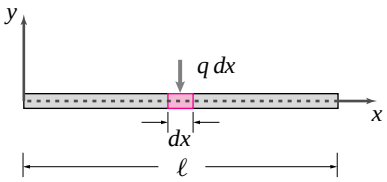


Figure 4.53: (Filename:sfig2.vec3.uniform.a)

$$d\vec{F} = q\,dx(-\hat{j}).$$

Therefore, the net force is,

$$\vec{F}_{\text{net}} = \int_0^{\ell} q\,dx(-\hat{j}) = q\,\ell\hat{j} = -100\,\text{N/m}\cdot 2\,\text{m}\hat{j} = -200\,\text{N}\hat{j}.$$

$$\boxed{\vec{F}_{\text{net}} = -200\,\text{N}\hat{j}}$$

(b) **The equivalent system at the mid-point:** We have already calculated the net force that can replace the uniformly distributed load. Now we need to calculate the couple at the mid-point of the beam to get the equivalent force-couple system. Again, consider a small segment of the beam of length dx located at distance x from the mid-point C (see Fig. 4.54). The moment about point C due to the load on dx is $(q\,dx)x(-\hat{k})$. But, we can find a similar segment on the other side of C with exactly the same length dx, at exactly the same distance x, that produces a moment of $(q\,dx)x(+\hat{k})$. The two contributions cancel each other and we have a net zero moment about C. Now, you can imagine the whole beam made up of these pairs that contribute equal and opposite moment about C and thus the net moment about the mid-point is zero. You can also find the same result by straight integration:

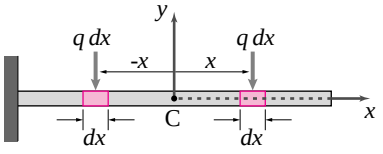


Figure 4.54: (Filename:sfig2.vec3.uniform.b)

$$\vec{M}_{\text{C}} = \int_{-\ell/2}^{+\ell/2} qx\,dx(-\hat{k}) = \frac{qx^2}{2}\bigg|_{-\ell/2}^{+\ell/2}(-\hat{k}) = \vec{0}.$$

$$\boxed{\vec{F}_{\text{net}} = -200\,\text{N}\hat{j}, \text{ and } \vec{M}_{\text{C}} = \vec{0}}$$

(c) **The equivalent system at the end:** The net force remains the same as above. We compute the net moment about the end point B, referring to Fig. 4.55, as follows.

$$\begin{aligned}\vec{M}_{\text{B}} &= \int_0^{\ell}(-x\hat{\imath}) \times (-q\,dx\,\hat{j}) = -q\int_0^{\ell} x\,dx\hat{k} \\ &= -\frac{q\ell^2}{2}\hat{k} = -\frac{100\,\text{N/m}\cdot 4\,\text{m}^2}{2}\hat{k} = -200\,\text{N}\cdot\text{m}\hat{k}.\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = -200\,\text{N}\hat{j} \text{ and } \vec{M}_{\text{B}} = -200\,\text{N}\cdot\text{m}\hat{k}}$$

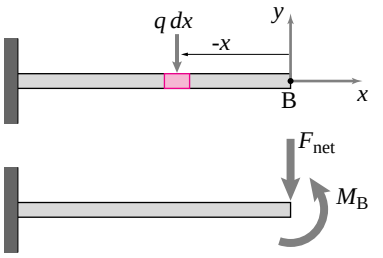


Figure 4.55: (Filename:sfig2.vec3.uniform.c)

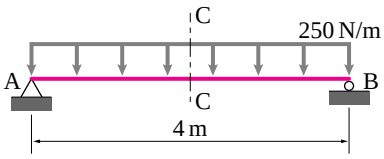


Figure 4.56: (Filename:sfig4.intern.ssvm)

SAMPLE 4.13 For the uniformly loaded, simply supported beam shown in the figure, find the shear force and the bending moment at the mid-section c-c of the beam.

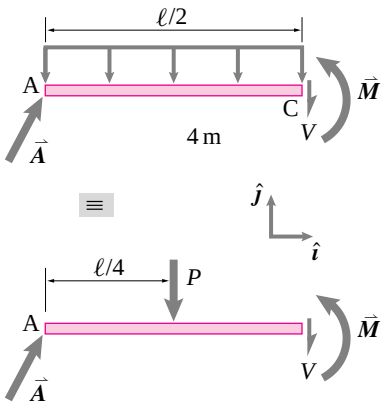


Figure 4.57: (Filename:sfig4.intern.ssvm.a)

Solution To determine the shear force V and the bending moment M at the mid-section c-c, we cut the beam at c-c and draw its free body diagram as shown in Fig. 4.57. For writing force and moment balance equations we use the second figure where we have replaced the distributed load with an equivalent single load $F = (q\ell)/2$ acting vertically downward at distance $\ell/4$ from end A.

The force balance, $\sum \vec{F} = \vec{0}$, implies that

$$A_x\hat{\imath} + A_y\hat{\jmath} - V\hat{\jmath} - F\hat{\jmath} = \vec{0}$$

Dotting with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$\begin{aligned} A_x &= 0 \\ V &= A_y - F \qquad (4.34) \\ &= A_y - \frac{q\ell}{2} \qquad (4.35) \end{aligned}$$

From the moment equilibrium about point A, $\sum \vec{M}_{\mathrm{A}} = \vec{0}$, we get

$$M\hat{\boldsymbol{k}} - \left(\frac{q\ell}{2}\cdot\frac{\ell}{4}\right)\hat{\boldsymbol{k}} - V\ell\hat{\boldsymbol{k}} = 0$$

$$\Rightarrow \quad M = \frac{q\ell^2}{8} + V\ell \qquad (4.36)$$

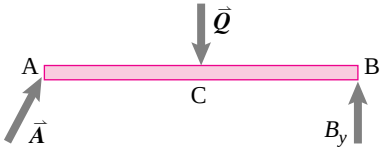


Figure 4.58: (Filename:sfig4.intern.ssvm.b)

Thus, to find V and M we need to know the support reaction $\vec{A}$. From the free body diagram of the beam in Fig. 4.58 and the moment equilibrium equation about point B, $\sum \vec{M}_{\mathrm{B}} = \vec{0}$, we get

$$\begin{aligned} \vec{r}_{\mathrm{A/B}} \times \vec{A} + \vec{r}_{\mathrm{C/B}} \times \vec{Q} &= \vec{0} \\ (-A_y\ell + q\ell\,\frac{\ell}{2})\hat{\boldsymbol{k}} &= \vec{0} \\ \Rightarrow \quad A_y = \frac{q\ell}{2} &= 500\,\mathrm{N} \end{aligned}$$

Thus $\vec{A} = 500\,\mathrm{N}\hat{\jmath}$. Substituting $\vec{A}$ in eqns. (4.35) and (4.36), we get

$$\begin{aligned} V &= 500\,\mathrm{N} - 500\,\mathrm{N} = 0 \\ M &= 250\,\mathrm{N}\cdot\frac{(4\,\mathrm{m})^2}{8} + 0 \\ &= 500\,\mathrm{N{\cdot}m} \end{aligned}$$

$$\boxed{V = 0, \quad M = 500\,\mathrm{N{\cdot}m}}$$

SAMPLE 4.14 The cantilever beam AD is loaded as shown in the figure where $W = 200\,\text{lbf}$. Find the shear force and bending moment on a section just left of point B and another section just right of point B.

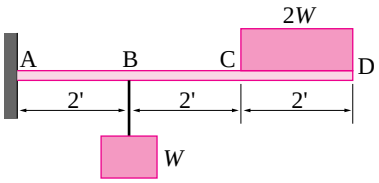


Figure 4.59: (Filename:sfig4.intern.cantvm)

Solution To find the desired internal forces, we need to make a cut at a section just to the left of B and one just to the right of B. We first take the one that is to the right of point B. The free body diagram of the right part of the cut beam is shown in Fig. 4.60. Note that if we selected the left part of the beam, we would need to determine support reactions at A. The uniformly distributed load $2W$ of the block sitting on the beam can be replaced by an equivalent concentrated load $2W$ acting at point E, at distance $a/2$ from the end D of the beam.

Let us denote the the shear force by V^+ and the bending moment by M^+ at the section of our interest. Now, from the force equilibrium of the part-beam BD we get

$$\begin{aligned} V^+\hat{\boldsymbol{j}} - 2W\hat{\boldsymbol{j}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad V^+ &= 2W \\ &= 400\,\text{lbf} \end{aligned}$$

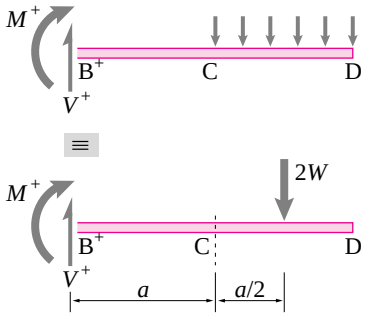


Figure 4.60: (Filename:sfig4.intern.cantvm.a)

The moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_\text{B} = \vec{\mathbf{0}}$, gives

$$\begin{aligned} -M^+\hat{\boldsymbol{k}} - 2W \cdot \frac{3a}{2}\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad M^+ &= -3Wa \\ &= -1200\,\text{lb·ft} \end{aligned}$$

Now, we determine the internal forces at a section just to the left of point B. Let the shear and bending moment at this section be V^- and M^-, respectively, as shown in the free body diagram (Fig. 4.61). Note that load W acting at B is now included in the free body diagram since the beam is now cut just a teeny bit left of this load.

From the force equilibrium of the part-beam, we have

$$\begin{aligned} V^-\hat{\boldsymbol{j}} - W\hat{\boldsymbol{j}} - 2W\hat{\boldsymbol{j}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad V^- &= 3W \\ &= 600\,\text{lbf} \end{aligned}$$

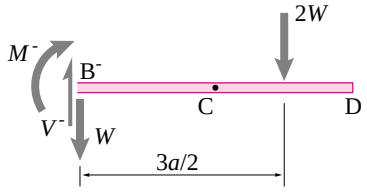


Figure 4.61: (Filename:sfig4.intern.cantvm.b)

and, from moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_\text{B} = \vec{\mathbf{0}}$, we get

$$\begin{aligned} -M^-\hat{\boldsymbol{k}} - 2W \cdot \frac{3a}{2}\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad M^- &= -3Wa \\ &= -1200\,\text{lb·ft} \end{aligned}$$

$$\boxed{M^+ = M^- = -1200\,\text{lb·ft}, \quad V^+ = 400\,\text{lbf}, \quad V^+ = 600\,\text{lbf}}$$

Note that the bending moment remains the same on either side of point B but the shear force jumps by $V^+ - V^- = 200\,\text{lbf} = W$ as we go from right to the left. This jump is expected because a concentrated load W acts at B, in between the two sections we consider. Concentrated external forces cause a jump in shear, and concentrated external moments cause a jump in the bending moment.

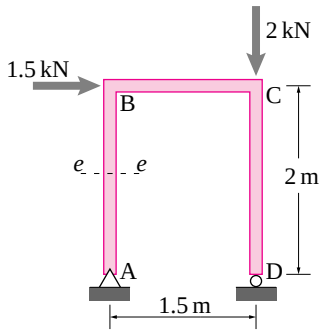


Figure 4.62: (Filename:sfig4.intern.frame)

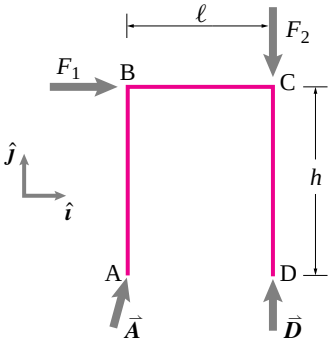


Figure 4.63: (Filename:sfig4.intern.frame.a)

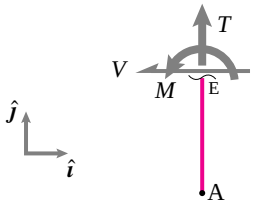


Figure 4.64: (Filename:sfig4.intern.frame.b)

SAMPLE 4.15 *A simple frame:* A 2 m high and 1.5 m wide rectangular frame ABCD is loaded with a 1.5 kN horizontal force at B and a 2 kN vertical force at C. Find the internal forces and moments at the mid-section e-e of the vertical leg AB.

Solution To find the internal forces and moments, we need to cut the frame at the specified section e-e and consider the free body diagram of either AE or EBCD. No matter which of the two we select, we will need the support reactions at A or D to determine the internal forces. Therefore, let us first find the support reactions at A and D by considering the free body diagram of the whole frame (Fig. 4.63). The moment balance about point A, $\sum \vec{M}_{\rm A} = \vec{0}$, gives

$$\begin{aligned}
\vec{r}_{\rm B} \times \vec{F}_1 + \vec{r}_{\rm C} \times \vec{F}_2 + \vec{r}_{\rm D} \times \vec{D} &= \vec{0} \\
h\hat{j} \times F_1\hat{i} + (h\hat{j} + \ell\hat{i}) \times (-F_2\hat{j}) + \ell\hat{i} \times D\hat{j} &= \vec{0} \\
-F_1 h\hat{k} - F_2\ell\hat{k} + D\ell\hat{k} &= \vec{0} \\
\Rightarrow \quad D &= F_1\frac{h}{\ell} + F_2 \\
&= 1.5\,\text{kN} \cdot \frac{2}{1.5} + 2\,\text{kN} \\
&= 4\,\text{kN}
\end{aligned}$$

From force equilibrium, $\sum \vec{F} = \vec{0}$, we have

$$\begin{aligned}
\vec{A} &= -\vec{F}_1 - \vec{F}_2 - \vec{D} \\
&= -F_1\hat{i} + F_2\hat{j} - D\hat{j} \\
&= -1.5\,\text{kN}\hat{i} - 2\,\text{kN}\hat{j}
\end{aligned}$$

Now we draw the free body diagram of AE to find the shear force V, axial (tensile) force T, and the bending moment M at section e-e.

From the force equilibrium of part AE, we get

$$\begin{aligned}
\vec{A} - V\hat{i} + T\hat{j} &= \vec{0} \\
(A_x - V)\hat{i} + (A_y + T)\hat{j} &= \vec{0} \\
\Rightarrow \quad V = A_x &= -1.5\,\text{kN} \\
T = -A_y &= 2\,\text{kN}
\end{aligned}$$

From the moment equilibrium about point A, $\sum \vec{M}_{\rm A} = \vec{0}$, we have

$$\begin{aligned}
M\hat{k} + \frac{h}{2}\hat{j} \times (-V\hat{i}) &= \vec{0} \\
M\hat{k} + V\frac{h}{2}\hat{k} &= \vec{0} \\
\Rightarrow \quad M &= -V\frac{h}{2} \\
&= -(-1.5\,\text{kN}) \cdot \frac{2\,\text{m}}{2} \\
&= 1.5\text{k N·m}
\end{aligned}$$

$$\boxed{V = 1.5\,\text{kN}, \quad T = 2\,\text{kN}, \quad M = 1.5\text{k N·m}}$$

SAMPLE 4.16 *Shear force and bending moment diagrams:* A simply supported beam of length $\ell = 2\,\text{m}$ carries a concentrated vertical load $F = 100\,\text{N}$ at a distance a from its left end. Find and plot the shear force and the bending moment along the length of the beam for $a = \ell/4$.

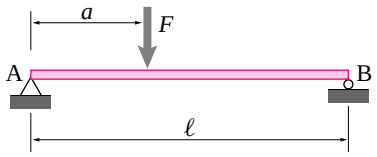


Figure 4.65: (Filename:sfig4.intern.ssvmx)

Solution We first find the support reactions by considering the free body diagram of the whole beam shown in Fig. 4.66. By now, we have developed enough intution to know that the reaction at A will have no horizontal component since there is no external force in the horizontal direction. Therefore, we take the reactions at A and B to be only vertical. Now, from the moment equilibrium about point B, $\sum \vec{M}_{\text{B}} = \vec{0}$, we get

$$\begin{aligned} F(\ell - a)\hat{k} - A_y \ell \hat{k} &= \vec{0} \\ \Rightarrow \quad A_y &= \frac{F(\ell - a)}{\ell} \\ &= F\left(1 - \frac{a}{\ell}\right) \end{aligned}$$

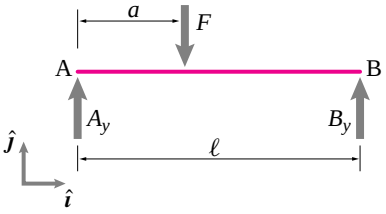


Figure 4.66: (Filename:sfig4.intern.ssvmx.a)

and from the force equilibrium in the vertical direction, $(\sum \vec{F} = \vec{0}) \cdot \hat{j}$, we get

$$B_y = F - A_y = F\frac{a}{\ell}$$

Now we make a cut at an arbitrary (variable) distance x from A where $x < a$ (see Fig. 4.67). Carrying out the force balance and the moment balance about point A, we get, for $0 \le x < a$,

$$V = A_y = F\left(1 - \frac{a}{\ell}\right) \tag{4.37}$$

$$M = Vx = F\left(1 - \frac{a}{\ell}\right)x \tag{4.38}$$

Thus V is constant for all $x < a$ but M varies linearly with x.

Now we make a cut at an arbitrary x to the right of load F, *i.e.*, $a < x \le \ell$. Again, from the force balance in the vertical direction, we get

$$V = -F + F\left(1 - \frac{a}{\ell}\right) = -F\frac{a}{\ell} \tag{4.39}$$

and from the moment balance about point A,

$$\begin{aligned} M &= Fa + Vx \\ &= Fa - F\frac{a}{\ell}x \\ &= Fa\left(1 - \frac{x}{\ell}\right) \end{aligned} \tag{4.40}$$

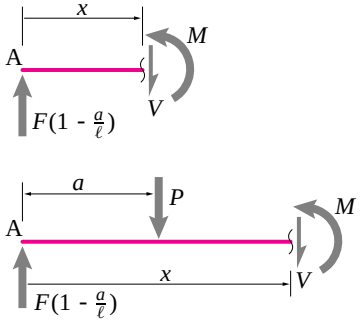


Figure 4.67: (Filename:sfig4.intern.ssvmx.b)

Although eqn. (4.38) is strictly valid for $x < a$ and eqn. (4.40) is strictly valid for $x > a$, sustituting $x = a$ in these two equations gives the same value for $M(= Fa(1 - a/\ell))$ as it must because there is no reason to have a jump in the bending moment at any point along the length of the beam. The shear force V, however, does jump because of the concentrated load F at $x = a$.

Now, we plug in $a = \ell/4 = 0.5\,\text{m}$, and $F = 100\,\text{N}$, in eqns. (4.37)–(4.40) and plot V and M along the length of the beam by varying x. The plots of $V(x)$ and $M(x)$ are shown in Fig. 4.68.

□

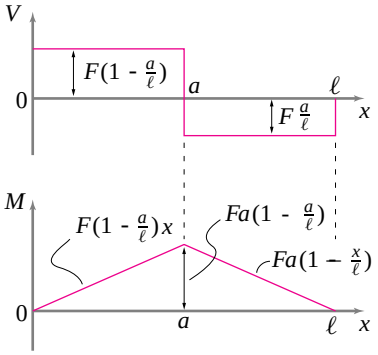


Figure 4.68: (Filename:sfig4.intern.ssvmx.c)

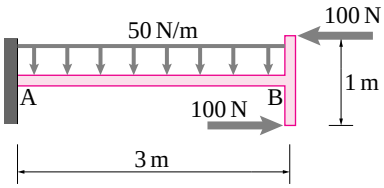


Figure 4.69: (Filename:sfig4.intern.cantvmx)

SAMPLE 4.17 *Shear force and bending moment diagrams by superposition:* For the cantilever beam and the loading shown in the figure, draw the shear force and the bending moment diagrams by

(a) considering all the loads together, and
(b) considering each load (of one type) at a time and using superposition.

Solution

(a) *$V(x)$ and $M(x)$ with all forces considered together:* The horizontal forces acting at the end of the cantilever are equal and opposite and, therefore, produce a couple. So, we first replace these forces by an equivalent couple $M_{\text{applied}} = 100\,\text{N} \cdot 1\,\text{m} = 100\,\text{N·m}$. Since we have a cantilever beam, we can consider the right hand side of the beam after making a cut anywhere for finding V and M without first finding the support reactions.
Let us cut the beam at an arbitrary distance x from the right hand side. The free body diagram of the right segment of the beam is shown in Fig. 4.70. From the force balance, $\sum \vec{F} = \vec{0}$, we find that

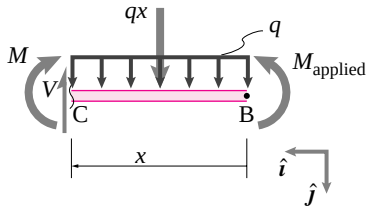


Figure 4.70: (Filename:sfig4.intern.cantvmx.a)

$$\begin{aligned} -V\hat{\boldsymbol{j}} + qx\hat{\boldsymbol{j}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad V &= qx \\ &= (50\,\text{N/m})x \end{aligned} \tag{4.41}$$

Thus the shear force varies linearly along the length of the beam with

$$\begin{aligned} V(x=0) &= 0, \\ \text{and} \quad V(x=3\,\text{m}) &= 150\,\text{N} \end{aligned}$$

The moment balance about point C, $\sum \vec{M}_{\text{C}} = \vec{0}$, gives

$$-M\hat{\boldsymbol{k}} - qx \cdot \frac{x}{2}\hat{\boldsymbol{k}} + M_{\text{applied}}\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}$$

where the moment due to the distributed load is most easily computed by considering an equivalent concentrated load qx acting at $x/2$ from the end B. Thus,

$$\Rightarrow \quad M = M_{\text{applied}} - q\frac{x^2}{2} \tag{4.42}$$

$$= 100\,\text{N·m} - 50\,\text{N/m} \cdot \frac{x^2}{2} \tag{4.43}$$

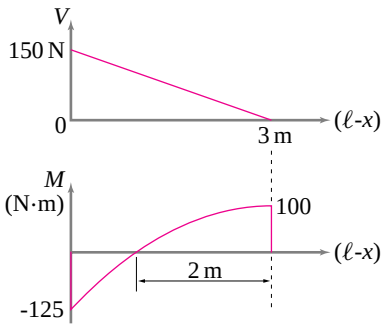


Figure 4.71: (Filename:sfig4.intern.cantvmx.b)

Thus, the bending moment varies quadratically with x along the length of the beam. In particular, the values at the ends are

$$\begin{aligned} M(x=0) &= 100\,\text{N·m} \\ \text{and} \quad M(x=3\,\text{m}) &= -125\,\text{N·m} \end{aligned}$$

The shear force and the bending moment diagrams obtained from eqns. (4.41) and (4.42) are shown in Fig. 4.71. Note that $M = 0$ at $x = 2\,\text{m}$ as given by eqn. (4.42).

(b) *$V(x)$ and $M(x)$ by superposition:* Now we consider the cantilever beam with only one type of load at a time. That is, we first consider the beam only with the uniformly distributed load and then only with the end couple. We draw the shear force and the bending moment diagrams for each case separately and then just *add them up*. That is superposition.

So, first let us consider the beam with the uniformly distributed load. The free body diagram of a segment CB, obtained by cutting the beam at a distance x from the end B, is shown in Fig. 4.72. Once again, from force balance, we get

$$V = qx \qquad \text{for } 0 \le x \le \ell \tag{4.44}$$

and from the moment balance about point C, $\sum \vec{M}_C = \vec{0}$, we get

$$M = -qx \cdot \frac{x}{2} = -q\frac{x^2}{2} \qquad \text{for } 0 \le x \le \ell \tag{4.45}$$

Figure 4.73 shows the plots of V and M obtained from eqns. (4.44) and (4.45), respectively, with the values computed from $x = 0$ to $x = 3\,\text{m}$ with $q = 50\,\text{N/m}$ as given.

Now we take the beam with only the end couple and repeat our analysis. A cut section of the beam is shown in Fig. 4.74. In this case, it should be obvious that from force balance and moment balance about any point, we get

$$\begin{aligned} V &= 0 \\ \text{and} \quad M &= M_{\text{applied}} \end{aligned}$$

Thus, both the shear force and the bending moment are constant along the length of the beam as shown in Fig. 4.74.
Now superimposing (adding) the shear force diagrams from Figs. 4.73 and 4.74, and similarly, the bending moment diagrams from Figs. 4.73 and 4.74, we get the same diagrams as in Fig. 4.75.

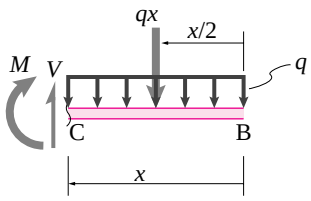


Figure 4.72: (Filename:sfig4.intern.cantvmx.c)

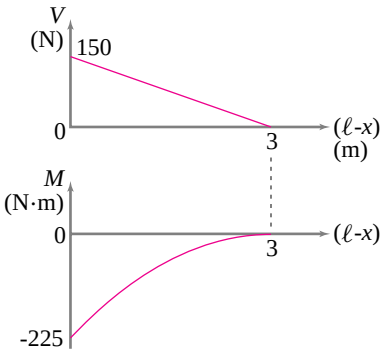


Figure 4.73: (Filename:sfig4.intern.cantvmx.d)

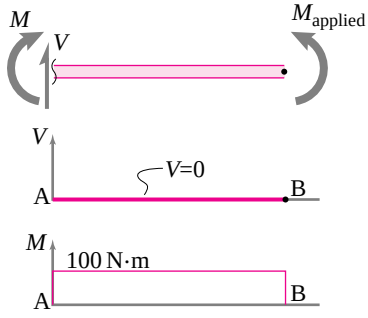


Figure 4.74: (Filename:sfig4.intern.cantvmx.e)

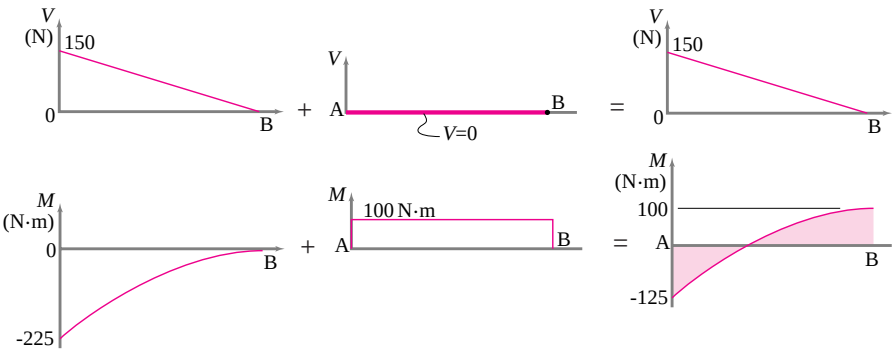


Figure 4.75: (Filename:sfig4.intern.cantvmx.f)

□

4.5 Springs

In the same way that machines and buildings are built from bricks, gears, beams, bolts and other standard pieces, elementary mechanics models of the world are made from a few elementary building blocks. Conspicuous so far, roughly categorized, are:

- Special objects:
  - Point masses.
  - Rigid bodies:
    - Two force bodies,
    - Three force bodies,
    - Pulleys, and
    - Wheels.
- Special connections:
  - Hinges,
  - Welds,
  - Sliding contact, and
  - Rolling contact.

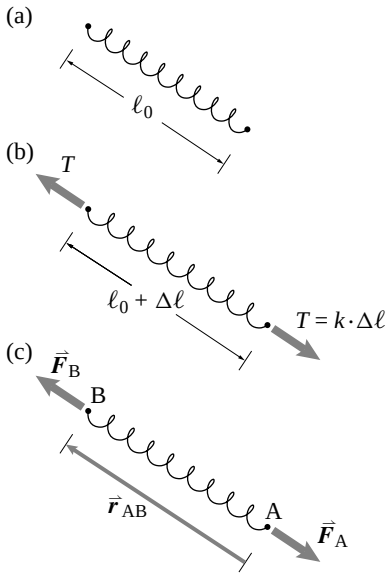


Figure 4.76: An ideal spring with rest length ℓ_0 and stretched length $\ell_o + \Delta\ell$. The tension in the spring is T and the vector forces at the ends are $\vec{F}_A$ and $\vec{F}_B$.

(Filename:tfigure2.spring2)

Each of these things has a dual life. On the one hand a mechanical hinge corresponds to a product you can buy in a hardware store called a hinge. On the other hand a hinge in mechanics represents a constraint that restricts certain motions and freely allows others. A hinge in a mechanics model may or may not correspond to hardware called a hinge. When considering a box balanced on an edge, we may *model* the contact as a hinge meaning we would use the same equations for the forces of contact as we would use for a hinge. We might buy a pulley, but we might model a rope sliding around a post as a rope on a pulley even though there was no literal pulley in sight. We might buy a brick because it is fairly rigid, and model it in mechanics as a rigid body. But a rigid body model might well be used for human body parts that we know deform noticeably. Thus the mechanics model for these things may correspond more or less with the properties of physical objects with the same names.

This section is devoted to a new building block that similarly has a dual personality: a spring.

Springs, in various forms but most characteristically as helices made of steel wire, can be purchased from hardware stores and mechanical parts suppliers. Springs are used to hold things in place (a clothes pin), to store energy (a clock or toy spring), to reduce contact forces (spring bumpers), and to isolate something from vibrations (a car suspension spring). You will find springs in most any complicated machine. Take apart a disposable camera, an expensive printer, a gas lawn mower, or a washing machine and you will find springs.

On the other hand, springs are used in mechanical 'models' of many things that are not explicitly springs. For much of this book we approximate solids as rigid. But sometimes the flexibility or *elasticity* of an object is an important part of its mechanics. The simplest way of accounting for this is to use a spring. So a tire may be modeled as a spring as might be the near-surface-material of a bouncing ball, a strut in a truss, the snap-back of the earth's crust in an earthquake, your achilles tendon, or the give of soil under a concrete slab. Engineer Tom McMahon idealized the give of a running track as that of a spring when he designed the record breaking track used in the Harvard stadium.

In this section we consider an ideal spring (see also page 86 in section 3.1). You may view an ideal spring as an approximation to a hardware product or as an idealized building block for mechanical models.

An *ideal spring* is a massless two-force body characterized by its *rest length* ℓ_0 (also called the 'relaxed length', or 'reference length'), its stiffness k, and defining equation (or constitutive law):

$$T = k \cdot (\ell - \ell_0) \quad \text{or} \quad T = k \cdot \Delta\ell$$

where ℓ is the present length and $\Delta\ell$ is the increase in length or *stretch* (see Fig. 4.76). This model of a spring goes by the name *Hooke's law*.

This spring is *linear* because of the formula $k\Delta\ell$ and not, say, $k(\Delta\ell)^3$. It is *elastic* because the tension only depends on length and not on, say, rate of extension. The spring formula is sometimes quoted as '$F = kx$'①. A plot of tension verses length for an ideal spring is shown in Fig. 4.77a.

A comment on notation. Often in engineering we write $\Delta(something)$ to mean the change of '*something*.' Most often one also has in mind a small change. In the context of springs, however, $\Delta\ell$ is allowed to be a rather large change. We use the notation $\delta\ell$ for small increments to avoid confusion. A useful way to think about springs is that increments of force are proportional to increments of length change, whether the force or length is already large or small:

$$\delta T = k\delta\ell \quad \text{or} \quad \frac{\delta T}{\delta\ell} = k \quad \text{or} \quad \frac{\delta\ell}{\delta T} = \frac{1}{k}.$$

The reciprocal of stiffness $\frac{1}{k}$ is called the *compliance*. A compliant spring stretches a lot when the tension is changed. A compliant spring is not stiff. A stiff spring has small stretch when the tension is changed. A stiff spring is not compliant.

Because the spring force is along the spring, we can write a vector formula for the force on the B (say) end of the spring as (see Fig. 4.76)

$$\vec{F}_B = k \cdot \underbrace{\left(|\vec{r}_{AB}| - \ell_0\right)}_{\Delta\ell} \underbrace{\left(\frac{\vec{r}_{AB}}{|\vec{r}_{AB}|}\right)}_{\hat{\lambda}_{AB}}. \tag{4.46}$$

where $\hat{\lambda}_{AB}$ is a unit vector along the spring. This explicit formula is useful for, say, numerical calculations.

Zero length springs

A special case of linear springs that has remarkable mechanical consequences is a *zero-length* spring that has rest length $\ell_0 = 0$. The defining equations in scalar and vector form are thus simplified to

$$T = k\ell \quad \text{and} \quad \vec{F}_B = k \cdot \vec{r}_{AB}.$$

The tension verses length curve for a zero-length spring is shown in Fig. 4.77b.

At first blush such a spring seems *non-physical*, meaning that it seems to represent something which is not a reasonable approximation to any real thing. If you take a coil spring all the metal gets in the way of the spring possibly relaxing to the point of the ends coinciding. In fact, however, there are many ways to build things which act something like zero length springs. For example, the tension verses length curve of a rubber band (or piece of surgical tubing) looks something like that shown in Fig. 4.77c. Over some portion of the curve the zero-length spring approximation may be reasonable. For other physical implementations of zero-length springs see box 4.5 on page 164.

① The form '$F = kx$' can lead to sign errors because the direction of the force is not evident. The safest way to avoid sign errors when dealing with springs is to
• Draw a free body diagram of the spring;
• Write the *increase* in length $\Delta\ell$ in terms of geometry variables in your problem;
• Use $T = k\Delta\ell$ to find the tension in the spring; and then
• Use the principle of action and reaction to find the forces on the objects to which the spring is connected.

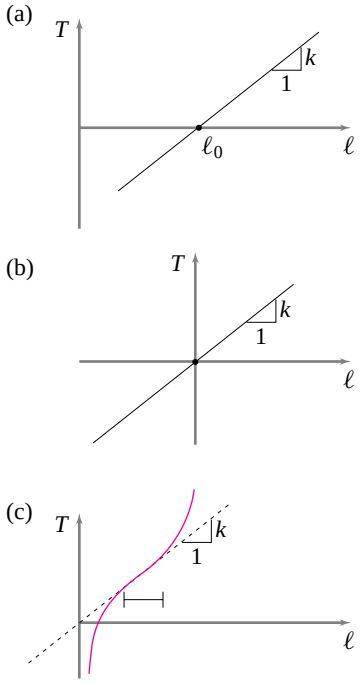


Figure 4.77: a) Tension verses length for an ideal spring, b) for a zero-length spring, and c) for a strip of rubber.

(Filename:tfigure.tensionvslength)

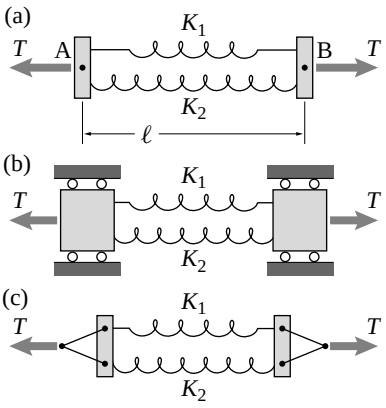


Figure 4.78: a) Schematic of parallel springs, b) genuinely parallel springs, c) a reasonable approximation of parallel springs.

Assemblies of springs

Here and there throughout the rest of the book you will see how springs are put together with others of the basic building blocks in mechanics. Here we see how springs are put together with other springs.

In short, the result of attaching springs to each other in various ways is a new spring with a stiffness that depends on the stiffnesses of the components and on how the springs are connected.

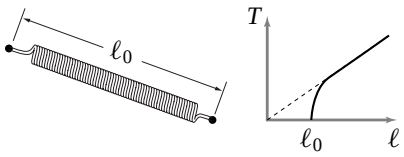
Springs in parallel

Two springs that share a load and stretch the same amount are said to be in *parallel*.

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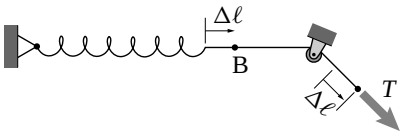
4.4 Examples of zero length springs

The mathematics in many mechanics problems is simplified by the zero-length spring approximation. When is it reasonable?



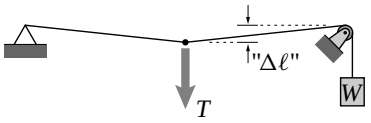
Rubber bands. As shown in Fig. 4.77c straps of rubber behave like zero-length springs over some of their length. If this is the working length of your mechanism then the zero-length spring approximation may be good.

A spring, string, and pulley. If a spring is connected to a string that is wrapped around a pulley then the end of the string can feel like a zero force spring if the attachment point is at the pulley when the spring is relaxed.



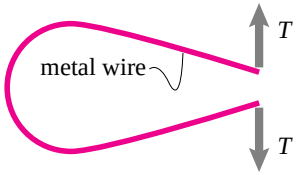
A stretchy conventional spring. Some springs are so stretchy that they are used at lengths much larger than their rest lengths. Thus the approximation that $k(\ell - \ell_0) = k\ell\left(1 - \frac{\ell_0}{\ell}\right) \approx k\ell$ may be reasonable.

A string pulled from the side. If a taught string is pulled from the side it acts like a zero-length spring in the plane orthogonal to the string.



A pre-stressed coil spring. Some door springs and many springs used in desk lamps are made tightly wound so that each coil of wire is pressed against the next one. It takes some tension just to start to stretch such a spring. The tension verses length curve for such springs can look very much like a zero-length spring once stretch has started. In fact, in the original elegant 1930's patent, which commonly seen present-day parallelogram-mechanism lamps imitate, specifies that the spring should behave as a zero-length spring. Such a pre-stressed zero-length coil spring was a central part of the design of the long period seismometer featured on a 1959 Scientific American cover.

A 'U' clip. If a springy piece of metal is bent so that its unloaded shape is a pinched 'U' then it acts very much like a zero length spring. This is perhaps the best example in that it needs no anchor (unlike the pulley) and can be relaxed to almost zero length (unlike a pre-stressed coil).



The standard schematic for this is shown in Fig. 4.78a where the springs are visibly parallel. This schematic is a non-physical cartoon since applied tension would cause the end-bars to rotate unless the attachment points A and B are located carefully. What is meant by the schematic in Fig. 4.78a is the somewhat clumsy constrained mechanism of Fig. 4.78b. In engineering practice one rarely builds such a structure. A simpler partial constraint against rotations is provided by the triangle of cables shown in Fig. 4.78c; rotations are quite limited if the triangles are much longer than wide. For the purposes of discussion here, we assume that any of Fig. 4.78abc represent a situation where the springs both stretch the same amount.

The stretches and tensions of the two springs are $\Delta\ell_1$, $\Delta\ell_2$, T_1, and T_2. For each spring we have the defining constitutive relation:

$$T_1 = k_1\Delta\ell_1 \quad \text{and} \quad T_2 = k_2\Delta\ell_2. \tag{4.47}$$

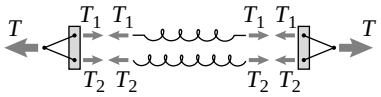


Figure 4.79: Free body diagrams of the components of a parallel spring arrangement.

(Filename:tfigure.parallelfbds)

As usual, they key to understanding the situation is through appropriate free body diagrams (see Fig. 4.79). Force balance for one of the end supports shows that

$$T = T_1 + T_2 \tag{4.48}$$

showing that the load is shared by the two springs. Springs in parallel stretch the same amount thus we have the kinematic relation:

$$\Delta\ell_1 = \Delta\ell_2 = \Delta\ell. \tag{4.49}$$

Determining the relation between T and $\Delta\ell$ is a matter of manipulating these equations:

$$\begin{aligned} T &= T_1 + T_2 \\ &= k_1\Delta\ell_1 + k_2\Delta\ell_2 \\ &= k_1\Delta\ell + k_2\Delta\ell \\ &= \underbrace{(k_1+k_2)}_{k}\Delta\ell \ . \end{aligned}$$

Thus we get that the effective spring constant of the pair of springs in parallel is, intuitively:

$$k = k_1 + k_2. \tag{4.50}$$

The loads carried by the springs are

$$T_1 = \frac{k_1}{k_1+k_2}T \quad \text{and} \quad T_2 = \frac{k_2}{k_1+k_2}T$$

which add up to T as they must.

Example: **Two springs in parallel.**

Take $k_1 = 99\,\mathrm{N/cm}$ and $k_2 = 1\,\mathrm{N/cm}$. The effective spring constant of the parallel combination is:

$$k = k_1 + k_2 = 99\,\mathrm{N/cm} + 1\,\mathrm{N/cm} = 100\,\mathrm{N/cm}.$$

Note that $T_1/T = .99$ so even though the two springs share the load, the stiffer one carries 99% of it. For practical purposes, or for the design of this system, it would be reasonable to remove the much less stiff spring. □

The reasoning above with two springs in parallel is easy enough to reproduce with 3 or more springs. The result is:

$$k_{\text{tot}} = k_1 + k_2 + k_3 + \ldots \quad \text{and} \quad T_1 = T k_1/k_{\text{tot}}, \quad T_2 = T k_2/k_{\text{tot}} \ldots$$

That is,

- The net spring constant is the sum of the constants of the separate springs; and
- The load carried by springs is in proportion to their spring constants.

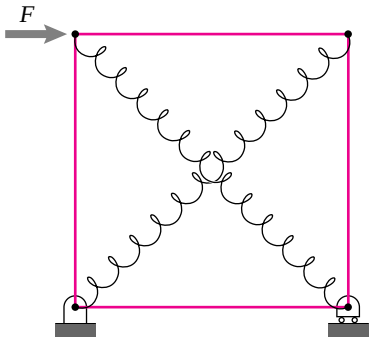


Figure 4.80: A mechanics joke to make a point. The bars in the open square above are rigid. The deformation into a diamond is resisted by the two springs shown. They share the load and they have stretches that are linked by the kinematics. Thus these two perpendicular springs are 'in parallel.' (By the way, you are not expected to be able to analyze the compliance of this structure at this point.)

(Filename:tfigure.paralleljoke)

Some comments on parallel springs

Once you understand the basic ideas and calculations for two side-by-side springs connected to common ends, there are a few things to think about for context.

For the purposes of drawing pictures (*e.g.*, Fig. 4.78a) parallel springs are drawn side by side. But in the mechanics analysis we treated them as if they were on top of each other. A pair of parallel springs is like a two bar truss where the bars are on top of each other but connected at their ends. With 2 bars and 2 joints we have $2j < b + 3$, and a redundant truss. In fact this is the simplest redundant truss, as one spring (read bar) does exactly the same job as the other (carries the same loads, resists the same motions). With statics alone we can not find the tensions in the springs since the statics equation $T_1 + T_2 = T$ has non-unique solutions.

In the context of trusses you may have had the following reasonable thought: The laws of statics allow multiple solutions to redundant problems. But a bar in a real physical structure has, at one instant of time, some unique bar tension. What determines this tension? Now we know the answer: the deformations and material properties. This is the first, and perhaps most conspicuous, occasion in this book that you see a problem where the three pillars of mechanics are assembled in such clear harmony, namely, material properties (eq. 4.47), the laws of mechanics (eq. 4.48), and the geometry of motion and deformation (eq. 4.49). In strength of materials calculations, where the distribution of stress is not determinable by statics alone, this threesome (geometry of deformation, material properties and statics) clearly come together in almost every calculation.

Finally, in the discussion above 'in parallel' corresponded to the springs being geometrically parallel. In common mechanics usage the words 'in parallel' are more general and mean that the net load is the sum of the loads carried by the two springs, and the stretches of the two springs are the same (or in a ratio restricted by kinematics). You will see cases where 'in parallel' springs are not the least bit parallel (*e.g.*, see Fig. 4.80).

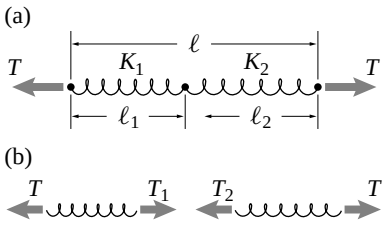


Figure 4.81: Schematic of springs in series.

(Filename:tfigure.seriessprings)

Springs in series

Two springs that share a displacement and carry the same load are *in series*.

A schematic of two springs in series is shown in Fig. 4.81a where the springs are aligned serially, one after the other. To determine the net stiffness of this simple

spring network we again assemble the three pillars of mechanics, using the free body diagram of Fig. 4.81b.

$$\begin{array}{lccc} \text{Constitutive law:} & T_1 = k_1(\ell_1 - \ell_{1_0}), & T_2 = k_2(\ell_2 - \ell_{2_0}), & \\ \text{Kinematics:} & \ell_0 = \ell_{1_0} + \ell_{2_0}, & \ell = \ell_1 + \ell_2, & (4.51) \\ \text{Force Balance:} & T_1 = T, \quad \text{and} & T_2 = T. & \end{array}$$

(where,*e.g.*, ℓ_{1_0} reads 'ell sub one zero' and is the rest length of spring 1). We can manipulate these equations much as we did for the similar equations for springs in parallel. The manipulation differs in structure the same way the equations do. For springs in parallel the tensions add and the displacements are equal. For springs in series the displacements add and the tensions are equal.

$$\begin{array}{rcccc} \Delta\ell & = & \ell & - & \ell_0 \\ & = & (\ell_1 + \ell_2) & - & (\ell_{1_0} + \ell_{2_0}) \\ & = & (\ell_1 - \ell_{1_0}) & + & (\ell_2 - \ell_{2_0}) \\ & = & \Delta\ell_1 & + & \Delta\ell_2 \\ & = & \frac{T_1}{k_1} & + & \frac{T_2}{k_2} \\ & = & \frac{T}{k_1} & + & \frac{T}{k_2} \\ & = & \underbrace{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}_{\frac{1}{k}} T & . & \end{array}$$

Thus we get that the net compliance is the sum of the compliances:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2},$$

which you might compare with springs in parallel (Eqn. 4.50). The sharing of the net stretch is in proportion to the compliances:

$$\Delta\ell_1 = \frac{1/k_1}{1/k_1 + 1/k_2}\Delta\ell \quad \text{and} \quad \Delta\ell_2 = \frac{1/k_2}{1/k_1 + 1/k_2}\Delta\ell$$

which add up to $\Delta\ell$ as they must.

Example: **Two springs in series.**

Take $1/k_1 = 99\,\text{cm/ N}$ and $1/k_2 = 1\,\text{cm/ N}$. The effective compliance of the parallel combination is:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = 99\,\text{cm/ N} + 1\,\text{cm/ N} = 100\,\text{cm/ N}.$$

Note that $\Delta\ell_1/\Delta\ell = .99$ so even though the two springs share the displacement, the more compliant one has 99% of it. For design purposes, or for modeling this system, it would be fair to replace the much more stiff spring with a rigid link. □

Much of what you need to know about the words 'in parallel' and 'in series' follows easily from these phrases:

In parallel,	forces	and	stiffnesses	add.
In series,	displacements	and	compliances	add.

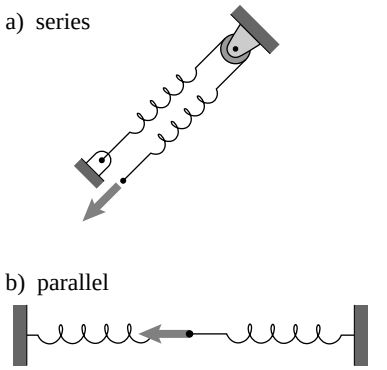


Figure 4.82: a) The two springs shown are in series because the carry the same load and their displacements add. b) These two springs are in parallel because the have a common displacement and their forces add.

(Filename:tfigure.parallelconfusion)

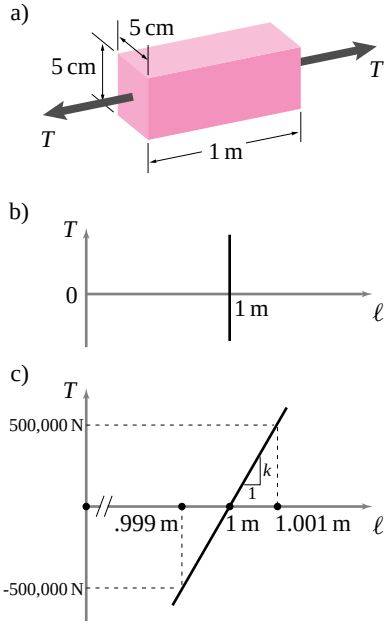


Figure 4.83: a) a steel rod in tension, b) tension verses length curve, c) zoom in on the tension verses length curve

(Filename:tfigure.steelrodspring)

① Because it is hard to picture steel deforming, your intuition may be helped by thinking of all solids as being rubber, or, if you want to look inside, like a chunk of Jello. (Jello is colored water held together by long gelatine molecules extracted from animal hooves. Those who are Kosher or vegetarian may substitute a sea-weed based Agar jell in their imagined deformation experiments.)

Rigid bodies, springs and air

As the previous two examples illustrate, springs can sometimes be replaced with 'air' (nothing) or with rigid links without changing the system or model behavior much. One way to think about this is that in the limit as $k \to \infty$ a spring becomes a rigid bar and in the limit $k \to 0$ a spring becomes air.

These ideas are used by engineers, often intuitively or even subconsciously and with no substantiating calculations, when making a model of a mechanical system. If one of several pieces in series is much stiffer than the others it is often replaced with a rigid link. If one of several pieces in parallel is much more compliant than the others it is often replaced with air. For example:

- When a coil spring is connected to a linkage, the other pieces in the linkage, though undoubtedly somewhat compliant, are typically modelled as rigid. They are stiffer than the spring and in series with it.
- A single hinge resists rotation about axes perpendicular to the hinge axis. But a door connected at two points along its edge is stiffly prevented against such rotations. Thus the hinge stiffness is in parallel with the greater rotational stiffness of the two connection points and is thus often neglected (see the discussion and figures in section 3.1 starting on page 84).
- Welded joints in a determinate truss are modeled as frictionless pins. The rotational stiffness of the welds is 'in parallel' with the axial stiffness of the bars. To see this look at two bars welded together at an angle. Imagine trying to break this weld by pulling the two far bar ends apart. Now imagine trying to break the weld if the two far ends are connected to each other with a third bar. The third bar is 'in parallel' with the weld material. See the first few sentences of section 4.2 for a do-it-yourself demonstration of the idea.
- Human bones are often modeled as rigid because, in part, when they interact with the world they are in series with more compliant flesh.

Note, again, that the mechanics usage of the words 'in parallel' and 'in series' don't always correspond to the geometric arrangement. For example the two springs in Fig. 4.82a are in series and the two springs in Fig. 4.82b are in parallel.

Solid bars are linear springs

When a structure or machine is built with literal springs (*e.g.*, a wire helix) it is common to treat the other parts as rigid. But when a structure has no literal springs the small amount of deformation in rigid looking objects can be important, especially for determining how loads are shared in redundant structures.

Let's consider a 1 m (about a yard) steel rod with a 5 cm square (about $(2\,\text{in})^2$) cross section (Fig. 4.83a). If we plot the tension verses length we get a curve like Fig. 4.83b. The length just doesn't visibly change (unless the tension got so large as to damage the rod, not shown.) But, when you pull on anything, it does deform at least a little. If we zoom in on the tension verses length plot we get Fig. 4.83c. To change the length by one part in a thousand we have to apply a tension of about 500, 000 N (about 60 tons). Nonetheless the plot reveals that the solid steel rod behaves like a (very stiff) linear spring.

Surprisingly perhaps this little bit of compliance is important to structural engineers who often like to think of solid metal rods as linear springs. How does their stiffness depend on their shape and composition?①.

Let's take a reference bar with cross sectional area A_0 and rest length ℓ_0 and pull it with tension T and measure the elongation $\Delta\ell_0$ (Fig. 4.84ab). The stiffness of this reference rod is $k_0 = T_0/\Delta\ell_0$. Now put two such rods side by side and you have parallel springs. You might imagine this sequence: two bars are near each other,

then side by side, then touching each other, then glued together, then melted together into one rod with twice the cross section. The same tension in each causes the same elongation, or it takes twice the tension to cause the same elongation when you have twice the cross sectional area. Likewise with three side by side bars and so on, so for bars of equal length

$$k = \frac{A}{A_0} k_0.$$

On the other hand we could put the reference rods end to end in series. Then the same tension causes twice the elongation. We could be three or more rods together in series thus for bars with equal cross sections:

$$k = \frac{\ell_0}{\ell} k_0.$$

Putting these together we get:

$$k = \left(\frac{A}{A_0}\right)\left(\frac{\ell_0}{\ell}\right) k_0 = \left(\frac{k_0 \ell_0}{A_0}\right)\frac{A}{\ell}.$$

Now presumably if we took a rod with a given material, length, and cross section the stiffness would be k, no matter what the dimensions of the reference rod. So $\left(\frac{k_0 \ell_0}{A_0}\right)$ has to be a material constant. It is called E, the *modulus of elasticity* or Young's modulus. For all steels $E \approx 30 \cdot 10^6 \,\mathrm{lbf/in^2} \approx 210 \cdot 10^9 \,\mathrm{N/m^2}$ (consistent with Fig. 4.83c). Aluminum has about a third this stiffness. So, a solid bar is a linear spring, obeying the spring equations:

$$k = \frac{EA}{\ell} \quad \text{or} \quad \Delta\ell = \frac{T\ell}{EA} \quad \text{or} \quad T = \frac{\Delta\ell \, EA}{\ell}$$

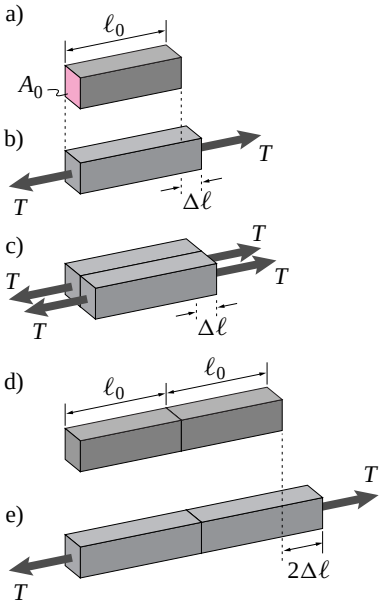


Figure 4.84: a) reference rod, b) reference rod in tension, c) two reference rods side by side, d) and e) two reference rods glued end to end.

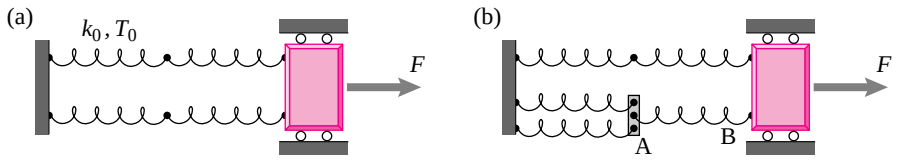
(Filename:tfigure.steelparallel)

Strength and stiffness

Most often when you build a structure you want to make it stiff and strong. The ideas of stiffness and strength are so intimately related that it is sometimes hard to untangle them. For example, you might examine a product in discount store by putting your hand on it, applying small forces and observing the motion. Then you might say: "pretty shaky, I don't think it will hold up" meaning that the stiffness is low so you think the thing may break if the loads get high.

Although stiffness and strength are often correlated, they are distinct concepts. Something is stiff if the force to cause a given motion is high. Something is strong if the force to cause any part of it to break is high. In fact, it is possible for a structure to be made weaker by making it stiffer.

Example: **Stiffer but weaker.**



Say all springs have stiffness k_0 and break when the tension in them reaches T_0. Because of the mixture of parallel and series springs, the net stiffness of the structure in (a) is $k_{\text{net}} = k_0$. Its strength is $2T_0$ because none of the springs reaches its breaking tension until $F = T_0$.

By doubling up one of the springs in (b) the structure is made 16% stiffer ($k_{\text{net}} = 7k_0/6$. But spring AB now reaches its breaking point T_0 when the applied load $F = 21T_0/12$, a 12.5% lower load than the $2T_0$ the structure could carry before the stiffening.

The structure is made stiffer by reducing the deflection of point A. But this causes spring AB to stretch more and thus break at a smaller load. In some approximate sense, the load is thus concentrated in spring AB. This concentration of load into one part of structure is one reason that stiffness and strength need to be considered separately. Load concentration (or *stress concentration*) is a major cause of structural failure. □

Why aren't springs in all mechanical models?

All things deform a little under load. Why don't we take this deformation into account in all mechanics calculations by, for example, modeling solids as elastic springs? Because many problems have solutions which would be little effected by such deformation. In particular, if a problem is statically determinate then very small deformations only have a very small effect on the equilibrium equations and calculated forces.

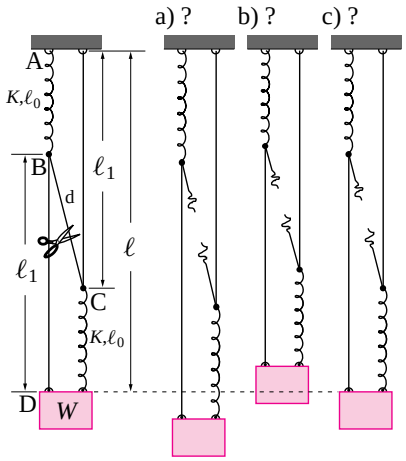
Linear springs are just one way to model 'give'

If it is important to consider the deformability of an object, the linear spring model is just one simple model. It happens to be a good model for the small deformation of many solids. But the linear spring model is defined by the two words 'linear' and 'elastic'. For some purposes one might want to model the force due to deformation as being *non-linear*, like $T = k_1(\Delta\ell) + k_2(\Delta\ell)^3$. And one may want to take account of the dissipative or *in-elastic* nature of something. The most common example being a linear dashpot $T = c\dot{\ell}$. Various mixtures of non-linearity and inelasticity may be needed to model the large deformations of a yielding metal, for example.

4.5 A puzzle with two springs and three ropes.

This is a tricky puzzle.

Consider a weight hanging from 3 strings (BD,BC, and AC) and 2 springs (AB and CD) as in the left picture below. Point B is above point C and all ropes and springs are somewhat taught (none is slack).



When rope BC is cut does the weight go (a) down?, (b) up?, or (c) stay put?

If you have the energy and curiosity you should stop reading and try thinking, experimenting, or calculating when you see three dots.

• • •

In 15 minutes or so you can set up this experiment with 3 pieces of string, 2 rubber bands and a soda bottle. Hang the partially filled soda bottle from a door knob (or the top corner of a door, or a ruler cantilevered over the top of a refrigerator). Adjust the string lengths and amount of weight so that no strings or rubber bands are slack and make sure point B is above point C. The two points A can coincide as can the two points D. You might want to separate them a little with, say, a small wad of paper so you can see which string is which.

• • •

Looking at your experimental setup, but not pulling and poking at it, try to predict whether your bottle will go up down or not move when you cut the middle string.

• • •

The answer is, by experiment, that: When you cut the middle string the weight goes up a little. This violates many people's intuitions. In fact, this puzzle was published as one of a class of problems for which people have poor intuitions.

• • •

Now try to figure out why the experiment comes out the way it does? Also, try to figure out the error in your thinking if you got it wrong (like most people do).

• • •

All simple explanations are based on the assumption that the lengths of the strings AC and BD are constant at ℓ_1.

• • •

To simplify the reasoning let's assume that springs AB and CD are identical and carry the same tension T_s and that the ropes AC and BD carry the same tension T_r. As usual, we need free body diagrams. (With the symmetry we have assumed diagrams (a) and (c) provide identical information.) The three free body diagrams can be considered before and after the middle string removal by having $T_d > 0$ or $T_d = 0$, respectively. Vertical force balance gives (approximating T_d as vertical):

$$T_s + T_r = W \quad \text{and} \quad 2T_r + T_d = W \quad \Rightarrow \quad T_s = (W + T_d)/2.$$

Because we approximate AC as rigid with length ℓ_1, the downwards position of the weight is the string length ℓ_1 plus the rest length of the spring ℓ_0 plus the stretch of the spring T_s/k:

$$\ell = \ell_1 + \ell_0 + T_s/k = \ell_1 + \ell_0 + (W + T_d)/(2k).$$

In the course of this experiment ℓ_1, ℓ_0, W and k are constants. So as the tension T_d goes from positive to zero (when the rope BC is cut) ℓ decreases. So the weight goes up.

• • •

More intuitively, start with the configuration with the rope already cut and apply a small upwards force at C. It has no effect on the tension in spring CD thus the weight does not move. Now apply a small downwards force at B. This does stretch spring AB and thus lower point B, thus lowering the weight since ℓ_1 is constant. Applying both simultaneously is like attaching the middle rope. Thus attaching the middle rope lowers the weight and cutting the middle rope raises it again.

• • •

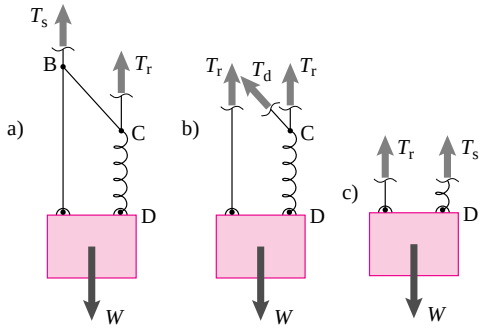
Here is another intuitive approach. Point C can't move. Point B moves up and down just as much as the weight does. Point B is a distance d above point C. Since the rope BC is taught, releasing it will allow B and C to separate, thus increasing d and raising the weight.

• • •

What about springs in parallel and series? Here is a quick but wrong explanation for the experimental result, though it happens to predict the right answer, or at least the right direction of motion.

> "Before rope BC is cut the two springs are more or less in series because the load is carried from spring through BC to spring. Afterwards they are more or less in parallel because they have the same stretch and share the load. Two springs in parallel have 4 times the stiffness of the same two springs in series. So in the parallel arrangement the deflection is less. So the weight goes up when the springs switch from series to parallel."

What is the error in this thinking? The position of the weight comes from spring deflection added to the position when there is no weight. For the argument just presented to make sense, the rest position of the mass (with gravity switched off) would have to be the same for the supposed 'series' and 'parallel' cases, which it is not ($\ell_1 + \ell_0 \neq \ell_0 + d + \ell_0$).



Another way to see the fallacy of this 'parallel verses series' argument is that the incremental stiffness of the system is, assuming inextensible ropes, infinite. That is, if you add or subtract a small load to the bottle it doesn't move until one or another rope goes slack. (The small deformation you do see has to do with the stretch of the ropes, something that none of the simple explanations take into account.) If the springs were in series or parallel we would expect an incremental stiffness that was related to spring stretch not rope stretch.

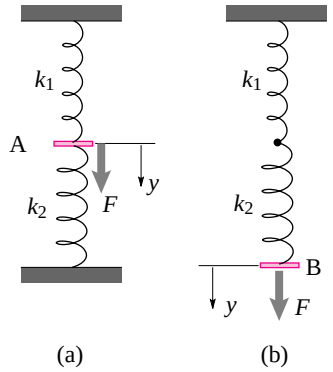


Figure 4.85: (Filename:sfig4.2springs)

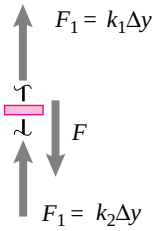


Figure 4.86: Free body diagram of point A.

(Filename:sfig4.2springs.a)

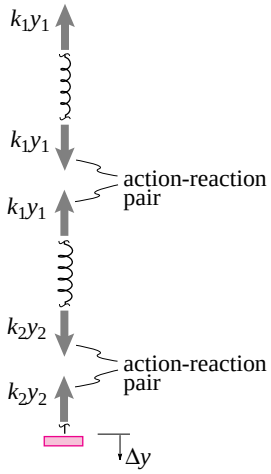


Figure 4.87: Free body diagrams

(Filename:sfig4.2springs.b)

SAMPLE 4.18 *Springs in series versus springs in parallel:* Two springs with spring constants $k_1 = 100\,\text{N}/m$ and $k_2 = 150\,\text{N}/m$ are attached together as shown in Fig. 4.85. In case (a), a vertical force $F = 10\,\text{N}$ is applied at point A, and in case (b), the same force is applied at the end point B. Find the force in each spring for static equilibrium. Also, find the equivalent stiffness for (a) and (b).

Solution In static equilibrium, let Δy be the displacement of the point of application of the force in each case. We can figure out the forces in the springs by writing force balance equations in each case.

- **Case (a):** The free body diagram of point A is shown in Fig. 4.86. As point A is displaced downwards by Δy, spring 1 gets stretched by Δy whereas spring 2 gets compressed by Δy. Therefore, the forces applied by the two springs, $k_1\Delta y$ and $\Delta k_2 y$, are in the same direction. Then, the force balance in the vertical direction, $\hat{\boldsymbol{j}} \cdot (\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}})$, gives:

$$\begin{aligned} F &= F_1 + F_2 = (k_1 + k_2)\Delta y \\ \Rightarrow \quad \Delta y &= \frac{F}{k_1 + k_2} = \frac{10\,\text{N}}{(100 + 150)\,\text{N}/m} = 0.04\,\text{m} \\ \Rightarrow \quad F_1 &= k_1\Delta y = 100\,\text{N}/m \cdot 0.04\,\text{m} = 4\,\text{N} \\ \Rightarrow \quad F_2 &= k_2\Delta y = 150\,\text{N}/m \cdot 0.04\,\text{m} = 6\,\text{N} \end{aligned}$$

The equivalent stiffness of the system is the stiffness of a single spring that will undergo the same displacement Δy under F. From the equilibrium equation above, it is easy to see that,

$$k_e = \frac{F}{\Delta y} = k_1 + k_2 = 250\,\text{N}/\,\text{m}.$$

$$\boxed{F_1 = 4\,\text{N}, \quad F_2 = 6\,\text{N}, \quad k_e = 250\,\text{N}/\,\text{m}}$$

- **Case (b):** The free body diagrams of the two springs is shown in Fig. 4.87 along with that of point B. In this case both springs stretch as point B is displaced downwards. Let the net stretch in spring 1 be y_1 and in spring 2 be y_2. y_1 and y_2 are unknown, of course, but we know that

$$y_1 + y_2 = \Delta y \tag{4.52}$$

Now, using the free body diagram of point B and writing the force balance equation in the vertical direction, we get $F = k_2 y_2$ and from the free body diagram of spring 2, we get $k_2 y_2 = k_1 y_1$. Thus the force in each spring is the same and equals the applied force, *i.e.*,

$$F_1 = k_1 y_1 = F = 10\,\text{N} \quad \text{and} \quad F_2 = k_2 y_2 = F = 10\,\text{N}.$$

The springs in this case are in series. Therefore, their equivalent stiffness, k_e, is

$$k_e = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1} = \left(\frac{1}{100\,\text{N}/m} + \frac{1}{150\,\text{N}/m}\right)^{-1} = 60\,\text{N}/m.$$

Note that the displacements y_1 and y_2 are different in this case. They can be easily found from $y_1 = F/k_1$ and $y_2 = F/k_2$.

$$\boxed{F_1 = F_2 = 10\,\text{N}, \quad k_e = 60\,\text{N}/m}$$

Comments: Although the springs attached to point A do not visually seem to be in parallel, from mechanics point of view they are parallel. Springs in *parallel* have the *same displacement* but *different forces*. Springs in *series* have *different displacements* but the *same force*.

SAMPLE 4.19 *Stiffness of three springs:* For the spring networks shown in Fig. 4.88(a) and (b), find the equivalent stiffness of the springs in each case, given that each spring has a stiffness of $k = 20\,\text{N/m}$.

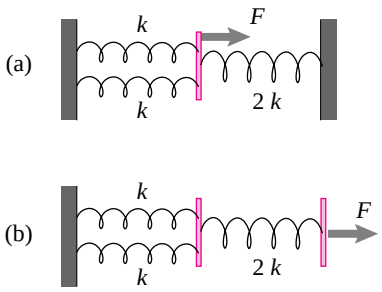


Figure 4.88: (Filename:sfig4.3springs)

Solution

(a) In Fig. 4.88(a), all springs are in parallel since all of them undergo the same displacement Δx in order to balance the applied force F. Each of the two springs on the left stretches by Δx and the spring on the right compresses by Δx. Therefore, the equivalent stiffness of the three springs is

$$k_p = k + k + 2k = 4k = 80\,\text{kN/ m}.$$

Pictorially,

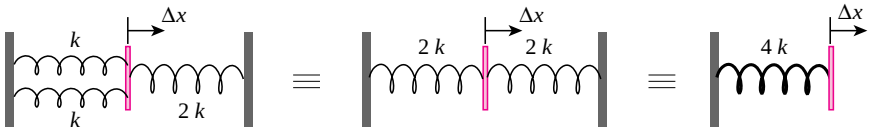


Figure 4.89: (Filename:sfig4.3springs.a)

$$\boxed{k_{equiv} = 80\,\text{kN/ m}}$$

(b) In Fig. 4.88(b), the first two springs (on the left) are in parallel but the third spring is in series with the first two. To see this, imagine that for equilibrium point A moves to the right by Δx_A and point B moves to the right by Δx_B. Then each of the first two springs has the same stretch Δx_A while the third spring has a net stretch $= \Delta x_B - \Delta x_A$. Therefore, to find the equivalent stiffness, we can first replace the two parallel springs by a single spring of equivalent stiffness $k_p = k + k = 2k$. Then the springs with stiffnesses k_p are $2k$ are in series and therefore their equivalent stiffness k_s is found as follows.

$$\begin{aligned} \frac{1}{k_s} &= \frac{1}{k_p} + \frac{1}{2k} = \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \\ \Rightarrow \quad k_s &= k = 20\,\text{kN/ m}. \end{aligned}$$

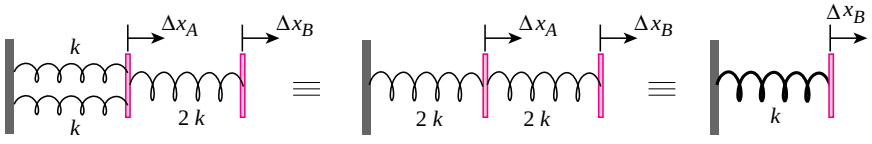


Figure 4.90: (Filename:sfig4.3springs.b)

$$\boxed{k_{equiv} = 20\,\text{kN/ m}}$$

SAMPLE 4.20 *Stiffness vs strength:* Which of the two structures (network of springs) shown in the figure is stiffer and which one has more strength if each spring in has stiffness $k = 10\,\text{kN}/m$ and strength $F_0 = 10\,\text{kN}$.

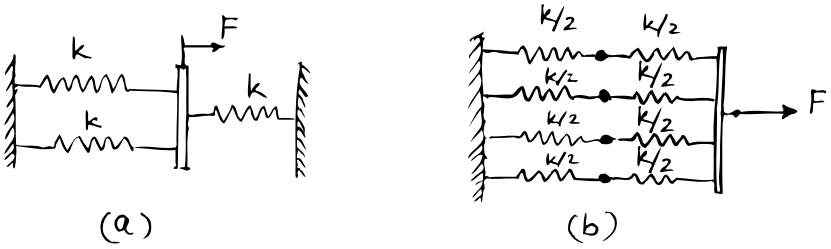


Figure 4.91: (Filename:sfig4.manysprings)

Solution In structure (a), all the three springs are in parallel. Therefore, the equivalent stiffness of the three springs is

$$k_a = k + k + k = 3k = 30\,\text{kN/m}.$$

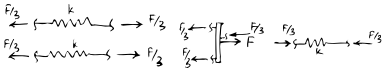


Figure 4.92: (Filename:sfig4.manysprings.a)

For figuring out the strength of the structure, we need to find the force in each spring. From the free body diagram in Fig. 4.92 we see that,

$$k\Delta x + k\Delta x + k\Delta x \quad = \quad F \quad \Rightarrow \quad \Delta x = \frac{F}{3k}$$

Therefore, the force in each spring is

$$F_\text{s} \quad = \quad k\Delta x = \frac{F}{3}$$

But the maximum force that a spring can take is $(F_\text{s})_\text{max} = F_0 = 10\,\text{kN}$. Therefore, the maximum force that the structure can take, *i.e.*, the strength of the structure, is

$$F_\text{max} \quad = \quad 3F_0 = 30\,\text{kN}.$$

$$\boxed{\text{Stiffness} = 30\,\text{kN}/m,\ \text{Strength} = 30\,\text{kN}}$$

Now we carry out a similar analysis for structure (b). There are four parallel chains in this structure, with each chain containing two springs in series. The stiffness of each chain, k_c, is found from

$$\frac{1}{k_c} = \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \quad \Rightarrow \quad k_c = \frac{k}{2} = 5\,\text{kN/m}.$$

Now the stiffness of the entire structure is

$$k_b = k_c + k_c + k_c + k_c = 4k_c = 20\,\text{kN/m}.$$

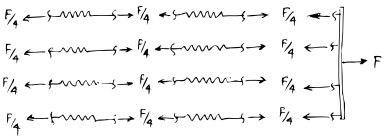


Figure 4.93: (Filename:sfig4.manysprings.b)

We find the force in each spring to be F/4 from the free body diagram shown in Fig. 4.93. Therefore, the maximum force that the structure can take is

$$F_\text{max} = 4F_0 = 40\,\text{kN}.$$

$$\boxed{\text{Stiffness} = 20\,\text{kN}/m,\ \text{Strength} = 40\,\text{kN}}$$

Thus, the structure in Fig. 4.91(a) is stiffer but the structure in (b) is stronger (more strength).

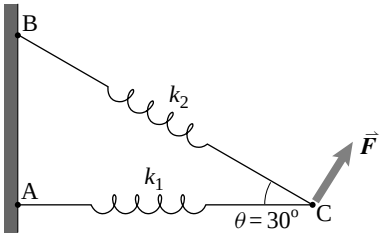


Figure 4.94: (Filename:sfig4.springs.compl)

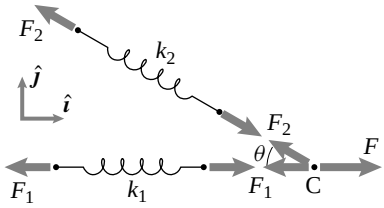


Figure 4.95: (Filename:sfig4.springs.compl.a)

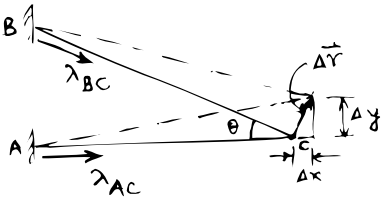


Figure 4.96: (Filename:sfig4.springs.compl.b)

SAMPLE 4.21 *Compliance matrix of a structure:* For the two-spring structure shown in the figure, find the deflection of point C when

(a) $\vec{F} = 1\,\text{N}\hat{\imath}$,
(b) $\vec{F} = 1\,\text{N}\hat{\jmath}$,
(c) $\vec{F} = 30\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath}$,

The spring stiffnesses are $k_1 = 10\,\text{kN}/m$ and $k_2 = 20\,\text{kN}/m$.

Solution

(a) *Deflections with unit force in the x-direction:* Let $\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$ be the displacement of point C of the structure due to the applied load. We can figure out the deflections in each spring as follows. Let $\hat{\boldsymbol{\lambda}}_{\text{AC}}$ and $\hat{\boldsymbol{\lambda}}_{\text{BC}}$ be the unit vectors along AC and BC, respectively. Then, the change in the length of spring AC due to the displacement of point C is

$$\begin{aligned}\Delta_{\text{AC}} &= \hat{\boldsymbol{\lambda}}_{\text{AC}} \cdot \Delta\vec{r} \\ &= \hat{\imath} \cdot (\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = \Delta x\end{aligned}$$

Similarly, the change in the length of spring BC is

$$\begin{aligned}\Delta_{\text{BC}} &= \hat{\boldsymbol{\lambda}}_{\text{AC}} \cdot \Delta\vec{r} \\ &= (cos\theta\hat{\imath} - \sin\theta\,\hat{\jmath}) \cdot (\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = \Delta x\cos\theta - \Delta y\sin\theta.\end{aligned}$$

Now we can find the force in each spring since we know the deflection in each spring.

$$\text{Force in spring AB} = F_1 = k_1\Delta x \tag{4.53}$$
$$\text{Force in spring BC} = F_2 = k_2(\Delta x\cos\theta - \Delta y\sin\theta). \tag{4.54}$$

The forces in the springs, however, depend on the applied force, since they must satisfy static equilibrium. Thus, we can determine the deflection by first finding F_1 and F_2 in terms of the applied load and substituting in the equations above to solve for the deflection components.
Let $\vec{F} = f_x\hat{\imath} = 1\,\text{N}\hat{\imath}$, (we have adopted a special symbol f_x for the unit load). Then, from the free body diagram of the springs and the end pin shown in Fig. 4.96 and the force equilibrium ($\sum\vec{F} = \vec{0}$), we have,

$$f_x\hat{\imath} - F_1\hat{\imath} + F_2(-\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) = \vec{0}$$

Dotting with $\hat{\jmath}$ and $\hat{\imath}$ we get,

$$\begin{aligned}F_2 &= 0 \\ F_1 &= f_x = 1\,\text{N}.\end{aligned}$$

Substituting the values of F_1 and F_2 from above in eqns. (4.53 and 4.54), and solving for Δx and Δy we get,

$$\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix}_{\vec{F}=f_x\hat{\imath}} = \begin{pmatrix}\frac{1}{k_1} \\ \frac{1}{k_1}\cot\theta\end{pmatrix} f_x. \tag{4.55}$$

Substituting the given values of θ and k_1 and $f_x = 1\,\text{N}$, we get

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} = (100\hat{\imath} + 173\hat{\jmath}) \times 10^{-6}\,\text{m}.$$

$$\boxed{\Delta\vec{r} = (100\hat{\imath} + 173\hat{\jmath}) \times 10^{-6}\,\text{m}}$$

(b) *Deflections with unit force in the y-direction:* We carry out a similar analysis for this case. We again assume the displacement of point C to be $\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$. Since the geometry of deformation and the associated results are the same, eqns. (4.53) and (4.54) remain valid. We only need to find the spring forces from the static equilibrium under the new load. From the free body diagram in Fig. 4.97 we have,

$$\begin{aligned}(-F_1 - F_2\cos\theta)\hat{\imath} + (F_2\sin\theta + F)\hat{\jmath} &= \vec{\mathbf{0}}\\ \Rightarrow \quad F_2 &= -\frac{F}{\sin\theta}\\ \text{and} \quad F_1 &= -F_2\cos\theta = F\cot\theta.\end{aligned}$$

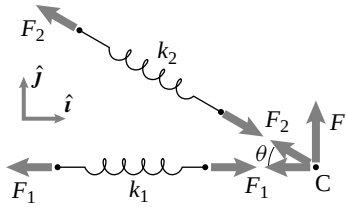


Figure 4.97: (Filename:sfig4.springs.compl.c)

Substituting these values of F_1 and F_2 in terms of f_y in eqns. (4.53) and (4.54), we get

$$\begin{aligned}f_y\cot\theta &= k_1\Delta x \quad \Rightarrow \quad \Delta x = \frac{f_y}{k_1}\cot\theta\\ -\frac{f_y}{\sin\theta} &= k_2(\Delta x\cos\theta - \Delta y\sin\theta)\\ \Rightarrow \quad \Delta y &= \frac{1}{\sin\theta}(\Delta x\cos\theta + \frac{f_y}{k_2\sin\theta})\\ &= f_y\left(\frac{1}{k_1}\cot^2\theta + \frac{1}{k_2}\csc^2\theta\right)\end{aligned}$$

Thus,

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{\boldsymbol{F}}=f_y\hat{\jmath}} = \begin{pmatrix}\frac{1}{k_1}\cot\theta\\ \frac{1}{k_1}\cot^2\theta + \frac{1}{k_2}\csc^2\theta\end{pmatrix} f_y. \qquad (4.56)$$

Substituting the values of θ, k_1, and k_2, and $f_y = 1\,\text{N}$, we get

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} = (173\hat{\imath} + 500\hat{\jmath})\times 10^{-6}\,\text{m}.$$

$$\boxed{\Delta\vec{r} = (173\hat{\imath} + 500\hat{\jmath})\times 10^{-6}\,\text{m}}$$

(c) *Deflection under general load:* Since we have already got expressions for deflections in the x and y-directions under unit loads in the x and y-directions, we can now combine the results to find the deflection under any general load $\vec{\boldsymbol{F}} = F_x\hat{\imath} + F_y\hat{\jmath}$ as follows.

$$\begin{aligned}\Delta\vec{r} &= \begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = F_x\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{\boldsymbol{F}}=1\hat{\imath}} + F_y\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{\boldsymbol{F}}=1\hat{\jmath}}\\ &= \begin{bmatrix}k_1^{-1} & k_1^{-1}\cot\theta\\ k_1^{-1}\cot\theta & k_1^{-1}\cot^2\theta + k_2^{-1}\csc^2\theta\end{bmatrix}\begin{pmatrix}F_x\\ F_y\end{pmatrix}.\end{aligned}$$

Once again, substituting all given values and $F_x = 30\,\text{N}$ and $F_y = 20\,\text{N}$, we get

$$\Delta\vec{r} = (6.4\hat{\imath} + 15.2\hat{\jmath})\times 10^{-3}\,\text{m}.$$

$$\boxed{\Delta\vec{r} = (6.4\hat{\imath} + 15.3\hat{\jmath})\times 10^{-3}\,\text{m}}$$

Note: The matrix obtained above for finding the deflection under general load is called the *compliance matrix* of the structure. Its inverse is known as the *stiffness matrix* of the structure and is used to find forces given deflections.

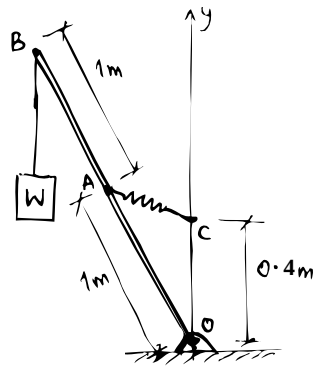


Figure 4.98: (Filename:sfig4.zerospring)

SAMPLE 4.22 *Zero length springs are special!* A rigid and massless rod AB of length 2 m supports a weight $W = 100$ kg hung from point B. The rod is pinned at O and supported by a zero length (in relaxed state) spring attached at mid-point A and point C on the vertical wall. Find the equilibrium angle θ and the force in the spring.

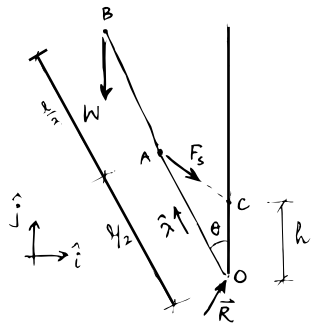


Figure 4.99: (Filename:sfig4.zerospring.a)

Solution The free body diagram of the rod is shown in Fig. 4.99 in an assumed equilibrium state. Let $\hat{\boldsymbol{\lambda}} = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$ be a unit vector along OB. The spring force can be written as $\vec{\boldsymbol{F}}_{\rm s} = k\vec{\boldsymbol{r}}_{\rm C/A}$. We need to determine θ and δ.

Let us write moment equilibrium equation about point O, *i.e.*, , $\sum \vec{\boldsymbol{M}}_{\rm O} = \vec{\boldsymbol{0}}$,

$$\vec{\boldsymbol{r}}_{\rm B/O} \times \vec{\boldsymbol{W}} + \vec{\boldsymbol{r}}_{\rm A/O} \times \vec{\boldsymbol{F}}_{\rm s} = \vec{\boldsymbol{0}}$$

Noting that

$$\begin{aligned} \vec{\boldsymbol{r}}_{\rm B/O} &= \ell\hat{\boldsymbol{\lambda}}, \quad \vec{\boldsymbol{r}}_{\rm A/O} = \frac{\ell}{2}\hat{\boldsymbol{\lambda}}, \\ \vec{\boldsymbol{F}}_{\rm s} &= k\vec{\boldsymbol{r}}_{\rm C/A} = k(\vec{\boldsymbol{r}}_{\rm C} - \vec{\boldsymbol{r}}_{\rm A}) \\ &= k(h\hat{\boldsymbol{\jmath}} - \frac{\ell}{2}\hat{\boldsymbol{\lambda}}), \end{aligned}$$

we get,

$$\begin{aligned} \ell\hat{\boldsymbol{\lambda}} \times (-W\hat{\boldsymbol{\jmath}}) + \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times k(h\hat{\boldsymbol{\jmath}} - \frac{\ell}{2}\hat{\boldsymbol{\lambda}}) &= \vec{\boldsymbol{0}} \\ -W\ell(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) + kh\frac{\ell}{2}(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{0}} \end{aligned}$$

Dotting this equation with $(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}})$, we get,

$$\begin{aligned} -W\ell + kh\frac{\ell}{2} &= 0 \\ \Rightarrow \quad kh &= 2W. \end{aligned}$$

Thus the result is independent of θ! As long as the spring stiffness k and the height of point C, h, are such that their product equals $2W$, the system will be in equilibrium at any angle. This is why zero length springs are special.

Equilibrium is satisfied at any angle if $kh = 2W$

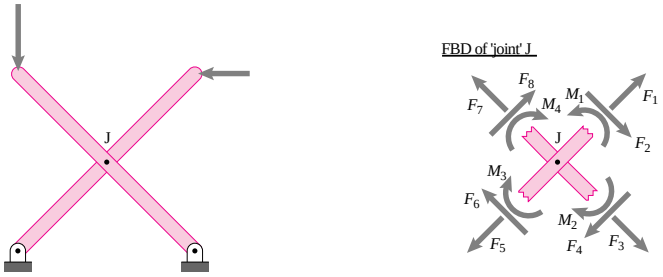
4.6 Structures and machines

The laws of mechanics apply to one body shown in one free body diagram. Yet engineers design things with many pieces each of which may be thought of as a body. One class of examples are trusses which you learned to analyze in section 4.2.

We would now like to analyze things built of pieces that are connected in a more complex way. These things include various structures which are designed to not move and various machines which are designed to move. Our general goal here is to find the interaction forces and the 'internal' forces in the components.

The secret to our success with trusses was that all of the pieces in a truss are two-force members. Thus free body diagrams of joints involved forces that were in known directions. Because now the pieces are not all two-force bodies, we will not know the directions of the interaction forces *a priori* and the method of joints will be nearly useless.

Example: **An X structure**



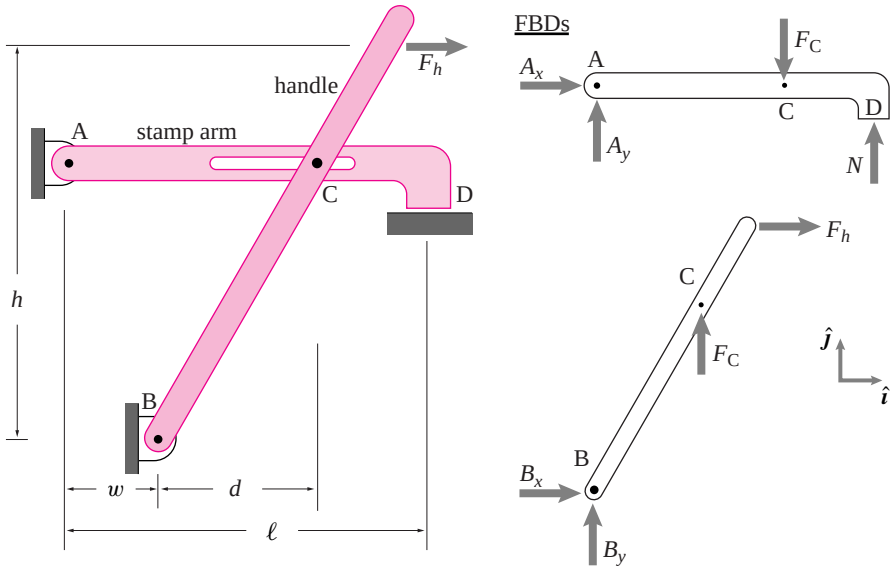
Two bars are joined in an 'X' by a pin at J. Neither of the bars is a two-force body so a free body diagram of the 'joint' at J, made by cutting and leaving stubs as we did with trusses, has 12 unknown force and moment components. □

Instead of drawing free body diagrams of the connections, our approach here is to draw free body diagrams of each of the structure or machine's parts. Sometimes, as was the case with trusses, it is also useful to draw a free body diagram of a whole structure or of some multi-piece part of the structure①.

① You might wonder why we didn't analyze trusses this way, by drawing free body diagrams of each of the bars. This seldom used approach to trusses, the 'method of bars and pins' is discussed in box 4.6 on 187.

Example: **Stamp machine**

Pulling on the handle (below) causes the stamp arm to press down with a force N at D. We can find N in terms of F_h by drawing free body diagrams of the handle and stamp arm, writing three equilibrium equations for each piece and then solving these 6 equations for the 6 unknowns (A_x, A_y, F_C, N, B_x, and B_y).



For this problem, the answer can be found more quickly with a judicious choice of equilibrium equations.

$$\text{For the handle, } \left\{\sum \vec{M}_{/\mathrm{B}} = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad -hF_h + dF_c = 0$$
$$\text{For the stamp arm, } \left\{\sum \vec{M}_{/\mathrm{A}} = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad -(d+w)F_c + \ell N = 0$$

$$\text{eliminating } F_c \quad \Rightarrow \quad N = \frac{h(d+w)}{d\ell} F_h.$$

Note that the stamp force N can be made very large by making d small and thus the handle nearly vertical. Often in structural or machine design one or another force gets extremely large or small as the design is changed to put pieces in near alignment. □

Static determinacy

A statically determinate structure has a solution for all possible applied loads, has only one solution, and this solution can be found by using equilibrium equations applied to each of the pieces. As for trusses, not all structures are statically determinate. The simple counting formula that is necessary for determinacy but does not guarantee determinacy is:

$$\text{number of equations} = \text{number of unknowns}$$

Where, in 2D, there are three equilibrium equations for each object. There are two unknown force components for every pin connection, whether to the ground or to another piece. And there is one unknown force component for every every roller connection whether to the ground or between objects. Applied forces do not count in this determinacy check, even if they are unknown.

Example: **'X' structure counting**

In the 'X' structure above we can count as follows.

$$\begin{aligned} \text{number of equations} &\overset{?}{=} \text{number of unkowns} \\ \text{(3 eqs per bar)} \cdot \text{(2 bars)} &\overset{?}{=} \text{(2 unkown force comps per pin)} \cdot \text{(3 pins)} \\ \text{6 eqs} &\overset{\surd}{=} \text{6 unkown force components} \end{aligned}$$

So the 'X' structure passes the counting test for static determinacy. □

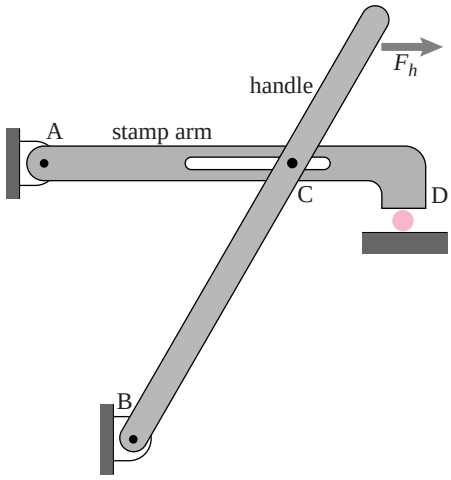
Indeterminate structures are mechanisms

An indeterminate structure cannot carry all loads and, if not also redundant, has more equilibrium equations than unknown reaction or interaction force components. Such a structure is also called a mechanism. The stamp machine above is a mechanism if there is assumed to be no contact at D. In particular the equilibrium equations cannot be satisfied unless $F_h = 0$. Mechanisms have variable configurations. That is, the constraints still allow relative motion.

An attempt to design a rigid structure that turns out to be a mechanism is a design failure. But for machine design, the mechanism aspect of a structure is essential. Even though mechanisms are called 'statically indeterminate' because they cannot carry all possible loads, the desired forces can often be determined using statics. For the stamp machine above the equilibrium equations are made solvable by treating one of the applied forces, say N, as an unknown, and the other, F in this case, as a known. This is a common situation in machine design where you want to determine the loads at one part of a mechanism in terms of loads at another part. For the purposes of analysis, a trick is to make a mechanism determinate by putting a pin on rollers connection to ground at the location of any forces with unknown magnitudes but known directions.

Example: **Stamp machine with roller**

Putting a roller at D, the location of the unknown stamp force, turns the stamp machine into a determinate structure.

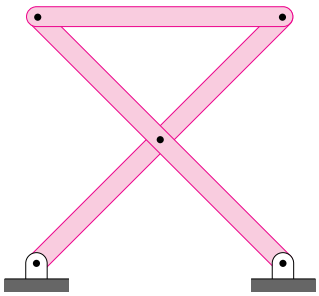


□

Redundant structures

A redundant structure can carry whatever loads it can carry in more than one way. If not also indeterminate, a redundant structure has fewer equilibrium equations than unknown reaction or interaction force components.

We generally avoid trying to find those force components which cannot be found uniquely from the equilibrium equations. Finding them depends on modeling the deformation, a topic emphasized in advanced structural courses.



Example: **Overbraced 'X'**

The structure above is evidently redundant because it has a bar added to a structure which was already statically determinate. By counting we get

$$\begin{aligned} \text{number of equations} &\stackrel{?}{=} \text{number of unkowns} \\ 3 \cdot \underbrace{(\text{number of bars})}_{3} &\stackrel{?}{=} 2 \cdot \underbrace{(\text{number of joints})}_{5} \\ 9 \text{ eqs} &< 10 \text{ unknown force components} \end{aligned}$$

thus demonstrating redundancy. □

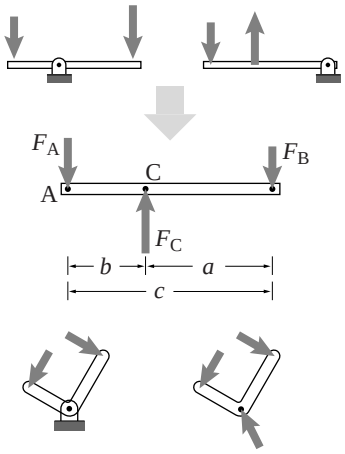


Figure 4.100: A lever can have the pivot in various places. The free body diagram looks the same in any case.

(Filename:tfigure.lever)

Some common mechanical designs

Rigid bodies can be connected in various arrangements for various purposes. Here we describe several basic machine fragments.

A lever

Maybe the simplest machine, and one we have mentioned several times, is a lever (fig. 4.100). An ideal lever is a rigid body held in place with a frictionless hinge and with two other applied loads. The hinge could be at point A, B or C and the free body diagram of fig. 4.100 is the same. The study of the lever precedes the change of mechanics from a taxonomic to a quantitative subject. So there is specialized antiquated vocabulary of levers, classifying them depending on where the pivot is located, and on which force you think of as input and which you think of as output. For historical curiosity: A 'class one' lever has the pivot in the middle; a 'class two' lever has the pivot at one end and the input force at the other; and a 'class three' lever has the pivot at one end and the input force in the middle.

Lots of things can be viewed as levers including, for example, a wheelbarrow, a hammer pulling a nail, a boat oar, one half of a pair of tweezers, a break lever, a gear,

and, most generally, any three-force body. Using the equilibrium relations on the free body diagram in fig. 4.100 you can find that

$$\frac{F_A}{a} = \frac{F_B}{b} = \frac{F_C}{c}$$

from which you can find the relation between any pair of the forces. In practice it is undoubtedly easier to use moment balance about an appropriate point than to memorize this formula.

Gears

A transmission is used to 'transmit' motion caused at one place to motion at another and, generally to also speed it up or slow it down. Simultaneously force or moment is transmitted from one point to another, generally being attenuated or amplified. One type of transmission is based on gears (Fig. 4.101a). If we think of the input and output as the moments on the two gears, we find from the free body diagram in Fig. 4.101b that

$$\begin{aligned} \text{For gear A, } \left\{\sum \vec{\boldsymbol{M}}_{\text{i/A}} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad -R_{\text{A}}F + M_{\text{A}} = 0 \\ \text{For gear B, } \left\{\sum \vec{\boldsymbol{M}}_{\text{i/B}} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad -R_{\text{B}}F + M_{\text{B}} = 0 \end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad M_B = \frac{R_B}{R_A} M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B} M_B$$

depending on which you want to think of input and which as output. The force amplification or attenuation ratio is just the radius ratio, just like for a lever.

Because the spacing of gear teeth for both of a meshed pair of gears is the same, a gears circumference, and hence its radius is proportional to the number of teeth. And formulas involving radius ratios can just as well be expressed in terms of ratios of numbers of teeth. The tooth ratio is not just used as an approximation to the radius ratio. Averaged over the passage of several teeth, it is exactly the reciprocal ratio of the turning rates of the meshed gears.

Two gears pulled out of a bigger transmission are shown in Fig. 4.101c. Gear A has an inner part with radius R_{A_i} welded to an outer part with radius R_{A_o}. Gear B also has an inner part welded to an outer part.

Moment balance about A in the first free body diagram in Fig. 4.101d gives that $R_{A_i} F_A = R_{A_o} F$. You can think of the one gear as a lever (see Fig. 4.102). Moment balance about B in the second free body diagram gives that $R_{B_i} F = R_{B_o} F_B$. Combining we get

$$F_B = \frac{R_{A_i}}{R_{A_o}} \frac{R_{B_i}}{R_{B_o}} F_A \quad \text{or} \quad F_A = \frac{R_{A_o}}{R_{A_i}} \frac{R_{B_o}}{R_{B_i}} F_B$$

depending on which force you want to find in terms of the other. The transmission attenuates the force if you think of F_A as the input and amplifies the force if you think of F_B as the input. If the inner gears have one tenth the radius of the outer gears than the multiplication or attenuation is a factor of 100.

An ideal wedge

Wedges are kind of machine. For an ideal wedge one neglects friction, effectively replacing sliding contact with rolling contact (see Fig. 4.103ab). Although this approximation may not be accurate, it is helpful for building intuition. For the free

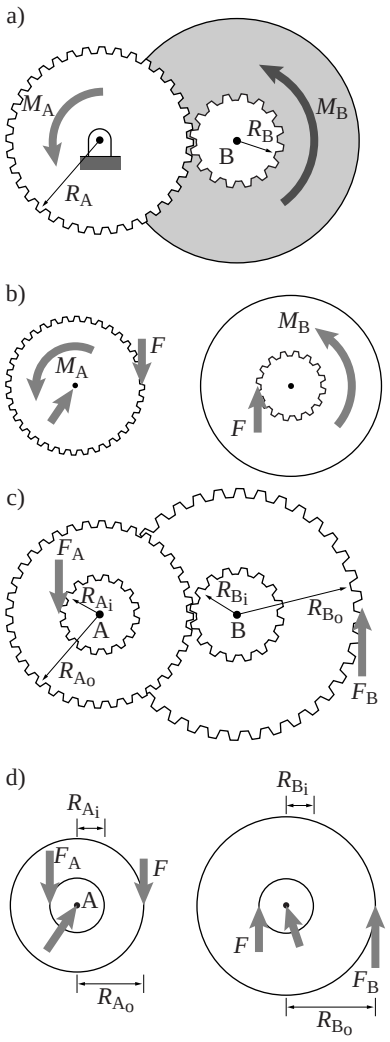


Figure 4.101: a) Two gear pairs pulled out of a transmission with forces on the teeth, b) Free body diagrams, c) The same gear pair, but loaded with tooth-forces from unseen gears, d) the consequent free body diagrams.

(Filename:tfigure.gears)

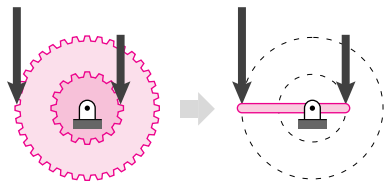


Figure 4.102: One gear may be thought of as a lever.

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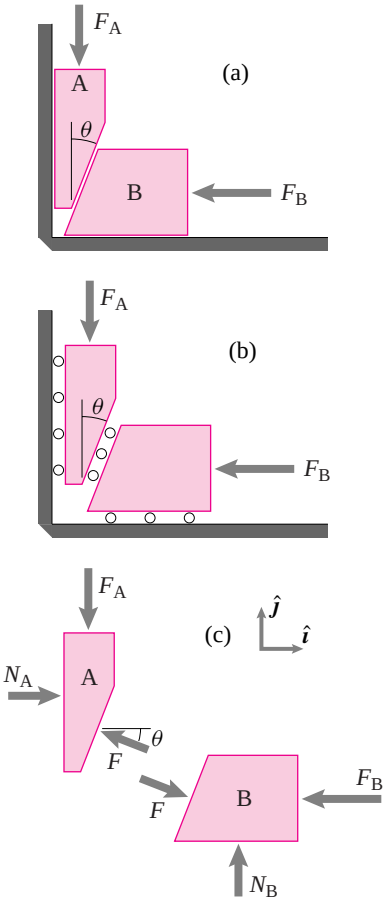


Figure 4.103: a) A wedge, b) treated as frictionless in the ideal case, c) free body diagrams

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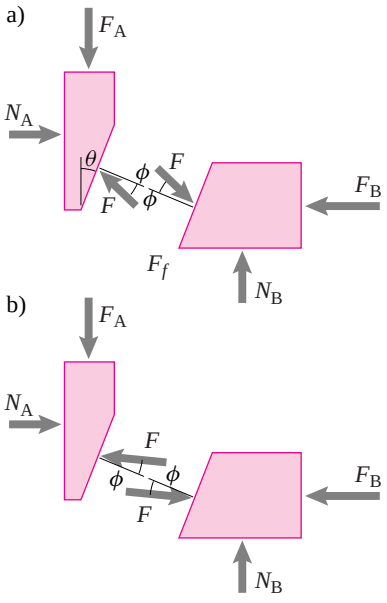


Figure 4.104: Free body diagrams of a wedge a) assuming A slides down, b) assuming A slides up

(Filename:tfigure.wedgefriction)

body diagrams of Fig. 4.103c we have not fussed over the exact location of the contact forces since the key idea depends on force balance and not moment balance. Neglecting gravity,

$$\begin{array}{lll} \text{For block A, } \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{j} & \Rightarrow & -F_A + F\sin\theta = 0 \\ \text{For block B, } \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{\imath} & \Rightarrow & -F_B + F\cos\theta = 0 \end{array}$$

$$\text{eliminating } F \quad \Rightarrow \quad F_B = \frac{1}{\tan\theta}F_A.$$

To multiply the force F_A by 10 takes a wedge with a taper of $\theta = \tan^{-1} 0.1 \approx 6^o$. With this taper, an ideal wedge could also be viewed as a device to attenuate the force F_B by a factor of 10, although wedges are never used for force attenuation in practice, as we now explain.

A wedge with friction

In the real world frictionless things are hard to find. Nonetheless the concept of a frictionless bearing can be a reasonable idealization because of rollers, grease, and the big lever-arm that the wheel periphery has compared to the axle radius. In the case of wedges, neglecting friction is not generally an accurate model.

Consideration of friction qualitatively changes the behavior of the machine. For simplicity we still take the wall and floor interactions to be frictionless.

Figure 4.104 shows free body diagrams of wedge blocks. We draw separate free body diagrams for the case when (a) block A is sliding down and block B to the right, and (b) block A is sliding up and block B to the right. In both cases the friction resists relative slip and obeys the sliding friction relation

$$F_f = \underbrace{\tan\phi}_{\mu} N$$

where Fig. 4.104 shows the resultant contact force (normal component plus frictional component) and its angle ϕ to the surface normal.

Assuming block A is sliding down we get from free body diagram 4.104a that

$$\begin{array}{lll} \text{For block A, } \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{j} & \Rightarrow & -F_A + F\sin(\theta+\phi) = 0 \\ \text{For block B, } \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{\imath} & \Rightarrow & -F_B + F\cos(\theta+\phi) = 0 \end{array}$$

$$\text{eliminating } F \quad \Rightarrow \quad F_B = \frac{1}{\tan(\theta+\phi)}F_A. \tag{4.57}$$

If we take a taper of 6^o and a friction coefficient of $\mu = .3$ ($\Rightarrow \phi \approx 17^o$) we get that $F_B/F_A \approx 2.5$ instead of 10 as we got when neglecting friction. The wedge still serves as a way to multiply force, but substantially less so than the frictionless idealization led us to believe. Now lets consider the case when force F_B is pushing block B to the left, pinching block A, and forcing it up. The only change in the calculation is the change in the direction of the friction interaction force. From free body diagram 4.104b

$$\begin{array}{lll} \text{For block A, } \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{j} & \Rightarrow & -F_A + F\sin(\theta-\phi) = 0 \\ \text{For block B, } \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{\imath} & \Rightarrow & -F_B + F\cos(\theta-\phi) = 0 \end{array}$$

$$\text{eliminating } F \quad \Rightarrow \quad F_A = \tan(\theta-\phi)F_B. \tag{4.58}$$

Again using $\theta = 6^o$ and $\phi = 17^o$ we see that if $F_B = 100\,\text{lbf}$ that $F_A = \tan(-11^o)\cdot 100\,\text{lbf} \approx -20\,\text{lbf}$. That is, the 100 pounds doesn't push block A up at all, but even with no gravity you need to pull up with a 20 pound force to get it to move. If we

insist that the downwards force F_A is positive or zero, that the pushing force F_B is positive, and that block A is sliding up then there is no solution to the equilibrium equations whenever $\phi > \theta$. (Actually we didn't need to do this second calculation at all. Eqn 4.57 shows the same paradox when $\theta + \phi > 90^o$. Trying to squeeze block B to the right for large θ is exactly like trying to squeeze block A up for small θ.)

This *self locking* situation is intuitive. In fact it's hard to picture the contrary, that pushing a block like B would lift block A. If you view this wedge mechanism as a transmission, it is said to be *non-backdrivable* whenever $\phi > \theta$. That is, pushing down on A can 'drive' block B to the right, but pushing to the left on block B cannot push block A 'back' up. Non-backdrivability is a feature or a defect depending on context.

The borderline case of backdrivability is when $\theta = \phi$ and $F_B = F_A / \tan 2\theta$. Assuming θ is a fairly small angle we get

$$F_B = \frac{F_A}{\tan 2\theta} \approx \frac{F_A}{2\theta} \approx \frac{1}{2}\frac{F_A}{\tan\theta} \approx \frac{1}{2} \cdot (\text{the value of } F_B \text{ had there been no friction}).$$

Thus the design guideline: non-back-drivable transmissions are generally 50% or less efficient, they transmit 50% or less of the force they would transmit if they were frictionless.

To use a wedge in this backwards way requires very low friction. A rare case where a narrow wedge is back drivable is with fresh wet watermelon seed squeezed between two pinched fingers.

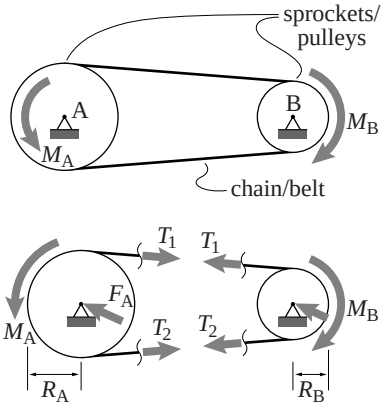


Figure 4.105: a) A chain or pulley drive involving two sprockets or pulleys and one chain or belt, b) free body diagrams of each of the sprockets/pulleys.

(Filename:tfigure.chainpulley)

Pulley and chain drives

Chain and pulley drives are kind of like spread out gears (Fig. 4.105). The rotation of two shafts is coupled not by the contact of gear teeth but by a belt around a pulley or a chain around a sprocket. For simple analysis one draws free body diagrams for each sprocket or pulley with a little bit of chain as in Fig. 4.105b. Note that $T_1 \neq T_2$, unlike the case of an ideal undriven pulley. Applying moment balance we find,

$$\text{For gear A, } \left\{\sum \vec{\boldsymbol{M}}_{i/A} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{k}} \quad \Rightarrow \quad -R_A(T_2 - T_1) + M_A = 0$$
$$\text{For gear B, } \left\{\sum \vec{\boldsymbol{M}}_{i/B} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{k}} \quad \Rightarrow \quad R_B(T_1 - T_2) - M_B = 0$$

$$\text{eliminating } (T_2 - T_1) \quad \Rightarrow \quad M_B = \frac{R_B}{R_A} M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B} M_B$$

exactly as for a pair of gears. Note that we cannot find T_2 or T_1 but only their difference. Typically in design if, say, M_A is positive, one would try to keep T_1 as small as possible without the belt slipping or the chain jumping teeth. If T_1 grows then so must T_2, to preserve their difference. This increase in tension increases the loads on the bearings as well as the chain or belt itself.

4-bar linkages

Four bar linkages often, confusingly, have 3 bars, the fourth piece is the something bigger. A planar mechanism with four pieces connected in a loop by hinges is a four bar linkage. Four bar linkages are remarkably common. After a single body connected at a hinge (like a gear or lever) a four bar linkage is one of the simplest mechanisms that can move in just one way (have just one degree of freedom).

A reasonable model of seated bicycle pedaling uses a 4-bar linkage (Fig. 4.106a). The whole bicycle frame is one bar, the human thigh is the second, the calf is the third, and the bicycle crank is the fourth. The four hinges are the hip joint, the knee joint, the pedal axle, and the bearing at the bicycle crank axle. A more sophisticated

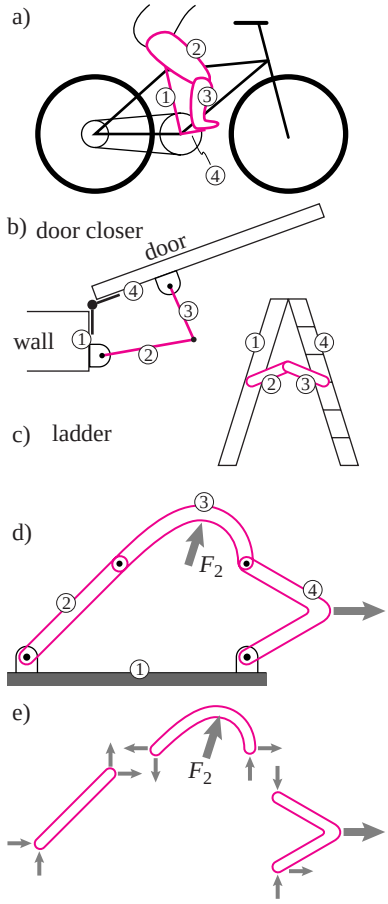


Figure 4.106: Four bar linkages. a) A bicycle, thigh, calf, and crank, b) a door closer, c) a folding ladder, d) a generic mechanism, e) free body diagrams of the parts of a generic mechanism.

(Filename:tfigure.fourbar)

model of the system would include the ankle joint and the foot would make up a fifth bar.

A standard door closing mechanism is part of a 4-bar linkage (Fig. 4.106b). The door jamb and door are two bars and the mechanism pieces make up the other two.

A standard folding ladder design is, until locked open, a 4-bar linkage (Fig. 4.106c).

An abstracted 4-bar linkage with two loads is shown in Fig. 4.106d with free body diagrams in Fig. 4.106e. If one of the applied loads is given, then the other applied load along with interaction and reaction forces make up nine unknown components (after using the principle of action and reaction). With three equilibrium equations for each of the three bars, all these unknowns can be found.

Slider crank

A mechanism closely related to a four bar linkage is a slider crank (Fig. 4.107a). An umbrella is one example (rotated 90^o in Fig. 4.107b). If the sliding part is replaced by a bar, as in Fig. 4.107c, the point C moves in a circle instead of a straight line. If the height h is very large then the arc traversed by C is nearly a straight line so the motion of the four-bar linkage is almost the same as the slider crank. For this reason, slider cranks are sometimes regarded as a special case of a four-bar linkage in the limit as one of the bars gets infinitely long.

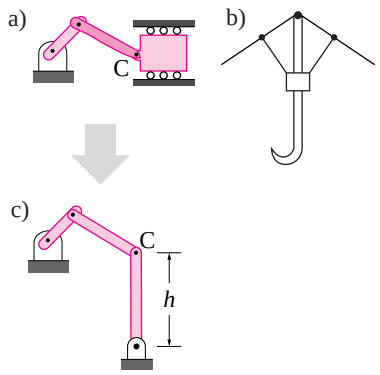


Figure 4.107: a) a slider crank, b) an umbrella has a slider-crank mechanism, c) the equivalent four-bar linkage, at least when $h \to \infty$.

(Filename:tfigure.slidercrank)

Summary of structures and machines

The basic approach to the statics of structures and machines in 2D is straightforward and involves no tricks:

(a) Draw free body diagrams for each of the components.
(b) On the free body diagram use the principle of action and reaction to relate the forces on interacting components.
(c) Write three independent equilibrium equations for each piece (Say, force balance and moment balance, or moment balance about three non-colinear points).
(d) Solve these equations for the desired unknowns.

If you are lazy and resourceful, you can sometimes save work by

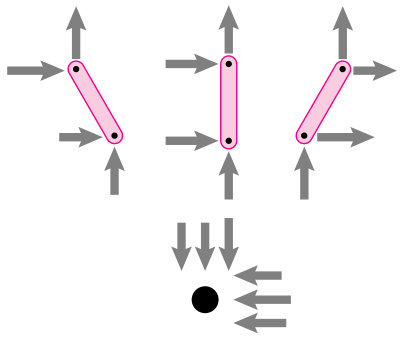
- drawing a free body diagram of the whole structure or some collection of pieces, or
- using appropriate equilibrium equations that avoid variables that you don't know and don't care about.

4.6 The ‘method of bars and pins’ for trusses

A statically determinate truss is a special case of the type of structure discussed in this section. So the methods of this section should work. They do and the resulting method, which is essentially never used in such detail, we will call ‘the method of bars and pins’.

In the method of bars and pins you treat a truss like any other structure. You draw a free body diagram of each bar and of each pin. You use the principle of action and reaction to relate the forces on the different bars and pins. Then you solve the equilibrium equations.

Assuming a frictionless round pin at the hinge, all the bar forces on the pin pass through its center.

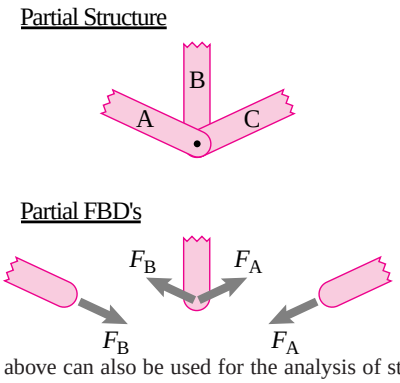


Thus, in 2D, you get two equilibrium equations for each pin and three for each bar. If you apply the three bar equations to a given bar you find that it obeys the two-force body relations. Namely, the reactions on the two bar ends are equal and opposite and along the connecting points. Now application of the pin equilibrium equations is identical to the joint equations we had previously. Thus, the ‘method of bars and pins’ reduces to the method of joints in the end.

Another approach is to ignore the pins and just think of a truss as bars that are connected with forces and no moments. Draw free body diagrams of each piece, use the principle of action and reaction, and write the equilibrium equations for each bar. This is the approach that is used in this section for other structures.

However this approach leads to a difficulty if more than two bars are connected at one hinge. The law of action and reaction is stated for pair-wise interactions not for triples or quadruples. Nonetheless, one can proceed by the following trick.

At each joint where say bars A, B, and C are connected, brake the connection into pair-wise interactions. For example, imagine a frictionless hinge connecting A to B and one connecting B to C but ignore the connection of A with C. That the two connections are spatially coincident is confusing but not a problem. On the free body diagram of A a force will show from B. On the free body diagram of B forces will show from A and C. And on the free body diagram of C a force will show from B. (Beware not to assume that the force from B onto A or C is along B.) The truss is thus analyzable by writing the equilibrium equations for these bars in terms of the unknown interaction forces.



The trick above can also be used for the analysis of structures and machines that have multiple pieces connected at one point. In the machines treated in this section we have avoided the difficulty above by only considering connections between pairs of bodies. This covers many mechanisms and structures but unfortunately does not cover many trusses. For trusses this trickiness can be avoided by use of the method of joints.

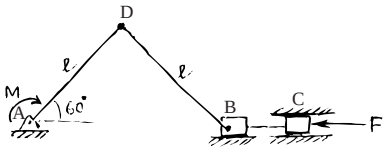


Figure 4.108: (Filename:sfig4.mech.slider)

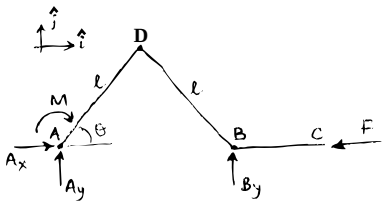


Figure 4.109: (Filename:sfig4.mech.slider.a)

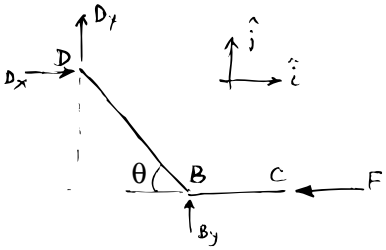


Figure 4.110: (Filename:sfig4.mech.slider.b)

SAMPLE 4.23 *A slider crank:* A torque $M = 20\,\text{N}\cdot\text{m}$ is applied at the bearing end A of the crank AD of length $\ell = 0.2\,\text{m}$. If the mechanism is in static equilibrium in the configuration shown, find the load F on the piston.

Solution The free body diagram of the whole mechanism is shown in Fig. 4.109. From the moment equilibrium about point A, $\sum \vec{\boldsymbol{M}}_{\text{A}} = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned} \vec{\boldsymbol{M}} + \vec{\boldsymbol{r}}_{\text{B/A}} \times (\vec{\boldsymbol{B}} + \vec{\boldsymbol{F}}) &= \vec{\boldsymbol{0}} \\ -M\hat{\boldsymbol{k}} + 2\ell\cos\theta\hat{\boldsymbol{\imath}} \times (B_y\hat{\boldsymbol{\jmath}} - F\hat{\boldsymbol{\imath}}) &= \vec{\boldsymbol{0}} \\ (-M + 2B_y\ell\cos\theta)\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad B_y &= \frac{M}{2\ell\cos\theta} \end{aligned}$$

The force equilibrium, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives

$$\begin{aligned} (A_x - F)\hat{\boldsymbol{\imath}} + (A_y + B_y)\hat{\boldsymbol{\jmath}} &= 0 \\ A_x &= F \\ A_y &= -B_y \end{aligned}$$

Note that we still need to find F or A_x. So far, we have had only three equations in four unknowns (A_x, A_y, B_y, F). To solve for the unknowns, we need one more equation. We now consider the free body diagram of the mechanism without the crank, that is, the connecting rod DB and the piston BC together. See Fig. 4.110. Unfortunately, we introduce two more unknowns (the reactions) at D. However, we do not care about them. Therefore, we can write the moment equilibrim equation about point D, $\sum \vec{\boldsymbol{M}}_{\text{D}} = \vec{\boldsymbol{0}}$ and get the required equation without involving D_x and D_y.

$$\begin{aligned} \vec{\boldsymbol{r}}_{\text{B/D}} \times (-F\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{0}} \\ \ell(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}) \times (-F\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{0}} \\ B_y\ell\cos\theta\hat{\boldsymbol{k}} - F\ell\sin\theta\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \end{aligned}$$

Dotting the last equation with $\hat{\boldsymbol{k}}$ we get

$$\begin{aligned} F &= B_y\frac{\cos\theta}{\sin\theta} \\ &= \frac{M}{2\ell\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} \\ &= \frac{M}{2\ell\sin\theta} \\ &= \frac{20\,\text{N}\cdot\text{m}}{2\cdot 0.2\,\text{m}\cdot\sqrt{3}/2} \\ &= 57.74\,\text{N}. \end{aligned}$$

$$\boxed{F = 57.74\,\text{N}}$$

Note that the force equilibrium carried out above is not really useful since we are not interested in finding the reactions at A. We did it above to show that just one free body diagram of the whole mechanism was not sufficient to find F. On the other hand, writing moment equations about A for the whole mechanism and about D for the connecting rod plus the piston is enough to determine F.

SAMPLE 4.24 *There is more to it than meets the eye!* A flyball governor is shown in the figure with all relevant masses and dimensions. The relaxed length of the spring is 0.15 m and its stiffness is 500 N/m.

(a) Find the static equilibrium position of the center collar.
(b) Find the force in the strut AB or CD.
(c) How does the spring force required to hold the collar depend on θ?

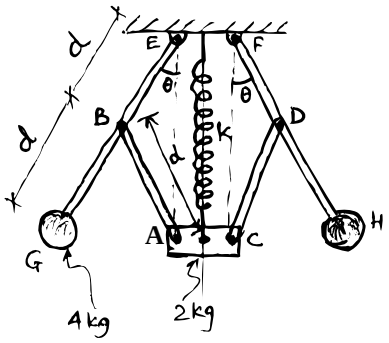


Figure 4.111: (Filename:sfig4.mech.gov)

Solution Let $\ell_0 (= 0.15\,\text{m})$ denote the relaxed length of the spring and let ℓ be the stretched length in the static equilibrium configuration of the flyball, *i.e.*, the collar is at a distance ℓ from the fixed support EF. Then the net stretch in the spring is $\delta \equiv \Delta\ell = \ell - \ell_0$. We need to determine ℓ, the spring force $k\delta$, and its dependence on the angle θ of the ball-arm.

The free body diagram of the collar is shown in Fig. 4.112. Note that the struts AB and CD are two-force bodies (forces act only at the two end points on each strut). Therefore, the force at each end must act along the strut. From geometry (AB = BE = d), then, the strut force F on the collar must act at angle θ from the vertical. Now, the force balance in the vertical direction, *i.e.*, $[\sum \vec{F} = \vec{0}] \cdot \hat{j}$, gives

$$-2F\cos\theta + k\delta = mg \tag{4.59}$$

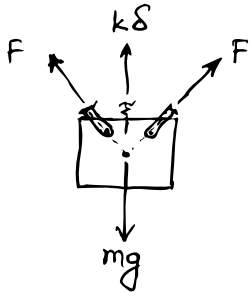


Figure 4.112: (Filename:sfig4.mech.gov.a)

Thus to find δ we need to find F and θ. Now we draw the free body diagram of arm EBG as shown in Fig. 4.113. From the moment balance about point E, we get

$$\begin{aligned}
\vec{r}_{\text{G/E}} \times (-2mg\hat{j}) + \vec{r}_{\text{B/E}} \times \vec{F} &= \vec{0} \\
2d\hat{\boldsymbol{\lambda}} \times (-2mg\hat{j}) + d\hat{\boldsymbol{\lambda}} \times F(-sin\theta\hat{\imath} + \cos\theta\hat{j}) &= \vec{0} \\
-4mgd(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{j}}_{-\sin\theta\hat{k}}) + Fd[-\sin\theta(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{\imath}}_{\cos\theta\hat{k}}) + \cos\theta(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{j}}_{-\sin\theta\hat{k}})] &= \vec{0} \\
4mgd\sin\theta\hat{k} + Fd(-\sin\theta\cos\theta\hat{k} - \cos\theta\sin\theta\hat{k}) &= \vec{0} \\
(4mgd\sin\theta - 2Fd\sin\theta\cos\theta)\hat{k} &= \vec{0}
\end{aligned}$$

Dotting this equation with $\hat{k}$ and assuming that $\theta \neq 0$, we get

$$2F\cos\theta = 4mg \tag{4.60}$$

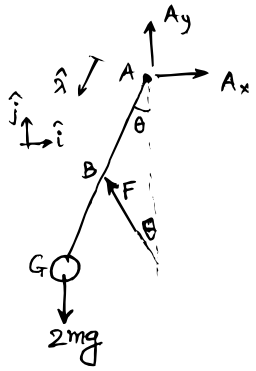


Figure 4.113: (Filename:sfig4.mech.gov.b)

Substituting eqn. (4.60) in eqn. (4.59) we get

$$\begin{aligned}
k\delta &= mg + 2F\cos\theta = mg + 4mg = 5mg \\
\Rightarrow \quad \delta &= \frac{5mg}{k} = \frac{5 \cdot 2\,\text{kg} \cdot 9.81\,\text{m/s}^2}{500\,\text{N/m}} = 0.196\,\text{m}
\end{aligned}$$

(a) The equilibrium configuration is specified by the stretched length ℓ of the spring (which specifies θ). Thus,

$$\ell = \ell_0 + \delta = 0.15\,\text{m} + 0.196\,\text{m} = 0.346\,\text{m}$$

Now, from $\ell = 2d\cos\theta$, we find that $\theta = 30.12^{\text{o}}$.

(b) The force in strut AB (or CD) is

$$F = 2mg/\cos\theta = 45.36\,\text{N}$$

(c) The force in the spring $k\delta = 5mg$ as shown above and thus, it does not depend on θ! In fact, the angle θ is determined by the relaxed length of the spring.

$$\boxed{\text{(a) } \ell = 0.346\,\text{m}, \quad \text{(b) } F = 45.36\,\text{N}, \quad \text{(c) } k\delta \neq f(\theta)}$$

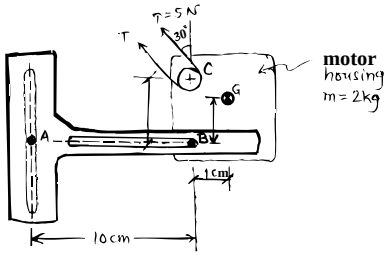


Figure 4.114: (Filename:sfig4.mech.motor)

SAMPLE 4.25 *: A motor housing support:* A slotted arm mechanism is used to support a motor housing that has a belt drive as shown in the figure. The motor housing is bolted to the arm at B and the arm is bolted to a solid support at A. The two bolts are tightened enough to be modeled as welded joints (*i.e.*, they can also take some torque). Find the support reactions at A.

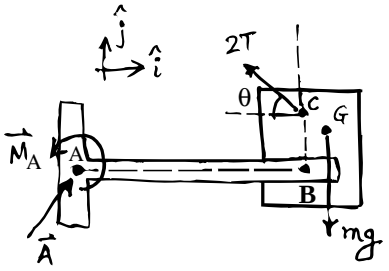


Figure 4.115: (Filename:sfig4.mech.motor.a)

Solution Although the mechanism looks complicated, the problem is straightforward. We cut the bolt at A and draw the free body diagram of the motor housing plus the slotted arm. Since the bolt, modeled as a welded joint, can take some torque, the unknowns at A are $\vec{A}(= A_x\hat{\imath} + A_y\hat{\jmath})$ and $\vec{M}_{\rm A}$. The free body diagram is shown in Fig. 4.115. Note that we have replaced the tension at the two belt ends by a single equivalent tension $2T$ acting at the center of the axle. Now taking moments about point A, we get

$$\vec{M}_{\rm A} + \vec{r}_{\rm C/A} \times 2\vec{T} + \vec{r}_{\rm G/A} \times m\vec{g} = \vec{0}$$

where

$$\begin{aligned}
\vec{r}_{\rm C/A} \times 2\vec{T} &= (\ell\hat{\imath} + h\hat{\jmath}) \times 2T(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
&= 2T(\ell\sin\theta + h\cos\theta)\hat{k} \\
\vec{r}_{\rm G/A} \times m\vec{g} &= [(\ell + d)\hat{\imath} + (\text{anything})\hat{\jmath}] \times (-mg\hat{\jmath}) \\
&= -mg(\ell + d)\hat{k}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{M}_{\rm A} &= -\vec{r}_{\rm C/A} \times 2\vec{T} - \vec{r}_{\rm G/A} \times m\vec{g} \\
&= -2T(\ell\sin\theta + h\cos\theta)\hat{k} + mg(\ell + d)\hat{k} \\
&= -2(5\,\text{N})(0.1\,\text{m}\cdot\sin 60^{\rm o} + 0.04\,\text{m}\cdot\cos 60^{\rm o})\hat{k} \\
&\quad + 2\,\text{kg}\cdot 9.81\,\text{m/s}^2\cdot(0.1 + 0.01)\,\text{m}\hat{k} \\
&= 1.092\,\text{N}\cdot\text{m}\hat{k}
\end{aligned}$$

The reaction force $\vec{A}$ can be determined from the force balance, $\sum \vec{F} = \vec{0}$ as follows.

$$\begin{aligned}
\vec{A} + 2\vec{T} + m\vec{g} &= \vec{0} \\
\Rightarrow \quad \vec{A} &= -2\vec{T} - m\vec{g} \\
&= -10\,\text{N}(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}) - (-19.62\,\text{N}\hat{\jmath}) \\
&= 5\,\text{N}\hat{\imath} + 10.96\,\text{N}\hat{\jmath}
\end{aligned}$$

$$\boxed{\vec{M}_{\rm A} = 1.092\,\text{N}\cdot\text{m}\hat{k} \text{ and } \vec{A} = 5\,\text{N}\hat{\imath} + 10.96\,\text{N}\hat{\jmath}}$$

SAMPLE 4.26 *A gear train:* In the compound gear train shown in the figure, the various gear radii are: $R_A = 10\,\text{cm}$, $R_B = 4\,\text{cm}$, $R_C = 8\,\text{cm}$ and $R_D = 5\,\text{cm}$. The input load $F_i = 50\,\text{N}$. Assuming the gears to be in static equilibrium find the machine load F_o.

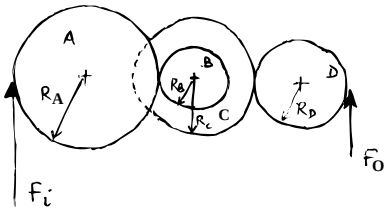


Figure 4.116: (Filename:sfig4.mech.gear)

Solution You may be tempted to think that a free body diagram of the entire gear train will do since we only need to find F_o. However, it is not so because there are unknown reactions at the axle of each gear and, therefore, there are too many unknowns. On the other hand, we can find the load F_o easily if we go gear by gear from the left to the right.

The free body diagram of gear A is shown in Fig. 4.117. Let F_1 be the force at the contact tooth of gear A that meshes with gear B. From the moment balance about the axle-center O, $\sum \vec{M}_\text{O} = \vec{0}$, we have

$$\begin{aligned} \vec{r}_\text{M} \times \vec{F}_\text{i} + \vec{r}_\text{N} \times \vec{F}_1 &= \vec{0} \\ -F_i R_A \hat{k} + F_1 R_A \hat{k} &= \vec{0} \\ \Rightarrow \quad F_1 &= F_i \end{aligned}$$

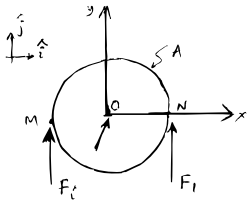


Figure 4.117: (Filename:sfig4.mech.gear.a)

Similarly, from the free body diagram of gear B and C (together) we can write the moment balance equation about the axle-center P as

$$\begin{aligned} F_1 R_B \hat{k} + F_2 R_C \hat{k} &= \vec{0} \\ \Rightarrow \quad F_2 &= \frac{R_B}{R_C} F_1 \\ &= \frac{R_B}{R_C} F_i \end{aligned}$$

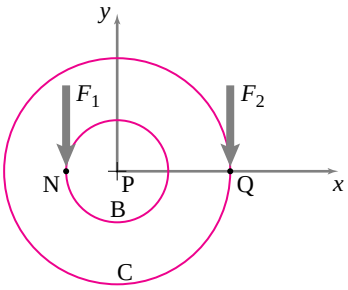


Figure 4.118: (Filename:sfig4.mech.gear.b)

Finally, from the free body diagram of the last gear D and the moment equilibrium about its center R, we get

$$\begin{aligned} -F_2 R_D \hat{k} + F_o R_D \hat{k} &= \vec{0} \\ \Rightarrow \quad F_o &= F_2 \\ &= \frac{R_B}{R_C} F_i \\ &= \frac{4\,\text{cm}}{8\,\text{cm}} \cdot 50\,\text{N} = 25\,\text{N} \end{aligned}$$

$$\boxed{F_o = 25\,\text{N}}$$

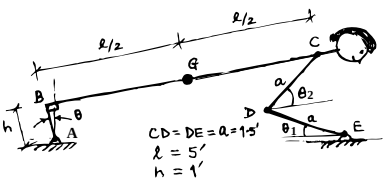


Figure 4.119: (Filename:sfig4.mech.pushup)

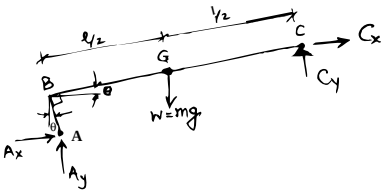


Figure 4.120: (Filename:sfig4.mech.pushup.a)

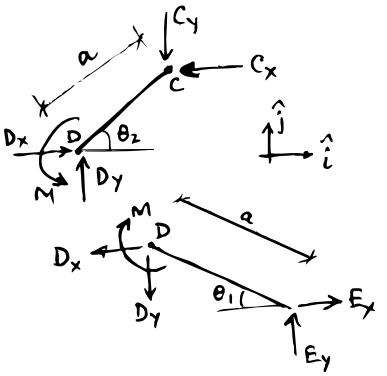


Figure 4.121: (Filename:sfig4.mech.pushup.b)

SAMPLE 4.27 *Push-up mechanics:* During push-ups the body, including the legs, usually moves as a single rigid unit; the ankle is almost locked, and the push-up is powered by the shoulder and the elbow muscles. A simple model of the body during push-ups is a four-bar linkage ABCDE shown in the figure. In this model, each link is a rigid rod, joint B is rigid (thus ABC can be taken as a single rigid rod), joints C, D, and E are hinges, but there is a motor at D that can supply torque. The weight of the person, $W = 150\,\text{lbf}$, acts through G. Find the torque at D for $\theta_1 = 30^o$ and $\theta_2 = 45^o$.

Solution The free body diagram of part ABC of the mechanism is shown in Fig. 4.120. Writing moment balance equation about point A, $\sum \vec{M}_A = \vec{0}$, we get

$$\vec{r}_C \times \vec{C} + \vec{r}_G \times \vec{W} = \vec{0}$$

Let $\vec{r}_C = r_{C_x}\hat{\imath} + r_{C_y}\hat{\jmath}$ and $\vec{r}_G = r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}$ for now (we can figure it out later). Then, the moment equation becomes

$$\begin{aligned}(r_{C_x}\hat{\imath} + r_{C_y}\hat{\jmath}) \times (C_x\hat{\imath} + C_y\hat{\jmath}) + (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) \times (-W\hat{\jmath}) &= \vec{0} \\ [(C_y r_{C_x} - C_x r_{C_y})\hat{k} - W r_{G_x}\hat{k} &= \vec{0}] \\ [\,] \cdot \hat{k} \quad \Rightarrow \qquad C_y r_{C_x} - C_x r_{C_y} &= W r_{G_x} \end{aligned} \tag{4.61}$$

We now draw free body diagrams of the links CD and DE seperately (Fig. 4.121) and write the moment and force balance equations for them.

For link CD, the force equilibrium $\sum \vec{F} = \vec{0}$ gives

$$(-C_x + D_x)\hat{\imath} + (D_y - C_y)\hat{\jmath} = \vec{0}$$

Dotting with $\hat{\imath}$ and $\hat{\jmath}$ gives

$$\begin{aligned} D_x &= C_x \\ D_y &= C_y \end{aligned} \tag{4.62}$$

and the moment equilibrium about point D, gives

$$\begin{aligned} M\hat{k} - a(\cos\theta_2\hat{\imath} + \sin\theta_2\hat{\jmath}) \times (-C_x\hat{\imath} - C_y\hat{\jmath}) &= \vec{0} \\ M\hat{k} + (C_y a\cos\theta_2 - C_x a\sin\theta_2)\hat{k} &= \vec{0} \end{aligned} \tag{4.63}$$

Similarly, the force equilibrium for link DE requires that

$$\begin{aligned} E_x &= D_x \\ E_y &= D_y \end{aligned} \tag{4.64}$$

and the moment equilibrium of link DE about point E gives

$$-M + D_x a\sin\theta_1 + D_y a\cos\theta_1 = 0. \tag{4.65}$$

Now, from eqns. (4.62) and (4.65)

$$-M + C_x a\sin\theta_1 + C_y a\cos\theta_1 = 0 \tag{4.66}$$

Adding eqns. (4.63) and (4.66) and solving for C_x we get

$$C_x = \frac{\cos\theta_1 + \cos\theta_1}{\sin\theta_2 - \sin\theta_1} C_y$$

For simplicity, let

$$f(\theta_1, \theta_2) = \frac{\cos\theta_1 + \cos\theta_1}{\sin\theta_2 - \sin\theta_1}$$

so that

$$C_x = f(\theta_1, \theta_2) C_y \tag{4.67}$$

Now substituting eqn. (4.67) in (4.61) we get

$$C_y = \frac{r_{G_x}}{r_{C_x} - r_{C_y} f} W$$

Now substituting C_y and C_x into eqn. (4.66) we get

$$M = \frac{r_{G_x} a(\cos\theta_1 + f \sin\theta_1)}{r_{C_x} - r_{C_y} f} W$$

where

$$\begin{aligned} r_{G_x} &= (\ell/2)\cos\theta - h\sin\theta \\ r_{C_x} &= \ell\cos\theta - h\sin\theta \\ r_{C_y} &= \ell\sin\theta + h\cos\theta \end{aligned}$$

Now plugging all the given values: $W = 160\,\text{lbf}$, $\theta_1 = 30^\text{o}$, $\theta_2 = 45^\text{o}$, $\ell = 5\,\text{ft}$, $h = 1\,\text{ft}$, $a = 1.5\,\text{ft}$, and, from simple geomtery, $\theta = 9.49^\text{o}$,

$$\begin{aligned} f &= 7.60 \\ r_{C_x} &= 4.77\,\text{ft}, \quad r_{C_y} = 1.81\,\text{ft}, \quad r_{G_x} = 2.30\,\text{ft} \\ \Rightarrow \quad M &= -269.12\,\text{lb·ft} \end{aligned}$$

$$\boxed{M = -269.12\,\text{lb·ft}}$$

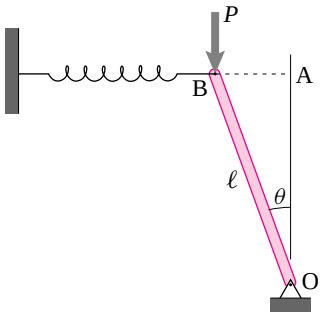


Figure 4.122: (Filename:sfig4.mech.buckling)

SAMPLE 4.28 *A spring and rod buckling model:* A simple model of sideways buckling of a rod can be constructed with a spring and a rod as shown in the figure. Assume the rod to be in static equilibrium at some angle θ from the vertical. Find the angle θ for a given vertical load P, spring stiffness k, and bar length ℓ. Assume that the spring is relaxed when the rod is vertical.

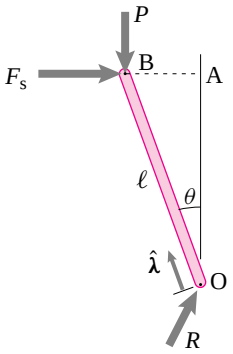


Figure 4.123: (Filename:sfig4.mech.buckling.a)

Solution When the rod is displaced from its vertical position, the spring gets compressed or stretched depending on which side the rod tilts. The spring then exerts a force on the rod in the opposite direction of the tilt. The free body diagram of the rod with a counterclockwise tilt θ is shown in Fig. 4.123. From the moment balance $\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \vec{\mathbf{0}}$ (about the bottom support point O of the rod), we have

$$\vec{\boldsymbol{r}}_{\mathrm{B}} \times \vec{\boldsymbol{P}} + \vec{\boldsymbol{r}}_{\mathrm{B}} \times \vec{\boldsymbol{F}}_{\mathrm{s}} = \vec{\mathbf{0}}$$

Noting that

$$\begin{aligned} \vec{\boldsymbol{r}}_{\mathrm{B}} &= \ell\hat{\boldsymbol{\lambda}}, \\ \vec{\boldsymbol{P}} &= -P\hat{\boldsymbol{\jmath}}, \\ \text{and } \vec{\boldsymbol{F}}_{\mathrm{s}} &= k(\vec{\boldsymbol{r}}_{\mathrm{A}} - \vec{\boldsymbol{r}}_{\mathrm{B}}) \\ &= k(\ell\hat{\boldsymbol{\jmath}} - \ell\hat{\boldsymbol{\lambda}}), \end{aligned}$$

we get

$$\begin{aligned} \ell\hat{\boldsymbol{\lambda}} \times (P\hat{\boldsymbol{\jmath}}) + \ell\hat{\boldsymbol{\lambda}} \times k\ell(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\lambda}}) &= \vec{\mathbf{0}} \\ -P\ell(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) + k\ell^2(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) &= \vec{\mathbf{0}} \end{aligned}$$

Dotting this equation with $(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}})$ we get

$$\begin{aligned} -P\ell + k\ell^2 &= 0 \\ \Rightarrow \quad P &= k\ell. \end{aligned}$$

Thus the equilibrium only requires that P be equal to $k\ell$ and it is independent of θ! That is, the system will be in static equilibrium at any θ as long as $P = k\ell$.

If $P = k\ell$, any θ is an equilibrium position.

4.7 Hydrostatics

Hydrostatics is primarily concerned with finding the net force and moment of still water on a surface. The surfaces are typically the sides of a pool, dam, container, or pipe, or the outer surfaces of a floating object such as a boat or of a submerged object like a toilet bowl float, or the imagined surface that separates some water of interest from the other water. Although the hydrostatics of air helps explain the floating of hot air balloons, dirigibles, and chimney smoke; and the hydrostatics of oil is important for hydraulics (hydraulic breaks for example), often the fluid of concern for engineers is water, and we will use the word 'water' as an informal synonym for 'fluid.'

Besides the basic laws of mechanics that you already know, elementary hydrostatics is based on the following two constitutive assumptions:

1) The force of water on a surface is perpendicular to the surface; and
2) The density of water, ρ (pronounced 'row') is a constant (doesn't vary with depth or pressure),

The first assumption, that all static water forces are perpendicular to surfaces on which they act, can be restated: still water cannot carry any shear stress. For near-still water this constitutive assumption is abnormally good (in the world of constitutive assumptions), approximately as good as the laws of mechanics.

The assumption of constant density is called *incompressibility* because it corresponds to the idea that water does not change its volume (compress) much under pressure. This assumption is reasonable for most purposes. At the bottom of the deepest oceans, for example, the extreme pressure (about 800 atmospheres) only causes water to increase its density about 4% from that of water at the surface. That water density does depend measurably on salinity and temperature is, however, important for some hydrostatic calculations, in particular for determining which water floats on which other water. Sometimes instead of talking about the mass per unit of volume ρ we will use the weight per unit volume $\gamma = g\rho$ ('gammuh = gee row').

Surface area A, outward normal $\hat{n}$, pressure p, and force $\vec{F}$

We are going to be generalizing the high-school physics fact

$$\text{force} = \text{pressure} \times \text{area}$$

to take account that force is a vector, that pressure varies with position, and that not all surfaces are flat. So we need a clear notation and sign convention. The area of a surface is A which we can think of as being the sum of the bits of area ΔA that compose it:

$$A = \int dA.$$

Every bit of surface area has an *outer* normal $\hat{\boldsymbol{n}}$ that points from the surface out into the fluid. The (scalar) force per unit area on the surface is called the pressure p, so that the force on a small bit of surface is

$$\Delta\vec{\boldsymbol{F}} = p\,(\Delta A)\,(-\hat{\boldsymbol{n}}),$$

pointing into the surface, assuming positive pressure, and with magnitude proportional to both pressure and area. Thus the total force and moment due to pressure forces on a surface :

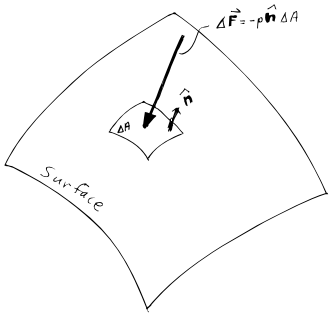


Figure 4.124: A bit of area ΔA on a surface on which pressure p acts. The outward (into the water) normal of the surface is $\hat{\boldsymbol{n}}$ so the increment of force is $\Delta\vec{\boldsymbol{F}} = -p\hat{\boldsymbol{n}}\Delta A$.
(Filename:tfigure.deltaA)

$$\begin{aligned} \vec{F} &= \int d\vec{F} &&= -\int_A p\,\hat{n}\; dA \\ \vec{M}_C &= \int_A d\vec{M}_{/C} &&= -\int_A \vec{r}_{/C} \times (p\,\hat{n})\; dA \end{aligned} \tag{4.68}$$

Hydrostatics is the evaluation of the (intimidating-at-first-glance) integrals 4.68 and their role in equilibrium equations. In the rest of this section we consider a variety of important special cases.

Water in equilibrium with itself

Before we worry about how water pushes on other things, lets first understand what it means for water to be in static equilibrium. These first important facts about hydrostatics follow from drawing free body diagrams of various chunks of water and assuming static equilibrium.

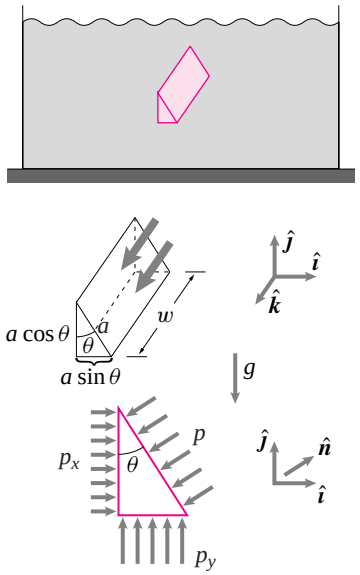


Figure 4.125: A small prism of water is isolated from some water in equilibrium. The free body diagram does not show the forces in the z direction.

(Filename:tfigure.waterprism)

Pressure doesn't depend on direction

We assume that the pressure p does not vary in wild ways from point to point, thus if we look at a small enough region we can think of the pressure as constant in that region. Now if we draw a free body diagram of a little triangular prism of water the net forces on the prism must add to zero (see Fig. 4.125). For each surface the magnitude of the force is the pressure times the area of the surface and the direction is minus the outward normal of the surface. We assume, for the time being, that the pressure is different on the differently oriented surfaces. So, for example, because the area of the left surface is $a\cos\theta w$ and the pressure on the surface is p_x, the net force is $a\cos\theta w p_x \hat{i}$. Calculating similarly for the other surfaces:

$$\begin{aligned} \vec{0} &= \sum \vec{F}_i \\ &= \underbrace{(a\cos\theta)w\; p_x\,\hat{i} + (a\sin\theta)w\; p_y\,\hat{j} - a\;w\;p\,\hat{n}}_{\text{pressure terms}} - \underbrace{\frac{a^2\cos\theta\sin\theta w}{2}\rho g\,\hat{j}}_{\text{weight}} \\ &= aw\left(\cos\theta\; p_x\,\hat{i} + \sin\theta\; p_y\,\hat{j} - p\,(\underbrace{\cos\theta\hat{i} + \sin\theta\,\hat{j}}_{\hat{n}}) - \tfrac{a\cos\theta\sin\theta}{2}\rho g\,\hat{j}\right) \end{aligned}$$

If a is arbitrarily small, the weight term drops out compared to the pressure terms. Dividing through by aw we get

$$\vec{0} = \cos\theta p_x\hat{i} + \sin\theta\; p_y\,\hat{j} - p\,(\cos\theta\hat{i} + \sin\theta)\,\hat{j}.$$

Taking the dot product of both sides of this equation with $\hat{i}$ and $\hat{j}$ gives that $p = p_x = p_y$. Since θ could be anything, force balance for the free body diagram of a small prism tells us that for a fluid in static equilibrium①

pressure is the same in every direction.

① That pressure has to be the same in any pair of directions could also be found by drawing a prism with a cross section which is an isosceles triangle. The prism is oriented so that two surfaces of the prism have equal area and have the desired orientations. Force balance along the base of the triangle gives that the pressures on the equal area surfaces are equal. The argument that pressure must not depend on direction is also sometimes based on equilibrium of a small tetrahedron.

Pressure doesn't vary with side to side position

Consider the equilibrium of a horizontally aligned box of water cut out of a bigger body of water (Fig. 4.126a). The forces on the end caps at A and B are the only forces along the box. Therefor they must cancel. Since the areas at the two ends are the same, the pressure must be also. This box could be anywhere and at any length and any horizontal orientation. Thus for a fluid in static equilibrium

pressure doesn't depend on horizontal position.

If we take the $\hat{\boldsymbol{j}}$ or y direction to be up, then we have

$$p(x, y, z) = p(y).$$

Pressure increases linearly with depth

Consider the vertically aligned box of Fig. 4.126b.

$$\begin{aligned} \left\{\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{j}} \quad &\Rightarrow \quad \underbrace{p(y)a^2 - p(y+h)a^2}_{\text{pressure terms}} - \underbrace{\rho g a^2 h}_{\text{weight}} \quad = \quad 0 \\ &\Rightarrow \qquad\qquad p_{\text{bottom}} - p_{\text{top}} \quad = \quad \rho g h. \end{aligned}$$

So the pressure increases linearly with depth. If the top of a lake, say, is at atmospheric pressure p_a then we have that

$$p = p_a + \rho g h = p_a + \gamma h = p_a + (H - y)\gamma$$

where h is the distance down from the surface, H is the depth to some reference point underwater and y is the distance up from that reference point (so that $h = H - y$). Neglecting atmospheric pressure at the top surface we have the useful and easy to remember formula:

$$p = \gamma h. \tag{4.69}$$

Because the pressure at equal depths must be equal and because the pressure at the top surface must be equal to atmospheric pressure, the top surface must be flat and level. Thus waves and the like are a definite sign of static disequilibrium as are any bumps on the water surface even if they don't seem to move (as for a bump in the water where a stream goes steadily over a rock).

The buoyant force of water on water.

In a place under water in a still swimming pool where there is nothing but water, imagine a chunk of water the shape of a sea monster. Now draw a free body diagram of that water. Because your sea monster is in equilibrium, force balance and moment balance must apply. The only forces are the complicated distribution of pressure forces and the weight of water. The pressure forces must exactly cancel the weight of the water and, to satisfy moment balance, must pass through the center of mass of the water monster. So, in static equilibrium:

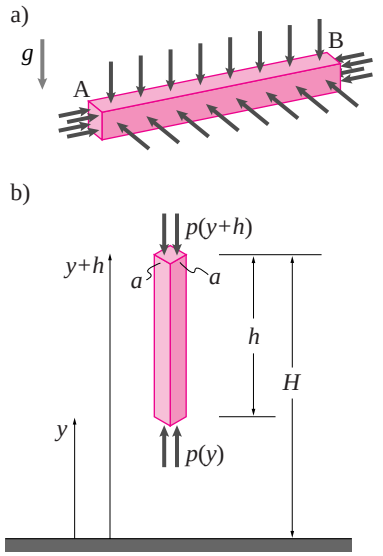


Figure 4.126: Free body diagram of a horizontally aligned box of water cut out of a bigger body of water.

(Filename:tfigure.waterbox)

The pressure forces acting on a surface enclosing a volume of water is equivalent to the negative weight passing through the center of mass of the water.

The force of water on submerged and floating objects

The net pressure force and moment on a still object surrounded by still water can be found by a clever argument credited to Archimedes. The pressure at any one point on the outside of the object does not depend on what's inside. The pressure is determined by how far the point of interest is below the surface by eqn. 4.69①. So if you can find the resultant force from any object that is the shape of the submerged object, but replacing the submerged object, it tells you what you want to know.

① If there is no column of water from the point up to the surface it is still true that the pressure is γh, as you can figure out by tracking the pressure changes along on a staircase-like path from the surface to that point.

The clever idea is to replace your object with water. In this new system the water is in equilibrium, so the pressure forces exactly balance the weight. We thus obtain Archimedes' Principle:

The resultant of all pressure forces on a a totally submerged object is minus the weight of the displaced water. The resultant acts at the centroid of the displaced volume:

$$\vec{\boldsymbol{F}}_{\text{bouyancy}} = \gamma V \hat{\boldsymbol{k}} \qquad \text{acting at} \qquad \vec{\boldsymbol{r}} = \frac{\int \vec{\boldsymbol{r}}_{/0}\, dV}{V}.$$

This explanation of Archimedes' principle depends on assuming the displaced volume is in equilibrium. The result can also be found by adding the effects of all the pressure forces on the outside surface (see box 4.7 on page 200).

For floating objects, the same argument can be carried out, but since the replaced fluid has to be in equilibrium we cannot replace the whole object with fluid, but only the part which is below the level of the water surface.

The force of constant pressure on a totally immersed object

When there is no gravity, or gravity is neglected, the pressure in a static fluid is the same everywhere. Exactly the same argument we have just used shows that the resultant of the pressure forces is zero. We could derive this result just by setting $\gamma = 0$ in the formulas above.

The force of constant pressure on a flat surface

The net force of constant pressure on one flat surface (not all the way around a submerged volume) is the pressure times the area acting normal to the surface at the centroid of the surface:

$$\begin{aligned}\vec{F}_{\text{net}} &= \int_A -p\,\hat{\boldsymbol{n}}\,dA \\ &= -pA\hat{\boldsymbol{n}}.\end{aligned}$$

That this force acts at the centroid can be checked by calculating the moment of the pressure forces relative to the centroid C.

$$\begin{aligned}\vec{M}_{/\text{C,net}} &= \int_A \vec{r}_{/\text{C}} \times \left(-p\,\hat{\boldsymbol{n}}\,dA\right) \\ &= \underbrace{\left(\int_A \vec{r}_{/\text{C}} dA\right)}_{0} \times \left(-p\,\hat{\boldsymbol{n}}\right) \\ &= 0\end{aligned}$$

where the zero follows from the position of the center of mass relative to the center of mass being zero.

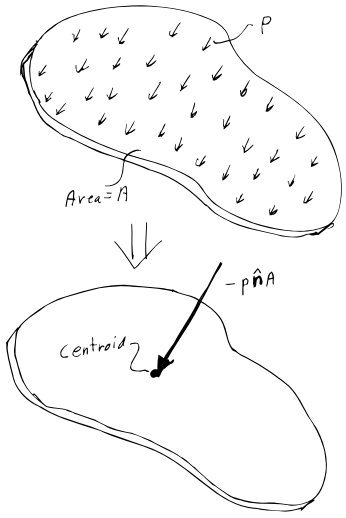


Figure 4.127: The resultant force from a constant pressure p on a flat plate is $\vec{F} = -pA\hat{\boldsymbol{n}}$ acting at the centroid of the plate.

(Filename:tfigure.centroidpressure)

The force of water on a rectangular plate

Consider a rectangular plate with width into the page w and length ℓ. Assume the water-side normal to the plate is $\hat{\boldsymbol{n}}$ and that the top edge of the plate is horizontal. Take $\hat{\boldsymbol{j}}$ to be the up direction with y being distance up from the bottom and the total depth of the water is H. Thus the area of the plate is $A = \ell w$. If the bottom and top of the plate are at y_1 and y_2 the net force on the plate can be found as:

$$\begin{aligned}\vec{F}_{\text{net}} &= -\int_A p\hat{\boldsymbol{n}}\,dA \\ &= -\int_A \gamma(H-y)\hat{\boldsymbol{n}}\,dA \\ &= -w\int_0^\ell \gamma(H-y(s))\hat{\boldsymbol{n}}\,ds \\ &= -w\int_0^\ell \gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{j}}\,s)\right)\hat{\boldsymbol{n}}\,ds \\ &= -w\gamma\left(H\ell - y_1\ell - \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{j}}\,\ell^2/2\right)\hat{\boldsymbol{n}} \\ &= -w\ell\gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{j}}\,\ell/2)\right)\hat{\boldsymbol{n}} \\ &= -w\ell\gamma\left(H-(y_1+(y_2-y_1)/2)\right)\hat{\boldsymbol{n}} \\ &= -w\ell\left(\gamma(H-y_1)/2+\gamma(H-y_2)/2\right)\hat{\boldsymbol{n}} \\ &= -w\ell\tfrac{p_1+p_2}{2}\hat{\boldsymbol{n}}. \\ &= -(\text{area})(\text{average pressure})(\text{outwards normal direction})\end{aligned}$$

The net water force is the same as that of the average pressure acting on the whole surface. To find where it acts it is easiest to think of the pressure distribution as the sum of two different pressure distributions. One is a constant over the plate at the pressure of the top of the plate. The other varies linearly from zero at the top to $\gamma(y_2-y_1)$ at the bottom.

$$p = \gamma(H-y) = \underbrace{\gamma(H-y_2)}_{\text{Constant pressure, the pressure at the top edge.}} + \underbrace{\gamma(y_2-y)}_{\text{Varies linearly from 0 at the top to } \gamma(y_2-y_1) \text{ at the bottom.}}$$

The first corresponds to a force of $w\ell\gamma(H-y_2)$ acting at the middle of the plate. The second corresponds to a force of $w\ell\gamma\frac{y_2-y_1}{2}$ acting a third of the way up from the bottom of the plate.

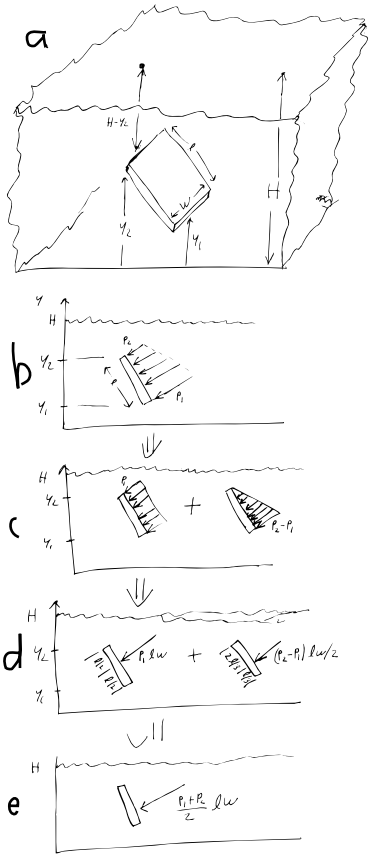


Figure 4.128: The resultant force from a constant depth-increasing pressure on a rectangular plate.

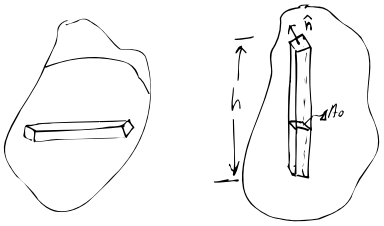
(Filename:tfigure.plateunder)

4.7 *THEORY*
Adding forces to derive Archimedes' principle

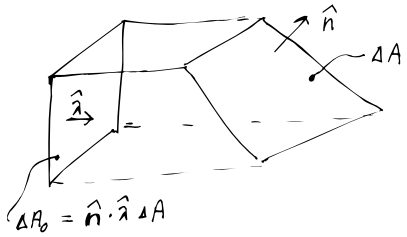
Archimedes' principle follows from adding up all the pressure forces on the outer surfaces of an arbitrarily shaped submerged solid, say something potato shaped.

First we find the answer by cutting the potato into french fries. This approach is effectively a derivation of a theorem in vector calculus. After that, for those who have the appropriate math background, we quote the vector calculus directly.

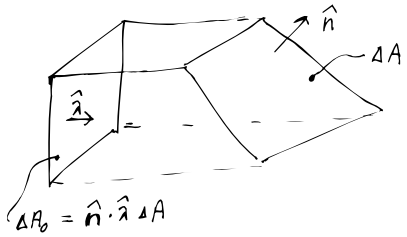
First cut the potato into horizontal french-fries (horizontal prisms) and look at the forces on the end caps (there are no water forces on the sides since those are inside the potato).



The pressure on two ends is the same (because they have the same water depth). The areas on the two ends are probably different because your potato is probably not box shaped. But the area is bigger at one end if the normal to the surface is more oblique compared to the axis of the prism. If the cross sectional area of the prism is ΔA_0 then the area of one of the prism caps is



$\Delta A = \Delta A_0/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})$ where $\hat{\boldsymbol{\lambda}}$ is along the axis of the prism and $\hat{\boldsymbol{n}}$ is the outer unit normal to the end cap (Note $\Delta A \geq \Delta A_0$ because $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}} \leq 1$).

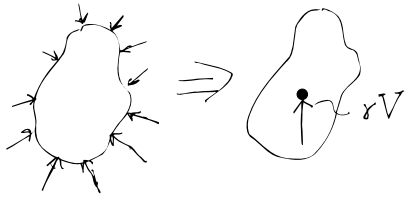


So the net force on the cap is $-p\Delta A_0 \hat{\boldsymbol{n}}/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})$. The component of the force along the prism is $\left[-p\Delta A_0 \hat{\boldsymbol{n}}/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})\right] \cdot \hat{\boldsymbol{\lambda}}$ which is $-p\Delta A_0$. An identical calculation at the other end of the french fry gives minus the same answer. So the net force of the water pressure along the prism is zero for this and every prism and thus the whole potato. Likewise for prisms with any horizontal orientation. Thus the net sideways force of water on any submerged object is zero.

To find the net vertical force on the potato we cut it into vertical french fries. The net forces on the end caps are calculated just as in the above paragraph but taking account that the pressure on the bottom of the french fry is bigger than at the top. The sum of the forces of the top and bottom caps is an upwards force that is

$$\begin{aligned} \text{net upwards force on vertical french fry} &= \Delta p \Delta A_0 \\ &= (\gamma h)\Delta A_0 \\ &= \gamma(h\Delta A_0) \\ &= \gamma\, \Delta V_0 \end{aligned}$$

where ΔV_0 is the volume of the french fry. Adding up over all the french fries that make up the potato one gets that the net upwards force is γV The net result, summarized by the figure below, is that the resultant of the pressure forces on a submerged solid is an upwards force whose magnitude is the weight of the displaced water. The location of the force is the centroid of the displaced volume. (Note that the centroid of the displaced volume is not necessarily at the center of mass of the submerged object.)



A vector calculus derivation

Here is a derivation of Archimedes' principle, at least the net force part, using multivariable integral calculus. Only read on if you have taken a math class that covers the divergence theorem. The net pressure force on a submerged object is

$$\begin{aligned} \vec{\boldsymbol{F}}_{\text{bouyancy}} &= -\int_A p\, \hat{\boldsymbol{n}}\, dA \\ &= -\int_S p\, \hat{\boldsymbol{n}}\, dS \\ &= -\int_S (H-z)\gamma\, \hat{\boldsymbol{n}}\, dS \\ &= -\int_V \vec{\nabla}((H-z)\gamma)\, dV \\ &= -\int_V (-\hat{\boldsymbol{k}})\,\gamma\, dV \\ &= \int_V \gamma\, dV\, \hat{\boldsymbol{k}} \\ &= (\text{weight of displaced water})\, \hat{\boldsymbol{k}}. \end{aligned}$$

In this derivation we first changed from calling bits of surface area dA to dS because that is a common notation in calculus books. The depth from the surface, of a point with vertical component z from the bottom, is $H - z$. The $\vec{\nabla}$ symbol indicates the gradient and its place in this equation is from the divergence theorem:

$$\int_S (\text{any scalar})\hat{\boldsymbol{n}}\, dS = \int_V \vec{\nabla}(\text{the same scalar})\, dV.$$

The gradient of $(H - z)\gamma$ is $-\hat{\boldsymbol{k}}\gamma$ because H and γ are constants. Note, where we write $\int_S$ some books would write $\iint_S$, and where we write $\int_V$ some books would write $\iiint_V$.

SAMPLE 4.29 *The force due to the hydrostatic pressure:* The hydrostatic pressure distribution on the face of a wall submerged in water upto a height $h = 10\,\text{m}$ is shown in the figure. Find the net force on the wall from water. Take the length of the wall (into the page) to be unit.

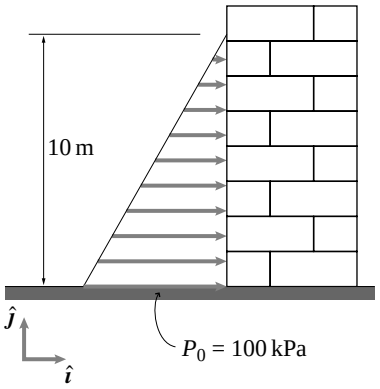
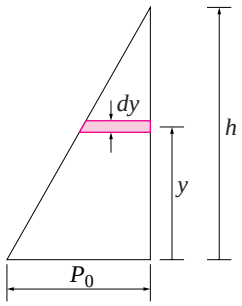


Figure 4.129: (Filename:sfig4.hydro.force1)

Solution Since the pressure varies across the height of the submerged part of the wall, let us take an infinitesimal strip of height dy along the full length ℓ of the wall as shown in Fig. 4.130. Since the height of the strip is infinitesimal, we can treat the water pressure on this strip to be essentially constant and to be equal to $p_0\frac{y}{h}$. Then the force on the string due to the water pressure is

$$\begin{aligned} d\vec{F} &= p(y)\cdot y\cdot \ell\hat{i} \\ &= p_0\frac{y}{h}\,\ell\,dy\hat{i} \end{aligned}$$



The net force due to the pressure distribution on the whole wall can now be found by integrating $d\vec{F}$ along the wall.

$$\begin{aligned} \vec{F} &= \int d\vec{F} \\ &= \int_0^h p_0\frac{y}{h}\,\ell\,dy\hat{i} \\ &= \frac{p_0}{h}\,\ell\int_0^h y\,dy\hat{i} \\ &= \frac{p_0}{h}\,\ell\,\frac{h^2}{2}\hat{i} \\ &= \frac{1}{2}p_0h\ell\hat{i} \\ &= \frac{1}{2}\cdot(100\frac{\text{kN}}{\text{m}^2})\cdot(10\,\text{m})\cdot(1\,\text{m})\hat{i} \\ &= 500\,\text{kN}\hat{i} \end{aligned}$$

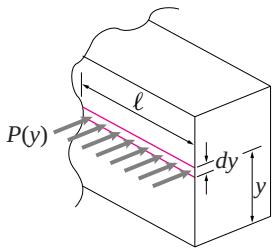


Figure 4.130: (Filename:sfig4.hydro.force1.a)

$$\boxed{\vec{F} = 500\,\text{kN}\hat{i}}$$

Alternatively, the net force can be computed by calculating the area of the pressure triangle and multiplying by the unit length ($\ell = 1\,\text{m}$), *i.e.*,

$$\begin{aligned} \vec{F} &= (\frac{1}{2}\cdot h\cdot p_0\hat{i})\ell \\ &= \frac{1}{2}\cdot 10\,\text{m}\cdot 100\frac{\text{kN}}{\text{m}^2}\cdot 1\,\text{m}\hat{i} \\ &= 500\,\text{kN} \end{aligned}$$

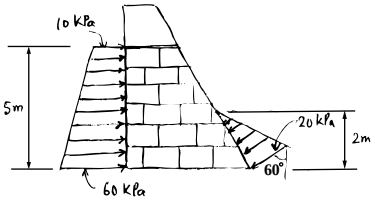


Figure 4.131: (Filename:sfig4.hydro.force2)

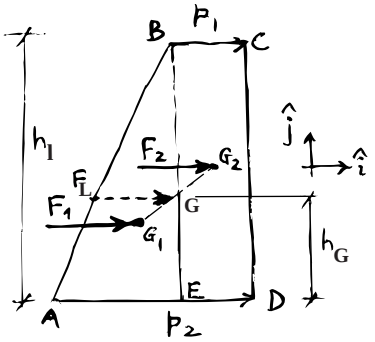


Figure 4.132: (Filename:sfig4.hydro.force2.a)

SAMPLE 4.30 *The equivalent force due to hydrostatic pressure:* Find the net force and its location on each face of the dam due to the pressure distributions shown in the figure. Take unit length of the dam (into the page).

Solution We can determine the net force on each face of the dam by considering the given pressure distribution on one face at a time and finding the net force and its point of action.

On the left face of the dam we are given a trapezoidal pressure distribution. We break the given distribution into two parts — a trigular distribution given by ABE, and a rectangular distribution given by EBCD. We find the net force due to each distribution by finding the area of the distribution and multiplying by the unit length of the dam.

$$\begin{aligned}
\vec{F}_1 &= (\text{area of ABE}) \cdot \ell\hat{\imath} = \frac{1}{2}(p_2 - p_1)h_l\ell\hat{\imath} \\
&= \frac{1}{2}(60\text{kPa} - 10\text{kPa}) \cdot 5\,\text{m} \cdot 1\,\text{m}\hat{\imath} \\
&= 125\,\text{kN}\hat{\imath} \\
\vec{F}_2 &= (\text{area of EBCD}) \cdot \ell\hat{\imath} = p_1 h_l \ell\hat{\imath} \\
&= 10\text{kPa} \cdot 5\,\text{m} \cdot 1\,\text{m}\hat{\imath} \\
&= 50\,\text{kN}\hat{\imath}
\end{aligned}$$

The two forces computed above act through the centroids of the triangle ABE and the rectangle EBCD, respectively. The centroid are marked in Fig. 4.132. Now the net force on the left face is the vector sum of these two forces, *i.e.*,

$$\vec{F}_\text{L} = \vec{F}_1 + \vec{F}_2 = 175\,\text{kN}\hat{\imath}$$

The net force $\vec{F}_\text{L}$ acts through point G which is determined by the moment balance of the two forces $\vec{F}_1$ and $\vec{F}_2$ about point G:

$$\begin{aligned}
\vec{r}_{\text{G}_1/\text{G}} \times \vec{F}_1 &= -\vec{r}_{\text{G}_2/\text{G}} \times \vec{F}_2 \\
F_1(h_G - \frac{h_l}{3})\hat{k} &= -F_2(\frac{h_l}{2} - h_G)(-\hat{k}) \\
\Rightarrow \quad h_G &= \frac{F_1\frac{h_l}{3} + F_2\frac{h_l}{2}}{F_1 + F_2} \\
&= \frac{125\,\text{kN} \cdot 1.667\,\text{m} + 50\,\text{kN} \cdot 2.5\,\text{m}}{175\,\text{kN}} \\
&= 1.905\,\text{m}
\end{aligned}$$

Similarly, we compute the force on the right face of the dam by calculating the area of the tringular distribution shown in Fig. 4.133.

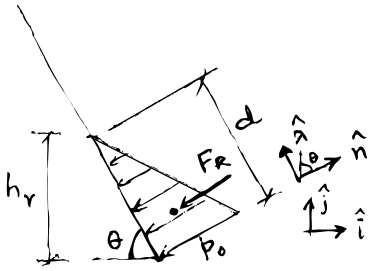


Figure 4.133: (Filename:sfig4.hydro.force2.b)

$$\begin{aligned}
\vec{F}_\text{R} &= \frac{1}{2}p_0(\underbrace{h_r/\sin\theta}_{d})(\underbrace{-\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}}_{-\hat{n}}) \\
&= \frac{1}{2}p_0 h_r(-\hat{\imath} - \tan\theta\hat{\jmath}) \\
&= -20(\hat{\imath} + \sqrt{3}\hat{\jmath})\,\text{kN}
\end{aligned}$$

and this force acts though the centroid of the triangle as shown in Fig. 4.133.

$$\boxed{\vec{F}_\text{L} = 175\,\text{kN}\hat{\imath},\ \text{and}\ \vec{F}_\text{R} = -20(\hat{\imath} + \sqrt{3}\hat{\jmath}\,\text{kN}}$$

SAMPLE 4.31 *Forces on a submerged sluice gate:* A rectangular plate is used as a gate in a tank to prevent water from draining out. The plate is hinged at A and rests on a frictionless surface at B. Assume the width of the plate to be 1 m. The height of the water surface above point A is h. Ignoring the weight of the plate, find the forces on the hinge at A as a function of h. In particular, find the vertical pull on the hinge for $h = 0$ and $h = 2\,\text{m}$.

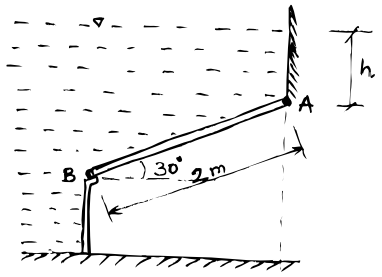


Figure 4.134: (Filename:sfig4.hydro.gate)

Solution Let $\gamma = \rho g$ be the weight density (weight per unit volume) of water. Then the pressure due to water at point A is $p_A = \gamma h$ and at point B is $p_B = \gamma(h + \ell \sin\theta)$. The pressure acts perpendicular to the plate and varies linearly from p_A at A to p_B at B. The free body diagram of the plate is shown in Fig. 4.135. Let $\hat{\boldsymbol{\lambda}}$ be a unit vector along BA and $\hat{\boldsymbol{n}}$ be a unit vector normal to BA. For computing the reaction forces on the plate at points A and B, we first replace the distributed pressure on the plate by two equivalent concentrated forces F_1 and F_2 by dividing the pressure distribution into a rectangular and a triangular region and finding their resultants.

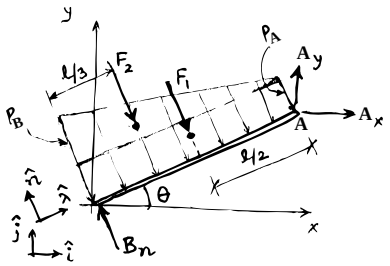


Figure 4.135: (Filename:sfig4.hydro.gate.a)

$$F_1 = p_A \ell = \gamma h \ell, \quad F_2 = (p_B - p_A)\,\frac{\ell}{2} = \frac{1}{2}\gamma \ell^2 \sin\theta$$

Now, we carry out moment balance about point A, $\sum \vec{\boldsymbol{M}}_{\text{A}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{B/A}} \times \vec{\boldsymbol{B}} + \vec{\boldsymbol{r}}_{\text{D/A}} \times \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{r}}_{\text{C/A}} \times \vec{\boldsymbol{F}}_1 &= \vec{\boldsymbol{0}} \\
-\ell\hat{\boldsymbol{\lambda}} \times B_n\hat{\boldsymbol{n}} - \frac{2\ell}{3}\hat{\boldsymbol{\lambda}} \times (-F_1\hat{\boldsymbol{n}}) - \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times (-F_2\hat{\boldsymbol{n}}) &= \vec{\boldsymbol{0}} \\
-B_n\ell\hat{\boldsymbol{k}} + F_1\frac{2\ell}{3}\hat{\boldsymbol{k}} + F_2\frac{\ell}{2}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad B_n = \frac{2F_1}{3} + \frac{F_2}{2} &= \gamma\ell(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta)
\end{aligned}$$

and, from force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned}
\vec{\boldsymbol{A}} &= -B_n\hat{\boldsymbol{n}} + F_1\hat{\boldsymbol{n}} + F_2\hat{\boldsymbol{n}} \\
&= \left(-\gamma\ell(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta) + \gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} \\
&= \left(\frac{1}{3}\gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} = \gamma\ell(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta)\hat{\boldsymbol{n}}
\end{aligned}$$

The force $\vec{\boldsymbol{A}}$ computed above is the force exerted by the hinge at A on the plate. Therefore, the force on the hinge, exerted by the plate, is $-\vec{\boldsymbol{A}}$ as shown in Fig. 4.136. From the expression for this force, we see that it varies linearly with h.

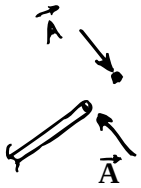


Figure 4.136: (Filename:sfig4.hydro.gate.b)

Let the vertical pull on the hinge be A_{hinge_y}. Then

$$A_{\text{hinge}_y} = -\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{j}} = -\gamma\ell(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta)\overbrace{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{j}}}^{\cos\theta} = \frac{1}{4}\gamma\ell\sin 2\theta + (\frac{1}{3}\gamma\ell\cos\theta)h$$

Now, substituting $\gamma = 9.81\,\text{kN/m}^3$, $\ell = 2\,\text{m}$, $\theta = 30^{\text{o}}$, the two specified values of h, and multiplying the result (which is force per unit length) with the width of the plate (1 m) we get,

$$A_{\text{hinge}_y}(h = 0) = 4.25\,\text{kN}, \quad A_{\text{hinge}_y}(h = 2\,\text{m}) = 15.58\,\text{kN}$$

$$\boxed{A_{\text{hinge}_y}|_{h=0} = 4.25\,\text{kN}, \quad A_{\text{hinge}_y}|_{h=2\,\text{m}} = 15.58\,\text{kN}}$$

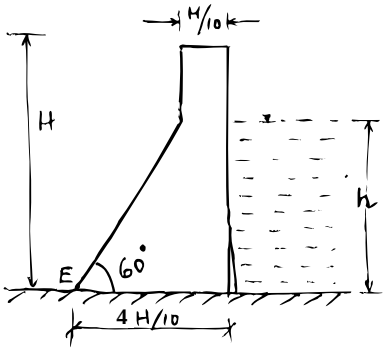


Figure 4.137: (Filename:sfig4.hydro.dam1)

① This assumption is valid only if water does not leak through the edge B to the bottom of the dam. If it does, there would be some force on the bottom due to the water pressure. See the following sample where we include the water pressure at the bottom in the analysis.

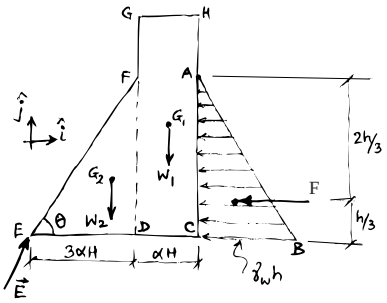


Figure 4.138: (Filename:sfig4.hydro.dam1.a)

SAMPLE 4.32 *Tipping of a dam:* The cross section of a concrete dam is shown in the figure. Take the weight-density $\gamma(=\rho g)$ of water to be $10\,\text{kN/m}^3$ and that of concrete to be $25\,\text{kN/m}^3$. For the given design of the cross-section, find the ratio h/H that is safe enough for the dam to not tip over (about the downstream edge E).

Solution Let us imagine the critical situation when the dam is just about to tip over about edge E. In such a situation, the dam bottom would almost lose contact with the ground except along edge E. In that case, there is no force along the bottom of the dam from the ground except at E.① With this assumption, the free body diagram of the dam is shown in Fig. 4.138.

To compute all the forces acting on the dam, we assume the width w (into the paper) to be unit (*i.e.*, $w = 1\,\text{m}$). Let γ_w and γ_c denote the weight-densities of water and concrete, respectively. Then the resultant force from the water pressure is

$$F = \frac{1}{2}\,\gamma_w h \cdot h \cdot w = \frac{1}{2}\gamma_w h^2 w$$

This is the horizontal force (in the $-\hat{\boldsymbol{\imath}}$ direction) that acts through the centroid of triangle ABC.

To compute the weight of the dam, we divide the cross-section into two sections — the rectangular section CDGH and the triangular section DEF. We compute the weight of these sections separately by computing their respective volumes:

$$\begin{aligned} W_1 &= \underbrace{\alpha H^2 \cdot w}_{\text{volume}} \cdot \gamma_c = \gamma_c \alpha H^2 w \\ W_2 &= \underbrace{\frac{1}{2} \cdot 3\alpha H \cdot 3\alpha H \tan\theta \cdot w}_{\text{volume}} \cdot \gamma_c = \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta \end{aligned}$$

Now we apply moment balance about point E, $\sum \vec{\boldsymbol{M}}_{\text{E}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned} \vec{\boldsymbol{r}}_{\text{G}_1} \times \vec{\boldsymbol{W}}_1 + \vec{\boldsymbol{r}}_{\text{G}_2} \times \vec{\boldsymbol{W}}_2 + \vec{\boldsymbol{r}}_{\text{G}_3} \times \vec{\boldsymbol{F}} &= \vec{\boldsymbol{0}} \\ -(3\alpha H + \frac{1}{2}\alpha H) W_1 \hat{\boldsymbol{k}} - \frac{2}{3}(3\alpha H) W_2 \hat{\boldsymbol{k}} + \frac{h}{3} F \hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \end{aligned}$$

Dotting this equation with $\hat{\boldsymbol{k}}$, we get

$$\begin{aligned} \frac{h}{3}F &= (3\alpha H + \frac{1}{2}\alpha H) \cdot \gamma_c \alpha H^2 w + \frac{2}{3}(3\alpha H) \cdot \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= 9\gamma_c \alpha^3 H^3 \tan\theta + \frac{7}{2}\gamma_c \alpha^2 H^3 \\ \Rightarrow \quad \left(\frac{h}{H}\right)^3 &= \frac{\gamma_c}{\gamma_w}(54\alpha^3 \tan\theta + 21\alpha^2) \\ &= 2.5(54 \cdot 0.1^3 \cdot \sqrt{3} + 21 \cdot 0.1^2) = 0.7588 \\ \Rightarrow \quad \frac{h}{H} &= 0.91 \end{aligned}$$

Thus, for the dam to not tip over, $h \le 0.91H$ or 91% of H.

$$\boxed{\tfrac{h}{H} \le 0.91}$$

SAMPLE 4.33 *Dam design:* You are to design a dam of rectangular cross section ($b \times H$), ensuring that the dam does not tip over even when the water level h reaches the top of the dam ($h = H$). Take the specific weight of concrete to be 3. Consider the following two scenarios for your design.

(a) The downstream bottom edge of the dam is plugged so that there is no leakage underneath.

(b) The downsteram edge is not plugged and the water leaked under the dam bottom has full pressure across the bottom.

Solution Let γ_c and γ_w denote the weight densities of concrete and water, respectively. We are given that $\gamma_c/\gamma_w = 3$. Also, let $b/H = \alpha$ so that $b = \alpha H$. Now we consider the two scenarios and carry out analysis to find appropriate cross-section of the dam. In the calculations below, we consider unit length (into the paper) of the dam.

(a) *No water pressure on the bottom:* When there is no water pressure on the bottom of the dam, then the water pressure acts only on the downstream side of the dam. The free body diagram of the dam, considering critical tipping (just about to tip), is shown in Fig. 4.139 in which F is the resultant force of the triangular water pressure distribution. The known forces acting on the dam are $W = \gamma_c \alpha H^2$, and $F = (1/2)\gamma_w h^2$. The moment balance about point A gives

$$\begin{aligned} F \cdot \frac{h}{3} &= W \cdot \frac{\alpha H}{2} \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= \gamma_c \frac{\alpha^2 H^3}{2} \\ \Rightarrow \quad \alpha^2 &= (1/3)(\gamma_w/\gamma_c)(h/H)^3 \end{aligned}$$

Considering the case of critical water level up to the height of the dam, *i.e.*, $h/H = 1$, and substituting $\gamma_c/\gamma_w = 3$, we get

$$\alpha^2 = 1/9 \quad \Rightarrow \quad \alpha = 1/3 = 0.333$$

Thus the width of the cross-section needs to be at least one-third of the height. For example, if the height of the dam is 9 m then it needs to be at least 3 m wide.

$$\boxed{b/H = 0.33}$$

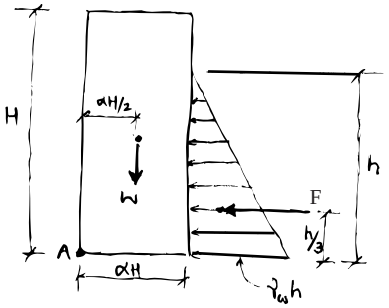


Figure 4.139: (Filename:sfig4.hydro.dam2.a)

(b) *Full water pressure on the bottom:* In this case, the water pressure on the bottom is uniformly distributed and its intensity is the same as the lateral pressure at B, *i.e.*, $p = \gamma_w h$. The free body diagram diagram is shown in Fig. 4.140 where the known forces are $W = \gamma_c \alpha H^2$, $F = (1/2)\gamma_w h^2$, and $R = \gamma_w \alpha h H$. Again, we carry out moment balance about point A to get

$$\begin{aligned} F \cdot \frac{h}{3} &= (W - R) \cdot \frac{\alpha h}{2} \\ \gamma_w h^3 &= 3(\gamma_c \alpha H^2 - \gamma_w \alpha h H)\alpha H \\ \alpha^2 &= \frac{(h/H)^3}{3(\gamma_c/\gamma_w - h/H)} \end{aligned}$$

Once again, substituting the given values and $h/H = 1$, we get

$$\alpha^2 = 1/6 \quad \Rightarrow \quad \alpha = 0.408$$

Thus the width in this case needs to be at least 0.41 times the height H, slightly wider than the previous case.

$$\boxed{b/H \geq 0.41}$$

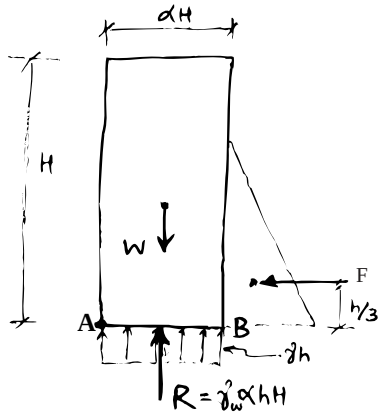


Figure 4.140: (Filename:sfig4.hydro.dam2.b)

4.8 Advanced statics

We now continue our study of statics, but with the goal of developing facility at some harder problems. One way that the material is expanded here is to take the three dimensionality of the world a little more seriously. Each subsection here corresponds to one of the six previous sections, namely, statics of one body, trusses, internal forces, springs, machines and mechanisms, and hydrostatics.

Primarily, the subject of 3D statics is the same as for 2D. However, generally one needs to take more care with vectors when working problems.

Statics of one body in 3D

The consideration of statics of one body in 3D follows the same general principles as for 2D.

- Draw a free body diagram.
- Using the forces and moments shown, write the equilibrium equations
  - force balance (one 3D vector equation, three scalar equations), and
  - moment balance (one 3D vector equation, three scalar equations).

As for the case in 2D when one could use moment balance about 3 non-colinear points and not use force balance at all, in 3D one can use moment balance about 6 sufficiently different axes. If a is a distance, then one such set is, for example: the $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ axis through $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$, the $\hat{\boldsymbol{\jmath}}$ axis through $\vec{\boldsymbol{r}} = a\hat{\boldsymbol{\imath}}$, the $\hat{\boldsymbol{k}}$ axis through $\vec{\boldsymbol{r}} = a\hat{\boldsymbol{\jmath}}$, and the $\hat{\boldsymbol{\imath}}$ axis through $\vec{\boldsymbol{r}} = a\hat{\boldsymbol{k}}$. Other combinations of force balance and moment balance are also sufficient. One can test the sufficiency of the equations by seeing if they imply that, if a force at the origin and a couple are the only forces applied to a system, that they must be zero.

For 2D problems we used the phrase ‘moment about a point’ to be short for ‘moment about an axis in the z direction that passes through the point. In 3D moment about a point is a vector and moment about an axis is a scalar.

Two- and three-force bodies

The concepts of two-force and three-force bodies are identical in 3D. If there are only two forces applied to a body in equilibrium they must be equal and opposite and acting along the line connecting the points of application. If there are only three force applied to a body they must all be in the plane of the points of application and the three forces must have lines of action that intersect at one point.

What does it mean for a problem to be ‘2D’?

The world we live in is three dimensional, all the objects to which we wish to study mechanically are three dimensional, and if they are in equilibrium they satisfy the three-dimensional equilibrium equations. How then can an engineer justify doing 2D mechanics? There are a variety of overlapping justifications.

- The 2D equilibrium equations are a subset of the 3D equations. In both 2D and 3D, $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \vec{\boldsymbol{M}}_{/0} \cdot \hat{\boldsymbol{k}} = 0$. So, if when doing 2D mechanics, one just neglects the z component of any applied forces and the x and y components of any applied couples, one is doing correct 3D mechanics, just not all of 3D mechanics. If the forces or conditions of interest to you are contained in the 2D equilibrium equations then 2D mechanics is really 3D mechanics, ignoring equations you don’t need.

- If the xy plane is a plane of symmetry for the object and any applied loading, then the three dimensional equilibrium equations not covered by the two dimensional equations, are automatically satisfied. For a car, say, the assumption of symmetry implies that the forces in the z direction will automatically add to zero, and the moments about the x and y axis will automatically be zero.
- If the object is thin and there are constraint forces holding it near the xy plane, and these constraint forces are not of interest, then 2D statics is also appropriate. This last case is caricatured by all the poor mechanical objects you have drawn so. They are constrained to lie in your flat paper by invisible slippery glass in front of and behind the paper. The 2D equations describe the forces between the slippery glass plates.

Trusses

The basic theory of trusses is the same in 3D as 2D. The method of joints is the primary basic approach. In ideal 3D truss theory the connections are 'ball and socket' not pins. That is the joints cannot carry any moments. For each joint the force balance equation can be reduced to three (rather than two for 2D trusses) scalar equations.

For the whole structure and for sections of the structure, the equilibrium equations can be reduced to two (rather than three for 2D trusses) scalar equations. The method of sections is less likely to be as useful a short-cut as in 2D because it is unlikely to find a section cut and equilibrium equation where only one bar force is unknown.

The counts for determinacy by matching the number of equations and number of unknowns change as follows. Instead of the 2D eqn. 4.28 from page 139 we have

$$3j = b + r \tag{4.70}$$

where j is the number of joints, including joints at reaction points, b is the number of bars, and r is the number of reaction components that shows on a free body diagram of the whole structure.

Example: **A tripod**

A tripod is the simplest rigid 3D structure. With four joints ($j = 4$), three bars ($b = 3$), and nine unknown reaction components ($r = 3 \times 3 = 9$), it exactly satisfies the equation $3j = b + r$, a check for determinacy of rigidity of 3D structures.

A tripod is the 3D equivalent of the two-bar truss shown in Fig. 4.34a on page 141. □

The check for determinacy of a floating (unattached) structure is

$$3j = b + 6. \tag{4.71}$$

There are various ways to think about the number six in the equation above. Assuming the structure is more than a point, six is the number of ways a structure can move in three dimensional space (three translations and three rotations), six is the number of equilibrium equations for the whole structure (one 3D vector moment, and one 3D vector force, and six is the number of constraints needed to hold a structure in place.

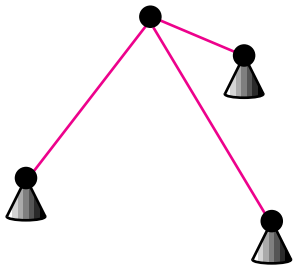


Figure 4.141: A tripod is the simplest rigid 3D truss.

(Filename:tfigure.tripod)

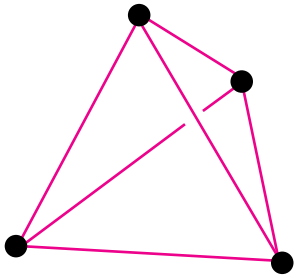


Figure 4.142: A tetrahedron is the simplest rigid truss in 3D that does not depend on grounding.

(Filename:tfigure.tetrahedron)

Example: **A tetrahedron**

The simplest 3D rigid floating structure is a tetrahedron. With four joints ($j = 4$) and six bars ($b = 6$) it exactly satisfies the equation $3j = b + 6$ which is a check for determinacy of rigidity of 3D structures.

A tetrahedron is thus, in some sense the 3D equivalent of a triangle in 2D. □

Internal forces

At a free body diagram cut on a long narrow structural piece in 2D there showed two force components, tension and shear, and one scalar moment. In 3D such a cut shows a force $\vec{\boldsymbol{F}}$ and a moment $\vec{\boldsymbol{M}}$ each with three components. If one picks a coordinate system with the x axis aligned with the bar at the cut, the concept of tension remains the same. Tension is the force component along the bar.

$$T = F_x = \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{\imath}}.$$

The two other force components, F_x and F_y, are two components of shear. The net shear force is a vector in the plane orthogonal to $\hat{\boldsymbol{\imath}}$.

The new concept, often called *torsion* is the component of $\vec{\boldsymbol{M}}$ along the axis:

$$\text{torsion} = M_x = \vec{\boldsymbol{M}} \cdot \hat{\boldsymbol{\imath}}$$

Torsion is the part of the moment that twists the shaft.

The remaining part of the $\vec{\boldsymbol{M}}$, in the yz plane, is the bending moment. It has two components M_x and M_y.

Springs

Ideal springs are simple two force bodies, whether in 2D or 3D. The equation describing the force *on* end B of a spring, in terms of the relative positions of the ends $\vec{\boldsymbol{r}}_{\text{AB}}$, the rest length of the spring ℓ_0, and the spring constant k is still eqn. 4.46 from page 163, namely,

$$\vec{\boldsymbol{F}}_{\text{B}} = k \cdot \underbrace{\left(|\vec{\boldsymbol{r}}_{\text{AB}}| - \ell_0\right)}_{\Delta\ell} \underbrace{\left(\frac{\vec{\boldsymbol{r}}_{\text{AB}}}{|\vec{\boldsymbol{r}}_{\text{AB}}|}\right)}_{\hat{\boldsymbol{\lambda}}_{\text{AB}}}. \tag{4.72}$$

Machines and structures

The approach to analysis of general machines and structures in 3D is the same as in 2D. One should draw a free body diagrams of the whole machine and of each of its parts, taking advantage of the principle of action and reaction. For each free body diagram the two vector equilibrium equations now lead to 6 scalar equations. Thus, for any but the simplest of 3D structures and machines one either tries to make a two dimensional model or one must resort to numerical solution.

Hydrostatics

The basic results of hydrostatics are 3D results.

SAMPLE 4.34 *Can a stack of three balls be in static equilibrium?* Three identical spherical balls, each of mass m and radius R, are stacked such that the top ball rests on the lower two balls. The two balls at the bottom do not touch each other. Let the coefficient of friction at each contact surface be μ. Find the minimum value of μ so that the three balls are in static equilibrium.

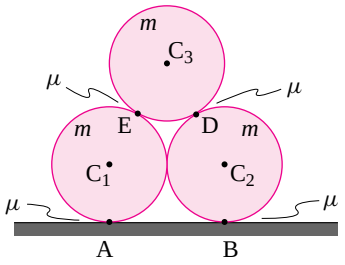


Figure 4.143: (Filename:sfig4.single.3balls)

Solution Let us assume that the three balls are in equilibrium. We can then find the forces required on each ball to maintain the equilibrium. If we can find a plausible value of the friction coefficient μ from the required friction force on any of the balls, then we are done, otherwise our initial assumption of static equilibrium is wrong.

The free body diagrams of the upper ball and the lower right ball (why the right ball? No particular reason) are shown in Fig. 4.144. The contact forces, $\vec{F}_{\rm E}$ and $\vec{F}_{\rm D}$, act on the upper ball at points E and D, respectively. Each contact force is the resultant of a tangential friction force and a normal force acting at the point of contact. From the free body diagrams, we see that each ball is a three-force-body. Therefore, all the three forces — the two contact forces and the force of gravity — must be concurrent. This requires that the two contact forces must intersect on the vertical line passing through the center of the ball (the line of action of the force of gravity). Now, if we consider the free body diagram of the lower right ball, we find that force $\vec{F}_{\rm D}$ has to pass through point B since the other two forces intersect at point B. Thus, we know the direction of force $\vec{F}_{\rm D}$.

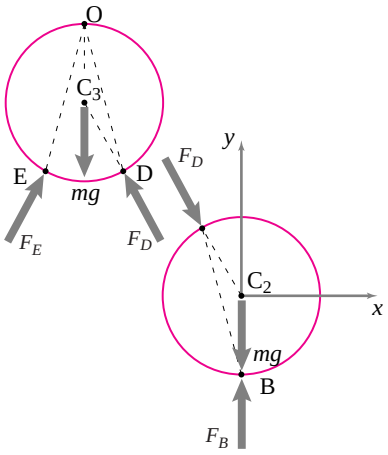


Figure 4.144: (Filename:sfig4.single.3balls.a)

Let α be the angle between the contact force $\vec{F}_{\rm D}$ and the normal to the ball surface at D. Now, from geometry, $\angle C_3DO + \angle C_3OD + OC_3D = 180^o$. But, $\alpha = \angle C_3DO = \angle C_3OD$. Therefore,

$$\begin{aligned} \alpha &= \frac{1}{2}(180^o - \angle OC_3D) = \frac{1}{2}(\angle GC_3D) \\ &= \frac{1}{2}30^o = 15^o \end{aligned}$$

where $\angle GC_3D = 30^o$ follows from the fact that $C_1C_2C_3$ is an equilateral triangle and C_3G bisects $\angle\, C_1C_3C_2$.

Now, from Fig. 4.145, we see that

$$\tan\alpha = \frac{F_s}{N}$$

But, the force of friction $F_s \leq \mu N$. Therefore, it follows that

$$\mu \geq \tan\alpha = \tan 15^o = 0.27$$

Thus, the friction coefficient must be at least 0.27 if the three balls have to be in static equilibrium.

$$\boxed{\mu \geq 0.27}$$

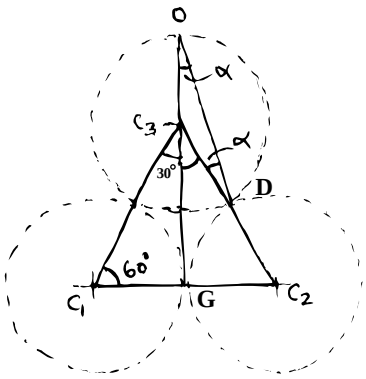


Figure 4.145: (Filename:sfig4.single.3balls.b)

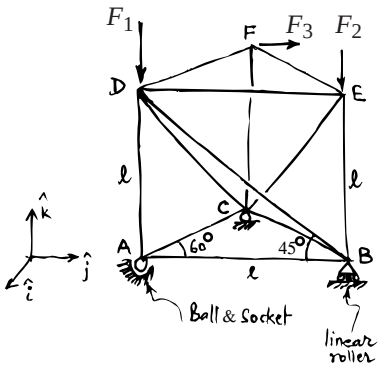


Figure 4.146: (Filename:sfig4.3d.truss1)

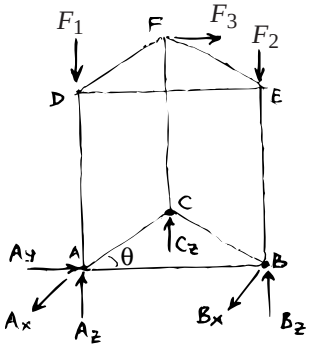


Figure 4.147: (Filename:sfig4.3d.truss1.a)

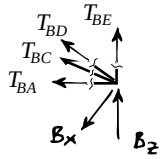


Figure 4.148: (Filename:sfig4.3d.truss1.b)

SAMPLE 4.35 *A simple 3-D truss:* The 3-D truss shown in the figure has 12 bars and 6 joints. Nine of the 12 bars that are either horizontal or vertical have length $\ell = 1\,\text{m}$. The truss is supported at A on a ball and socket joint, at B on a linear roller, and at C on a planar roller. The loads on the truss are $\vec{F}_1 = -50\,\text{N}\hat{k}$, $\vec{F}_2 = -60\,\text{N}\hat{k}$, and $\vec{F}_3 = 30\,\text{N}\hat{j}$. Find all the support reactions and the force in the bar BC.

Solution The free body diagram of the entire structure is shown in Fig. 4.147. Let the support reactions at A, B, and C be $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, $\vec{B} = B_x\hat{\imath} + B_z\hat{k}$, and $\vec{C} = C_z\hat{k}$. Then the moment balance about point A, $\sum \vec{M}_\text{A} = \vec{0}$ gives

$$\vec{r}_{\text{B/A}} \times \vec{B} + \vec{r}_{\text{C/A}} \times \vec{C} + \vec{r}_{\text{E/A}} \times \vec{F}_2 + \vec{r}_{\text{F/A}} \times \vec{F}_3 = \vec{0} \tag{4.73}$$

Note that $\vec{F}_1$ passes through A and, therefore, produces no moment about A. Now we compute each term in the equation above.

$$\begin{aligned}
\vec{r}_{\text{B/A}} \times \vec{B} &= \ell\hat{\jmath} \times (B_x\hat{\imath} + B_z\hat{k}) &&= -B_x\ell\hat{k} + B_z\ell\hat{\imath},\\
\vec{r}_{\text{C/A}} \times \vec{C} &= \ell(\cos 60^\text{o}\,\hat{\jmath} - \sin 60^\text{o}\hat{\imath}) \times C_z\hat{k} &&= C_z\tfrac{\ell}{2}\hat{\imath} + C_z\tfrac{\sqrt{3}\ell}{2}\hat{\jmath}\\
\vec{r}_{\text{E/A}} \times \vec{F}_2 &= (\ell\hat{\jmath} + \ell\hat{k}) \times (-F_2\hat{k}) &&= -F_2\ell\hat{\imath},\\
\vec{r}_{\text{F/A}} \times \vec{F}_3 &= [\ell(\cos 60^\text{o}\,\hat{\jmath} - \sin 60^\text{o}\hat{\imath}) + \ell\hat{k}] \times F_3\hat{\jmath} &&= -F_3\ell\hat{\imath} - F_3\tfrac{\sqrt{3}\ell}{2}\hat{k}
\end{aligned}$$

Substituting these products in eqn. (4.73), and dotting the resulting equation with $\hat{\jmath}$, $\hat{k}$, and $\hat{\imath}$, respectively, we get

$$\begin{aligned}
C_z &= 0\\
B_x &= -\frac{\sqrt{3}}{2}F_3 = -15\sqrt{3}\,\text{N}\\
B_z &= -\frac{1}{2}C_z + F_2 + F_3 = 90\,\text{N}
\end{aligned}$$

Thus, $\vec{B} = B_x\hat{\imath} + B_z\hat{k} = -15\sqrt{3}\,\text{N}\hat{\imath} + 30\,\text{N}\hat{k}$ and $\vec{C} = \vec{0}$. Now from the force balance, $\sum \vec{F} = \vec{0}$, we find $\vec{A}$ as

$$\begin{aligned}
\vec{A} &= -\vec{B} - \vec{C} - \vec{F}_1 - \vec{F}_2 - \vec{F}_3\\
&= -(-15\sqrt{3}\,\text{N}\hat{\imath} + 90\,\text{N}\hat{k}) - (-50\,\text{N}\hat{k}) - (-60\,\text{N}\hat{k}) - (30\,\text{N}\hat{\jmath})\\
&= 15\sqrt{3}\,\text{N}\hat{\imath} - 30\,\text{N}\hat{\jmath} + 20\,\text{N}\hat{k}
\end{aligned}$$

To find the force in bar BC, we draw a free body diagram of joint B (which connects BC) as shown in Fig. 4.148. Now, writing the force balance for the joint in the x-direction, *i.e.*, $[\sum \vec{F} = \vec{0}] \cdot \hat{\imath}$, gives

$$\begin{aligned}
B_x + \vec{T}_{\text{BC}} \cdot \hat{\imath} &= 0\\
\text{or} \qquad B_x + T_{BC}\sin 60^\text{o} &= 0\\
\Rightarrow \quad T_{BC} &= -\frac{B_x}{\sin 60^\text{o}}\\
&= -\frac{-15\sqrt{3}\,\text{N}}{\sqrt{3}/2} = 30\,\text{N}
\end{aligned}$$

Thus the force in bar BC is $T_{BC} = 30\,\text{N}$ (tensile force).

$$\boxed{\vec{A} = 15\sqrt{3}\,\text{N}\hat{\imath} - 30\,\text{N}\hat{\jmath} + 20\,\text{N}\hat{k},\ \vec{B} = -15\sqrt{3}\,\text{N}\hat{\imath} + 90\,\text{N}\hat{k},\ \vec{C} = \vec{0},\ T_{BC} = 30\,\text{N}}$$

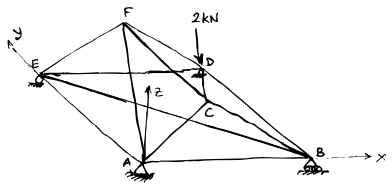
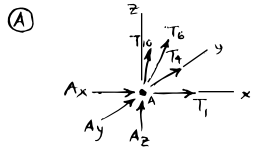


Figure 4.149: (Filename:sfig4.3d.truss2)

SAMPLE 4.36 *A 3-D truss solved on the computer:* The 3-D truss shown in the figure is fabricated with 12 bars. Bars 1–5 are of length $\ell = 1\,\text{m}$, bars 6–9 have length $\ell/\sqrt{2}(\approx 0.71\,\text{m})$, and bars 10–12 are cut to size to fit between the joints they connect. The truss is supported at A on a ball and socket, at B on a linear roller, and at C on a planar roller. A load $F = 2\,\text{kN}$ is applied at D as shown. Write all equations required to solve for all bar forces and support reactions and solve the equations using a computer.

Solution There are 12 bars and 6 joints in the given truss. The unknowns are 12 bar forces and six support reactions (3 at A (A_x, A_y, A_z), 2 at B (B_y, B_z), and 1 at E (E_z)). Therefore, we need 18 independent equations to solve for all the unknowns. Since the force equilibrium at each joint gives one vector equation in 3-D, *i.e.*, three scalar equations, the 6 joints in the truss can generate the required number ($6 \times 3 = 18$) of equations. Therefore, we go joint by joint, draw the free body diagram of the joint, write the force equilibrium equation, and extract the 3 scalar equations from each vector equation.

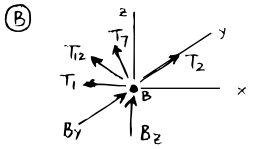
At each joint we use the following convention for force labels. At joint A, the force from bar AB is $\vec{F}_{\text{AB}} = T_1\hat{\boldsymbol{\lambda}}_{\text{AB}}$ and at joint B, the force from the same bar AB is $\vec{F}_{\text{BA}} = T_1\hat{\boldsymbol{\lambda}}_{\text{BA}} = T_1(-\hat{\boldsymbol{\lambda}}_{\text{AB}}) = -\vec{F}_{\text{AB}}$. We switch from the letters to denote the bars in the force vectors to numbers in its scalar representation (T_1, T_2, *etc.*) to facilitate computer solution.



Joint A:

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{F}_{\text{AB}} + \vec{F}_{\text{AC}} + \vec{F}_{\text{AF}} + \vec{F}_{\text{AE}} + \vec{A} = \vec{0}$$

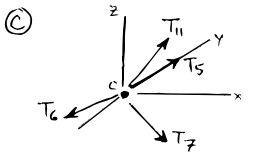
$$T_1\hat{\boldsymbol{\imath}} + \frac{T_6}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + \frac{T_{10}}{\sqrt{6}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) + T_4\hat{\boldsymbol{\jmath}} + A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}} = \vec{0} \quad (4.74)$$



Joint B:

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{F}_{\text{BA}} + \vec{F}_{\text{BC}} + \vec{F}_{\text{BD}} + \vec{F}_{\text{BE}} + \vec{B} = \vec{0}$$

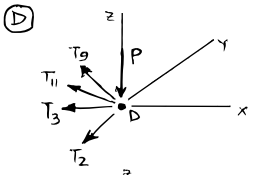
$$-T_1\hat{\boldsymbol{\imath}} + \frac{T_7}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + T_2\hat{\boldsymbol{\jmath}} + \frac{T_{12}}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) + B_y\hat{\boldsymbol{\jmath}} + B_z\hat{\boldsymbol{k}} = \vec{0} \quad (4.75)$$



Joint C:

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{F}_{\text{CA}} + \vec{F}_{\text{CB}} + \vec{F}_{\text{CF}} + \vec{F}_{\text{CD}} = \vec{0}$$

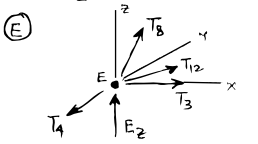
$$-\frac{T_6}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) - \frac{T_7}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + T_5\hat{\boldsymbol{\jmath}} + \frac{T_{11}}{\sqrt{6}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = \vec{0} \quad (4.76)$$



Joint D:

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{F}_{\text{DB}} + \vec{F}_{\text{DC}} + \vec{F}_{\text{DE}} + \vec{F}_{\text{DF}} + \vec{F} = \vec{0}$$

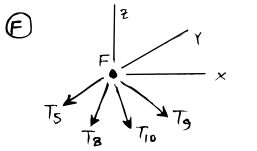
$$-T_2\hat{\boldsymbol{\jmath}} - \frac{T_{11}}{\sqrt{6}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) - T_3\hat{\boldsymbol{\imath}} + \frac{T_9}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) - F\hat{\boldsymbol{k}} = \vec{0} \quad (4.77)$$



Joint E:

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{F}_{\text{EA}} + \vec{F}_{\text{EB}} + \vec{F}_{\text{ED}} + \vec{F}_{\text{EF}} + \vec{E} = \vec{0}$$

$$-T_4\hat{\boldsymbol{\jmath}} + \frac{T_{12}}{\sqrt{2}}(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}) + T_3\hat{\boldsymbol{\imath}} + \frac{T_8}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + E_z\hat{\boldsymbol{k}} = \vec{0} \quad (4.78)$$



Joint F:

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{F}_{\text{FC}} + \vec{F}_{\text{FE}} + \vec{F}_{\text{FA}} + \vec{F}_{\text{FD}} = \vec{0}$$

$$-T_5\hat{\boldsymbol{\jmath}} - \frac{T_8}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) - \frac{T_{10}}{\sqrt{6}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) - \frac{T_9}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) = \vec{0} \quad (4.79)$$

Figure 4.150: (Filename:sfig4.3d.truss2.a)

Now we can separate out 3 scalar equations from each of the vector equations from eqn. (4.74)–eqn. (4.79) by dotting them with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$.

Eqn.	$[\text{Eqn.}] \cdot \hat{\boldsymbol{\imath}}$	$[\text{Eqn.}] \cdot \hat{\boldsymbol{\jmath}}$	$[\text{Eqn.}] \cdot \hat{\boldsymbol{k}}$
(1)	$T_1 + \frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + A_x = 0,$	$\frac{2}{\sqrt{6}}T_{10} + T_4 + A_y = 0,$	$\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + A_z = 0$
(2)	$-T_1 - \frac{1}{\sqrt{2}}T_7 - \frac{1}{\sqrt{2}}T_{12} = 0,$	$T_2 + \frac{1}{\sqrt{2}}T_{12} + B_y = 0,$	$\frac{1}{\sqrt{2}}T_7 + B_z = 0$
(3)	$-\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0,$	$T_5 + \frac{2}{\sqrt{6}}T_{11} = 0,$	$\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0$
(4)	$-\frac{1}{\sqrt{6}}T_{11} - T_3 - \frac{1}{\sqrt{2}}T_9 = 0,$	$-T_2 - \frac{2}{\sqrt{6}}T_{11} = 0,$	$\frac{1}{\sqrt{6}}T_{11} + \frac{1}{\sqrt{2}}T_9 = F$
(5)	$\frac{1}{\sqrt{2}}T_{12} + T_3 + \frac{1}{\sqrt{2}}T_8 = 0,$	$-T_4 - \frac{1}{\sqrt{2}}T_{12} = 0,$	$\frac{1}{\sqrt{2}}T_8 + E_z = 0$
(6)	$-\frac{1}{\sqrt{2}}T_8 - \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0,$	$-T_5 - \frac{2}{\sqrt{6}}T_{10} = 0,$	$\frac{1}{\sqrt{2}}T_8 + \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0$

Thus, we have 18 required equations for the 18 unknowns. Before we go to the computer, we need to do just one more little thing. We need to order the unknowns in some way in a one-dimensional array. So, let

$$x = [A_x \quad A_y \quad A_z \quad B_x \quad B_y \quad E_z \quad T_1 \ldots T_{12}]$$

Thus $x_1 = A_x,\ x_2 = A_y,\ \ldots, x_7 = T_1,\ x_8 = T_2,\ \ldots, x_{18} = T_{12}$. Now we are ready to go to the computer, feed these equations, and get the solution. We enter each equation as part of a matrix [A] and a vector {b} such that `[A]{x} = {b}`. Here is the pseudocode:

```
sq2i = 1/sqrt(2)            % define a constant
sq6i = 1/sqrt(6)            % define another constant
F = 2                       % specify given load

A(1,[1 7 12 16]) = [1 1 sq2i sq6i]
A(2,[2 10 16]) = [1 1 2*sq6i]
.
.
A(18,[14 15 16]) = [sq2i sq2i sq6i]
b(12,1) = F
form A and b setting all other entries to zero
solve A*x = b for x
```

The solution obtained from the computer is the one-dimensional array `x` which after decoding according to our numbering scheme gives the following answer.

$$\begin{gathered} A_x = A_y = 0, \quad A_z = -2\,\text{kN}, \quad B_y = 0, \quad B_z = 2\,\text{kN}, \quad E_z = 2\,\text{kN}, \\ T_1 = T_3 = -2\,\text{kN}, \quad T_2 = T_4 = T_5 = -4\,\text{kN}, \quad T_6 = 0, \\ T_7 = T_8 = -2.83\,\text{kN}, \quad T_9 = 0, \quad T_{10} = T_{11} = 4.9\,\text{kN}, \quad T_{12} = 5.66\,\text{kN}, \end{gathered}$$

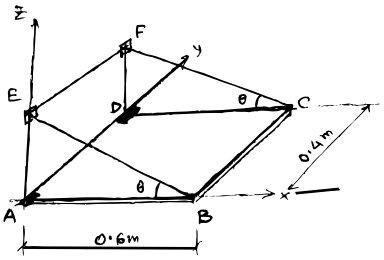


Figure 4.151: (Filename:sfig4.3d.plate)

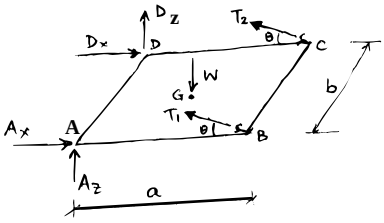


Figure 4.152: (Filename:sfig4.3d.plate.a)

SAMPLE 4.37 *An unsolvable problem?* A 0.6 m × 0.4 m uniform rectangular plate of mass $m = 4$ kg is held horizontal by two strings BE and CF and linear hinges at A and D as shown in the figure. The plate is loaded uniformly with books of total mass 6 kg. If the maximum tension the strings can take is 100 N, how much more load can the plate take?

Solution The free body diagram of the plate is shown in Fig. 4.152. Note that we model the hinges at A and D with no resistance in the y-direction. Since the plate has uniformly distributed load (including its own weight), we replace the distributed load with an equivalent concentrated load $\vec{W}$ acting vertically through point G.

The various forces acting on the plate are

$$\vec{W} = -W\hat{k},\ \vec{T}_1 = T_1\hat{\lambda}_{\rm BE},\ \vec{T}_2 = T_2\hat{\lambda}_{\rm CF},\ \vec{A} = A_x\hat{\imath} + A_z\hat{k},\ \vec{D} = D_x\hat{\imath} + D_z\hat{k}$$

Here $\hat{\lambda}_{\rm BE} = \hat{\lambda}_{\rm CF} = -\cos\theta\hat{\imath} + \sin\theta\hat{k} = \hat{\lambda}$(let). Now, we apply moment equilibrium about point A, *i.e.*, $\sum \vec{M}_{\rm A} = \vec{0}$.

$$\vec{r}_{\rm B} \times \vec{T}_1 + \vec{r}_{\rm C} \times \vec{T}_2 + \vec{r}_{\rm G} \times \vec{W} + \vec{r}_{\rm D} \times \vec{D} = \vec{0} \tag{4.80}$$

where,

$$\begin{aligned}
\vec{r}_{\rm B} \times \vec{T}_1 &= a\hat{\imath} \times T_1\hat{\lambda} = -aT_1\sin\theta\,\hat{\jmath} \\
\vec{r}_{\rm C} \times \vec{T}_2 &= (a\hat{\imath} + b\hat{\jmath}) \times T_2\hat{\lambda} = T_2 b\sin\theta\hat{\imath} - T_2 a\sin\theta\,\hat{\jmath} + T_2 b\cos\theta\hat{k} \\
\vec{r}_{\rm G} \times \vec{W} &= \frac{1}{2}(a\hat{\imath} + b\hat{\jmath}) \times (-W\hat{k}) = -\frac{Wa}{2}\hat{\imath} + \frac{Wa}{2}\hat{\jmath} \\
\vec{r}_{\rm D} \times \vec{D} &= b\hat{\jmath} \times (D_x\hat{\imath} + D_z\hat{k}) = D_z b\hat{\imath} - D_x b\hat{k}
\end{aligned}$$

Substituting these products in eqn. (4.80) and dotting with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, we get

$$T_2\sin\theta + D_z = \frac{W}{2} \tag{4.81}$$

$$T_2\cos\theta - D_x = 0 \tag{4.82}$$

$$(T_1 + T_2)\sin\theta = \frac{W}{2} \tag{4.83}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\vec{A} + \vec{D} + \vec{T}_1 + \vec{T}_2 + \vec{W} = \vec{0}$$

Again, substituting the forces in their component form and dotting with $\hat{\imath}$ and $\hat{k}$ (there are no $\hat{\jmath}$ components), we get

$$\begin{aligned}
A_x + D_x - (T_1 + T_2)\cos\theta &= 0 \\
\Rightarrow \quad A_x - T_1\cos\theta &= 0 \qquad (4.84) \\
A_z + D_z + (T_1 + T_2)\sin\theta &= 0 \\
\Rightarrow \quad A_z + T_1\sin\theta &= \frac{W}{2} \qquad (4.85)
\end{aligned}$$

These are all the equations that we can get. Now, note that we have five independent equations (eqns. (4.81) to (4.85)) but six unknowns. Thus we cannot solve for the unknowns uniquely. This is an indeterminate structure! No matter which point we use for our moment equilibrium equation, we will always have one more unknown than the number of independent equations. We can, however, solve the problem with an extra assumption (see comments below) — the structure is symmetric about the

axis passing thorugh G and parallel to x-axis. From this symmetry we conclude that $T_1 = T_2$. Then, from eqn. (4.84) we have

$$2T \sin\theta = \frac{W}{2} \quad \Rightarrow \quad T = \frac{W}{4\sin\theta}$$

We can now find the maximum load that the plate can take subject to the maximum allowable tension in the strings.

$$\begin{aligned} W &= 4T \sin\theta \\ \Rightarrow \quad W_{\max} &= 4T_{\max} \sin\theta \\ &= 4(100\,\text{N}) \cdot \frac{1}{2} = 200\,\text{N} \end{aligned}$$

The total load as given is $(6+4)\,\text{kg} \cdot 9.81\,\text{m/s}^2 = 98.1\,\text{N} \approx 100\,\text{N}$. Thus we can double the load before the strings reach their break-points. Now the reactions at D and A follow from eqns. (4.81), (4.82), (4.84), and (4.85).

$$\begin{aligned} D_z = A_z = \frac{W}{2} - T\sin\theta &= \frac{W}{2} \\ D_x = A_x = T\cos\theta &= \frac{W}{4}\cot\theta \end{aligned}$$

$$\boxed{W_{\max} = 200\,\text{N}}$$

Comments:

(a) We got only five independent equations (instead of the usual 6) because the force equilibrium in the y-direction gives a zero identity (0 = 0). There are no forces in the y-direction. The structure seems to be unstable in the y-direction — if you push a little, it will move. Remember, however, that it is so because we chose to model the hinges at A and D that way keeping in mind the only vertical loading. The actual hinges used on a bookshelf will not allow movement in the y-direction either. If we model the hinges as ball and socket joints, we introduce two more unknowns, one at each joint, and get just one more scalar equation. Thus we are back to square one. There is no way to determine A_y and D_y from equilibrium equations alone.

(b) The assumption of symmetry and the consequent assumption of equality of the two string tensions is, mathematically, an extra independent equation based on deformations (strength of materials). At this point, you may not know any strength of material calculations or deformation theory, but your intuition is likely to lead you to make the same assumption. Note, however, that this assumption is sensitive to accuracy in fabrication of the structure. If the strings were slightly different in length, the angles were slightly off, or the wall was not perfectly vertical, the symmetry argument would not hold and the two tensions would not be the same.

Most real problems are like this — indeterminate. Our modelling, which requires understanding of mechanics, makes them determinate and solvable.

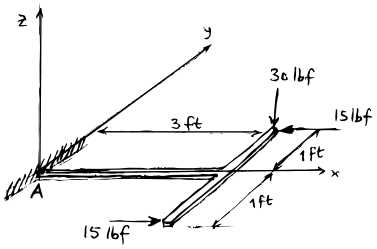


Figure 4.153: (Filename:sfig4.intern.cant3D)

SAMPLE 4.38 *3-D moment at the support:* A 'T' shaped cantilever beam is loaded as shown in the figure. Find all the support reactions at A.

Solution The free body diagram of the beam is shown in Fig. **??**. Note that the forces acting on the beam can produce in-plane as well as out of plane moments. Therefore, we show the unknown reactions $\vec{A}$ and $\vec{M}_{\rm A}$ as general 3-D vectors at A. The moment equilibrium about point A, $\sum \vec{M}_{\rm A} = \vec{0}$, gives

$$\vec{M}_{\rm A} + \vec{r}_{\rm C/A} \times (\vec{F}_1 + \vec{F}_2) + \vec{r}_{\rm D/A} \times \vec{F}_3 = \vec{0}$$

$$\begin{aligned} \Rightarrow \quad \vec{M}_{\rm A} &= (\vec{r}_{\rm B/A} + \vec{r}_{\rm C/B}) \times (\vec{F}_1 + \vec{F}_2) + (\vec{r}_{\rm B/A} + \vec{r}_{\rm D/B}) \times \vec{F}_3 \\ &= (\ell\hat{\imath} + a\hat{\jmath}) \times (-F_1\hat{k} - F_2\hat{\imath}) + (\ell\hat{\imath} - a\hat{\jmath}) \times F_3\hat{\imath} \end{aligned}$$

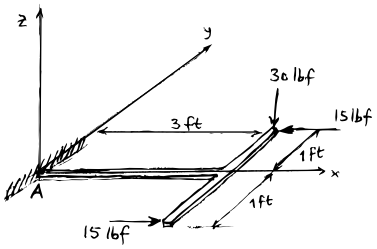


Figure 4.154: (Filename:sfig4.intern.cant3D)

But $F_3 = -F_2 = F$ (say). Therefore,

$$\begin{aligned} &= (\ell\hat{\imath} + a\hat{\jmath}) \times (-F_1\hat{k} - F\hat{k}) + (\ell\hat{\imath} - a\hat{\jmath}) \times F\hat{\imath} \\ &= F_1\ell\hat{\jmath} - F_1 a\hat{\imath} - 2Fa\hat{k} \\ &= 30\,\text{lbf}\cdot 3\,\text{ft}\,\hat{\jmath} - 30\,\text{lbf}\cdot 1\,\text{ft}\hat{\imath} - 2(30\,\text{lbf}\cdot 1\,\text{ft})\hat{k} \\ &= (-30\hat{\imath} + 90\hat{\jmath} - 60\hat{k})\,\text{lb·ft} \end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned} \vec{A} &= -\vec{F}_1 - \vec{F}_2 - \vec{F}_3 \\ &= -\vec{F}_1 - \vec{F} + \vec{F} \\ &= -(-F_1\hat{k}) = F_1\hat{k} \\ &= 30\,\text{lbf}\hat{k} \end{aligned}$$

$$\boxed{\vec{A} = 30\,\text{lbf}\hat{k},\ \text{and}\ \vec{M}_{\rm A} = (-30\hat{\imath} + 90\hat{\jmath} - 60\hat{k})\,\text{lb·ft}}$$

5 Dynamics of particles

We now progress from statics to dynamics. Although we treated statics as an independent topic, statics is really a special case of dynamics. In statics we neglected the inertial terms (the terms involving acceleration times mass) in the linear and angular momentum balance equations. In dynamics these terms are of central interest. In statics all the forces and moments cancel each other. In dynamics the forces and moments add to cause the acceleration of mass. As the names imply, statics is generally concerned with things that don't move, or at least don't move much, whereas dynamics with things that move a lot. How to quantify what is 'still' (statics) *vs* 'moving' (dynamics) is itself a dynamics question.

A big part of learning dynamics is learning to keep track of motion, *kinematics*. In addition, kinematic analysis is also useful for work and energy methods in statics. We are going to develop our understanding of dynamics by considering progressively harder-to-understand motions.

This chapter is limited to the dynamics of particles. A *particle* is a system totally characterized by its position (as a function of time) and its (fixed) mass. Often one imagines that a particle is something small. But the particle idealization is used, for example, to describe a galaxy in the context of its motion in a cluster of galaxies. Rather, a particle is something whose spatial extent is neglected in the evaluation of mechanics equations. An object's spatial extent might be neglected because the object is small compared to other relevant distances, or because distortion and rotation happen to be of secondary interest.

In this chapter we further limit our study of the dynamics of particles to cases where the applied forces are given as a function of time or can be determined from the positions and velocities of the particles. The time-varying thrust from an engine might be thought of as a force given as a function of time. Gravity and springs cause forces which are functions of position. And the drag on a particle as it moves through

air or water can be modeled as a force depending on velocity. Discussion of geometric constraints, as for particles connected with strings or rods, where some of the forces depend on finding the accelerations, begins in chapter 6.

The most important equation in this chapter is linear momentum balance applied to one particle. If we start with the general form in the front cover, discussed in general terms in chapter 1, we get:

$$\begin{aligned} \sum \vec{F}_{\mathrm{i}} &= \dot{\vec{L}} && \text{Linear momentum balance for any system} \\ &= \sum m_i \vec{a}_{\mathrm{i}} && \text{for a system of particles} \\ &= m\vec{a} && \text{for one particle} \end{aligned}$$

If we define $\vec{F}$ to be the net force on the particle ($\vec{F} = \sum \vec{F}_{\mathrm{i}}$) then we get

$$\vec{F} = m\vec{a} \tag{5.1}$$

which is sometimes called 'Newton's second law of motion.' In his words,

> *"Any change of motion is proportional to the force that acts, and it is made in the direction of the straight line in which that force is acting."*

In modern language, explicitly including the role of mass, the net force on a particle is its mass times its acceleration. Intuitively people think of this law as saying force causes motion, and, more precisely, that force causes acceleration of mass. Actually, what causes what, *causality*, is just a philosophical question. The important fact is that when there is a net force there is acceleration of mass, and when there is acceleration of mass there is a net force. When a car crashes into a pole there is a big force and a big deceleration of the car. You could think of the force on the bumper as causing the car to slow down rapidly. Or you could think of the rapid car deceleration as necessitating a force. It is just a matter of personal taste because in both cases equation 5.1 applies.

Acceleration is the second derivative of position

What is the acceleration of a particle? Lets assume that $\vec{r}(t)$ is the position of the particle as a function of time relative to some origin. Then its acceleration is

$$\begin{aligned} \vec{a} \equiv \frac{d}{dt}\vec{v} &= \frac{d}{dt}\left(\frac{d}{dt}\vec{r}\right) = \frac{d^2}{dt^2}\vec{r} \\ &= \dot{\vec{v}} = \frac{d}{dt}(\dot{\vec{r}}) = \ddot{\vec{r}} \end{aligned}$$

where one or two dots over something is a short hand notation for the first or second time derivative.

Newton's laws are accurate in a Newtonian reference frame

When the acceleration is calculated from position it is calculated using a particular coordinate system. A *reference frame* is, for our purposes at the moment, a coordinate system. The calculated acceleration of a particle depends on how the coordinate system itself is moving. So the simple equation

$$\vec{F} = m\vec{a}$$

has as many different interpretations as there are differently moving coordinate systems (and there are an infinite number of those). Sir Isaac was standing on earth measuring position relative to the ground when he noticed that his second law accurately described things like falling apples. So the equation $\vec{F} = m\vec{a}$ is valid using coordinate systems that are fixed to the earth. Well, not quite. Isaac noticed that the motion of the planets around the sun only followed his law if the acceleration was calculated using a coordinate system that was still relative to 'the fixed stars.' With a fixed-star coordinate system you calculate *very* slightly different accelerations for things like falling apples than you do using a coordinate system that is stuck to the earth. And nowadays when astrophysicists try to figure out how the laws of mechanics explain the shapes of spiral galaxies they realize that none of the so-called 'fixed stars' are so totally fixed. They need even more care to pick a coordinate system where eqn. 5.1 is accurate.

Despite all this confusion, it is generally agreed that there exists some coordinate system for which Newton's laws are incredibly accurate. Further, once you know one such coordinate system there are rules (which we will discuss in later chapters) to find many others. Any such reference frame is called a *Newtonian reference frame*. Sometimes people also call such a frame a *Fixed frame*, as in 'fixed to the earth' or 'fixed to the stars'.

For most engineering purposes, not counting, for example, trajectory control of interplanetary missions, a coordinate system attached to the ground under your feet is good approximation to a Newtonian frame. Fortunately. Or else apples would fall differently. Newton might not have discovered his laws. And this book would be much shorter.

The organization of this chapter

In the first four sections of this chapter we give a thorough introduction to the one-dimensional mechanics of single particle. This is a review and deepening of material covered in freshman physics. These sections introduce you to the time-varying nature of dynamics without the complexity of vector geometry. The later sections concern dynamics with more particles or more spatial dimensions or both.

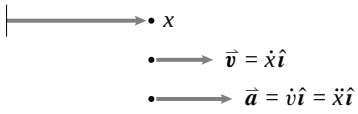


Figure 5.1: One-dimensional position, velocity, and acceleration in the x direction.

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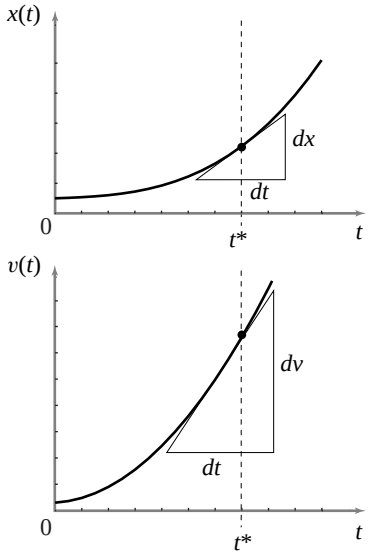


Figure 5.2: Graphs of $x(t)$ and $v(t) = \frac{dx}{dt}$ versus time. The slope of the position curve dx/dt at t^* is $v(t^*$. And the slope of the velocity curve dv/dt at t^* is $a(t^*$.

(Filename:tfigure3.4.1a)

5.1 Force and motion in 1D

We now limit our attention to the special case where one particle moves on a given straight line. We postpone until Chapter 6 issues about what forces might be required to keep the particle on that line. For problems with motion in only one direction, the kinematics is particularly simple. Although we use vectors here because of their help with signs, they are really not needed.

Position, velocity, and acceleration in one dimension

If, say, we call the direction of motion the $\hat{\imath}$ direction, then we can call x the position of the particle we study (see figure 5.1). Even though we are neglecting the spatial extent of the particle, to be precise we can define x to be the x coordinate of the particle's center. We can write the position $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ as

$$\vec{r} = x\hat{\imath} \qquad \text{and} \qquad \vec{v} = v\hat{\imath} = \frac{dx}{dt}\hat{\imath} = \dot{x}\hat{\imath} \qquad \text{and} \qquad \vec{a} = a\hat{\imath} = \frac{dv}{dt}\hat{\imath} = \frac{d^2x}{dt^2}\hat{\imath} = \ddot{x}\hat{\imath},$$

Figure 5.2 shows example graphs of $x(t)$ and $v(t)$ versus time. When we don't use vector notation explicitly we will take v and a to be positive if they have the same direction as increasing x (or y or whatever coordinate describes position).

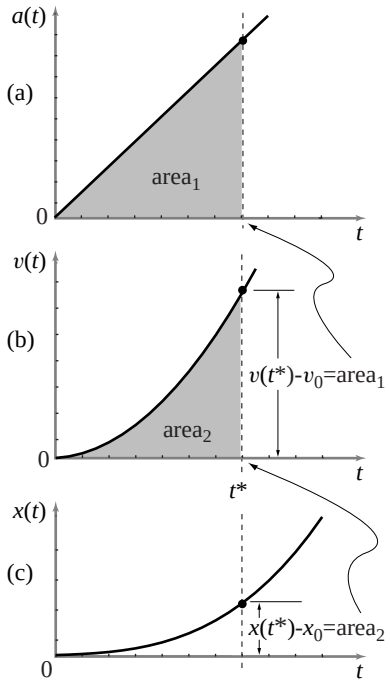


Figure 5.3: One-dimensional kinematics of a particle: (a) is a graph of the acceleration of a particle $a(t)$; (b) is a graph of the particle velocity $v(t)$ and the integral of $a(t)$ from $t_0 = 0$ to t^*, the shaded area under the acceleration curve; (c) is the position of the particle $x(t)$ and the integral of $v(t)$ from $t_0 = 0$ to t^*, the shaded area under the velocity curve.

(Filename:tfigure3.4.1b)

① To cover the range of calculus problems you need to be a very good rider, however, able to ride frontwards, backwards, at zero speed and infinitely fast.

Example: **Position, velocity, and acceleration in one dimension**

If position is given as $x(t) = 3e^{4t/\mathrm{s}}\,\mathrm{m}$ then $v(t) = dx/dt = 12e^{4t/\mathrm{s}}(\mathrm{m/s})$ and $a(t) = dv/dt = 48e^{4t/\mathrm{s}}(\mathrm{m/s^2})$. So at, say, time $t = 2\,\mathrm{s}$ the acceleration is $a|_{t=2\,\mathrm{s}} = 48e^{4\cdot 2\,\mathrm{s}/\mathrm{s}}(\mathrm{m/s^2}) = 48\cdot e^8\,\mathrm{m/s^2} \approx 1.43\cdot 10^5\,\mathrm{m/s^2}$. □

We can also, using the fundamental theorem of calculus, look at the integral versions of these relations between position, velocity, and acceleration (see Fig. 5.3).

$$x(t) = x_0 + \int_{t_o}^{t} v(\tau)\,d\tau \qquad \text{with} \qquad x_0 = x(t_0), \text{ and}$$
$$v(t) = v_0 + \int_{t_o}^{t} a(\tau)\,d\tau \qquad \text{with} \qquad v_0 = v(t_0).$$

With more informal notation, these equations can also be written as:

$$x = \int v\,dt$$
$$v = \int a\,dt.$$

So one-dimensional kinematics includes almost all elementary calculus problems. You are given a function and you have to differentiate it or integrate it. To put it the other way around, almost every calculus question could be phrased as a question about your bicycle speedometer. On your bicycle speedometer (which includes an odometer) you can read your speed and distance travelled as functions of time. Given one of those two functions, find the other.① As of this writing, common bicycle computers don't have accelerometers. But acceleration as a function of time is also of interest. For example, if you are given the (scalar part of) velocity $v(t)$ as a function of time and are asked to find the acceleration $a(t)$ you have to differentiate. If instead you were asked to find the position $x(t)$, you would be asked to calculate an integral (see figure 5.3).

If acceleration is given as a function of time, then position is found by integrating twice.

Differential equations

A *differential equation* is an equation that involves derivatives. Thus the equation relating position to velocity is

$$\frac{dx}{dt} = v \qquad \text{or, more explicitly} \qquad \frac{dx(t)}{dt} = v(t),$$

is a differential equation. An *ordinary differential equation* (ODE) is one that contains ordinary derivatives (as opposed to partial differential equations which we will not use in this book).

Example: **Calculating a derivative solves an ODE**

Given that the height of an elevator as a function of time on its 5 seconds long 3 meter trip from the first to second floor is

$$y(t) = (3\,\mathrm{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}$$

we can solve the differential equation $v = \frac{dy}{dt}$ by differentiating to get

$$v = \frac{dy}{dt} = \frac{d}{dt}\left[(3\,\mathrm{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}\right] = \frac{3\pi}{10}\sin\left(\frac{\pi t}{5\,\mathrm{s}}\right)\,\mathrm{m/s}$$

(Note: this would be considered a harsh elevator because of the jump in the acceleration at the start and stop.) □

A little less trivial is the case when you want to find a function when you are given the derivative.

Example: **Integration solves a simple ODE**

Given that you start at home ($x = 0$) and, over about 30 seconds, you accelerate towards a steady-state speed of 4 m/s according to the function

$$v(t) = 4(1 - e^{-t/(30\,\mathrm{s})})\,\mathrm{m/s}$$

and your whole ride lasts 1000 seconds (about 17 minutes). You can find how far you travel by solving the differential equation

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0$$

which can be accomplished by integration. Say, after 1000 seconds

$$\begin{aligned} x(t = 1000\,\mathrm{s}) &= \int_0^{1000\,\mathrm{s}} v(t)\,dt = \int_0^{1000\,\mathrm{s}} 4(1 - e^{-t/(30\,\mathrm{s})})(\,\mathrm{m/s})\,dt \\ &= \left(4t + (120\,\mathrm{s})e^{-t/(30\,\mathrm{s})}\right)\Big|_0^{1000\,\mathrm{s}}\,\mathrm{m/s} \\ &= \left((4\cdot 1000\,\mathrm{s} + (120\,\mathrm{s})e^{-100/3}) - (0 + (120\,\mathrm{s})e^{0})\right)\,\mathrm{m/s} \\ &= \left(4000 - 120 + 120e^{-100/3})\right)\,\mathrm{m} \\ &\approx 3880\,\mathrm{m} \qquad \text{(to within an angstrom or so)} \end{aligned}$$

The distance travelled is only 120 m less than would be travelled if the whole trip was travelled at a steady 4 m/s (4 m/s × 1000 s = 4000 m). □

Unlike the integral above, many integrals cannot be evaluated by hand.

Example: **An ODE that leads to an intractable integral**

If you were told that the velocity as a function of time was

$$v(t) = \frac{4t}{t + e^{-t/(30\,\mathrm{s})}\,\mathrm{s}}\,\mathrm{m}$$

you would, as for the previous example, be describing a bike trip where you started at zero speed and exponentially approached a steady speed of 4 m/s. Thus your position as a function of time should be similar. But what is it? Let's proceed as for the last example to solve the equation

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0$$

and the given $v(t)$. We can set up the integral to get

$$\begin{aligned} x(t = 1000\,\mathrm{s}) &= \int_0^{1000\,\mathrm{s}} v(t)\,dt = \int_0^{1000\,\mathrm{s}} \frac{4t}{t+e^{-t/(30\,\mathrm{s})}\,\mathrm{s}}\,\mathrm{m}\,dt \\ &= \ldots \end{aligned} \tag{5.2}$$

which is the kind of thing you have nightmares about seeing on an exam. This is an integral that you couldn't do if your life depended on it. No-one could. There is no formula for $x(t)$ that solves the differential equation $\dot{x} = v(t)$, with the given $v(t)$, unless you regard eqn. 5.2 as a formula. In days of old they would say 'the problem has been reduced to quadrature' meaning that all that remained was to evaluate an integral, even if they didn't know how to evaluate it. But you can always resort to numerical integration. One of many ways to evaluate the integral numerically is by the following pseudo code (note that the problem is formulated with consistent units so they can be dropped for the numerics).

```
ODE = { xdot = 4 t / (t+e^(-t/30)) }
IC  = { x(0) = 0  }
solve ODE with IC and evaluate at t=1000
```

The result is $x \approx 3988\,\text{m}$ which is also, as expected because of the similarity with the previous example, only slightly shy of the steady-speed approximation of 4000 m. □

The equations of dynamics

Linear momentum balance

For a particle moving in the x direction the velocity and acceleration are $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\imath}}$. Thus the linear momentum and its rate of change are

$$\begin{aligned} \vec{\boldsymbol{L}} &\equiv \textstyle\sum m_i \vec{\boldsymbol{v}}_i &&= m\vec{\boldsymbol{v}} = mv\hat{\boldsymbol{\imath}}, \text{ and} \\ \dot{\vec{\boldsymbol{L}}} &\equiv \textstyle\sum m_i \vec{\boldsymbol{a}}_i &&= m\vec{\boldsymbol{a}} = \text{m}a\hat{\boldsymbol{\imath}}. \end{aligned}$$

Thus the equation of linear momentum balance①, eqn. I from the front inside cover, or equation 5.1 reduces to:

$$F\hat{\boldsymbol{\imath}} = ma\hat{\boldsymbol{\imath}} \quad \text{or} \quad F = ma \tag{5.3}$$

where F is the net force to the right and a is the acceleration to the right.

Now the force could come from a spring, or a fluid or from your hand pushing the particle to the right or left. The most general case we want to consider here is that the force is determined by the position and velocity of the particle as well as the present time. Thus

$$F = f(x, v, t).$$

Special cases would be, say,

F	$=$	$f(x)$	$= -kx$	for a linear spring,
F	$=$	$f(v)$	$= -cv$	for a linear viscous drag,
F	$=$	$f(t)$	$= F_0 \sin(\beta t)$	for an oscillating load, and
F	$=$	$f(x, v, t)$	$= -kx - cv + F_0 \sin(\beta t)$	for all three forces at once.

So all elementary 1D particle mechanics problems can be reduced to the solution of this pair of coupled first order differential equations,

$$\begin{aligned} \tfrac{dv}{dt} &= \underbrace{f(x, v, t)/m}_{a(t)} \quad (a) \\ \tfrac{dx}{dt} &= v(t) \quad (b) \end{aligned} \tag{5.4}$$

where the function $f(x, v, t)$ is given and $x(t)$ and $v(t)$ are to be found.

① We do not concern ourselves with angular momentum balance in this section. Assuming we pick an origin on the line of travel, all terms on both sides of all angular momentum balance equations are zero. The angular momentum balance equations are thus automatically satisfied and have nothing to offer here.

Example: **viscous drag**

If the only applied force is a viscous drag, $F = -cv$, then linear momentum balance would be $-cv = ma$ and Eqns. 5.4 are

$$\begin{aligned} \frac{dv}{dt} &= -cv/m \\ \frac{dx}{dt} &= v \end{aligned}$$

where c and m are constants and $x(t)$ and $v(t)$ are yet to be determined functions of time. Because the force does nothing but slow the particle down there will be no motion unless the particle has some initial velocity. In general, one needs to specify the initial position and velocity in order to determine a solution. So we complete the problem statement by specifying the initial conditions that

$$x(0) = x_0 \quad \text{and} \quad v(0) = v_0$$

where x_0 and v_0 are given constants. Before worrying about how to solve such a set of equations, on should first know how to recognize a solution set. In this case the two functions

$$\begin{aligned} v(t) &= v_0 e^{-ct/m}, \qquad \text{and} \\ x(t) &= x_0 + m v_0 (1 - e^{-ct/m})/c \end{aligned}$$

solve the equations. You can check that the initial conditions are satisfied by evaluating the expressions at $t = 0$. To check that the differential equations are satisfied, you plug the candidate solutions into the equation and see if an identity results. □

Just like the case of integration (or equivalently the solution for x of $\dot{x} = v(t)$), one often cannot find formulas for the solutions of differential equations.

Example: **A dynamics problem with no pencil and paper solution**

Consider the following case which models a particle in a sinusoidal force field with a second applied force that oscillates in time. Using the dimensional constants c, d, F_0, β, and m,

$$\begin{aligned} \frac{dv}{dt} &= \left(c \sin(x/d) + F_0 \sin(\beta t)\right)/m \\ \frac{dx}{dt} &= v \end{aligned}$$

with initial conditions $x(0) = 0 \quad \text{and} \quad v(0) = 0.$

There is no known formula for $x(t)$ that solves this ODE. □

Just writing the ordinary differential equations and initial conditions is quite analogous to setting up an integral in freshman calculus. The solution is reduced to quadrature. Because numerical solution of sets of ordinary differential equations is a standard part of all modern computation packages you are in some sense done when you get to this point. You just ask your computer finish up.

Special methods and special cases in 1D mechanics

In some problems, the acceleration can be found as a function of *position* (as opposed to time) easily. In this case, one can find velocity as a function of position by the following formula (see Box 5.1:

$$(v(x))^2 = (v(x_0))^2 + 2\int_{x_0}^{x} a(x^*)\,dx^*. \tag{5.5}$$

An especially simple case is constant acceleration. Then we get the following kinematics formulas which are greatly loved and hated in high school and freshman physics:

$$\begin{aligned} a = const \Rightarrow \quad x &= x_0 + v_0 t + at^2/2 \\ a = const \Rightarrow \quad v &= v_0 + at \\ a = const \Rightarrow \quad v &= \pm\sqrt{v_0^2 + 2ax}. \end{aligned}$$

Some of these equations are also discussed in box 5.2 about the solution of the simplest ordinary differential equations on page 226.

Example: **Ramping up the acceleration at the start**

If you get a car going by gradually depressing the 'accelerator' so that its acceleration increases linearly with time, we have

$$\begin{array}{rcccc} a & = & ct & & (\text{take } t = 0 \text{ at the start}) \\ \Rightarrow \quad v(t) & = & \int_0^t a\,d\tau + v_0 & = & \int_0^t c\tau\,d\tau \quad = \quad ct^2/2 \\ & & & & (\text{since } v_0 = 0) \\ \Rightarrow \quad x(t) & = & \int_0^t v\,d\tau + x_0 & = & \int_0^t (c\tau^2/2)d\tau \quad = \quad ct^3/6 \\ & & & & (\text{since } x_0 = 0). \end{array}$$

The distance the car travels is proportional to the cube of the time that has passed from dead stop. □

5.1 *THEORY*
Finding $v(x)$ from $a(x)$

Equation 5.5 for velocity as a function of position can be derived as follows. Two derivations are given.

Derivation 1:

$$\begin{aligned}
\frac{dv}{dt} &= a \\
\Rightarrow v\frac{dv}{dt}\frac{dt}{dx} &= a, \quad \text{since } \left(\frac{1}{v} = \frac{dt}{dx}\right) \\
\Rightarrow \left[\frac{d}{dt}\left(\frac{1}{2}v^2\right)\right]dt &= a\,dx, \\
&\quad \text{since } \left(\frac{d}{dt}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dt}\right) \\
\Rightarrow \frac{1}{2}v^2 - \frac{1}{2}v_0^2 &= \int_{x_0}^{x} a(x^*)dx^*.
\end{aligned}$$

Derivation 2:

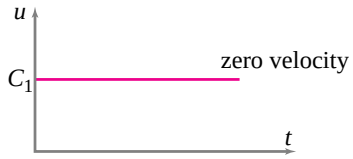
$$\begin{aligned}
\frac{dv}{dt} &= a \\
\Rightarrow \frac{dv}{dx}\cdot\frac{dx}{dt} &= a, \\
\Rightarrow v\,dv &= a\,dx, \\
\Rightarrow \int_{v_0}^{v} v^*\,dv^* &= \int_{x_0}^{x} a(x^*)\,dx^*, \\
\Rightarrow \frac{1}{2}v^2 - \frac{1}{2}v_0^2 &= \int_{x_0}^{x} a(x^*)dx^*.
\end{aligned}$$

5.2 The simplest ODEs, their solutions, and heuristic explanations

Sometimes differential equations you want to solve are simple enough that you might quickly find their solution. This table presents some of the simplest ODEs for $u(t)$ and their general solution. Each of these solutions can be used to solve one or another simple mechanics problem. In order to make these simplest ordinary differential equations (ODE's) feel like more than just a group of symbols, we will try to make each of them intuitively plausible. For this purpose, we will interpret the variable u as the distance an object has moved to the right of its 'home', the origin at 0. The velocity of motion to the right is thus $\dot{u}$ and its acceleration to the right is $\ddot{u}$. If $\dot{u} < 0$ the particle is moving to the left. If $\ddot{u} < 0$ the particle is accelerating to the left.

In all cases we assume that A and B are constants and that λ is a positive constant. C_1, C_2, C_3, and C_4 are arbitrary constants in the solutions that may be chosen to satisfy any initial conditions.

a)

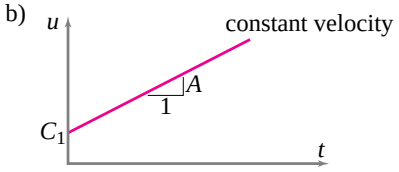


a) ODE: $\dot{u} = 0 \Rightarrow$ Soln: $u = C_1$.

$\dot{u} = 0$ means that the velocity is zero. This equation would arise in dynamics if a particle has no initial velocity and no force is applied to it. The particle doesn't move. Its position must be constant. But it could be anywhere, say at position C_1. Hence the general solution $u = C_1$, as can be found by direct integration.

• • •

b)

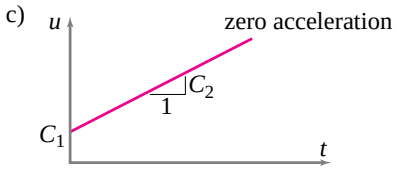


b) ODE: $\dot{u} = A \Rightarrow$ Soln: $u = At + C_1$.

$\dot{u} = A$ means the object has constant speed. This equation describes the motion of a particle that starts with speed $v_0 = A$ and because it has no force acting on it continues to move at constant speed. How far does it go in time t? It goes $v_0 t$. Where was it at time $t = 0$? It could have been anywhere then, say C_1. So where is it at time t? It's at its original position plus how far it has moved, $u = v_0 t + C_1$, as can also be found by direct integration.

• • •

c)

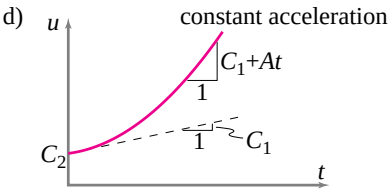


c) ODE: $\ddot{u} = 0 \Rightarrow$ Soln: $u = C_1 t + C_2$.

$\ddot{u} = 0$ means the acceleration is zero. That is, the rate of change of velocity is zero. This constant-velocity motion is the general equation for a particle with no force acting on it. The velocity, if not changing, must be constant. What constant? It could be anything, say C_1. Now we have the same situation as in case (b). So the position as a function of time is anything consistent with an object moving at constant velocity: $u = C_1 t + C_2$, where the constants C_1 and C_2 depend on the initial velocity and initial position. If you know that the position at $t = 0$ is u_0 and the velocity at $t = 0$ is v_0, then the position is $u = u_0 + v_0 t$.

• • •

d)

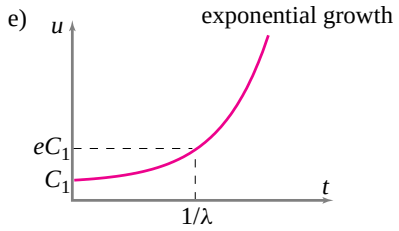


d) ODE: $\ddot{u} = A$

$\Rightarrow$ Soln: $u = At^2/2 + C_1 t + C_2$.

This constant acceleration A, constant rate of change of velocity, is the classic (all-too-often studied) case. This situation arises for vertical motion of an object in a constant gravitational field as well as in problems of constant acceleration or deceleration of vehicles. The velocity increases in proportion to the time that passes. The change in velocity in a given time is thus At and the velocity is $v = \dot{u} = v_0 + At$ (given that the velocity was v_0 at $t = 0$). Because the velocity is increasing constantly over time, the average velocity in a trip of length t occurs at $t/2$ and is $v_0 + At/2$. The distance traveled is the average velocity times the time of travel so the distance of travel is $t \cdot (v_0 + At/2) = v_0 t + At^2/2$. The position is the position at $t = 0$, u_0, plus the distance traveled since time zero. So $u = u_0 + v_0 t + At^2/2 = C_2 + C_1 t + At^2/2$. This solution can also be found by direct integration.

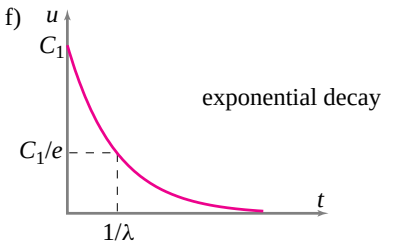
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e) ODE: $\dot{u} = \lambda u \;\Rightarrow$ Soln: $u = C_1 e^{\lambda t}$.

The displacement u grows in proportion to its present size. This equation describes the initial falling of an inverted pendulum in a thick viscous fluid. The bigger the u, the faster it moves. Such situations are called exponential growth (as in population growth or monetary inflation) for a good mathematical reason. The solution u is an exponential function of time: $u(t) = C_1 e^{\lambda t}$, as can be found by separating variables or guessing.

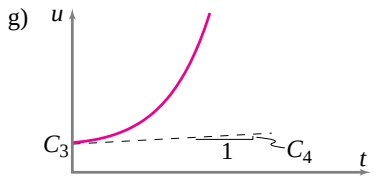
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f) ODE: $\dot{u} = -\lambda u \;\Rightarrow$ Soln: $u = C_1 e^{-\lambda t}$.

The smaller u is, the more slowly it gets smaller. u gradually tapers towards nothing: u decays exponentially. The solution to the equation is: $u(t) = C_1 e^{-\lambda t}$. This expression is essentially the same equation as in (e) above.

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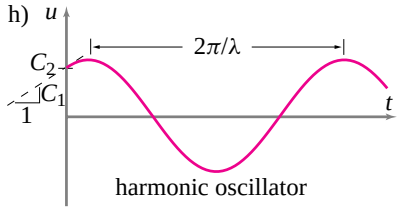
g) ODE: $\ddot{u} = \lambda^2 u$

$\Rightarrow$ Soln: $: u = C_1 e^{\lambda\, t} \;+\; C_2 e^{-\lambda\, t}$

$\Rightarrow u = C_3 \cosh(\lambda\, t) \;+\; C_4 \sinh(\lambda\, t)$.

Note, sinh and cosh are just combinations of exponentials. For $\ddot{u} = \lambda^2 u$, the point accelerates more and more away from the origin in proportion to the distance from the origin. This equation describes the falling of a nearly vertical inverted pendulum when there is no friction. Most often, the solution of this equation gives roughly exponential growth. The pendulum accelerates away from being upright. The reason there is also an exponentially decaying solution to this equation is a little more subtle to understand intuitively: if a not quite upright pendulum is given just the right initial velocity it will slowly approach becoming just upright with an exponentially decaying displacement. This decaying solution is not easy to see experimentally because without the perfect initial condition the exponentially growing part of the solution eventually dominates and the pendulum accelerates away from being just upright.

• • •



h) ODE: $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$

$\Rightarrow$ Soln: $u = C_1 \sin(\lambda\, t) + C_2 \cos(\lambda\, t)$.

This equation describes a mass that is restrained by a spring which is relaxed when the mass is at $u = 0$. When u is positive, $\ddot{u}$ is negative. That is, if the particle is on the right side of the origin it accelerates to the left. Similarly, if the particle is on the left it accelerates to the right. In the middle, where $u = 0$, it has no acceleration, so it neither speeds up nor slows down in its motion whether it is moving to the left or the right. So the particle goes back and forth: its position oscillates. A function that correctly describes this oscillation is $u = \sin(\lambda\, t)$, that is, sinusoidal oscillations. The oscillations are faster if λ is bigger. Another solution is $u = \cos(\lambda\, t)$. The general solution is $u = C_1 \sin(\lambda\, t) + C_2 \cos(\lambda\, t)$. A plot of this function reveals a sine wave shape for any value of C_1 or C_2, although the phase depends on the relative values of C_1 and C_2. The equation $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$ is called the 'harmonic oscillator' equation and is important in almost all branches of science. The solution may be found by guessing or other means (which are usually guessing in disguise).

• • •

i) There are a few other not-too-hard ODEs besides those listed in the box. For example, the general second order, constant coefficient ODE with sinusoidal forcing: $A\ddot{u} + B\dot{u} + Cu = F\sin(Dt)$. But the solution is a little more complicated and not quite so easily verified. So we save it for chapter 10 on vibrations. Most engineers, when confronted with an equation not on this list, will resort to a *numerical* computer solution.

• • •

SAMPLE 5.1 *Time derivatives:* The position of a particle varies with time as $\vec{r}(t) = (C_1 t + C_2 t^2)\hat{\imath}$, where $C_1 = 4\,\mathrm{m/s}$ and $C_2 = 2\,\mathrm{m/s^2}$.

(a) Find the velocity and acceleration of the particle as functions of time.

(b) Sketch the position, velocity, and acceleration of the particle against time from $t = 0$ to $t = 5\,\mathrm{s}$.

(c) Find the position, velocity, and acceleration of the particle at $t = 2\,\mathrm{s}$.

Solution

(a) We are given the position of the particle as a function of time. We need to find the velocity (time derivative of position) and the acceleration (time derivative of velocity).

$$\vec{r} = (C_1 t + C_2 t^2)\hat{\imath} = (4\,\mathrm{m/s}\,t + 2\,\mathrm{m/s^2}\,t^2)\hat{\imath} \tag{5.6}$$

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(C_1 t + C_2 t^2)\hat{\imath}$$

$$= (C_1 + C_2 t)\hat{\imath} = (4\,\mathrm{m/s} + 2\,\mathrm{m/s^2}\,t)\hat{\imath} \tag{5.7}$$

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d}{dt}(C_1 + C_2 t)\hat{\imath}$$

$$= C_2\hat{\imath} = (2\,\mathrm{m/s^2})\hat{\imath} \tag{5.8}$$

$$\boxed{\vec{v} = (4\,\mathrm{m/s} + 2\,\mathrm{m/s^2}\,t)\hat{\imath},\ \vec{a} = (2\,\mathrm{m/s^2})\hat{\imath}.}$$

Thus, we find that the velocity is a linear function of time and the acceleration is time-independent (a constant).

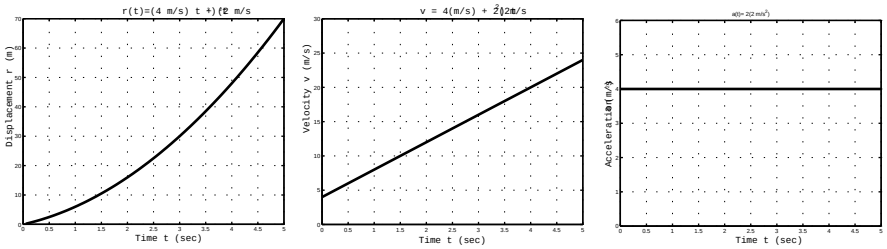


Figure 5.4: (Filename:sfig5.1.vraplot)

(b) We plot eqns. (5.6,5.7, and 5.8) against time by taking 100 points between $t = 0$ and $t = 5\,\mathrm{s}$, and evaluating $\vec{r}$, $\vec{v}$ and $\vec{a}$ at those points. The plots are shown below.

(c) We can find the position, velocity, and acceleration at $t = 2\,\mathrm{s}$ by evaluating their expressions at the given time instant:

$$\vec{r}(t = 2\,\mathrm{s}) = [(4\,\mathrm{m/s}) \cdot (2\,\mathrm{s}) + (2\,\mathrm{m/s^2}) \cdot (2\,\mathrm{s})^2]\hat{\imath}$$
$$= (16\,\mathrm{m})\hat{\imath}$$
$$\vec{v}(t = 2\,\mathrm{s}) = [(4\,\mathrm{m/s}) + (2\,\mathrm{m/s^2}) \cdot (2\,\mathrm{s})]\hat{\imath}$$
$$= (8\,\mathrm{m/s})\hat{\imath}$$
$$\vec{a}(t = 2\,\mathrm{s}) = (2\,\mathrm{m/s^2})\hat{\imath} = \vec{a}(\text{at all t})$$

$$\boxed{\text{At } t = 2\,\mathrm{s},\ \vec{r} = (16\,\mathrm{m})\hat{\imath},\ \vec{v} = (8\,\mathrm{m/s})\hat{\imath},\ \vec{a} = (2\,\mathrm{m/s^2})\hat{\imath}.}$$

SAMPLE 5.2 *Math review: Solving simple differential equations.* For the following differential equations, find the solution for the given initial conditions.

(a) $\frac{dv}{dt} = a,\ v(t=0) = v_0$, where a is a constant.

(b) $\frac{d^2x}{dt^2} = a,\ x(t=0) = x_0,\ \dot{x}(t=0) = \dot{x}_0$, where a is a constant.

Solution

(a)

$$\begin{aligned} \frac{dv}{dt} &= a \quad \Rightarrow \quad dv = a\,dt \\ \text{or} \qquad \int dv &= \int a\,dt = a\int dt \\ \text{or} \qquad v &= at + C, \quad \text{where C is a constant of integration} \end{aligned}$$

Now, substituting the initial condition into the solution, $v(t=0) = v_0 = a\cdot 0 + C \quad \Rightarrow \quad C = v_0$. Therefore,

$$v = at + v_0.$$

$$\boxed{v = at + v_0}$$

Alternatively, we can use definite integrals:

$$\int_{v_0}^{v} dv = \int_0^t a\,dt \quad \Rightarrow \quad v - v_0 = at \quad \Rightarrow \quad v = v_0 + at.$$

(b) This is a second order differential equation in x. We can solve this equation by first writing it as a first order differential equation in $v \equiv dx/dt$, solving for v by integration, and then solving again for x in the same manner.

$$\begin{aligned} \frac{d^2x}{dt^2} &= a \quad \text{or} \quad \frac{dv}{dt} = a \\ \text{or} \qquad \int dv &= \int a\,dt \\ \Rightarrow \quad v \equiv \dot{x} &= at + C_1 \qquad (5.9) \\ \text{but, } v \equiv \frac{dx}{dt}, \quad \Rightarrow \quad \int dx &= \int at\,dt + \int C_1\,dt \\ \text{or} \qquad x &= \frac{1}{2}at^2 + C_1 t + C_2, \qquad (5.10) \end{aligned}$$

where C_1 and C_2 are constants of integration. Substituting the initial condition for $\dot{x}$ in Eqn. (5.9), we get

$$\dot{x}(t=0) = \dot{x}_0 = a\cdot 0 + C_1 \quad \Rightarrow \quad C_1 = \dot{x}_0.$$

Similarly, substituting the initial condition for x in Eqn. (5.10), we get

$$x(t=0) = x_0 = \frac{1}{2}a\cdot 0 + \dot{x}_0\cdot 0 + C_2 \quad \Rightarrow \quad C_2 = x_0.$$

Therefore,

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}at^2.$$

$$\boxed{x(t) = x_0 + \dot{x}_0 t + \tfrac{1}{2}at^2}$$

SAMPLE 5.3 *Constant velocity motion:* A particle travels with constant velocity $\vec{v} = 5\,\mathrm{m/s}\hat{\imath}$. The initial position of the particle is $\vec{r}_0 = 2\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\hat{\jmath}$. Find the position of the particle at $t = 3\,\mathrm{s}$.

Solution Here, we are given the velocity, *i.e.*, the time derivative of position:

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = v_0\hat{\imath}, \qquad \text{where } v_0 = 5\,\mathrm{m/s}.$$

We need to find $\vec{r}$ at $t = 3\,\mathrm{s}$, given that $\vec{r}$ at $t = 0$ is $\vec{r}_0$.

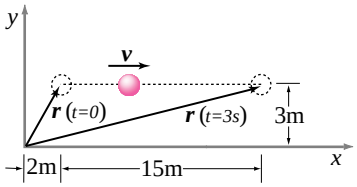


Figure 5.5: (Filename:sfig5.1.new1)

$$\begin{aligned}
d\vec{r} &= v_0\hat{\imath}\,dt \\
\Rightarrow \quad \int_{\vec{r}_0}^{\vec{r}(t)} d\vec{r} &= \int_0^t v_0\hat{\imath}\,dt = v_0\hat{\imath}\int_0^t dt \\
\vec{r}(t) - \vec{r}_0 &= v_0\hat{\imath}t \\
\vec{r}(t) &= \vec{r}_0 + v_0 t\hat{\imath} \\
\vec{r}(3\,\mathrm{s}) &= (2\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\hat{\jmath}) + (5\,\mathrm{m/s})\cdot(3\,\mathrm{s})\hat{\imath} \\
&= 17\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r} = 17\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\hat{\jmath}}$$

SAMPLE 5.4 A ship cruises at a constant speed of 15 knots per hour due Northeast. It passes a lighthouse at 8:30 am. The next lighthouse is approximately 35 knots straight ahead. At what time does the ship pass the next lighthouse?

Solution We are given the distance s and the speed of travel v. We need to find how long it takes to travel the given distance.

$$\begin{aligned}
s &= vt \\
\Rightarrow \quad t &= \frac{s}{v} = \frac{35\text{knots}}{15\text{knots/hour}} = 2.33\text{ hrs}.
\end{aligned}$$

Now, the time at $t = 0$ is 8:30 am. Therefore, the time after 2.33 hrs (2 hours 20 minutes) will be 10:50 am.

$$\boxed{10:50\text{ am}}$$

SAMPLE 5.5 *Constant acceleration:* A 0.5 kg mass starts from rest and attains a speed of $20\,\mathrm{m/s}\hat{\boldsymbol{\imath}}$ in 4 s. Assuming that the mass accelerates at a constant rate, find the force acting on the mass.

Solution Here, we are given the initial velocity $\vec{\boldsymbol{v}}(0) = \vec{\boldsymbol{0}}$ and the final velocity $\vec{\boldsymbol{v}}$ after $t = 4\,\mathrm{s}$. We have to find the force acting on the mass. The net force on a particle is given by $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. Thus, we need to find the acceleration $\vec{\boldsymbol{a}}$ of the mass to calculate the force acting on it. Now, the velocity of a particle under constant acceleration is given by

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{v}}_0 + \vec{\boldsymbol{a}}t$$

. Therefore, we can find the acceleration $\vec{\boldsymbol{a}}$ as

$$\begin{aligned}\vec{\boldsymbol{a}} &= \frac{\vec{\boldsymbol{v}}(t) - \vec{\boldsymbol{v}}(0)}{t} \\ &= \frac{20\,\mathrm{m/s}\hat{\boldsymbol{\imath}} - \vec{\boldsymbol{0}}}{4\,\mathrm{s}} \\ &= 5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}.\end{aligned}$$

The force on the particle is

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} = (0.5\,\mathrm{kg}) \cdot (5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}) = 2.5\,\mathrm{N}\hat{\boldsymbol{\imath}}.$$

$$\boxed{\vec{\boldsymbol{F}} = 2.5\,\mathrm{N}\hat{\boldsymbol{\imath}}}$$

SAMPLE 5.6 *Time of travel for a given distance:* A ball of mass 200 gm falls freely under gravity from a height of 50 m. Find the time taken to fall through a distance of 30 m, given that the acceleration due to gravity $g = 10\,\mathrm{m/s^2}$.

Solution The entire motion is in one dimension — the vertical direction. We can, therefore, use scalar equations for distance, velocity, and acceleration. Let y denote the distance travelled by the ball. Let us measure y vertically downwards, starting from the height at which the ball starts falling (see Fig. 5.6). Under constant acceleration g, we can write the distance travelled as

$$y(t) = y_0 + v_0 t + \frac{1}{2}gt^2.$$

Note that at $t = 0$, $y_0 = 0$ and $v_0 = 0$. We are given that at some instant t (that we need to find) $y = 30\,\mathrm{m}$. Thus,

$$\begin{aligned}y &= \frac{1}{2}gt^2 \\ t &= \sqrt{\frac{2y}{g}} = \sqrt{\frac{2 \times 30\,\mathrm{m}}{10\,\mathrm{m/s^2}}} = 2.45\,\mathrm{s}\end{aligned}$$

$$\boxed{t = 2.45\,\mathrm{s}}$$

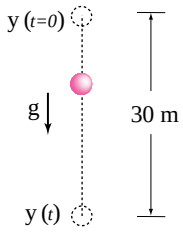


Figure 5.6: (Filename:sfig5.2.new3)

SAMPLE 5.7 *Numerical integration of ODE's:*

(a) Write the second order linear nonhomogeneous differential equation, $\ddot{x} + c\dot{x} + kx = a_0 \sin \omega t$, as a set of first order equations that can be used for numerical integration.

(b) Write the second order nonlinear homogeneous differential equation, $\ddot{x} + c\dot{x}^2 + kx^3 = 0$, as a set of first order equations that can be used for numerical integration.

(c) Solve the nonlinear equation given in (b) by numerical integration taking $c = 0.05,\ k = 1,\ x(0) = 0$, and $\dot{x}(0) = 0.1$. Compare this solution with that of the linear equation in (a) by setting $a_0 = 0$ and taking other values to be the same as for (b).

Solution

(a)

$$\begin{aligned} \text{If we let} \quad \dot{x} &= y, \\ \text{then} \quad \dot{y} &= \ddot{x} = -c\dot{x} - kx + a_0 \sin \omega t \\ &= -cy - kx + a_0 \sin \omega t \end{aligned}$$

$$\text{or} \qquad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ a_0 \sin \omega t \end{Bmatrix}. \tag{5.11}$$

Equation (5.11) is written in matrix form to show that it is a set of *linear* first-order ODE's. In this case linearity means that the dependent variables only appear linearly, not as powers *etc*.

(b)

$$\begin{aligned} \text{If} \quad \dot{x} &= y \\ \text{then} \quad \dot{y} &= \ddot{x} = c\dot{x} - kx^3 = -cy - kx^3 \end{aligned}$$

$$\text{or} \qquad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ -cy - k^3 \end{Bmatrix}. \tag{5.12}$$

Equation (5.12) is a set of *nonlinear* first order ODE's. It cannot be arranged as Eqn. 5.11 because of the nonlinearity in x and x. It is, however, in an appropriate form for numerical integration.

(c) Now we solve the set of first order equations obtained in (b) using a numerical ODE solver with the following pseudocode.

```
ODEs = {xdot = y,  ydot = -c y - k x^3}
IC   = {x(0) = 0, y(0) = 0.1}
Set  k=1, c=0.05
Solve ODEs with IC for t=0 to t=200
Plot x(t) and y(t)
```

The plot obtained from numerical integration using a Runge-Kutta based integrator is shown in Fig. 5.7. A similar program used for the equation in (a) with $a_0 = 0$ gives the plot shown in Fig. 5.8. The two plots show how a simple nonlinearity changes the response drastically.

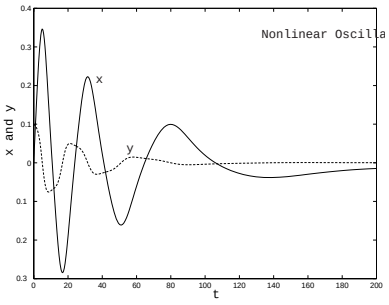


Figure 5.7: Numerical solution of the nonlinear ODE $\ddot{x} + c\dot{x}^2 + kx^3 = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

(Filename:sfig5.1.nonlin)

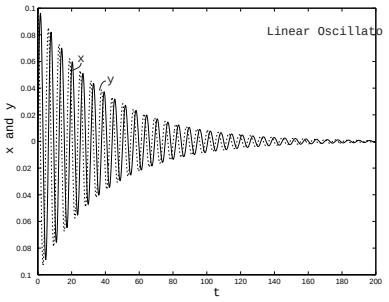


Figure 5.8: Numerical solution of the linear ODE $\ddot{x} + c\dot{x}^2 + kx = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

(Filename:sfig5.1.lin)

5.2 Energy methods in 1D

Energy is an important concept in science and is even a kind of currency in human trade. But for us now, an energy equations is primarily a short-cut for solving some mechanics problems.

The work-energy equation

On the inside cover the third basic law of mechanics is energy balance. Energy balance takes a number of different forms, depending on context. The kinetic energy of a particle is defined as

$$E_{\mathrm{K}} = \frac{1}{2} m_{\mathrm{tot}} v^2.$$

The power balance equation is thus, in rate form,

$$P = \frac{d}{dt}\left(\frac{1}{2} m v^2\right),$$

where $P = Fv$ is the power of the applied force F. Integrating in time we get, using that $v = dx/dt$,

$$\begin{aligned} \Rightarrow \quad \int F v dt &= \int \tfrac{d}{dt}\left(\tfrac{1}{2} m v^2\right) dt \\ \Rightarrow \quad \int F\, dx &= \Delta\left(\tfrac{1}{2} m v^2\right). \\ \Rightarrow \quad W &= \Delta E_{\mathrm{K}} \end{aligned} \tag{5.13}$$

The integral $W = \int F dx$ is called the work. The derivations above, from the general equations to the particle equations, are the opposite of historical. As Box 5.1 on page 225 shows, in this case the work-energy equation can be derived from the momentum-balance equation. In fact it is this one-dimensional mechanical case that first led to the discovery of energy as a concept. But now that we know that $F = ma$ implies that work is change in kinetic energy, we can use the result without deriving it every time.

Conservation of energy

One of the most useful intuitive concepts for simple mechanics problems is conservation of energy. So far we know that the work of a force on a particle gives its change of energy (eqn. 5.13). But some forces come from a source that has associated with it a potential energy. If, for example, the force to the right on a particle is a function of x (and not, say, of $\dot{x}$) then we have a *force field*. In one dimension we can define a new function of x that we will call $E_{\mathrm{P}}(x)$ as the integral of the force with respect to x:

$$E_{\mathrm{P}}(x) = -\int_0^x F(x')dx' = -(\text{Work done by the force in moving from 0 to } x.) \tag{5.14}$$

Note also, by the fundamental theorem of calculus, that given $E_{\mathrm{P}}(x)$ we can find $F(x)$ as

$$F(x) = -\frac{dE_{\mathrm{P}}(x)}{dx}.$$

Now let's consider the work done by the force on the particle when the particle moves from point x_1 to x_2. It is

$$\text{Work done by force from } x_1 \text{ to } x_2 = -(E_{\mathrm{P}2} - E_{\mathrm{P}1}) = -\Delta E_{\mathrm{P}}.$$

That is, the decrease in E_P is the amount of work that the force does. Or, in other words, E_P represents a potential to do work. Because work causes an increase in kinetic energy, E_P is called the *potential energy* of the force field. Now we can compare this result with the work-energy equation 5.13 to find that

$$-\Delta E_P = \Delta E_K \qquad \Rightarrow \qquad 0 = \Delta \underbrace{(E_P + E_K)}_{E_T}.$$

The *total energy* E_T doesn't change ($\Delta E_T = 0$) and thus is a constant. In other words,

> as a particle moves in the presence of a force field with a potential energy, the total energy $E_T = E_K + E_P$ is constant.

This fact goes by the name of *conservation of energy*.

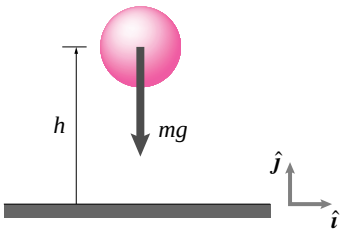


Figure 5.9: Free body diagram of a falling ball, assuming gravity is the only significant external force acting on the ball.

(Filename:tfigure1.falling.ball)

Example: **Falling ball**

Consider the ball in the free body diagram 5.9. If we define gravitational potential energy as minus the work gravity does on a ball while it is lifted from the ground, then

$$E_P = -\int_0^y (-mg)\, dy' = mgy = mgh.$$

For vertical motion

$$E_K = \frac{1}{2} m\dot{y}^2.$$

So conservation of energy says that in free fall:

$$\text{Constant} = E_P + E_K = mgy + m\dot{y}^2$$

which you can also derive directly from $m\ddot{y} = -mg$. Alternatively, we could start with conservation of energy and differentiate to get

$$\begin{aligned} E_T = \text{constant} \quad \Rightarrow \quad 0 &= \tfrac{d}{dt} E_T \\ &= \tfrac{d}{dt}(E_P + E_T) \\ &= \tfrac{d}{dt}(mgy + m\dot{y}^2/2) \\ &= (mg\dot{y} + m\dot{y}\ddot{y}) \\ \Rightarrow \quad m\ddot{y} &= -mg \end{aligned}$$

where we had to assume (and this is just a technical point) that $\dot{y} \neq 0$ in one of the cancellations. Thus, for this problem, energy balance can be used to derive linear-momentum balance.

We could also start with the power-balance equation,

$$\begin{aligned} \text{power of gravity force} &= \text{rate of change in particle's kinetic energy} \\ P &= \frac{d}{dt}(E_K) \\ \vec{F} \cdot \vec{v} &= \frac{d}{dt}(E_K) \\ (-mg\hat{j}) \cdot (\dot{y}\hat{j}) &= \frac{d}{dt}\left[\frac{1}{2} m v^2\right] \end{aligned}$$

$$\begin{aligned} -mg\dot{y} &= \frac{1}{2}m\frac{d}{dt}\left(v^2\right) \\ -mg\dot{y} &= \frac{1}{2}m\frac{d}{dt}\left(\dot{y}^2\right) \\ m\ddot{y} &= -mg, \end{aligned}$$

and again get the same result. Thus, for one dimensional particle motion, momentum balance, power balance, and energy balance can each be derived from either of the others. □

5.3 *THEORY*
Derivation of the work energy equation

Because $F = ma$, all our kinematics calculations above turn into dynamics calculations by making the substitution F/m every place that a appears. Equation 5.5, for example, becomes

$$(v(x))^2 = (v(x_0))^2 + \frac{2}{m}\int_{x_0}^{x} F(x^*)\,dx^*.$$

In box 5.1 on page 225 we found that

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \int_{x_0}^{x} a(x^*)dx^*.$$

If we multiply both sides of all equations in the above derivation by m and substitute F for ma the derivation above shows that

$$\underbrace{\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2}_{\Delta E_{\mathrm{K}}} = \underbrace{\int_0^x \overbrace{F(x^*)}^{ma(x^*)}\,dx^*}_{\text{work done by a force}}$$

For straight-line motion with a force in only one direction on a particle, we have no heat flow, dissipation, or internal energy to fuss over so that the energy equation (III) from the inside front cover has been derived.

Alternatively, if you can remember the work-energy equation ('The positive work of a force on a particle is the positive change in kinetic energy'), you can use it to recall the related kinematics equation. For example, if F and a are constant, and x is the total displacement,

$$\begin{aligned} \Delta(\frac{1}{2}mv^2) &= F\Delta x, \text{ and} \\ \Rightarrow \Delta(\frac{1}{2}v^2) &= a\Delta x. \end{aligned}$$

SAMPLE 5.8 How much time does it take for a car of mass 800 kg to go from 0 mph to 60 mph, if we assume that the engine delivers a constant power P of 40 horsepower during this period. (1 horsepower = 745.7 W)

Solution

$$\begin{aligned} P &= \dot{W} \equiv \frac{dW}{dt} \\ dW &= P\,dt \\ W_{12} &= \int_{t_0}^{t_1} P\,dt = P(t_1 - t_0) = P\,\Delta t \\ \Delta t &= \frac{W_{12}}{P}. \end{aligned}$$

Now, from IIIa in the inside front cover,

$$\begin{aligned} W_{12} &= (E_{\mathrm{K}})_2 - (E_{\mathrm{K}})_1 \\ &= \frac{1}{2}m(v_2^2 - v_1^2) \\ &= \frac{800\,\mathrm{kg}[(60\,\mathrm{mph})^2 - 0]}{2} \\ &= \frac{1}{2}\cdot 800\,\mathrm{kg}\left(60\frac{\mathrm{mi}}{\mathrm{hr}}\cdot\frac{1.61\times 10^3\,\mathrm{m}}{1\,\mathrm{mi}}\cdot\frac{1\,\mathrm{hr}}{3600\,\mathrm{s}}\right)^2 \\ &= 288.01\times 10^3\ \mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{m/s^2} \\ &= 288\ \mathrm{KJoule}. \end{aligned}$$

Therefore,

$$\Delta t = \frac{288\times 10^3\,\mathrm{J}}{40\times 745.7\,\mathrm{W}} = 9.66\,\mathrm{s}.$$

Thus it takes about 10 s to accelerate from a standstill to 60 mph.

$$\boxed{\Delta t = 9.66\,\mathrm{s}}$$

Note 1: This model gives a roughly realistic answer *but* it is not a realistic model, at least at the start, at time t_0. In the model here, the acceleration is infinite at the start (the power jumps from zero to a finite value at the start, when the velocity is zero), something the finite-friction tires would not allow.

Note 2: We have been a little sloppy in quoting the energy equation. Since there are no external forces doing work on the car, somewhat more properly we should perhaps have written

$$0 = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{int}} + \dot{E}_{\mathrm{P}}$$

and set $-(\dot{E}_{\mathrm{int}} + \dot{E}_{\mathrm{P}}) =$ 'the engine power' where the engine power is from the decrease in gasoline potential energy ($-\dot{E}_{\mathrm{P}}$ is positive) less the increase in 'heat' ($\dot{E}_{\mathrm{int}}$) from engine inefficiencies.

SAMPLE 5.9 *Energy of a mass-spring system.* A mass $m = 2\,\text{kg}$ is attached to a spring with spring constant $k = 2\,\text{kN/m}$. The relaxed (unstretched) length of the spring is $\ell = 40\,\text{cm}$. The mass is pulled up and released from rest at position A shown in Fig. 5.10. The mass falls by a distance $h = 10\,\text{cm}$ before reaching position B, which is the relaxed position of the spring. Find the speed at point B.

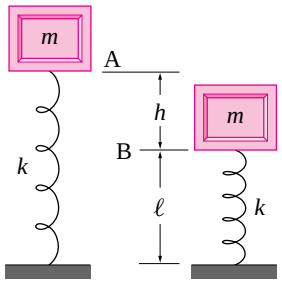


Figure 5.10: (Filename:sfig2.6.1)

Solution The total energy of the mass-spring system at any instant or position consists of the energy stored in the spring and the sum of potential and kinetic energies of the mass. For potential energy of the mass, we need to select a datum where the potential energy is zero. We can select any horizontal plane to be the datum. Let the ground support level of the spring be the datum. Then, at position A,

$$\begin{aligned}\text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = \frac{1}{2}kh^2\\ \text{Energy of the mass} &= E_\text{K} + E_\text{P} = \frac{1}{2}m\underbrace{v_A^2}_{0} + mg(\ell+h) = mg(\ell+h).\end{aligned}$$

Therefore, the total energy at position A

$$E_A = \frac{1}{2}kh^2 + mg(\ell+h).$$

Let the speed of the mass at position B be v_B. When the mass is at B, the spring is relaxed, i.e., there is no stretch in the spring. Therefore, at position B,

$$\begin{aligned}\text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = 0\\ \text{Energy of the mass} &= E_\text{K} + E_\text{P} = \frac{1}{2}mv_B^2 + mg\ell,\end{aligned}$$

and the total energy

$$E_B = \frac{1}{2}mv_B^2 + mg\ell.$$

Because the net change in the total energy of the system from position A to position B is

$$\begin{aligned}0 &= \Delta E\\ &= E_A - E_B = \frac{1}{2}kh^2 + mg(\ell+h) - \frac{1}{2}mv_B^2 - mg\ell\\ &= \frac{1}{2}(kh^2 - mv_B^2) + mgh\\ \Rightarrow\quad v_B^2 &= kh^2/m + 2gh\\ \Rightarrow\quad |v_B| &= \left(kh^2/m + 2gh\right)^{1/2}\\ &= \left((2000\,\text{N/m}\cdot(0.1\,\text{m})^2/2\,\text{kg}) + 2\cdot 9.81\,\text{m/s}^2\cdot 0.1\,\text{m}\right)^{1/2}\\ &= 3.46\,\text{m/s}.\end{aligned}$$

$$\boxed{|v_B| = 3.46\,\text{m/s}}$$

SAMPLE 5.10 *Which is the best bicycle helmet?* Assume a bicyclist moves with speed v_0 when her head hits a brick wall. Assume her head is rigid and that it has constant deceleration as it travels through the 2 inches of the bicycle helmet. What is the deceleration? What force is required? (Neglect force from the neck on the head.)

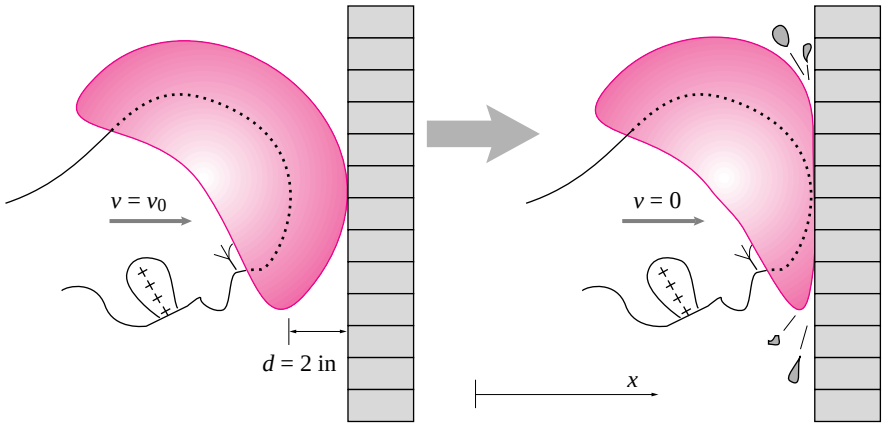


Figure 5.11: (Filename:sfig3.2.ouchie)

Solution 1 – Kinematics method 1: We are given the initial speed of V_0, a final speed of 0, and a constant acceleration a (which is negative) over a given distance of travel d. If we call t_c the time when the helmet is fully crushed,

$$
\begin{aligned}
v(t) &= v_0 + \int_0^{t_c} a(t')dt' \\
&= v_0 + at_c \\
0 = v(t_c) &= v_0 + at_c \quad \Rightarrow \quad t_c = -v_0/a \qquad (5.15) \\
x(t) &= x_0 + \int_0^{t_c} v(t')dt' \\
&= 0 + \int_0^{t_c} (v_0 + at)dt \\
d = x(t_c) &= 0 + v_0 t_c + a t_c^2/2 \\
d &= v_0\left(\frac{-v_0}{a}\right) + a\left(\frac{v_0}{a}\right)^2/2 \quad \Rightarrow \quad d = \frac{-v_0^2}{2a} \qquad \text{(using (5.15))} \\
\Rightarrow \quad a &= \frac{-v_0^2}{2d} \\
&= \frac{-(25\,\text{mph})^2}{2\cdot(2\,\text{in})} \\
&= \frac{-25^2}{4}\cdot\frac{\text{mi}^2}{\text{hr}^2\cdot\text{in}}\cdot\underbrace{\left(\frac{5280\,\text{ft}}{\text{mi}}\right)^2}_{1}\cdot\underbrace{\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)^2}_{1}\cdot\underbrace{\left(\frac{12\,\text{in}}{\text{ft}}\right)}_{1}\cdot\underbrace{\left(\frac{1g}{32.2\,\text{ft/s}^2}\right)}_{1} \\
&= \frac{-25}{4}\cdot\frac{5280^2}{3600^2}\cdot 12\cdot\frac{1}{32.2}g \\
a &= -125g
\end{aligned}
$$

To stop from 25 mph in 2 inches requires an acceleration that is 125 times that of gravity.

Solution 2 – Kinematics method 2:

$$\begin{aligned}
\frac{dv}{dt} &= a \quad \Rightarrow \quad dv = a\,dt \\
\Rightarrow \quad v\,dv &= a\,v\,dt \quad \Rightarrow \quad v\,dv = a\frac{dx}{dt}dt \\
\Rightarrow \quad v\,dv &= a\,dx \\
\Rightarrow \quad \int v\,dv &= \int a\,dx \\
\Rightarrow \quad \Delta\frac{v^2}{2} &= ax \qquad \text{(since } a = \text{constant)} \\
\Rightarrow \quad 0 - \frac{v_0^2}{2} &= ad \quad \Rightarrow \quad a = \frac{-v_0^2}{2d} \qquad \text{(as before)}
\end{aligned}$$

Solution 3 – Quote formulas:

$$\begin{aligned}
\text{“}v &= \sqrt{2ad}\text{”} \\
\Rightarrow \quad a &= \frac{v^2}{2d} \qquad \text{which is right if you know how to interpret it!}
\end{aligned}$$

Solution 4 – Work-Energy:
Constant acceleration $\Rightarrow$ constant force

$$\begin{aligned}
\text{Work in} &= \Delta E_{\text{K}} \\
-Fd &= 0 - \frac{mv_0^2}{2} \\
F &= \frac{mv_0^2}{2d} \\
\text{But } \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \quad &\Rightarrow \quad -F\hat{\boldsymbol{\imath}} = -ma\hat{\boldsymbol{\imath}} \\
&\Rightarrow \quad a = \frac{-F}{m} \\
\text{So } a &= \frac{-v_0^2}{2d} \qquad \text{(again)}
\end{aligned}$$

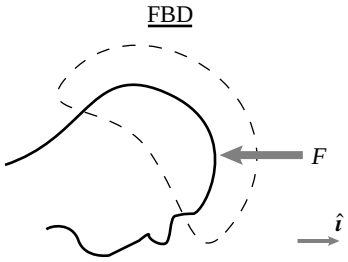


Figure 5.12: F is the force of the helmet on the moving head.
(Filename:sfig3.2.ouchie.fbd)

Assuming a head mass of 8 lbm, the force on the head during impact is

$$\begin{aligned}
|F| &= \tfrac{mv_0^2}{2d} = ma = 8\,\text{lbm} \cdot 125g \\
\boxed{|F| &= 1000\,\text{lbf}}
\end{aligned}$$

During a collision in which an 8 lbm head decelerates from 25 mph to 0 in 2 inches, the force applied to the head is 1000 lbf.

Note 1: The way to minimize the peak acceleration when stopping from a given speed over a given distance is to have constant acceleration. The ‘best’ possible helmet, the one we assumed, causes constant deceleration. There is no helmet of any possible material with 2 in thickness that could make the deceleration for this collision less than $125g$ or the peak force less than 1000 lbf.

Note 2: Collisions with head decelerations of $250g$ or greater are often fatal. Even $125g$ usually causes brain injury. So, the best possible helmet does not insure against injury for fast riders hitting solid objects.

Note 3: Epidemiological evidence suggests that, on average, chances of serious brain injury are decreased by about a factor of 5 by wearing a helmet.

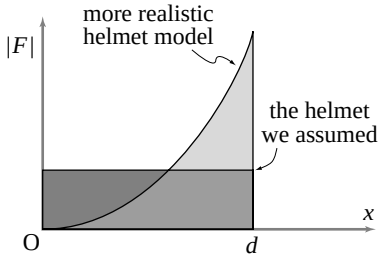


Figure 5.13: (Filename:sfig3.2.ouchie.graph)

5.3 The harmonic oscillator

Most engineering materials are nearly elastic under working conditions. And, of course, all real things have mass. These ingredients, elasticity and mass, are what make vibration possible. Even structures which are fairly rigid will vibrate if encouraged to do so by the shaking of a rotating motor, the rough rolling of a truck, or the ground motion of an earthquake. The vibrations of a moving structure can also excite oscillations in flowing air which can in turn excite the structure further. This mutual excitement of fluids and solids is the cause of the vibrations in a clarinet reed, and may have been the source of the wild oscillations in the famous collapse of the Tacoma Narrows bridge. Mechanical vibrations are not only the source of most music but also of most annoying sounds. They are the main function of a vibrating massager, and the main defect of a squeaking hinge. Mechanical vibrations in pendula or quartz crystals are used to measure time. Vibrations can cause a machine to go out of control, or a buildings to collapse. So the study of vibrations, for better or for worse, is not surprisingly one of the most common applications of dynamics.

When an engineer attempts to understand the oscillatory motion of a machine or structure, she undertakes a *vibration* analysis. A vibration analysis is a study of the motions that are associated with vibrations. Study of motion is what dynamics is all about, so vibration analysis is just a part of dynamics.

A vibration analysis could mean the making of a dynamical model of the structure one is studying, writing equations of motion using the momentum balance or energy equations and then looking at the solution of these equations. But, in practice, the motions associated with vibrations have features which are common to a wide class of structures and machines. For this reason, a special vocabulary and special methods of approach have been developed for vibration analysis. For example, one can usefully discuss *resonance*, *normal modes*, and *frequency response*, concepts which we will soon discuss, without ever writing down any equations of motion. We will first approach these concepts within the framework of the differential equations of motion and their solutions. But after the concepts have been learned, we can use them without necessarily referring directly to the governing differential equations.

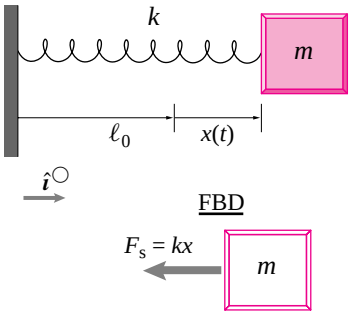


Figure 5.14: A spring mass system.

(Filename:tfigure3.MS)

> The unforced oscillations of a spring and mass is the basic model for all vibrating systems.

So it is worth knowing well.

We start with a free body diagram of a mass which is cut from a spring in an extended state, as shown in figure 5.14. The mass slides on a frictionless surface. The spring is relaxed at $x = 0$. The spring is thus stretched from ℓ_0 to $\ell_0 + \Delta\ell$, a stretch of $\Delta\ell = x$. The free body diagram at the bottom shows the force on the mass. Gravity is neglected.

Linear momentum balance in the x direction ($\{\sum \vec{F} = \dot{\vec{L}}\} \cdot \hat{\imath}$) gives:

$$\begin{aligned} \sum F_x &= \dot{L}_x \\ -kx &= m\ddot{x}. \end{aligned}$$

Rearranging this equation we get one of the most famous and useful differential equations of all time:

$$\ddot{x} + \frac{k}{m}x = 0. \tag{5.16}$$

This equation appears in many contexts both in and out of dynamics. In non-mechanical contexts the variable x and the parameter combination k/m are replaced by other physical quantities. In an electrical circuit, for example, x might represent a voltage and the term corresponding to k/m might be $1/LC$, where C is a capacitance and L an inductance. But even in dynamics the equation appears with other physical quantities besides k/m multiplying the x, and x itself could represent rotation, say, instead of displacement. In order to avoid being specific about the physical system being modeled, the harmonic oscillator equation is often written as

$$\ddot{x} + \lambda^2 x = 0. \tag{5.17}$$

The constant in front of the x is called λ^2 instead of just, say, λ ('lambda')①, for two reasons:

(a) This convention shows that λ^2 is positive,
(b) In the solution we need the square root of this coefficient, so it is convenient to have $\sqrt{\lambda^2} = \lambda$.

For the spring-block system, λ^2 is k/m and in other problems λ^2 is some other combination of physical quantities.

① **Caution:** Most books use p^2 or ω^2 in the place we have put λ^2. Using ω ('omega') can lead to confusion because we will later use ω for angular velocity. If one is studying vibrations of a rotating shaft then there would be two very different ω's in the problem. One, the coefficient of a differential equation and, the other, the angular velocity. To add to the confusion, this coincidence of notation is not accidental. Simple harmonic oscillations and circular motion have a deep connection. Despite this deep connection, the ω in the differential equation is not the same thing as the ω describing angular motion of a physical object. We avoid this confusion by using λ instead of ω. Note that this λ is unrelated to the unit vector $\hat{\boldsymbol{\lambda}}$ that we use in some problems.

Solution of the harmonic oscillator differential equation

Finding solutions to the harmonic oscillator differential equation 5.17 from first principles is a topic for a math class. Here we content ourselves with remembering its general solution, namely

$$\begin{aligned} x(t) &= A\cos(\lambda t) + B\sin(\lambda t), \\ \text{or}\quad x(t) &= C_1\cos(\lambda t) + C_2\sin(\lambda t). \end{aligned} \tag{5.18}$$

This sum of sine waves②is a solution of differential equation 5.17 for any values of the constants A (or C_1) and B (or C_2).

What does it means to say "$u = C_1\sin(\lambda t) + C_2\cos(\lambda t)$ solves the equation: $\ddot{u} = -\lambda^2 u$?" The solution is a function that has the property that its second derivative is the same as minus the original function multiplied by the constant λ^2. That is, the function $u(t) = C_1\sin(\lambda t) + C_2\cos(\lambda t)$ has the property that its second derivative is the original function multiplied by $-\lambda^2$. You need not take this property on faith.

② A cosine function is also a sine wave.

To check if a function is a solution, plug it into the differential equation and see if an identity is obtained.

Is this equality correct for the proposed $u(t)$?

$$\begin{aligned}
\frac{d^2}{dt^2}u &= -\lambda^2 u \\
\frac{d^2}{dt^2}\underbrace{[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)]}_{u(t)} &\stackrel{?}{=} -\lambda^2\underbrace{[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)]}_{u(t)} \\
\frac{d}{dt}\left(\frac{d}{dt}[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)]\right) &\stackrel{?}{=} -\lambda^2[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)] \\
\frac{d}{dt}[C_1\lambda\cos(\lambda\,t) - C_2\lambda\sin(\lambda\,t)] &\stackrel{?}{=} -\lambda^2[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)] \\
\underbrace{-C_1\lambda^2\sin(\lambda t) - C_2\lambda^2\cos(\lambda\,t)}_{\ddot{u}} &\stackrel{\surd}{=} -\lambda^2\underbrace{[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)]}_{u(t)}
\end{aligned}$$

The equation $\ddot{u} = -\lambda^2 u$ *does* hold with the given $u(t)$

This calculation verifies that, no matter what the constants C_1 and C_2, the proposed solution satisfies the given differential equation.

Although we have checked the solution, we have not proved its uniqueness. That is, there might be other solutions to the differential equation. There are not. We leave discussion of uniqueness to your math classes.

Interpreting the solution of the harmonic oscillator equation

The solution above means that if we built a system like that shown in figure 5.14 and watched how the mass moved, it would move (approximately) so that $x(t) = A\cos(\lambda\,t) + B\sin(\lambda\,t)$, as shown in the graph in figure 5.15.

This back and forth motion is called *vibration*. One might think that vibrations are fast oscillations. But in mechanics anything that oscillates a vibration. For example, the slow rocking of a ship might be called a vibration.

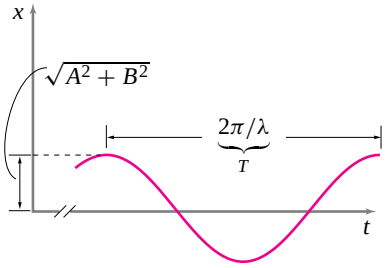


Figure 5.15: **Position versus time for an undamped, unforced harmonic oscillator.** x is the position of the mass, t is time.

(Filename:tfigure12.sinewave)

Angular frequency, period, and frequency

Three related measures of the rate of oscillation are angular frequency, period, and frequency. The simplest of these is *angular frequency* $\lambda = \sqrt{(k/m)}$, sometimes called *circular frequency*. The period T is the amount of time that it takes to complete one oscillation. One oscillation of both the *sine* function and the *cosine* function occurs when the argument of the function advances by 2π, that is when

$$\lambda T = 2\pi, \qquad \text{so} \qquad T = \frac{2\pi}{\lambda} = \frac{2\pi}{\sqrt{(k/m)}},$$

formulas often memorized in elementary physics courses. The *natural frequency* f is the reciprocal of the period

$$f = \frac{1}{T} = \frac{\lambda}{2\pi} = \frac{\sqrt{(k/m)}}{2\pi}.$$

Typically, natural frequency f is measured in cycles per second or *Hertz* and the angular frequency λ in radians per second. Mechanical vibrations can have frequencies from millions of cycles per second, for the vibrations of a microscopic quartz timing crystal, to thousandths of a cycle per second (i.e. thousands of seconds per cycle), say, for the free vibrations of the whole earth.

The amplitude of the sine wave that results from the addition of the *sine* function and the *cosine* function is given by the square root of the sum of the squares of the two amplitudes. That is, the amplitude of the resulting sine wave is $\sqrt{A^2 + B^2}$. Another way of describing this sum is through the trigonometric identity:

$$A\cos(\lambda\, t) + B\sin(\lambda\, t) = R\cos(\lambda\, t - \phi), \tag{5.19}$$

where $R = \sqrt{A^2 + B^2}$ and $\tan\phi = B/A$. So,

the only possible motion of a spring and mass is a sinusoidal oscillation which can be thought of either as the sum of a *cosine* function and a *sine* function or as a single *cosine* function with *phase shift* ϕ.

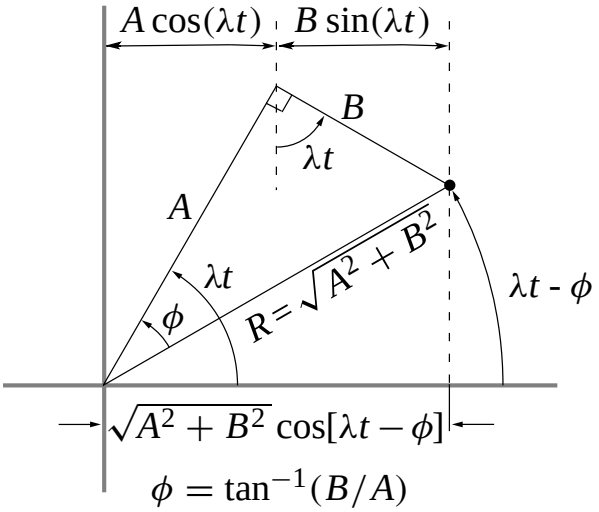
What are the constants A and B in the solution?

The general motion of the harmonic oscillator, equation 5.18, has the constants A and B which could have any value. Or, equivalently, the amplitude R and phase ϕ in equation 5.19 could be anything. They are determined by the way motion is started, the *initial conditions*. Two special initial conditions are worth getting a feel for: release from rest and initial velocity with no spring stretch.

5.4 *THEORY*

Visualization of $A\cos(\lambda t) + B\sin(\lambda t) = R\cos(\lambda t - \phi)$

Here is a demonstration that the sum of a *cosine* function and a *sine* function is a new sine wave. By sine wave we mean a function whose shape is the same as the sine function, though it may be displaced along the time axis. First, consider the line segment A spinning in circles about the origin at rate λ; that is, the angle the segment makes with the positive x axis is λt. The projection of that segment onto the x axis is $A\cos(\lambda t)$. Now consider the segment labeled B in the figure, glued at a right angle to A. The length of its projection on the x-axis is $B\sin(\lambda t)$. So, the sum of these two projections is $A\cos(\lambda t) + B\sin(\lambda t)$. The two segments A and B make up a right triangle with diagonal $R = \sqrt{A^2 + B^2}$.



The projection or 'shadow' of R on the x axis is the same as the sum of the shadows of A and B. The angle it makes with the x axis is $\lambda t - \phi$ where one can see from the triangle drawn that $\phi = \arctan(B/A)$. So, by adding the shadow lengths, we see $A\cos(\lambda t) + B\sin(\lambda t) = \sqrt{A^2 + B^2}\cos(\lambda t - \phi)$.

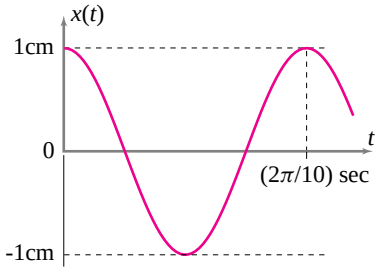


Figure 5.16: The position of a mass as a function of time if $k = 50\,\mathrm{N/m}$, $m = 0.5\,\mathrm{kg}$, $x(0) = 1\,\mathrm{cm}$ and $v(0) = 0$.

(Filename:tfigure12.cosine)

①**Caution:** It is tempting, but wrong, to evaluate $x(t)$ at $t = 0$ and then differentiate to get $v(0)$. This procedure is wrong because $x(0)$ is just a number, differentiating it would always give zero, even when the initial velocity is not zero.

Release from rest

The simplest motion to consider is when the spring is stretched a given amount and the mass is *released from rest*, meaning the initial velocity of the mass is zero. For example, if the mass in figure 5.14 is 0.5 kg, the spring constant is $k = 50\,\mathrm{N/m}$, and the initial displacement is 2 cm, we find the motion by looking at the general solution

$$x(t) = A\cos(\sqrt{(k/m)}\,t) + B\sin(\sqrt{(k/m)}\,t).$$

At $t = 0$, this general solution has to agree with the initial condition that the displacement is 1 cm, so

$$x(0) = A\underbrace{\cos(0)}_{1} + B\underbrace{\sin(0)}_{0} = A \qquad \Rightarrow \qquad A = 2\,\mathrm{cm}.$$

The initial velocity must also match, so first we find the velocity by differentiating the position to get

$$v(t) = \dot{x}(t) = -A\sqrt{(k/m)}\sin(\sqrt{(k/m)}\,t) + B\sqrt{(k/m)}\cos(\sqrt{(k/m)}\,t).$$

Now, we evaluate this expression at $t = 0$ and set it equal to the given initial velocity which in this case was zero:①

$$v(0) = -A\sqrt{(k/m)}\underbrace{\sin(0)}_{0} + B\sqrt{(k/m)}\underbrace{\cos(0)}_{1} = B\sqrt{(k/m)} \qquad \Rightarrow \qquad B = 0.$$

Substituting in the values for $k = 5\,\mathrm{N/m}$ and $m = 0.5\,\mathrm{kg}$, we get

$$x(t) = 2\cos\left(\sqrt{\underbrace{\left(\frac{0.5\,\mathrm{kg}}{50\,\mathrm{N/m}}\right)}_{0.01\,\mathrm{s}^{-1}}}\,t\right)\,\mathrm{cm} = 2\cos(0.1t/\,\mathrm{s})\,\mathrm{cm}$$

which is plotted in figure 5.16.

Initial velocity with no spring stretch

Another simple case is when the spring has no initial stretch but the mass has some initial velocity. Such might be the case just after a resting mass is hit by a hammer. Using the same 0.5 kg mass and $k = 50\,\mathrm{N/m}$ spring, we now consider an initial displacement of zero but an initial velocity of 10 cm/s. We can find the motion for this case from the general solution by the same procedure we just used. We get

$$x(t) = B\sin(\sqrt{(k/m)}\,t)$$

with $B\sqrt{(k/m)} = 10\,\mathrm{cm/s} \qquad \Rightarrow \qquad B = 1\,\mathrm{cm}$. The resulting motion, $x(t) = (1\,\mathrm{cm}) \cdot \sin(\frac{0.1t}{s})$, is shown in figure 5.17.

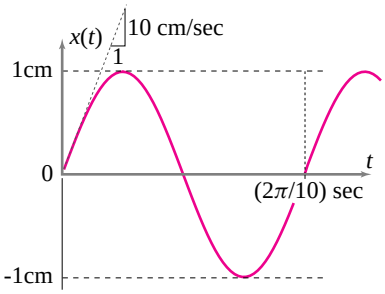


Figure 5.17: The position of a mass as a function of time if $k = 50\,\mathrm{N/m}$, $m = 0.5\,\mathrm{kg}$, $x(0) = 0$ and $v(0) = 10\ \mathrm{cm/s}$.

(Filename:tfigure12.sine)

Work, energy, and the harmonic oscillator

Various energy concepts give another viewpoint for looking at the harmonic oscillator. We can derive energy balance from momentum balance. Or, if we already trust energy balance, we can use it instead of momentum balance to derive the governing differential equation. Energy balance can be used as a check of a solution. Energy accounting gives an extra intuitive way to think about what happens in an oscillator.

The work of a spring

Associated with the force of a spring on a mass is a potential energy. Because the force of a spring on a mass is $-kx$, and the work of a force on a mass is $\int_0^x F(x')dx'$ we find the potential for work, measured from the relaxed state $x = 0$, on the mass to be

$$E_{\rm P} = -\int_0^x F(x')\,dx' - \int_0^x -kx'\,dx' = \frac{1}{2}kx^2.$$

Conservation of energy

Because there is no damping or dissipation, the total mechanical energy of the harmonic oscillator is constant in time. That is, the sum of the kinetic energy $E_{\rm K} = \frac{1}{2}mv^2$ and the potential energy $E_{\rm P} = \frac{1}{2}k(\Delta L)^2$ is constant.

$$E_{\rm T} = E_{\rm K} + E_{\rm P} = \text{constant}.$$

As the oscillation progresses, energy is exchanged back and forth between kinetic and potential energy. At the extremes in the displacement, the spring is most stretched, the potential energy is at a maximum, and the kinetic energy is zero. When the mass passes through the center position the spring is relaxed, so the potential energy is at a minimum (zero), the mass is at its maximum speed, and the kinetic energy reaches its maximum value.

Although energy conservation is a basic principle, this is a case where it can be derived, or more easily, checked. Using the special case where the motion starts from rest (i.e., $x(t) = A\cos(\sqrt{k/m}\,t)$), we can make sure that the total energy really is constant.

$$\begin{aligned}
E_{\rm T} &= E_{\rm K} + E_{\rm P} \\
&= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\
&= \frac{1}{2}k(\underbrace{A\cos(\sqrt{k/m}t)}_{x})^2 + \frac{1}{2}m(\underbrace{A\sqrt{k/m}\sin(\sqrt{k/m}t)}_{v})^2 \\
&= \frac{1}{2}kA^2\underbrace{\{\cos^2(\sqrt{k/m}t) + \sin^2(\sqrt{k/m}t)\}}_{1} \\
&= \frac{1}{2}kA^2 = \quad \text{initial energy in spring}
\end{aligned}$$

which does not change with time.

Using energy to derive the oscillator equation

Rather than just checking the energy balance, we could use the energy balance to help us find the equations of motion. As for all one-degree-of-freedom systems, the equations of motion can be derived by taking the time derivative of the energy balance equation. Starting from $E_T = \text{constant}$, we get

$$\begin{aligned}
0 &= \frac{d}{dt}E_{\rm T} \\
&= \frac{d}{dt}(E_{\rm P} + E_{\rm K}) \\
&= \frac{d}{dt}(\frac{1}{2}kx^2 + \frac{1}{2}mv^2) \\
&= kx\underbrace{\dot{x}}_{v} + mv\underbrace{\dot{v}}_{a}
\end{aligned}$$

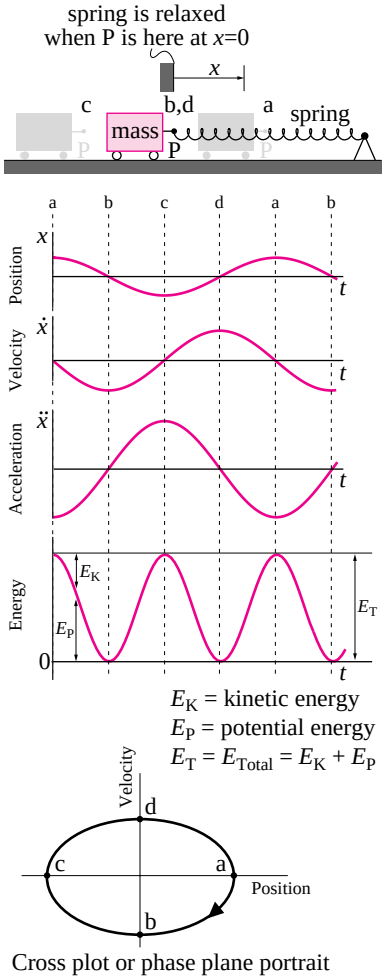


Figure 5.18: Various plots of the motion of the harmonic oscillator. Points a,b,c,d show what is happening at different parts of the motion. The spring is relaxed at $x = 0$. Some things to note are the following: The acceleration curve is proportional to the negative of the displacement curve. The displacement is at a maximum or minimum when the velocity is zero. The velocity is at a maximum or minimum when the displacement is zero. The kinetic and potential energy fluctuate at twice the frequency as the position. The motion is an ellipse in the cross plot of velocity *vs.* position.

(Filename:tfigure12.oscplots)

$$= \quad kx\not{v} + m\not{v}\underbrace{a}_{\ddot{x}}$$
$$0 \quad = \quad kx + m\ddot{x}$$

which is the differential equation for the harmonic oscillator. (A technical defect of this derivation is that it does not apply at the instant when $v = 0$.)

Power balance can also be used as a starting point to find the harmonic oscillator equation. Referring to the FBD in figure 5.14, the equation of energy balance for the block during its motion after release is:

$$\underbrace{P}_{\text{Power in}} = \underbrace{\dot{E}_K}_{\text{Rate of change of internal energy}}$$

$$\vec{F}_{spring} \cdot \vec{v}_A = \frac{d}{dt}(\frac{1}{2} m \vec{v}_A \cdot \vec{v}_A)$$
$$-kx_A\hat{\imath} \cdot \dot{x}_A\hat{\imath} = \frac{d}{dt}(\frac{1}{2} m \dot{x}_A^2)$$
$$-kx_A\dot{x}_A = m\dot{x}_A\ddot{x}_A$$

Dividing both sides by $\dot{x}_A$ (assuming it is not zero), we again get

$$-kx_A = m\ddot{x}_A \qquad \text{or} \qquad m\ddot{x}_A + kx_A = 0,$$

the familiar equation of motion for a spring-mass system.

We can now talk through a cycle of oscillation in terms of work and energy. Let's assume the block is released from rest at $x = x_A > 0$.

After the mass is released, the mass begins to move to the left and the spring does positive work on the mass since the motion and the force are in the same direction. After the block passes through the rest point $x = O$, it does work on the spring until it comes to rest at its left extreme. The spring then commences to do work on the block again as the block gains kinetic energy in its rightward motion. The block then passes through the rest position and does work on the spring until its kinetic energy is all used up and it is back in its rest position.

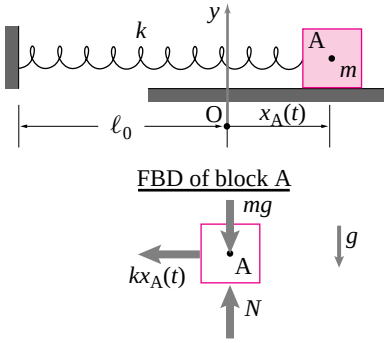


Figure 5.19: (Filename:t.ex.2.7.1)

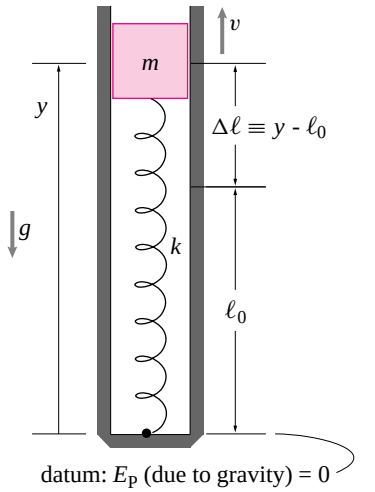


Figure 5.20: Spring and mass with gravity.

(Filename:t.ex.2.6.1)

A spring-mass system with gravity

When a mass is attached to a spring but gravity also acts one has to take some care to get things right (see fig. 5.20). Once a good free body diagram is drawn using well defined coordinates, all else follows easily.

SAMPLE 5.11 *Math review: Solution of a second order ODE:* Solve the equation:

$$\ddot{x} + k^2 x = 0, \quad \text{with initial conditions } x(0) = x_0, \;\; \dot{x}(0) = u_0. \tag{5.20}$$

Solution Let us guess a solution. We need a function $x(t)$ whose second derivative is equal to $-k^2$ times the function itself. We know at least two such functions: *sine* and *cosine*. To check, let

$$\begin{aligned} x(t) &= \sin kt \\ \Rightarrow \quad \ddot{x} &= -k^2 \sin kt = -k^2 x. \end{aligned}$$

Similarly, let

$$\begin{aligned} x(t) &= \cos kt \\ \Rightarrow \quad \ddot{x} &= -k^2 \cos kt = -k^2 x. \end{aligned}$$

Thus both functions satisfy the equation. Because Eqn. (5.20) is a linear differential equation, a linear combination of the two solutions will also satisfy it. Therefore, let

$$x(t) = A \sin kt + B \cos kt. \tag{5.21}$$

Substituting in Eqn. (5.20), we get

$$\ddot{x} + k^2 x = -Ak^2 \sin kt - Bk^2 \cos kt + k^2 (A \sin kt + B \cos kt) = 0,$$

which shows that the solution in Eqn. (5.21) satisfies the given differential equation. Now we evaluate the two constants A and B using the given initial conditions.

$$\begin{aligned} x(0) = x_0 &= A \cdot 0 + B \cdot 1 \\ \Rightarrow \quad B &= x_0 \\ \dot{x}(0) = u_0 &= (Ak \cos kt - Bk \sin kt)|_{t=0} \\ &= Ak \cdot 1 - Bk \cdot 0 \\ \Rightarrow \quad A &= \frac{u_0}{k}. \end{aligned}$$

Therefore, the solution is

$$x(t) = \frac{u_0}{k} \sin kt + x_0 \cos kt.$$

$$\boxed{x(t) = \tfrac{u_0}{k} \sin kt + x_0 \cos kt}$$

Alternatively, you could also guess $x(t) = e^{rt}$, plug it into the given equation, and find that you must have $r = \pm ik$ satisfy the equation. Now take a linear combination of the two solutions, say $x(t) = A\,e^{ikt} + B\,e^{-ikt}$, and find the constants A and B from the given initial conditions.

SAMPLE 5.12 A block of mass $m = 20\,\text{kg}$ is attached to two identical springs each with spring constant $k = 1\,\text{kN/m}$. The block slides on a horizontal surface without any friction.

(a) Find the equation of motion of the block.
(b) What is the oscillation frequency of the block?
(c) How much time does the block take to go back and forth 10 times?

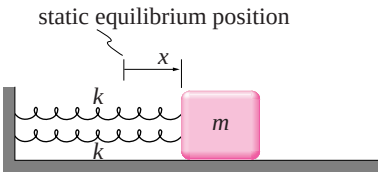


Figure 5.21: (Filename:sfig10.1.1.1)

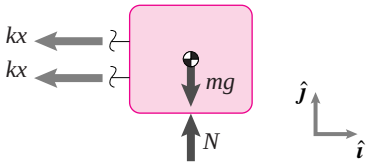


Figure 5.22: (Filename:sfig10.1.1.1a)

Solution

(a) The free body diagram of the block is shown in Figure 5.22. The linear momentum balance, $\sum \vec{F} = m\vec{a}$, for the block gives

$$-2kx\hat{\imath} + (N - mg)\hat{\jmath} = m\vec{a}$$

Dotting both sides with $\hat{\imath}$ we have,

$$\begin{aligned} -2kx = ma_x &= m\ddot{x} && (5.22)\\ \text{or}\quad m\ddot{x} + 2kx &= 0 && (5.23)\\ \text{or}\quad \ddot{x} + \frac{2k}{m}x &= 0. && (5.24) \end{aligned}$$

$$\boxed{\ddot{x} + \tfrac{2k}{m}x = 0}$$

(b) Comparing Eqn. (5.24) with the standard harmonic oscillator equation, $\ddot{x} + \lambda^2 x = 0$, where λ is the oscillation frequency, we get

$$\begin{aligned} \lambda^2 &= \frac{2k}{m}\\ \Rightarrow\quad \lambda &= \sqrt{\frac{2k}{m}}\\ &= \sqrt{\frac{2\cdot(1\,\text{kN/m})}{20\,\text{kg}}}\\ &= 10\,\text{rad/s}. \end{aligned}$$

$$\boxed{\lambda = 10\,\text{rad/s}}$$

(c) Time period of oscillation $T = \frac{2\pi}{\lambda} = \frac{2\pi}{10\,\text{rad/s}} = \frac{\pi}{5}\,\text{s}$. Since the time period represents the time the mass takes to go back and forth just once, the time it takes to go back and forth 10 times (*i.e.*, to complete 10 cycles of motion) is

$$t = 10T = 10\cdot\frac{\pi}{5}\,\text{s} = 2\pi\,\text{s}.$$

$$\boxed{t = 2\pi\,\text{s}}$$

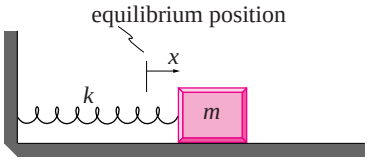


Figure 5.23: (Filename:sfig10.1.3)

SAMPLE 5.13 A spring-mass system executes simple harmonic motion: $x(t) = A\cos(\lambda t - \phi)$. The system starts with initial conditions $x(0) = 25\,\text{mm}$ and $\dot{x}(0) = 160\,\text{mm/s}$ and oscillates at the rate of 2 cycles/sec.

(a) Find the time period of oscillation and the oscillation frequency λ.
(b) Find the amplitude of oscillation A and the phase angle ϕ.
(c) Find the displacement, velocity, and acceleration of the mass at $t = 1.5\,\text{s}$.
(d) Find the maximum speed and acceleration of the system.
(e) Draw an accurate plot of displacement *vs*. time of the system and label all relevant quantities. What does ϕ signify in this plot?

Solution

(a) We are given $f = 2$ Hz. Therefore, the time period of oscillation is

$$T = \frac{1}{f} = \frac{1}{2\text{ Hz}} = 0.5\,\text{s},$$

and the oscillation frequency $\lambda = 2\pi f = 4\pi$ rad/s.

$$\boxed{T = 0.5\,\text{s}, \quad \lambda = 4\pi\text{ rad/s}.}$$

(b) The displacement $x(t)$ of the mass is given by

$$x(t) = A\cos(\lambda t - \phi).$$

Therefore the velocity (actually the speed) is

$$\dot{x}(t) = -A\lambda\sin(\lambda t - \phi)$$

At $t = 0$, we have

$$x(0) = A\cos(-\phi) = A\cos\phi \tag{5.25}$$
$$\dot{x}(0) = -A\lambda\sin(-\phi) = A\lambda\sin\phi \tag{5.26}$$

By squaring Eqn (5.25) and adding it to the square of [Eqn (5.26) divided by λ], we get

$$A^2\cos^2\phi + \frac{A^2\lambda^2\sin^2\phi}{\lambda^2} = A^2 = x^2(0) + \frac{\dot{x}^2(0)}{\lambda^2}$$
$$\Rightarrow \quad A = \sqrt{(25\,\text{mm})^2 + \frac{(160\,\text{mm/s})^2}{(4\pi\text{ rad/s})^2}}$$
$$= 28.06\,\text{mm}.$$

Substituting the value of A in Eqn (5.25), we get

$$\phi = \cos^{-1}\frac{x(0)}{A}$$
$$= \cos^{-1}\frac{25\,\text{mm}}{28.06\,\text{mm}}$$
$$= 0.471\,\text{rad} \approx 27^{\text{o}}.$$

$$\boxed{A = 28.06\,\text{mm}. \quad \phi = 0.471\,\text{rad}.}$$

(c) The displacement, velocity, and acceleration of the mass at any time t can now be calculated as follows

$$\begin{aligned} x(t) &= A\cos(\lambda t - \phi) \\ \Rightarrow \quad x(1.5\,\text{s}) &= 28.06\,\text{mm}\cdot\cos(6\pi - 0.471) \\ &= 25\,\text{mm}. \end{aligned}$$

$$\begin{aligned} \dot{x}(t) &= -A\lambda\sin(\lambda t - \phi) \\ \Rightarrow \quad \dot{x}(1.5\,\text{s}) &= 28.06\,\text{mm}\cdot(4\pi\ \text{rad/s})\cdot\sin(6\pi - 0.471) \\ &= 160\,\text{mm/s}. \end{aligned}$$

$$\begin{aligned} \ddot{x}(t) &= -A\lambda^2\cos(\lambda t - \phi) \\ \Rightarrow \quad \ddot{x}(1.5\,\text{s}) &= 28.06\,\text{mm}\cdot(4\pi\ \text{rad/s})^2\cdot\cos(6\pi - 0.471) \\ &= -3.95\times 10^3\,\text{mm/s}^2 \\ &= -3.95\,\text{m/s}^2. \end{aligned}$$

①

$$\boxed{x(1.5\,\text{s}) = 25\,\text{mm}. \quad \dot{x}(1.5\,\text{s}) = 160\,\text{mm/s}. \quad \ddot{x}(1.5\,\text{s}) = -3.93\,\text{m/s}^2.}$$

① We can find the displacement and velocity at $t = 1.5$ s without any differentiation. Note that the system completes 2 cycles in 1 second, implying that it will complete 3 cycles in 1.5 seconds. Therefore, at $t = 1.5$ s, it has the same displacement and velocity as it had at $t = 0$ s.

(d) Maximum speed:

$$|\dot{x}_{\max}| = A\lambda = (28.06\,\text{mm})\cdot(4\pi\ \text{rad/s}) = 0.35\,\text{m/s}.$$

Maximum acceleration:

$$|\ddot{x}_{\max}| = A\lambda^2 = (28.06\,\text{mm})\cdot(4\pi\ \text{rad/s})^2 = 4.43\,\text{m/s}^2.$$

$$\boxed{|\dot{x}_{\max}| = 0.35\,\text{m/s}, \quad |\ddot{x}_{\max}| = 4.43\,\text{m/s}^2.}$$

(e) The plot of $x(t)$ versus t is shown in Fig. 5.24. The phase angle ϕ represents the shift in $\cos(\lambda t)$ to the right by an amount $\frac{\phi}{\lambda}$.

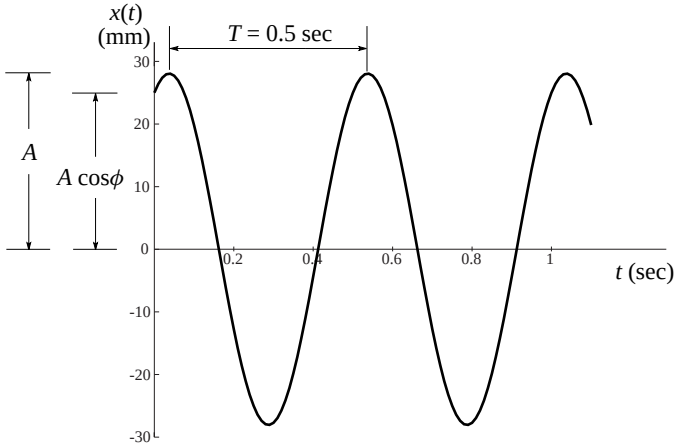


Figure 5.24: (Filename:sfig10.1.3a)

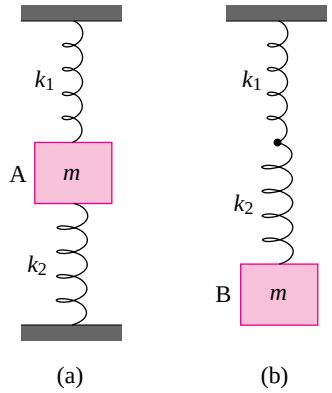


Figure 5.25: (Filename:sfig3.4.2)

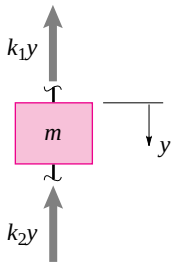


Figure 5.26: Free body diagram of the mass.

(Filename:sfig3.4.2a)

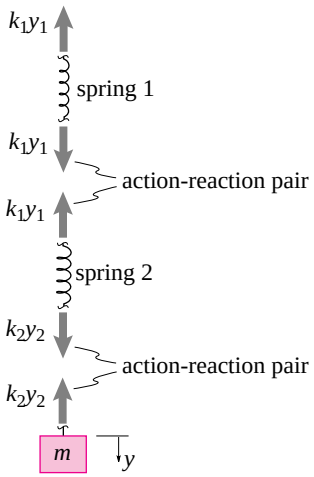


Figure 5.27: Free body diagrams

(Filename:sfig3.4.2b)

SAMPLE 5.14 *Springs in series versus springs in parallel:* Two massless springs with spring constants k_1 and k_2 are attached to mass A *in parallel* (although they look superficially as if they are in series) as shown in Fig. 5.25. An identical pair of springs is attached to mass B *in series*. Taking $m_A = m_B = m$, find and compare the natural frequencies of the two systems. Ignore gravity.

Solution Let us pull each mass downwards by a small vertical distance y and then release. Measuring y to be positive downwards, we can derive the equations of motion for each mass by writing the Balance of Linear Momentum for each as follows.

- **Mass A:** The free body diagram of mass A is shown in Fig. 5.26. As the mass is displaced downwards by y, spring 1 gets stretched by y whereas spring 2 gets compressed by y. Therefore, the forces applied by the two springs, $k_1 y$ and $k_2 y$, are in the same direction. The LMB of mass A in the vertical direction gives:

$$\begin{aligned} \sum F &= ma_y \\ \text{or} \quad -k_1 y - k_2 y &= m\ddot{y} \\ \text{or} \quad \ddot{y} + \left(\frac{k_1 + k_2}{m}\right) y &= 0. \end{aligned}$$

Let the natural frequency of this system be ω_p. Comparing with the standard simple harmonic equation $\ddot{x} + \lambda^2 x = 0$ we get the natural frequency (λ) of the system:

$$\omega_p = \sqrt{\frac{k_1 + k_2}{m}} \tag{5.27}$$

$$\boxed{\omega_p = \sqrt{\frac{k_1+k_2}{m}}}$$

- **Mass B:** The free body diagram of mass B and the two springs is shown in Fig. 5.27. In this case both springs stretch as the mass is displaced downwards. Let the net stretch in spring 1 be y_1 and in spring 2 be y_2. y_1 and y_2 are unknown, of course, but we know that

$$y_1 + y_2 = y \tag{5.28}$$

Now, using the free body diagram of spring 2 and then writing linear momentum balance we get,

$$\begin{aligned} k_2 y_2 - k_1 y_1 &= \underbrace{m}_{0}\, a = 0 \\ y_1 &= \frac{k_2}{k_1} y_2 \end{aligned} \tag{5.29}$$

Solving (5.28) and (5.29) we get

$$y_2 = \frac{k_1}{k_1 + k_2} y.$$

Now, linear momentum balance of mass B in the vertical direction gives:

$$\begin{aligned} -k_2 y_2 = ma_y &= m\ddot{y} \\ \text{or} \quad m\ddot{y} + k_2 \overbrace{\frac{k_1}{k_1 + k_2} y}^{y_2} &= 0 \\ \text{or} \quad \ddot{y} + \frac{k_1 k_2}{m(k_1 + k_2)} y &= 0. \end{aligned} \tag{5.30}$$

Let the natural frequency of this system be denoted by ω_s. Then, comparing with the standard simple harmonic equation as in the previous case, we get

$$\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}. \tag{5.31}$$

$$\boxed{\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}}$$

From (5.27) and (5.31)

$$\frac{\omega_p}{\omega_s} = \frac{k_1 + k_2}{\sqrt{k_1 k_2}}.$$

Let $k_1 = k_2 = k$. Then, $\omega_p/\omega_s = 2$, *i.e.*, the natural frequency of the system with two identical springs in parallel is twice as much as that of the system with the same springs in series. Intuitively, the restoring force applied by two springs in parallel will be more than the force applied by identical springs in series. In one case the forces add and in the other they don't and each spring is stretched less. Therefore, we do expect mass A to oscillate at a faster rate (higher natural frequency) than mass B.

Comments:

(a) Although the springs attached to mass A do not visually seem to be in parallel, from mechanics point of view they are parallel. You can easily check this result by putting the two springs visually in parallel and then deriving the equation of mass A. You will get the same equations. For springs in parallel, each spring has the *same displacement* but *different forces*. For springs in series, each has *different displacements* but the *same force*.

(b) When many springs are connected to a mass in series or in parallel, sometimes we talk about their *effective* spring constant, *i.e.*, the spring constant of a single imaginary spring which could be used to replace all the springs attached in parallel or in series. Let the effective spring constant for springs in parallel and in series be represented by k_{pe} and k_{se} respectively. By comparing eqns. (5.27) and (5.31) with the expression for natural frequency of a simple spring mass system, we see that

$$k_{pe} = k_1 + k_2 \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

These expressions can be easily extended for any arbitrary number of springs, say, N springs:

$$k_{pe} = k_1 + k_2 + \ldots + k_N \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_N}.$$

SAMPLE 5.15 Figure 5.28 shows two responses obtained from experiments on two spring-mass systems. For each system

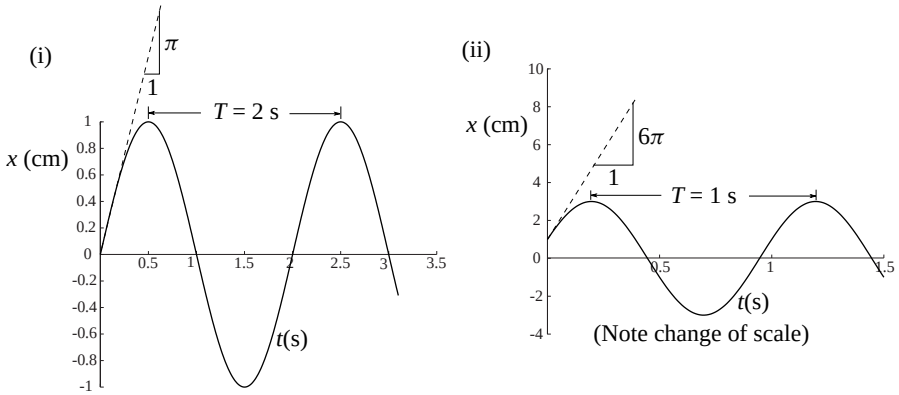


Figure 5.28: (Filename:sfig10.1.4)

(a) Find the natural frequency.
(b) Find the initial conditions.

Solution

(a) **Natural frequency:** By definition, the natural frequency f is the number of cycles the system completes in one second. From the given responses we see that:

Case(i): the system completes $\frac{1}{2}$ a cycle in 1 s.

$$\Rightarrow \qquad f = \frac{1}{2} \text{ Hz.}$$

Case(ii): the system completes 1 cycle in 1 s.

$$\Rightarrow \qquad f = 1 \text{ Hz.}$$

It is usually hard to measure the fraction of cycle occurring in a short time. It is easier to first find the time period, *i.e.*, the time taken to complete 1 cycle. ①Then the natural frequency can be found by the formula $f = \frac{1}{T}$. From the given responses, we find the time period by estimating the time between two successive peaks (or troughs): From Figure 5.28 we find that for

① To estimate the frequency of some repeated motion in an experiment, it is best to measure the time for a large number of cycles, say 5, 10 or 20, and then divide that time by the total number of cycles to get an average value for the time period of oscillation.

Case (i):

$$f = \frac{1}{T} = \frac{1}{2\,\text{s}} = \frac{1}{2} \text{ Hz,}$$

Case (ii):

$$f = \frac{1}{T} = \frac{1}{1\,\text{s}} = 1 \text{ Hz}$$

case (i) $f = \frac{1}{2}$ Hz. case (ii) $f = 1$ Hz.

(b) **Initial conditions:** Now we are to find the displacement and velocity at $t = 0$ s for each case. Displacement is easy because we are given the displacement plot, so we just read the value at $t = 0$ from the plots:

Case (i): $x(0) = 0.$

Case (ii): $x(0) = 1\,\text{cm}$.

The velocity (actually the speed) is the time-derivative of the displacement. Therefore, we get the initial velocity from the slope of the displacement curve at $t = 0$.

Case (i): $\dot{x}(0) = \frac{dx}{dt}(t = 0) = \frac{\pi\,\text{cm}}{1\,\text{s}} = 3.14\,\text{cm/s}$.

Case (ii): $\dot{x}(0) = \frac{dx}{dt}(t = 0) = \frac{6\pi\,\text{cm}}{1\,\text{s}} = 18.85\,\text{cm/s}$.

Thus the initial conditions are

Case (i) $x(0) = 0,\ \dot{x}(0) = 3.14\,\text{cm/s}$. Case (ii) $x(0) = 1\,\text{cm},\ \dot{x}(0) = 18.85\,\text{cm/s}$.

Comments: Estimating the speed from the initial slope of the displacement curve at $t = 0$ is not a very good method because it is hard to draw an accurate tangent to the curve at $t = 0$. A slightly different line but still seemingly tangential to the curve at $t = 0$ can lead to significant error in the estimated value. A better method, perhaps, is to use the known values of displacement at different points and use the energy method to calculate the initial speed. We show sample calculations for the first system:

Case(i): We know that $x(0) = 0$. Therefore the entire energy at $t = 0$ is the kinetic energy $= \frac{1}{2}mv_0^2$. At $t = 0.5\,\text{s}$ we note that the displacement is maximum, *i.e.*, the speed is zero. Therefore, the entire energy is potential energy $= \frac{1}{2}kx^2$, where $x = x(t = 0.5\,\text{s}) = 1\,\text{cm}$.

Now, from the conservation of energy:

$$\begin{aligned} \frac{1}{2}mv_0^2 &= \frac{1}{2}k(x_{t=0.5\,\text{s}})^2 \\ \Rightarrow \qquad v_0 &= \sqrt{\frac{k}{m}}\cdot(x_{t=0.5\,\text{s}}) \\ &= \underbrace{\sqrt{\frac{k}{m}}}_{\lambda}\cdot(1\,\text{cm}) \\ &= 2\pi f\cdot(1\,\text{cm}) \\ &= 2\pi\cdot\frac{1}{2}\,\text{Hz}\cdot 1\,\text{cm} \\ &= 3.14\,\text{cm/s}. \end{aligned}$$

Similar calculations can be done for the second system.

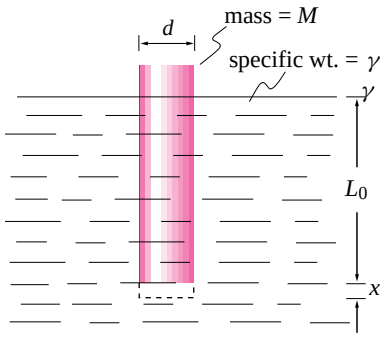


Figure 5.29: (Filename:sfig3.4.1)

SAMPLE 5.16 *Simple harmonic motion of a buoy.* A cylinder of cross sectional area A and mass M is in static equilibrium inside a fluid of specific weight γ when L_o length of the cylinder is submerged in the fluid. From this position, the cylinder is pushed down vertically by a small amount x and let go. Assume that the only forces acting on the cylinder are gravity and the buoyant force and assume that the buoy's motion is purely vertical. Derive the equation of motion of the cylinder using Linear Momentum Balance. What is the period of oscillation of the cylinder?

Solution The free body diagram of the cylinder is shown in Fig. 5.30 where F_B represents the buoyant force. Before the cylinder is pushed down by x, the linear momentum balance of the cylinder gives

$$F_B - Mg = M \underbrace{a}_{0} = 0 \qquad \Rightarrow \qquad F_B = Mg$$

Now F_B = (volume of the displaced fluid)· (its specific weight) $= AL_o\gamma$. Thus,

$$AL_o\gamma = Mg. \tag{5.32}$$

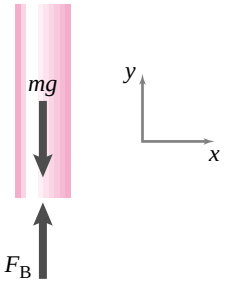


Figure 5.30: (Filename:sfig3.4.1a)

Now, when the cylinder is pushed down by an amount x,

$$F_B' = \text{ new buoyant force } = (L_o + x)A\gamma.$$

Therefore, from LMB we get

$$\begin{aligned} F_B' - Mg &= -M\ddot{x} \\ \text{or} \quad (L_o + x)A\gamma - Mg &= -M\ddot{x} \\ &= \overbrace{-AL_o\gamma + Mg}^{=0 \text{ from (5.32).}} \\ \text{or} \quad M\ddot{x} + A\gamma x &= 0 \\ \text{or} \quad \ddot{x} + \frac{A\gamma}{M}x &= 0. \end{aligned}$$

$$\boxed{\ddot{x} + \tfrac{A\gamma}{M}x = 0}$$

Comparing this equation with the standard simple harmonic equation (*e.g.*, eqn.(g), in the box on ODE's on page 226).

$$\text{The circular frequency } \lambda = \sqrt{\frac{A\gamma}{M}},$$

$$\text{Therefore, the period of oscillation } T = \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{M}{A\gamma}}$$

.

$$\boxed{T = 2\pi\sqrt{\tfrac{M}{A\gamma}}}$$

Comments: Note this calculation neglects the fluid mechanics. The common way of making a correction is to use 'added mass' to account for fluid that moves more-or-less with the cylinder. The added mass is usually something like one-half the mass of the fluid with volume equal to that of the cylinder. Another way to see the error is to realize that the pressure used in this calculation assumes fluid statics when in fact the fluid is moving.

5.4 More on vibrations: damping

The mother of all vibrating machines is the simple harmonic oscillator from the previous section. With varying degrees of approximation, car suspensions, violin strings, buildings responding to earthquakes, earthquake faults themselves, and vibrating machines are modeled as mass-spring-dashpot systems. Almost all of the concepts in vibration theory are based on concepts associated with the behavior of the harmonic oscillator. The harmonic oscillator has no friction or inelastic deformation so that mechanical energy is conserved. Such vibrations will, once started, persist forever even with no pushing, pumping, or energy supply of any kind. Total lack of friction does not describe any real system perfectly, but it is a useful approximation if one is trying to understand the oscillations of a system and not the decay of those oscillations.

But for any real system the oscillations will decay in time due to friction. We would now like to study this decay.

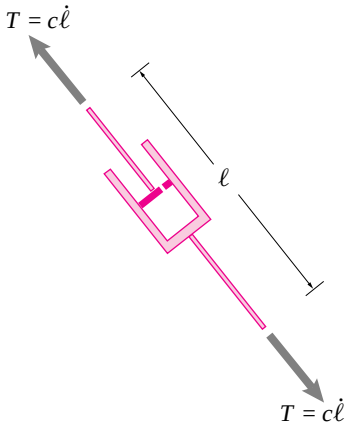


Figure 5.31: **A damper or dashpot.** The symbol shown represents a device which resists the relative motion of its endpoints. The schematic is supposed to suggest a plunger in a cylinder. For the plunger to move, fluid must leak around the cylinder. This leakage happens for either direction of motion. Thus the damper resists relative motion in either direction; i. e., for $\dot{L} > 0$ and $\dot{L} < 0$.

(Filename:tfigure12.dashpot)

Damping

The simplest system to study is the *damped harmonic oscillator* and the motions that are of interest are *damped* oscillations.

Again the simplest model, and also the prototype of all models, is a spring and mass system. But now we add a component called a *damper* or *dashpot*, shown in figure 5.31. The dashpot provides resistance to motion by drawing air or oil in and out of the cylinder through a small opening. Due to the viscosity of the air or oil, a pressure drop is created across the opening that is related to the speed of the fluid flowing through. Ideally, this viscous resistance produces linear damping, meaning that the force is exactly proportional to the velocity. In a physical dashpot nonlinearities are introduced from the fluid flow and from friction between the piston and the cylinder. Also, dashpots that use air as a working fluid may have compressibility that introduces non-negligible springiness to the system in addition to that of any metallic springs.

Adding a dashpot in parallel with the spring of a mass-spring system creates a *mass-spring-dashpot* system, or *damped harmonic oscillator*. The system is shown in figure 5.32. Figure 5.33 is a free body diagram of the mass. It has two forces acting on it, neglecting gravity:

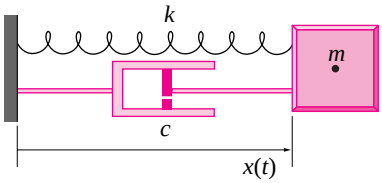


Figure 5.32: A mass spring dashpot system, or damped harmonic oscillator.

(Filename:tfigure12.MSD)

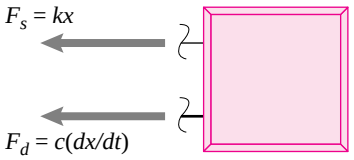


Figure 5.33: Free body diagram of the mass spring dashpot system.

(Filename:tfigure12.MSDFBD)

$$\begin{aligned} F_s &= kx && \text{is the spring force, assuming a linear spring, and} \\ F_d &= c\,dx/dt = c\dot{x} && \text{is the dashpot force assuming a linear dashpot.} \end{aligned}$$

The system is a one degree of freedom system because a single coordinate x is sufficient to describe the complete motion of the system. The equation of motion for this system is

$$m\ddot{x} = -F_d - F_s \quad \text{where} \quad \ddot{x} = d^2x/dt^2. \tag{5.33}$$

Assuming a linear spring and a linear dashpot this expression becomes

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{5.34}$$

We have taken care with the signs of the various terms. You should check that you can derive equation 5.34 without introducing any sign errors. ①

①**Caution:** When push comes to shove, so to speak, many students have trouble deriving equations like 5.34 without getting sign errors from figures like 5.32.

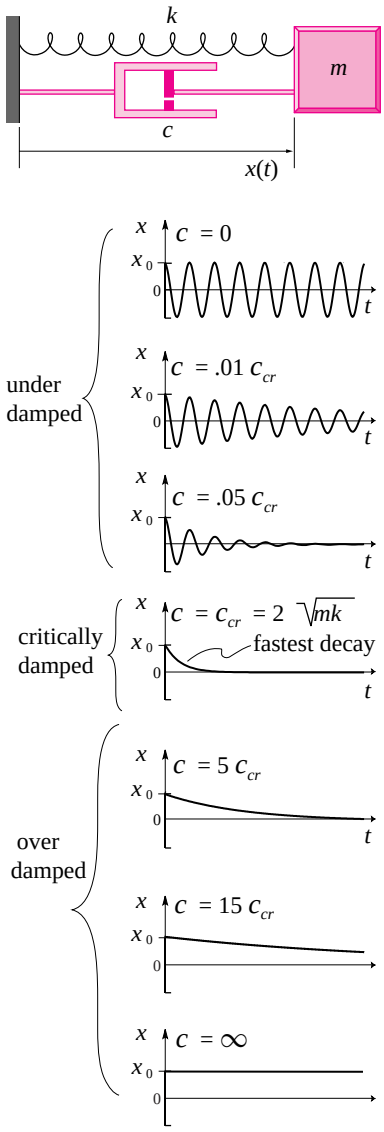


Figure 5.34: The effect of varying the damping with a fixed mass and spring. In all the plots the mass is released from rest at $x = x_0$. In the case of under-damping, oscillations persist for a long time, forever if there is no damping. In the case of over-damping, the dashpot doesn't relax for a long time; it stays locked up forever in the limit of $c \to \infty$. The fastest relaxation occurs for critical damping.

(Filename:tfigure12.damping)

Solution of the damped-oscillator equations

The governing equation 5.34 has a solution which depends on the values of the constants. There are cases where one wants to consider negative springs or negative dashpots, but for the purposes of understanding classical vibration theory we can assume that m, c, and k are all positive. Even with this restriction the solution depends on the relative values of m, c, and k. You can learn all about these solutions in any book that introduces ordinary differential equations; most freshman calculus books have such a discussion.

The three solutions are categorized as follows:

- *Under-damped*: $c^2 < 4mk$. In this case the damping is small and oscillations persist forever, though their amplitude diminishes exponentially in time. The general solution for this case is:

$$x(t) = e^{(-\frac{c}{2m})t}[A\cos(\lambda_d t) + B\sin(\lambda_d t)], \tag{5.35}$$

  where λ_d is the damped natural frequency and is given by $\lambda_d = \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$.
- *Critically damped*: $c^2 = 4mk$. In this case the damping is at a critical level that separates the cases of under-damped oscillations from the simply decaying motion of the over-damped case. The general solution is:

$$x(t) = Ae^{(-\frac{c}{2m})t} + Bte^{(-\frac{c}{2m})t}. \tag{5.36}$$

- *Over-damped*: $c^2 > 4mk$. Here there are no oscillations, just a simple return to equilibrium with at most one crossing through the equilibrium position on the way to equilibrium. The general solution in the over-damped case is:

$$x(t) = Ae^{(-\frac{c}{2m}+\sqrt{(\frac{c}{2m})^2-\frac{k}{m}})t} + Be^{(-\frac{c}{2m}-\sqrt{(\frac{c}{2m})^2-\frac{k}{m}})t}. \tag{5.37}$$

  The solution 5.37 actually includes equations 5.36 and 5.35 as special cases. To interpret equation refoverdampe as the general solution you need to know the relation between complex exponentials and trigonometric functions for the cases when the argument of the square root term is negative.

For a given mass and spring we can imagine the damping as a variable to adjust. A system which has small damping (small c) is *under-damped* and does not come to equilibrium quickly because oscillations persist for a long time. A system which has a lot of damping (big c) is *over-damped* does not come to equilibrium quickly because the dashpot holds it away from equilibrium. A system which is *critically-damped* comes to equilibrium most quickly. In many cases, the purpose of damping is to purge motions after disturbance from equilibrium. If the only design variable available for adjustment is the damping, then the quickest purge is accomplished by picking $c = \sqrt{(4km)}$ and achieving critical damping. This damping design is commonly employed.

Measurement of damping: logarithmic decrement method

In the under-damped case, the viscous damping constant c may be determined experimentally by measuring the rate of decay of unforced oscillations. This decay can be quantified using the logarithmic decrement. The logarithmic decrement is the natural logarithm of the ratio of any two successive amplitudes. The larger the damping, the greater will be the rate of decay of oscillations and the bigger the logarithmic decrement:

$$\text{logarithmic decrement} \equiv D = \ln(\frac{x_n}{x_{n+1}}) \tag{5.38}$$

where x_n and x_{n+1} are the heights of two successive peaks in the decaying oscillation pictured in figure 5.35. Because of the exponential envelope that this curve has, $x_n = (\text{const.})e^{-(\frac{c}{2m})t_1}$ and $x_{n+1} = (\text{const.})e^{-(\frac{c}{2m})t_1+T}$.

$$D = \ln[(e^{-(\frac{c}{2m})t_1})/(e^{-(\frac{c}{2m})t_1+T})]$$

Simplifying this expression, we get that

$$D = \frac{cT}{2m}$$

where T is the period of oscillation. Thus, the damping constant c can be measured by measuring the logarithmic decrement D and the period of oscillation T as

$$c = \frac{2mD}{T}.$$

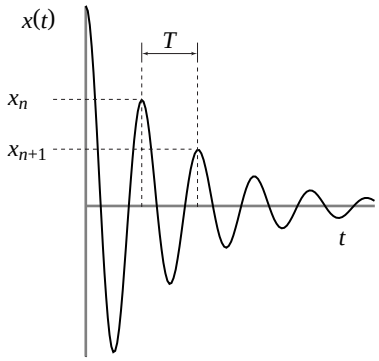


Figure 5.35: **The logarithmic decrement method.** $D = \ln(x_n/x_{n+1})$

(Filename:tfigure12.decrement)

Summary of equations for the unforced harmonic oscillator

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- $\ddot{x} + \frac{k}{m}x = 0$, mass-spring equation
- $\ddot{x} + \lambda^2 x = 0$, harmonic oscillator equation
- $x(t) = A\cos(\lambda t) + B\sin(\lambda t)$, general solution to harmonic oscillator equation
- $x(t) = R\cos(\lambda t - \phi)$, amplitude-phase version of solution to harmonic oscillator solution, $R = \sqrt{A^2 + B^2}$, $\phi = \tan^{-1}(\frac{B}{A})$
- $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$, mass-spring-dashpot equation (see equations 5.35-5.37 for solutions)
- $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$, logarithmic decrement. $c = \frac{2mD}{T}$.

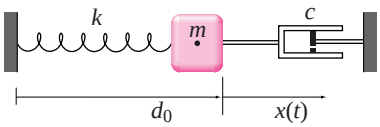


Figure 5.36: Spring-mass dashpot.

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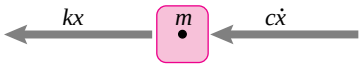


Figure 5.37: free body diagram of system at instant t goes here!

(Filename:sfig10.2.1a)

SAMPLE 5.17 A block of mass 10 kg is attached to a spring and a dashpot as shown in Figure 5.36. The spring constant $k = 1000\,\text{N/m}$ and a damping rate $c = 50\,\text{N}\cdot\text{s/m}$. When the block is at a distance d_0 from the left wall the spring is relaxed. The block is pulled to the right by 0.5 m and released. Assuming no initial velocity, find

(a) the equation of motion of the block.
(b) the position of the block at $t = 2\,\text{s}$.

Solution

(a) Let x be the position of the block, measured positive to the right of the static equilibrium position, at some time t. Let $\dot{x}$ be the corresponding speed. The free body diagram of the block at the instant t is shown in Figure 5.37.
Since the motion is only horizontal, we can write the linear momentum balance in the x-direction ($\sum F_x = m\,a_x$):

$$\underbrace{-kx - c\dot{x}}_{\sum F_x} = m\,\underbrace{\ddot{x}}_{a_x}$$

$$\text{or} \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{5.39}$$

which is the desired equation of motion of the block.

$$\boxed{\ddot{x} + \tfrac{c}{m}\dot{x} + \tfrac{k}{m}x = 0.}$$

(b) To find the position and velocity of the block at any time t we need to solve Eqn (5.39). Since the solution depends on the relative values of m, k, and c, we first compute c^2 and compare with the *critical value* $4mk$.

$$\begin{aligned} & c^2 &=& \ 2500(\,\text{N}\cdot\text{s/m})^2 \\ \text{and} \quad & 4mk &=& \ 4{\cdot}10\,\text{kg}{\cdot}1000\,\text{N/m} = 4000(\,\text{N}\cdot\text{s/m})^2. \\ \Rightarrow \quad & c^2 &<& \ 4mk. \end{aligned}$$

Therefore, the system is underdamped and we may write the general solution as

$$x(t) = e^{-\frac{c}{2m}t}\,[A\cos\lambda_D t + B\sin\lambda_D t] \tag{5.40}$$

where

$$\lambda_D \ = \ \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 9.682\,\text{rad/s}.$$

Substituting the initial conditions $x(0) = 0.5\,\text{m}$ and $\dot{x}(0) = 0$ m/s in Eqn (5.40) (we need to differentiate Eqn (5.40) first to substitute $\dot{x}(0)$), we get

$$\begin{aligned} x(0) &= 0.5\,\text{m} = A. \\ \dot{x}(0) &= 0 = -\frac{c}{2m}\cdot A + \lambda_D{\cdot}B \\ \Rightarrow \quad B &= \frac{A\,c}{2m\lambda_D} = \frac{(0.5\,\text{m})\cdot(50\,\text{N}\cdot\text{s/m})}{2\cdot(10\,\text{kg})\cdot(9.682\,\text{rad/s})} = 0.13\,\text{m}. \end{aligned}$$

Thus, the solution is

$$x(t) = e^{(-2.5\frac{1}{\text{s}})t}\,[0.50\cos(9.68\,\text{rad/s}\ t) + 0.13\sin(9.68\,\text{rad/s}\ t)]\ \text{m}.$$

Substituting $t = 2\,\text{s}$ in the above expression we get $x(2\,\text{s}) = 0.003\,\text{m}$.

$$\boxed{x(2\,\text{s}) = 0.003\,\text{m}.}$$

SAMPLE 5.18 A structure, modeled as a single degree of freedom system, exhibits characteristics of an underdamped system under free oscillations. The response of the structure to some initial condition is determined to be $x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t)$ where $A = 0.3\,\text{m}$, $\xi \equiv$ damping ratio $= 0.02$, $\lambda \equiv$ undamped circular frequency $= 1\,\text{rad/s}$, and $\lambda_D \equiv$ damped circular frequency $= \lambda\sqrt{1-\xi^2} \approx \lambda$.

(a) Find an expression for the ratio of energies of the system at the $(n+1)$th displacement peak and the nth displacement peak.

(b) What percent of energy available at the first peak is lost after 5 cycles?

Solution

(a) We are given that

$$x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t).$$

The structure attains its first displacement peak when $\sin\lambda_D t$ is maximum, *i.e.*, $\lambda_D t = \frac{\pi}{2}$, or $t = \frac{\pi}{2\lambda_D}$. At this instant,

$$x(t) \quad = \quad Ae^{-\xi\cdot\lambda\cdot\frac{\pi}{2\lambda_D}} = Ae^{-\frac{\pi}{2}\cdot\frac{\xi}{\sqrt{1-\xi^2}}} = (0.3\,\text{m})\cdot e^{-0.0314} = 0.29\,\text{m}.$$

Let x_n and x_{n+1} be the values of the displacement at the nth and the $(n+1)$th peak, respectively. Since x_n and x_{n+1} are peak displacements, the respective velocities are zero at these points. Therefore, the energy of the system at these peaks is given by the potential energy stored in the spring. That is

$$E_n = \frac{1}{2}kx_n^2 \quad \text{and} \quad E_{n+1} = \frac{1}{2}kx_{n+1}^2. \tag{5.41}$$

Let t_n be the time at which the nth peak displacement x_n is attained, *i.e.*,

$$x_n = Ae^{-\xi\lambda t_n} \tag{5.42}$$

Since x_{n+1} is the next peak displacement, it must occur at $t = t_n + T_D$ where T_D is the time period of damped oscillations. Thus

$$x_{n+1} = Ae^{-\xi\lambda(t_n+T_D)} \tag{5.43}$$

From Eqns (5.41), (5.42), and (5.43)

$$\frac{E_{n+1}}{E_n} = \frac{\frac{1}{2}k(Ae^{-\xi\lambda(t_n+T_D)})^2}{\frac{1}{2}k(Ae^{-\xi\lambda t_n})^2} = e^{-2\xi\lambda T_D}.$$

$$\boxed{\frac{E_{n+1}}{E_n} = e^{-2\xi\lambda T_D}.}$$

(b) Noting that $T_D = \frac{2\pi}{\lambda_D}$ and $\lambda_D = \lambda\sqrt{1-\xi^2}$, we get

$$E_{n+1} = E_n\, e^{-2\xi\cancel{\lambda}\cdot\frac{2\pi}{\cancel{\lambda}\sqrt{1-\xi^2}}} \approx e^{-4\pi\xi} \quad \Rightarrow \quad E_{n+1} = e^{-4\pi\xi}\,E_n.$$

Applying this equation recursively for $n = n-1, n-2, \ldots, 1, 0,$, we get

$$E_n = e^{-4\pi\xi}\cdot E_{n-1} = e^{-4\pi\xi}\cdot(e^{-4\pi\xi}\cdot E_{n-2}) = (e^{-4\pi\xi})^3\cdot E_{n-3}\ldots = (e^{-4\pi\xi})^n\cdot E_0.$$

Now we use this equation to find the percentage of energy of the first peak $(n=0)$ lost after 5 cycles $(n=5)$:

$$\Delta E_5 \quad = \quad \frac{E_0 - E_5}{E_0}\times 100 = \left(1 - e^{-4\pi\xi\,\cdot\,5}\right)\times 100 = 71.5\%.$$

$$\boxed{\Delta E_5 = 71.5\%.}$$

SAMPLE 5.19 The following table is obtained for successive peaks of displacement from the simulation of free vibration of a mechanical system. Make a single degree of freedom mass-spring-dashpot model of the system choosing appropriate values for mass, spring stiffness, and damping rate.

Data:

peak number n	0	1	2	3	4	5	6
time (s)	0.0000	0.6279	1.2558	1.8837	2.5116	3.1395	3.7674
peak disp. (m)	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443

Solution Since the data provided is for successive peak displacements, the time between any two successive peaks represents the period of oscillations. It is also clear that the system is underdamped because the successive peaks are decreasing. We can use the logarithmic decrement method to determine the damping in the system.

First, we find the time period T_D from which we can determine the damped circular frequency λ_D. From the given data we find that

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots = 0.6279\,\text{s}$$

Therefore,

$$\begin{aligned} & T_D = 0.6279\,\text{s}. \\ \Rightarrow \quad & \lambda_D = \frac{2\pi}{T_D} = 10\,\text{rad/s}. \end{aligned} \tag{5.44}$$

Now we make a table for the logarithmic decrement of the peak displacements:

peak disp. x_n (m)	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443
$\frac{x_n}{x_{n+1}}$	1.0658	1.0648	1.0647	1.0645	1,0637	1.0627	
$\ln\left(\frac{x_n}{x_{n+1}}\right)$	0.0637	0.0628	0.0627	0.0624	0.0618	0.0608	

Thus, we get several values of the logarithmic decrement $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$ ①.

① Theoretically, all of these values should be the same, but it is rarely the case in practice. When x_n's are measured from an experimental setup, the values of D may vary even more.

We take the average value of D:

$$D = \bar{D} = 0.0624. \tag{5.45}$$

Let the equivalent single degree of freedom model have mass m, spring stiffness k, and damping rate c. Then

$$\lambda_D = \lambda\sqrt{1-\xi^2} \approx \lambda = \sqrt{\frac{k}{m}}.$$

Thus, from Eqn (5.44),

$$\frac{k}{m} = \lambda^2 = 100(\,\text{rad/s})^2, \tag{5.46}$$

and, since $D = \frac{cT_D}{2m}$, from Eqn (5.45) we get

$$\begin{aligned} c &= \frac{2mD}{T_D} \\ &= \frac{2m(0.0624)}{0.6279\,\mathrm{s}} \\ &= (0.1988\frac{1}{\mathrm{s}})m. \end{aligned} \tag{5.47}$$

Equations (5.46) and (5.47) have three unknowns: k, m, and c. We cannot determine all three uniquely from the given information. So, let us pick an arbitrary mass $m = 5\,\mathrm{kg}$. Then

$$\begin{aligned} k &= (100\frac{1}{\mathrm{s}^2})\cdot(5\,\mathrm{kg}) = 500\,\mathrm{N/m}, \quad \text{and} \\ c &= (0.1988\frac{1}{\mathrm{s}})\cdot(5\,\mathrm{kg}) = 0.99\,\mathrm{N\cdot s/m}. \end{aligned}$$

$$\boxed{m = 5\,\mathrm{kg}, \quad k = 500\,\mathrm{N/m}, \quad c = 0.99\,\mathrm{N\cdot s/m}.}$$

Of course, we could choose many other sets of values for m, k, and c which would match the given response. In practice, there is usually a little more information available about the system, such as the mass of the system. In that case, we can determine k and c uniquely from the given response.

5.5 Forced oscillations and resonance

If the world of oscillators was as we have described them so far, there wouldn't be much to talk about. The undamped oscillators would be oscillating away and the damped oscillators (all the real ones) would be all damped out. The reason vibrations exist is because they are some how excited. This excitement is also called *forcing* whether or not it is due to a literal mechanical force.

The simplest example of a 'forced' harmonic oscillator is the mass-spring-dashpot system with an additional mechanical force applied to the mass. A picture of such a system is shown in figure 5.38. The governing equation for a forced harmonic oscillator is:

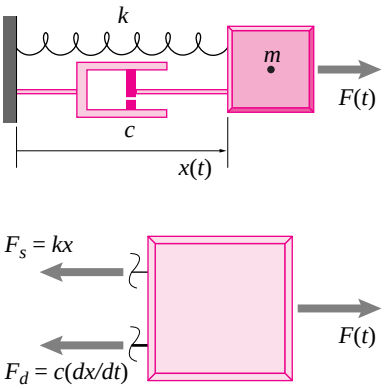


Figure 5.38: A forced mass-spring-dashpot is just a mass held in place by a spring and dashpot but pushed by a force F(t) from some external source.

(Filename:tfigure12.MSDforced)

$$m\ddot{x} + c\dot{x} + kx = F(t). \tag{5.48}$$

When $F(t) = 0$ there is no forcing and the governing equation reduces to that of the un-forced harmonic oscillator, eqn. (5.34). There are two special forcings of common interest:

- Constant force, and
- Sinusoidal forcing.

Constant force idealizes situations where the force doesn't vary much as due say, to gravity, a steady wind, or sliding friction. Sinusoidally varying forces are used to approximate oscillating forces as caused, say, by vibrating machine parts or earthquakes. Sums of sine waves can accurately approximate any force that varies with time①.

① The best approximation of a function with a sum of sine waves is a Fourier series, a topic we discuss no further here.

Forcing with a constant force

The case of constant forcing is both common and easy to analyze, so easy that it is often ignored. If $F = constant$, then the general solution of equation 5.48 for $x(t)$ is the same as the unforced case but with a constant added. The constant is F/k. The usual way of accommodating this case is to describe a new equilibrium point at $x = F/k$ and to pick a new deflection variable that is zero at that point. If we pick a new variable w and define it as $w = x - F/k$, the amount of motion away from equilibrium, then, substituting into equation 5.48 the forced oscillator equation becomes

$$m\ddot{w} + c\dot{w} + kw = 0, \tag{5.49}$$

which is the unforced oscillator equation. The case of constant forcing reduces to the case of no forcing if one merely changes what one calls the equilibrium point to be the place where the mass is in equilibrium, taking account of the constant applied force.

$$x(t) = \underbrace{A\cos(\lambda t) + B\sin(\lambda t)}_{x_h} + \underbrace{F/k}_{x_p}$$

An alternative approach is to use *superposition*. Here we say $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ satisfies $m\ddot{x} + c\dot{x} + kx = 0$ and $x_p(t)$ is any solution of $m\ddot{x} + c\dot{x} + kx = F$. Such a solution is $x_p = F/k$ if $F(t)$ is constant. So the net solution is F/k plus a solution to the 'homogeneous' equation 5.49.

Forcing with a sinusoidally varying force

The motion resulting from sinusoidal forcing is of central interest in vibration analysis. In this case we imagine that $F(t) = F_0 \cos(pt)$ where F_0 is the amplitude of forcing and p is the angular frequency of the forcing.

The general solution of equation 5.48 is given by the sum of two parts. One is the general solution of equation 5.34, $x_h(t)$, and the other is *any* solution of equation 5.48, $x_p(t)$. The solution $x_h(t)$ of the damped oscillator equation 5.34 is called the 'homogeneous' or 'complementary' solution. Any solution $x_p(t)$ of the forced oscillator equation 5.48 is called a 'particular' solution.

We already know the solution $x_h(t)$ of the undamped governing differential equation 5.34. This solution is equation 5.35, 5.36, or 5.37, depending on the values of the mass, spring and damping constants. So the new problem is to find any solution to the forced equation 5.48. The easiest way to solve this (or any other) differential equation is to make a fortuitous guess (you may learn other methods in your math classes). In this case if $F(t) = F_0 \cos(pt)$ we make the guess that

$$x_p(t) = A\cos(pt) + B\sin(pt). \tag{5.50}$$

If we plug this guess into the forced oscillator equation (5.48), we find, after much tedious algebra, that we do in fact have a solution if

$$\begin{aligned} A &= \frac{\frac{F_0}{k}\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)+\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)^2}, \\ \text{and} \qquad B &= \frac{\frac{F_0}{k}\left(\frac{cp}{k}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)+\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)^2} \quad . \end{aligned}$$

So the response to the cosine-wave forcing is the sum of a sine wave and a cosine wave.

$$\begin{aligned} x_p(t) = &\overbrace{\left(\frac{\frac{F_0}{k}\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)+\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)^2}\right)}^{A}\cos(pt) \\ &+\underbrace{\left(\frac{\frac{F_0}{k}\left(\frac{cp}{k}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)+\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)^2}\right)}_{B}\sin(pt) \end{aligned}$$

Alternatively sum of sine waves can be written as a cosine wave that has been shifted in phase as

$$x_p(t) = C\cos(pt-\phi),$$

$$\text{where} \qquad C = \sqrt{(A^2+B^2)} = \frac{\frac{F_0}{k}}{\sqrt{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)+\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)^2}}, \tag{5.51}$$

$$\text{and} \qquad \phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)}{\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)}\right). \tag{5.52}$$

The general solution, therefore, is

$$x(t) = x_h(t) + x_p(t). \tag{5.53}$$

Uses of resonance

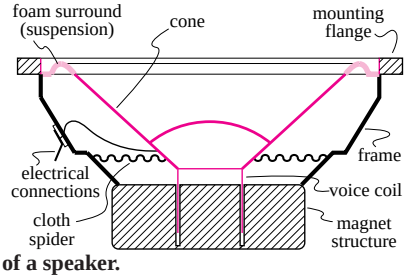
Though resonance is often a problem, it is also often of engineering use. Nuclear Magnetic Resonance imaging is used for medical diagnosis. The resonance of quartz crystals is used to time most watches now-a-days. In the old days, the resonant excitation of a clock pendulum was used to keep time. Self excited resonance is what makes musical instruments have such clear pitches.

Frequency response

One way to characterize a structures sensitivity to oscillatory loads is by a *frequency response* curve. The frequency response curve might be found by a physical experiment or from a calculation based on a simplified model of the structure. The curve somewhat describes the answer to the following question about a structure:

How does the size of the motion of a structure depend on the frequency and amplitude of an applied sinusoidal forcing?

5.5 A Loudspeaker cone is a forced oscillator.



of a speaker.

A speaker, similar to the ones used in many home and auto speaker systems, is one of many devices which may be conveniently modeled as a one-degree-of-freedom mass-spring-dashpot system. A typical speaker has a paper or plastic cone, supported at the edges by a roll of plastic foam (the surround), and guided at the center by a cloth bellows (the spider). It has a large magnet structure, and (not visible from outside) a coil of wire attached to the point of the cone, which can slide up and down inside the magnet. (The device described above is, strictly speaking, the speaker driver. A complete speaker system includes an enclosure, one or more drivers, and various electronic components.) When you turn on your stereo, it forces a current through the coil in time with the music, causing the coil to alternately attract and repel the magnet. This rapid oscillation of attraction and repulsion results in the vibration of the cone which you hear as sound.

In the speaker, the primary mass is comprised of the coil and cone, though the air near the cone also contributes as 'added mass.' The 'spring' and 'dashpot' effects in the system are due to the foam and cloth supporting the cone, and perhaps to various magnetic effects. Speaker system design is greatly complicated by the fact that the air surrounding the speaker must also be taken into account. Changing the shape of the speaker enclosure can change the effective values of all three mass-spring-dashpot parameters. (You may be able to observe this dependence by cupping your hands over a speaker (gently, without touching the moving parts), and observing amplitude or tone changes.) Nevertheless, knowledge of the basic characteristics of a speaker (e.g., resonance frequency), is invaluable in speaker system design.

Our approximate equation of motion for the speaker is identical to that of the ideal mass-spring-dashpot above, even though the forcing is from an electromagnetic force, rather a than a direct mechanical force:

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ with } F(t) = \alpha i(t) \tag{5.54}$$

where $i(t)$ is the electrical current flow through the coil in amps, and α is the electro-mechanical coupling coefficient, in force per unit current.

Here is how the method works. First, you must apply a sinusoidal force, say $F = F_0 \cos(pt)$, to the structure at a physical point of interest. Then you measure the motion of a part of the structure of interest. You might instead measure a strain or rotation, but for definiteness let's assume you measure the displacement of some point on the structure δ.

If the structure is linear and has some damping, the eventual motion of the structure will be a sinusoidal oscillation. In particular, you will measure that

$$\delta = C \cdot \cos(pt - \phi). \tag{5.55}$$

where C and ϕ have been defined previously in equation 5.51. If you had applied half as big a force, you would have measured half the displacement, still assuming the structure is linear, so the ratio of the displacement to the force C/F_0 is independent of the size of the force F_0. Let's define:

$$R = \frac{C}{F_0} \tag{5.56}$$

That is, the response variable R is the ratio of the amplitude of the displacement sine

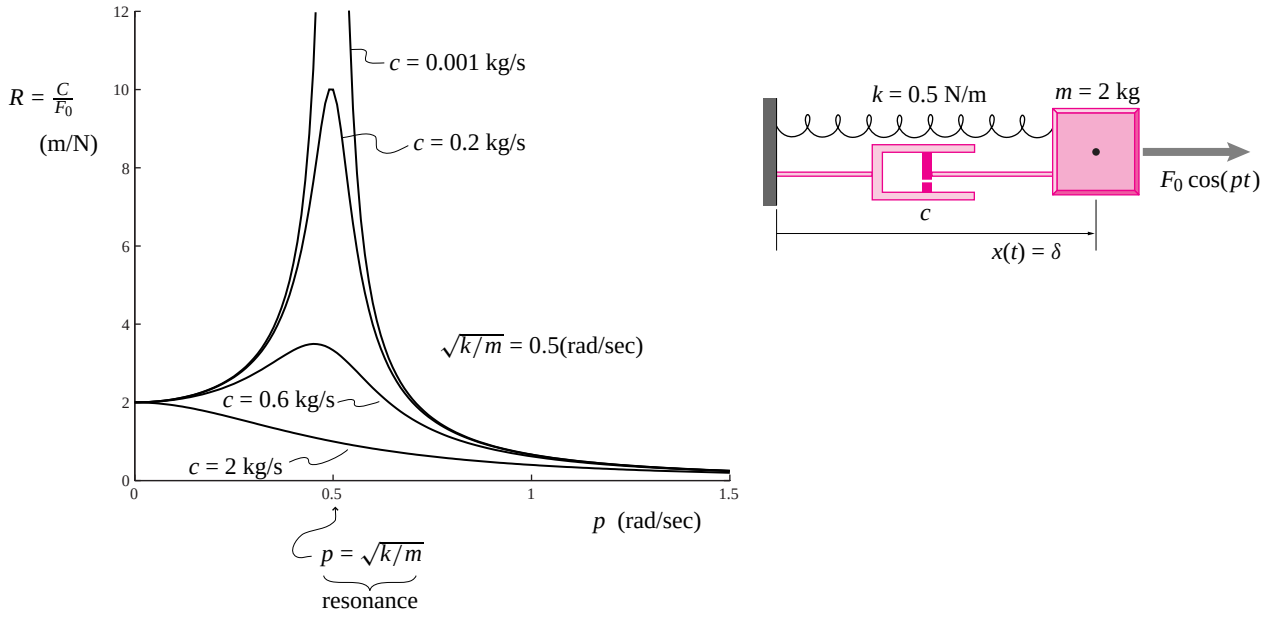


Figure 5.39: (Filename:tfigure12.ampl.vs.freq)

wave to the amplitude of the forcing sine wave.

Now, this experiment can be repeated for different values of of the angular forcing frequency p. The ratio of the vibrating displacement δ to that of the applied forcing F_0 will depend on p. The structure has different sensitivities to forcing at different frequencies. So the response ratio amplitude R depends on p. The function $R = R(p)$ is called the frequency response. A plot of the amplitude ratio R versus the driving frequency p is shown in figure 5.39 for various values of the damping coefficient c. Numerical values are shown for definiteness although the plot could be shown as dimensionless.

Experimental measurement

To measure the frequency response function experimentally, one can apply forcing at a whole range of forcing frequencies. Another approach is to apply a sudden,

'impulsive', force and look at the response. This second method is equivalent, it turns out, as you may learn in the context of Laplace transforms or Fourier analysis.

Why does on want to know the frequency response? The answer is because it is one way to think about structural response. A car suspension may never be tested on a sinusoidal road. But knowing how the suspension would respond to sine wave shaped roads of all possible wave lengths somehow characterizes the car's response to roads with any kind of bumpiness.

Example: **Resonance of a building**

A mildly damped structure has a natural frequency of 17 hz and is forced at 17 hz. Because the frequency response function has a peak at 17 hz, resonance, the structures motions will be very large. □

SAMPLE 5.20 *Particular solution:* Find a particular solution of the equation $xddot + \lambda^2 x = F(t)$ where

(a) $F(t) = mg$ (a constant),
(b) $F(t) = At$,
(c) $F(t) = C\sin(pt)$.

Solution The given differential equation is a second order linear ordinary differential equation with a non-zero right hand side. A particular solution of this equation must satisfy the entire equation. For such equations, we guess a particular solution to have the *same functional form* as the right hand side (the forcing function) and plug it into the equation to see if our guess works. We can usually determine the values of any unknown, assumed constants so that the assumed solution satisfies the equation. Let us see how it works here.

(a) The forcing function is a constant, mg. So, let us assume the particular solution to be a constant, *i.e.*, let $x_p = C$. Plugging it into the equation, we have

$$\underbrace{\ddot{C}}_{0} + \lambda^2 C = mg \quad \Rightarrow \quad C = mg/\lambda^2 \quad \Rightarrow \quad x_p = mg/\lambda^2$$

$$\boxed{x_p = mg/\lambda^2}$$

(b) The forcing function is linear in t. So, let us assume a linear function as a particular solution, $x_p = \alpha t$ where α is a constant. Now, noting that $\dot{x}_p = \alpha \quad \Rightarrow \quad \ddot{x_p} = 0$, and plugging back into the differential equation, we get

$$\lambda^2 \alpha t = At \quad \Rightarrow \quad \alpha = A/\lambda^2 \quad \Rightarrow \quad x_p(t) = (A/\lambda^2)t.$$

$$\boxed{x_p(t) = (A/\lambda^2)t}$$

(c) The forcing function is a harmonic function. So, let $x_p = \beta\sin(pt)$ where β is a constant to be determined later. Now, plugging x_p into the differential equation and noting that $\ddot{x}_p = -\beta\omega^2\sin(pt)$, we get

$$(-\beta p^2 + \beta\lambda^2)\sin(pt) = C\sin(pt) \quad \Rightarrow \quad \beta = \frac{C}{\lambda^2 - p^2}.$$

Thus the particular solution in this case is

$$x_p(t) = \frac{C}{\lambda^2 - p^2}\sin(pt).$$

$$\boxed{x_p(t) = \tfrac{C}{\lambda^2 - p^2}\sin(pt)}$$

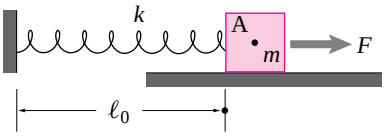


Figure 5.40: (Filename:sfig5.5.forcedosc)

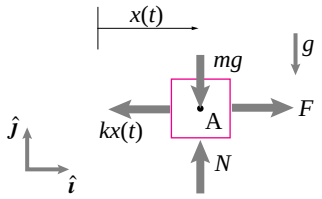


Figure 5.41: Free body diagram of the mass.

(Filename:sfig5.5.forcedosc.a)

SAMPLE 5.21 *Response to a constant force:* A constant force $F = 50\,\text{N}$ acts on a mass-spring system as shown in the figure. Let $m = 5\,\text{kg}$ and $k = 10\,\text{kN/}m$.

- Write the equation of motion of the system.
- If the system starts from the initial displacement $x_0 = 0.01\,\text{m}$ with zero velocity, find the displacement of the mass as a function of time.
- Plot the response (displacement) of the system against time and describe how it is different from the unforced response of the system.

Solution

(a) The free body diagram of the mass is shown in Fig. 5.41 at a displacement x (assumed positive to the right). Applying linear momentum balance in the x-direction, *i.e.*, $(\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}) \cdot \hat{\boldsymbol{\imath}}$, we get

$$\begin{aligned} F - kx &= m\ddot{x} \\ \Rightarrow \quad m\ddot{x} + kx &= F \end{aligned} \tag{5.57}$$

which is the equation of motion of the system.

(b) The equation of motion has a non-zero right hand side. Thus, it is a nonhomogeneous differential equation. A general solution of this equation is made up of two parts — the homogeneous solution x_h which is the solution of the unforced system (eqn. (5.57) with $F = 0$), and a particular solution x_p that satisfies the nonhomogeneous equation. Thus,

$$x(t) = x_h(t) + x_p(t). \tag{5.58}$$

Now, let us find $x_h(t)$ and $x_p(t)$.

Homogeneous solution: $x_h(t)$ has to satisfy $m\ddot{x} + kx = 0$. Let $\lambda = \sqrt{k/m}$. Then, from the solution of unforced harmonic oscillator, we know that

$$x_h(t) = A\sin(\lambda t) + B\cos(\lambda t)$$

where A and B are constants to be determined later from initial conditions.

Particular solution: x_p must satisfy eqn. (5.57). Since the nonhomogeneous part of the equation is a constant (F), we guess that x_p must be a constant too (of the same form as F). Let $x_p = C$. Now we substitute $x_p = C, \quad \Rightarrow \quad \ddot{x}_p = \ddot{(C)} = 0$ in eqn. (5.57) to determine C.

$$kC = F \quad \Rightarrow \quad C = F/k \quad \text{or} \quad x_p = F/k.$$

Substituting x_h and x_p in eqn. (5.58), we get

$$x(t) = A\sin(\lambda t) + B\cos(\lambda t) + F/k. \tag{5.59}$$

Now we use the given initial conditions to determine A and B.

$$\begin{aligned} x(t=0) &= B + F/k = x_0 \text{ (given)} \quad \Rightarrow \quad B = x_0 - F/k \\ \dot{x}(t) &= A\lambda\cos(\lambda t) - B\lambda\sin(\lambda t) \\ \Rightarrow \quad \dot{x}(t=0) &= A = 0 \text{ (given)} \quad \Rightarrow \quad A = 0. \end{aligned}$$

Thus,

$$x(t) = (x_0 - F/k)\cos(\lambda t) + F/k.$$

(c) Let us plug the given numerical values, $k = 10\,\text{kN}/m$, $m = 5\,\text{kg}$, $\Rightarrow \quad \lambda = \sqrt{k/m} = 44.72\,\text{rad/s}$, $F = 50\,\text{N}$ and $x0 = 0.01\,\text{m}$ in eqn. (5.60). The displacement is now given as

$$x(t) = -.04\,\text{m}\cos(44.72\,\text{rad/s}\cdot t) + .05\,\text{m}.$$

This response is plotted in Fig. 5.42 against time. Note that the oscillations of the mass are about a non-zero mean value, $x_{eq} = 0.04\,\text{m}$. A little thought should reveal that this is what we should expect. When a mass hangs from a spring under gravity, the spring elongates a little, by mg/k to be precise, to balance the mass. Thus, the new static equilibrium position is not at the relaxed length ℓ_0 of the spring but at $\ell_0 + mg/k$. Any oscillations of the mass will be about this new equilibrium.

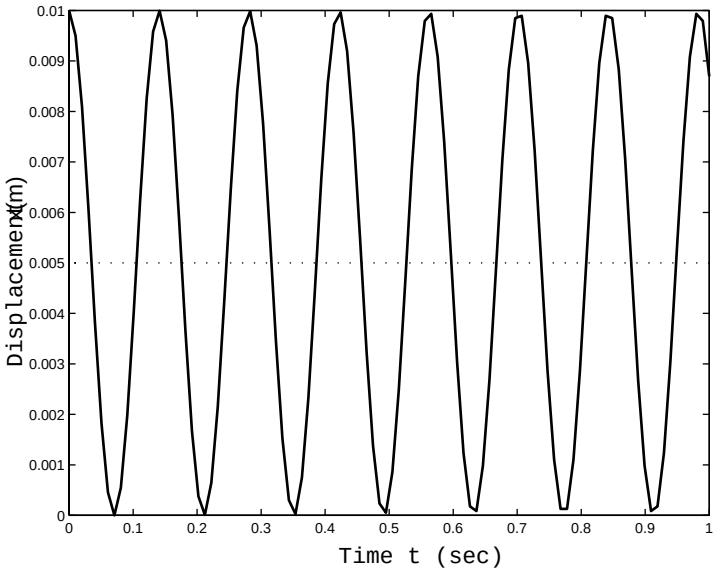


Figure 5.42: Displacement of the mass as a function of time. Note that the mass oscillates about a nonzero value of x.

(Filename:sfig5.5.forcedosc.b)

This problem is exactly like a mass hanging from a spring under gravity, a constant force, but just rotated by 90°. The new static equilibrium is at $x_eq = F/k$ and any oscillations of the mass have to be around this new equilibrium. We can rewrite the response of the system by measuring the displacement of the mass from the new equilibrium. Let $\tilde{x} = x - F/k$. Then, eqn. (5.60) becomes

$$\tilde{x} = \tilde{x}_0 \cos(\lambda t)$$

where $\tilde{x}_0 = x_0 - F/k$ is the initial displacement. Clearly, this is the response of an unforced harmonic oscillator. Thus the effect of a constant force on a spring-mass system is just a shift in its static equilibrium position.

□

SAMPLE 5.22 *Damping and forced response:* When a single-degree-of-freedom damped oscillator (mass-spring-dashpot system) is subjected to a periodic forcing $F(t) = F_0 \sin(pt)$, then the response of the system is given by

$$x(t) = C\cos(pt - \phi)$$

where $C = \frac{F_0/k}{\sqrt{(2\zeta r)^2+(1-r^2)^2}}$, $\phi = \tan^{-1}\frac{2\zeta r}{1-r^2}$, $r = \frac{p}{\lambda}$, $\lambda = \sqrt{k/m}$ and ζ is the damping ratio.

(a) For $r \ll 1$, *i.e.*, the forcing frequency p much smaller than the natural frequency λ, how does the damping ratio ζ affect the response amplitude C and the phase ϕ?

(b) For $r \gg 1$, *i.e.*, the forcing frequency p much larger than the natural frequency λ, how does the damping ratio ζ affect the response amplitude C and the phase ϕ?

Solution

(a) If the frequency ratio $r \ll 1$, then r^2 will be even smaller; so we can ignore r^2 terms with respect to 1 in the expressions for C and ϕ. Thus, for $r \ll 1$,

$$\begin{aligned} C &= \frac{F_0/k}{\sqrt{(2\zeta r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{1} = \frac{F_0}{k} \\ \phi &= \tan^{-1}(2\zeta r) \approx \tan^{-1} 0 = 0 \end{aligned}$$

that is, the response amplitude does not vary with the damping ratio ζ, and the phase also remains constant at zero. As an example, we use the full expressions for C and ϕ for plotting them against ζ for $r = 0.01$ in Fig. 5.43

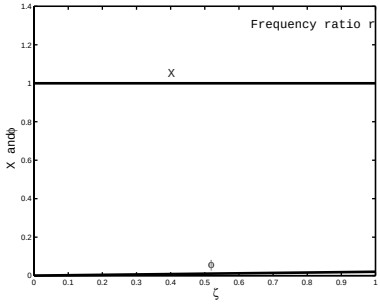


Figure 5.43: (Filename:sfig5.5.smallr)

$$\boxed{\text{For } r \ll 1,\ C \approx F_0/k,\ \text{and}\ \ \phi \approx 0}$$

(b) If $r \gg 1$, then the denominator in the expression for C, $4\zeta^2 r^2 + (1-r^2)^2 \approx r^4$ (because we can ignore all other terms with respect to r^4. Similarly, we can ignore 1 with respect to r^2 in the expression for ϕ. Thus, for $r \gg 1$,

$$\begin{aligned} C &= \frac{F_0/k}{\sqrt{(2\zeta r)^2 + (1-r^2)^2}} C \approx \frac{F_0/k}{r^2} = 0 \\ \phi &= \tan^{-1}\frac{2\zeta r}{-r^2} \approx \tan^{-1}\frac{2\zeta}{-r} \approx \tan^{-1}(-0) = \pi. \end{aligned}$$

Once again, we see that the response amplitude and phase do not vary with ζ. This is also evident from Fig. 5.44 where we plot C and ϕ using their full expressions for $r = 10$. The slight variation in ϕ around π goes away as we take higher values of r.

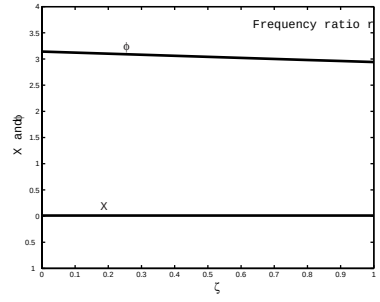


Figure 5.44: (Filename:sfig5.5.bigr)

$$\boxed{\text{For } r \gg 1,\ C \approx 0,\ \text{and}\ \ \phi \approx \pi}$$

Thus, we see that the damping in a system does not affect the response of the system much if the forcing frequency is far away from the natural frequency.

SAMPLE 5.23 *Energetics of resonance:* Consider the response of a damped harmonic oscillator to a periodic forcing. Find the work done on the system by the periodic force during a single cycle of the force and show how this work varies with the forcing frequency and the damping ratio.

Solution Let us consider the damped harmonic oscillator shown in Fig. 5.45 with $F(t) = F_0 \sin(pt)$. The equation of motion of the system is $m\ddot{x} + c\dot{x} + kx = F_0 \sin(pt)$ and the response of the system may be expressed as $X \sin(pt - \phi)$ where $X = (F_0/k)/\sqrt{(2\zeta r)^2 + (1-r^2)^2}$ and $\phi = \tan^{-1}(2\zeta r/(1-r^2))$, with $r = p/\lambda$, $\lambda = \sqrt{k/m}$ and $\zeta = c/\sqrt{2km}$.

We can compute the work done by the applied force on the system in one cycle by evaluating the integral

$$W = \int_{\text{onecycle}} F\,dx$$

But, $x = X\sin(pt-\phi) \quad \Rightarrow \quad dx = Xp\cos(pt-\phi)dt$. Therefore,

$$\begin{aligned}
W &= \int_0^{2\pi/\lambda} F_0 \sin(pt)\cdot Xp\cos(pt-\phi)\,dt \\
&= F_0 Xp \int_0^{2\pi/\lambda} \sin(pt)\cos(pt-\phi)\,dt \\
&= F_0 Xp \int_0^{2\pi/\lambda} \sin(pt)(\cos(pt)\cos\phi + \sin(pt)\sin\phi)\,dt \\
&= F_0 Xp \left[\cos\phi\cdot\frac{1}{2}\int_0^{2\pi/\lambda}\sin(2pt)\,dt + \sin\phi\cdot\frac{1}{2}\int_0^{2\pi/\lambda}(1-\cos(2pt))\,dt\right] \\
&= \frac{F_0 Xp}{2}\left[\cos\phi\left(-\frac{\cos(2pt)}{2p}\right)\Big|_0^{2\pi/\lambda} + \sin\phi\left(t-\frac{\sin(2pt)}{2p}\right)\Big|_0^{2\pi/\lambda}\right] \\
&= \frac{F_0 Xp}{2}\left[\frac{\cos\phi}{2p}(-1+1) + \frac{2\pi}{p}\sin\phi + 0\right] \\
&= \frac{F_0 Xp}{2}\cdot\frac{2\pi}{p}\sin\phi \\
&= F_0\pi X\sin\phi
\end{aligned}$$

Although the expression obtained above for W looks simple, we must substitute for X and ϕ to see the dependence of W on the damping ratio ζ and the frequency ratio r.

$$W = \frac{F_0^2\pi}{\pi\sqrt{(2\zeta r)^2 + (1-r^2)^2}}\cdot\sin\left(\tan^{-1}\frac{2\zeta r}{1-r^2}\right) \tag{5.60}$$

Unfortunately, this expression is too complicated to see the dependence of W on ζ and r. However, we know that for small $r(<1)$, $\phi \approx 0$ and for large $r(>1)$, $\phi = \approx \pi$, implying that W is almost zero in both these cases. On the other hand, for r close to one, that is, close to resonance, $\phi \approx \pi/2 \quad \Rightarrow \quad \sin\phi \approx 1$, but the response amplitude X is large (for small ζ), which makes W to be big near the resonance. Figure 5.46 shows a plot of W against r, using eqn. (5.60), for different values of ζ. It is clear from the plot that the work done on the system in a single cycle is much larger close to the resonance for lightly damped systems. This explains why the response amplitude keeps on growing near resonance.

$$\boxed{W = F_0\pi X\sin\phi}$$

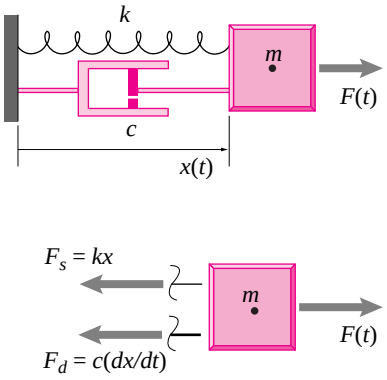


Figure 5.45: (Filename:sfig5.5.reswork)

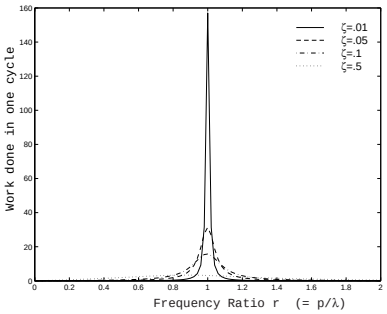


Figure 5.46: (Filename:sfig5.5.reswork.plot)

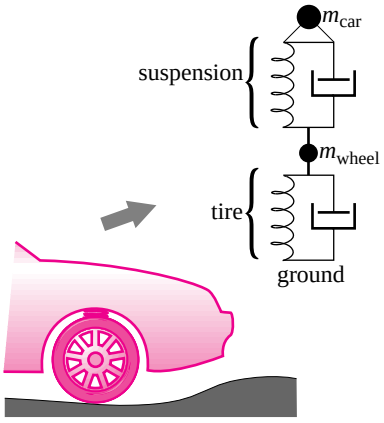


Figure 5.47: (Filename:tefig3.4.car.susp)

5.6 Coupled motions in 1D

Many important engineering systems have parts that move independently. A simple dynamic model using a single particle is not adequate. So here, still using one dimensional mechanics, we consider systems that can be modelled as two or more particles. Such one-dimensional coupled motion analysis is common in engineering practice in situations where there are connected parts that all move in about the same direction, but not the same amount at the same time.

Example: **Car suspension.**

A model of a car suspension treats the wheel as one particle and the car as another. The wheel is coupled to the ground by a tire and to the car by the suspension. In a first analysis the only motion to consider would be vertical for both the wheel and the car. □

The simplest way of dealing with the coupled motion of two or more particles is to write $\vec{F} = m\vec{a}$ for each particle and use the forces on the free body diagrams to evaluate the forces. Because the most common models for the interaction forces are springs and dashpots (see chapter 3), one needs to account for the relative positions and velocities of the particles.

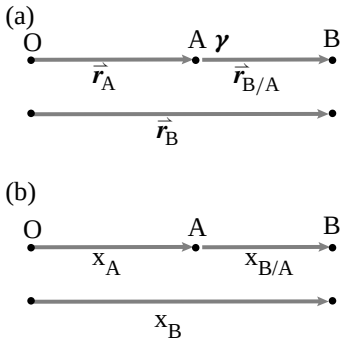


Figure 5.48: The relative position of points A and B in one dimension.
(Filename:tfigure.relpos1D)

Relative motion in one dimension

If the position of A is $\vec{r}_A$, and B's position is $\vec{r}_B$, then B's position *relative* to A is

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A.$$

Relative velocity and acceleration are similarly defined by subtraction, or by differentiating the above expression, as

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A \text{ and } \vec{a}_{B/A} = \vec{a}_B - \vec{a}_A.$$

In one dimension, the relative position diagram of Fig. 2.5 on page 11 becomes Fig. 5.48. $\vec{r} = x\hat{\imath}$, $\vec{v} = v\hat{\imath}$, and $\vec{a} = a\hat{\imath}$. So, we can write,

$$\begin{aligned} x_{B/A} &\equiv x_B - x_A, \\ v_{B/A} &\equiv v_B - v_A, \text{ and} \\ a_{B/A} &\equiv a_B - a_A. \end{aligned}$$

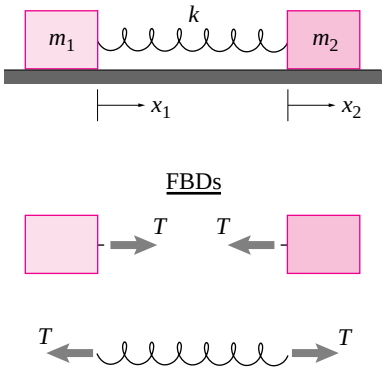


Figure 5.49: (Filename:tefig3.4.mass.springer)

Example: **Two masses connected by a spring.**

Consider the two masses on a frictionless support (Fig. 5.49). Assume the spring is unstretched when $x_1 = x_2 = 0$. After drawing free body diagrams of the two masses we can write $\vec{F} = m\vec{a}$ for each mass:

$$\begin{aligned} &\textbf{mass 1:} \quad \vec{F}_1 = m\vec{a}_1 \quad \Rightarrow \quad T\hat{\imath} = m_1\ddot{x}_1\hat{\imath} \\ &\textbf{mass 2:} \quad \vec{F}_2 = m\vec{a}_2 \quad \Rightarrow \quad -T\hat{\imath} = m_2\ddot{x}_2\hat{\imath} \end{aligned} \tag{5.61}$$

The stretch of the spring is

$$\Delta\ell = x_2 - x_1$$

$$\text{so} \qquad T = k\Delta\ell = k(x_2 - x_1). \tag{5.62}$$

Combining (5.61) and (5.62) we get

$$\begin{aligned} \ddot{x}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\ \ddot{x}_2 &= \left(\frac{1}{m_2}\right) (-k(x_2 - x_1)) \end{aligned} \tag{5.63}$$

Note: Take care with signs when setting up this type of problem. You can check for example that if $x_2 > x_1$, mass 1 accelerates to the right ($\ddot{x}_1 > 0$) and mass 2 accelerates to the left($\ddot{x}_2 < 0$).

□

The differential equations that result from writing $\vec{F} = m\vec{a}$ for the separate particles are coupled second-order equations. They are often solved by writing them as a system of first-order equations.

Example: **Writing second-order ODEs as first-order ODEs.**

Refer again to Fig. 5.49 If we define $v_1 = \dot{x}_1$ and $v_2 = \dot{x}_2$ we can rewrite equation 5.63 as

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{v}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\ \dot{x}_2 &= v_2 \\ \dot{v}_2 &= \left(\frac{1}{m_2}\right) (-k)\,(x_2 - x_1) \end{aligned}$$

or, defining $z_1 = x_1, z_2 = v_1, z_3 = x_2, z_4 = v_2$, we get

$$\begin{aligned} \dot{z}_1 &= & & z_2 & \\ \dot{z}_2 &= & -\left(\tfrac{k}{m}\right) z_1 & & +\left(\tfrac{k}{m_1}\right) z_3 \\ \dot{z}_3 &= & & & z_4 \\ \dot{z}_4 &= & \tfrac{k}{m_2} z_1 & & -\tfrac{k}{m_2} z_3\,. \end{aligned}$$

□

Most numerical solutions depend on specifying numerical values for the various constants and initial conditions.

Example: **computer solution**

If we take, in consistent units, $m_1 = 1, k = 1, m_2 = 1, x_1(0) = 0, x_2(0) = 0, v_1(0) = 1$, and $v_2(0) = 0$, we can set up a well defined computer problem (please see the preface for a discussion of the computer notation). This problem corresponds to finding the motion just after the left mass was hit on the left side with a hammer.:

```
ODEs = {z1dot =  z2
        z2dot = -z1 + z3
        z3dot =  z4
        z4dot =  z1 - z3}
ICs  = {z1(0) 0, z2(0)=1, z3(0)=0, z4(0)=0}
solve ODEs with ICs from t=0 to t=10
plot z1 vs t.
```

This yields the plot shown in Fig. 5.50. □

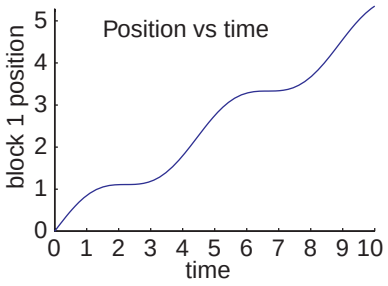


Figure 5.50: Plot of the position of the left mass *vs.* time.

(Filename:tfig.coupledmasses)

As the samples show, the same methods work for problems involving connections with dashpots.

Center of mass

For both theoretical and practical reasons it is often useful to pay attention to the motion of the average position of mass in the system. This average position is called the center of mass. For a collection of particles in one dimension the center of mass is

$$x_{\mathrm{CM}} = \frac{\sum x_i m_i}{m_{\mathrm{tot}}}, \tag{5.64}$$

where $m_{\mathrm{tot}} = \sum m_i$ is the total mass of the system. The velocity and acceleration of the center of mass are found by differentiation to be

$$v_{\mathrm{CM}} = \frac{\sum v_i m_i}{m_{\mathrm{tot}}} \quad \text{and} \quad a_{\mathrm{CM}} = \frac{\sum a_i m_i}{m_{\mathrm{tot}}}. \tag{5.65}$$

If we imagine a system of interconnected masses and add the $\vec{F} = m\vec{a}$ equations from all the separate masses we can get on the left hand side only the forces from the outside; the interaction forces cancel because they come in equal and opposite (action and reaction) pairs. So we get:

$$\sum F_{\mathrm{external}} = \sum a_i m_i = m_{\mathrm{tot}} a_{\mathrm{CM}}. \tag{5.66}$$

Thus, the center of mass of a system that may be deforming wildly, obeys the same simple governing equation as a single particle. Although our demonstration here was for particles in one dimension. The result holds for any bodies of any type in any number of dimensions.

5.6 *THEORY*
What saith Newton about collisions?

Page 25 of Newton Principia, Motte's translation revised, by Florian Cajori (Univ. of CA press, 1947) He discusses collisions of spheres as measured in pendulum experiments. He takes account of air friction. He has already discussed momentum conservation.

"In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of bodies are bruised by their impact, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and strongly compressed. For, first, by letting go the pendula's bodies, and measuring their reflection, I determined the quantity of their elastic force; and then, according to this force, estimated the reflections that ought to happen in other cases of impact. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity to which they met, as about 5 to 9. Balls of steel returned with almost the same velocity; those of cork with a velocity something less; but in balls of glass the proportion was as about 15 to 16. "

SAMPLE 5.24 For the given quantities and initial conditions, find $x_1(t)$. Assume the spring is unstretched when $x_1 = x_2$.

$$m_1 = 1\,\text{kg}, \quad m_2 = 2\,\text{kg}, \quad k = 3\,\text{N/m}, \quad c = 5\,\text{N/(m/s)}$$
$$x_1(0) = 1\,\text{m}, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 2\,\text{m}, \quad \dot{x}_2(0) = 0.$$

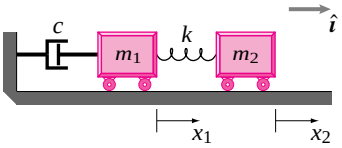


Figure 5.51: (Filename:sfig3.4.unstr.ini)

Solution

FBDs

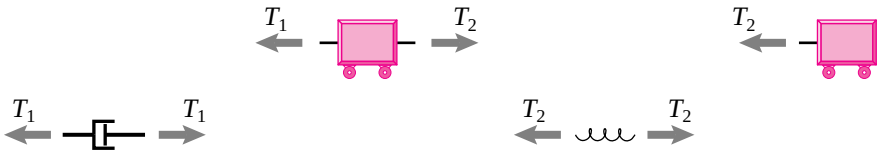


Figure 5.52: (Filename:sfig3.4.unstr.ini.fbd)

The spring and dashpot laws give

$$T_1 = c\dot{x}_1 \qquad T_2 = k(x_2 - x_1). \tag{5.67}$$

LMB

$$\begin{aligned} &\textstyle\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \\ \text{mass 1:}\quad &-T_1\hat{\boldsymbol{\imath}} + T_2\hat{\boldsymbol{\imath}} = m_1\ddot{x}_1\hat{\boldsymbol{\imath}} \\ \text{mass 2:}\quad &-T_2\hat{\boldsymbol{\imath}} = m_2\ddot{x}_2\hat{\boldsymbol{\imath}}. \end{aligned} \tag{5.68}$$

Applying the constitutive laws (5.67) to the momentum balance equations (5.68) gives

$$\begin{aligned} \ddot{x}_1 &= [k(x_2 - x_1) - c\dot{x}_1]/m_1 \\ \ddot{x}_2 &= [-k(x_2 - x_1)]/m_2. \end{aligned}$$

Defining $z_1 = x_1$, $z_2 = \dot{x}_1$, $z_3 = x_2$, $z_4 = \dot{x}_2$ gives

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= [k(z_3 - z_1) - cz_2]/m_1 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= [-k(z_3 - z_1)]/m_2. \end{aligned}$$

The initial conditions are

$$z_1(0) = 1\,\text{m}, \quad z_2(0) = 0, \quad z_3(0) = 2\,\text{m}, \quad z_4(0) = 0.$$

We are now set for numerical solution.

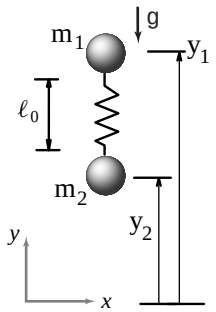


Figure 5.53: (Filename:sfig5.6.hopper)

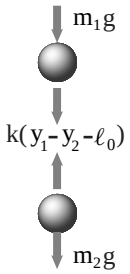


Figure 5.54: Free body diagram of the two masses m_1 and m_2

(Filename:sfig5.6.hopper.a)

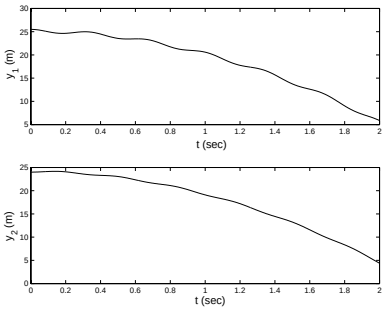


Figure 5.55: Numericaly obtained solutions $y_1(t)$ and $y_2(t)$

(Filename:sfig5.6.hopper.b)

SAMPLE 5.25 *Flight of a toy hopper.* A hopper model is made of two masses $m_1 = 0.4\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a spring with stiffness $k = 100\,\text{N/m}$ as shown in Fig. **??**. The unstretched length of the spring is $\ell_0 = 1\,\text{m}$. The model is released from rest from the configuration shown in the figure with $y_1 = 25.5\,\text{m}$ and $y_2 = 24\,\text{m}$.

(a) Find and plot $y_1(t)$ and $y_2(t)$ for $t = 0$ to $2\,\text{s}$.
(b) Plot the motion of m_1 and m_2 with respect to the center of mass of the hopper during the same time interval.
(c) Plot the motion of the center of mass of the hopper from the solution obtained for $y_1(t)$ and $y_2(t)$ and compare it with analytical values obtained by integrating the center of mass motion directly.

Solution The free body diagrams of the two masses are shown in Fig. 5.54. From the linear momentum balance in the y direction, we can write the equations of motion at once.

$$\begin{aligned} m_1\ddot{y}_1 &= -k(y_1 - y_2 - \ell_0) - m_1 g \\ \Rightarrow \quad \ddot{y}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \qquad (5.69) \\ m_2\ddot{y}_2 &= k(y_1 - y_2 - \ell_0) - m_2 g \\ \Rightarrow \quad \ddot{y}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g \qquad (5.70) \end{aligned}$$

(a) The equations of motion obtained above are coupled linear differential equations of second order. We can solve for $y_1(t)$ and $y_2(t)$ by numerical integration of these equations. As we have shown in previous examples, we first need to set up these equations as a set of first order equations.
Letting $\dot{y}_1 = v_1$ and $\dot{y}_2 = v_2$, we get

$$\begin{aligned} \dot{y}_1 &= v_1 \\ \dot{v}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \\ \dot{y}_2 &= v_2 \\ \dot{v}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g \end{aligned}$$

Now we solve this set of equations numerically using some ODE solver and the following pseudocode.

```
ODEs = {y1dot = v1,
        v1dot = -k/m1*(y1-y2-l0) - g,
        y2dot = v2,
        v1dot = k/m1*(y1-y2-l0) - g}
IC   = {y1(0)=25.5, v1(0)=0, y2(0)=24, v2(0)=0}
Set  k=100, m1=0.4, m2=1, l0=1
Solve ODEs with IC for t=0 to t=2
Plot y1(t) and y2(t)
```

The solution obtained thus is shown in Fig. 5.55.

(b) We can find the motion of m_1 and m_2 with respect to the center of mass by subtraction the motion of the center of mass, $y_{\rm cm}$ from y_1 and y_2. Since,

$$y_{\rm cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \tag{5.71}$$

we get,

$$\begin{aligned} y_{1/\rm cm} &= y_1 - y_{\rm cm} = \frac{m_2}{m_1+m_2}(y_1 - y_2) \\ y_{2/\rm cm} &= y_2 - y_{\rm cm} = -\frac{m_1}{m_1+m_2}(y_1 - y_2). \end{aligned}$$

The relative motions thus obtained are shown in Fig. 5.56. We note that the motions of m_1 and m_2, as seen by an observer sitting at the center of mass, are simple harmonic oscillations.

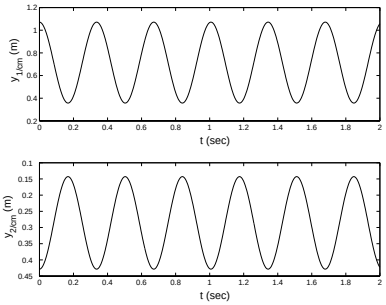


Figure 5.56: Numericaly obtained solutions $y_{1/\rm cm}(t)$ and $y_{1/\rm cm}(t)$.

(Filename:sfig5.6.hopper.c)

(c) We can find the center of mass motion $y_{\rm cm}(t)$ from y_1 and y_2 by using eqn. (5.71). The solution obtained thus is shown as a solid line in Fig. 5.58. We can also solve for the center of mass motion analytically by first writing the equation of motion of the cente of mass and then integrating it analytically.
The free body diagram of the hopper as a single system is shown in Fig. 5.57. The linear momentum balance for the system in the vertical direction gives

$$\begin{aligned} (m_1+m_2)\ddot{y}_{\rm cm} &= -m_1 g - m_2 g \\ \Rightarrow \quad \ddot{y}_{\rm cm} &= -g. \end{aligned}$$

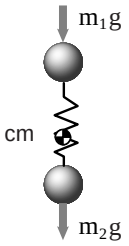


Figure 5.57: Free body diagram of the hopper as a single system. The spring force does not show up here since it becomes an internal force to the system

(Filename:sfig5.6.hopper.e)

We recognize this equation as the equation of motion of a freely falling body under gravity. We can integrate this equation twice to get

$$y_{\rm cm}(t) = y_{\rm cm}(0) + \dot{y}_{\rm cm}(0)t - \frac{1}{2}gt^2$$

Noting that $y_{\rm cm}(0) = 24.43\,\text{m}$ (from eqn. (5.71)), and $\dot{y}_{\rm cm}(0) = 0$ (the system is released from rest), we get

$$y_{\rm cm}(t) = 24.43\,\text{m} - \frac{1}{2}\cdot 9.81\,\text{m/s}^2 \cdot t^2$$

. The values obtained for the center of mass position from the above expression are shown in Fig. 5.58 by small circles.

□

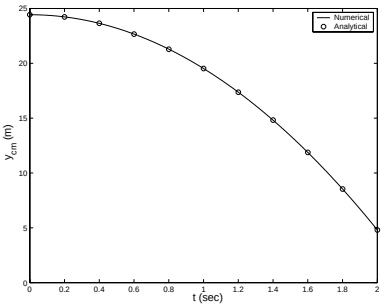


Figure 5.58: Numericaly obtained solution for the position of the center of mass, $y_{\rm cm}(t)$.

(Filename:sfig5.6.hopper.d)

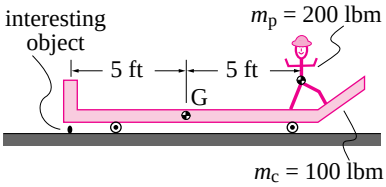


Figure 5.59: Mr. P spots an interesting object.

(Filename:sfig2.6.5)

SAMPLE 5.26 *Conservation of linear momentum.* Mr. P with mass $m_p = 200$ lbm is standing on a cart with frictionless and massless wheels. The cart weighs half as much as Mr. P. Standing at one end of the cart, Mr. P spots an interesting object at the other end of the cart. Mr. P decides to walk to the other end of the cart to pick up the object. How far does he find himself from the object after he reaches the end of the cart?

Solution From your own experience in small boats perhaps, you know that when Mr. P walks to the left the cart starts moves to the right. What we want to find is how far.

Consider the cart and Mr. P together to be the system of interest. The free body diagram of the system is shown in Fig. 5.60(a). From the diagram it is clear that

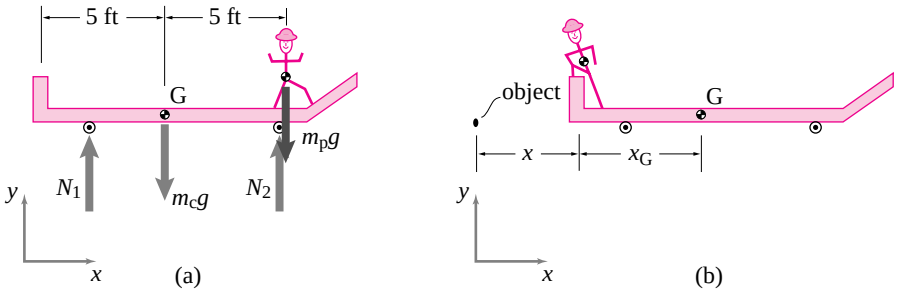


Figure 5.60: (a) Free body diagram of Mr. P-and-the-cart system. (b) The cart has moved to the right by distance x when Mr. P reaches the other end.

(Filename:sfig2.6.5a)

there are no external forces in the x-direction. Therefore,

$$\dot{L}_x = \sum F_x = 0 \quad \Rightarrow \quad L_x = \text{constant}$$

that is, the linear momentum of the system in the x-direction is 'conserved'. But the initial linear momentum of the system is zero. Therefore,

$$L_x = m_{tot}(v_{cm})_x = 0 \text{ all the time} \quad \Rightarrow \quad (v_{cm})_x = 0 \text{ all the time.}$$

Because the horizontal velocity of the center of mass is always zero, the center of mass does not change its horizontal position. Now let x_{cm} and x'_{cm} be the x-coordinates of the center of mass of the system at the beginning and at the end, respectively. Then,

$$x'_{cm} = x_{cm}.$$

Now, from the given dimensions and the stipulated position at the end in Fig. 5.60(b),

$$x_{cm} = \frac{m_c x_G + m_p x_p}{m_c + m_p} \quad \text{and} \quad x'_{cm} = \frac{m_c(x_G + x) + m_p x}{m_c + m_p}.$$

Equating the two distances we get,

$$\begin{aligned} m_c x_G + m_p x_p &= m_c(x_G + x) + m_p x \\ &= m_c x_G + x(m_c + m_p) \\ \Rightarrow \quad x &= \frac{m_p x_p}{m_c + m_p} \\ &= \frac{200\,\text{lbm} \cdot 10\,\text{ft}}{300\,\text{lbm}} = 6\frac{2}{3}\,\text{ft}. \end{aligned}$$

6.67 ft

[Note: if Mr. P and the cart have the same mass, the cart moves to the right the same distance Mr. P moves to the left.]

5.7 Time derivative of a vector: position, velocity and acceleration

So far in this chapter we have only considered things that move in a straight line. Of course we are interested also in things that move on more complicated paths. What are the paths of a hit baseball, a satellite, or a crashing plane? We now need to think about vector-valued functions of time. For example, the vectors linear momentum $\vec{L}$ and angular momentum $\vec{H}$ have a central place in the basic mechanics governing equations. Evaluation of these terms depends, in turn, on understanding the relation between position $\vec{r}$, its rate of change velocity $\vec{v}$, and between velocity $\vec{v}$ and its rate of change the acceleration $\vec{a}$.

What do we mean by the rate of change of a vector? The rate of change of any quantity, including vectors, is the ratio of the change of that quantity to the amount of time that passes, for very small amounts of time. ①The notation for the rate of change of a vector $\vec{r}$ is

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt}.$$

Or, in the short hand 'dot' notation invented by Newton for just this purpose, $\vec{v} = \dot{\vec{r}}$. The expression for the derivative of a vector $\frac{d\vec{r}}{dt}$ or $\dot{\vec{r}}$ has the same definition as the derivative of a scalar that one learns in elementary calculus. That is,

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}.$$

Vector differentiation is also sometimes needed for the calculation of the rate of change of linear momentum $\dot{\vec{L}}$ and rate of change of angular momentum $\dot{\vec{H}}_C$ for use in the momentum balance equations.

① Strictly speaking these words describe the average rate of change over the small time interval. Only in the mathematical limit of vanishing time intervals is this ratio not just approximately the rate of change, but exactly the rate of change.

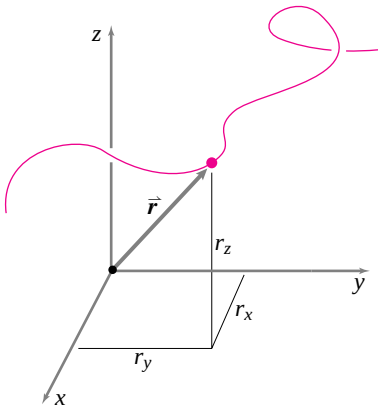


Figure 5.61: Cartesian coordinates

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Cartesian coordinates

The most primitive way to understand the motion of a system is to understand the motion of each of its parts using cartesian coordinates. That is each bit of mass in a system has a location $\vec{r}$, relative to the origin of a 'good' reference frame as shown in figure 5.61, which can be written as:

$$\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k} \qquad \text{or} \qquad \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}.$$

So velocity and acceleration are simply described by derivatives of $\vec{r}$. Since the base vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are constant, differentiation to get velocity and acceleration is simple:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} \qquad \text{and} \qquad \vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}.$$

So if x, y, and z are known functions of time for every particle in the system, we can evaluate the rate of change of linear and angular momentum just by differentiating the functions twice to get the acceleration and then summing (or integrating) to get $\dot{\vec{L}}$ and $\dot{\vec{H}}$.

The idea is illustrated in figure 5.62. Let's assume

$$\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}$$

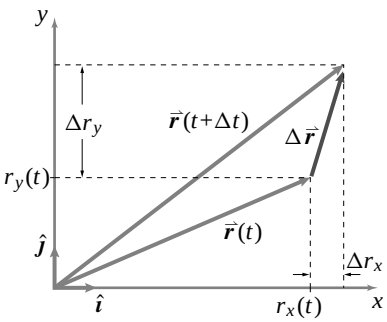
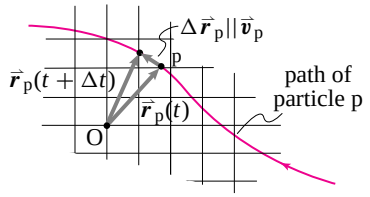


Figure 5.62: Change of position in Δt broken into components in 2-D . $\Delta\vec{r}$ is $\vec{r}(t+\Delta t) - \vec{r}(t)$. $\Delta\vec{r}$ has components Δr_x and Δr_y. So $\Delta\vec{r} = \Delta r_x\hat{\imath} + \Delta r_y\hat{\jmath}$. In the limit as Δt goes to zero, $\dot{\vec{r}}$ is the ratio of $\Delta\vec{r}$ to Δt.

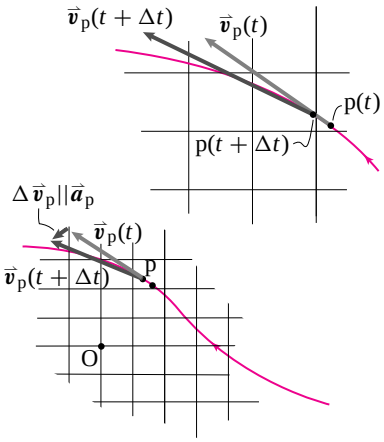
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①**Caution:** Later in the book we will use base vectors that change in time, such as polar coordinate base vectors, path basis vectors, or basis vectors attached to a rotating frame. For these vectors the components of the vector's derivative will *not* be the derivatives of its components.

(a) **Position**



(b) **Velocity** is tangent to the path. It is approximately in the direction of $\Delta\vec{r}$.



(c) **Acceleration** is generally *not* tangent to the path. It is approximately in the direction of $\Delta\vec{v}$.

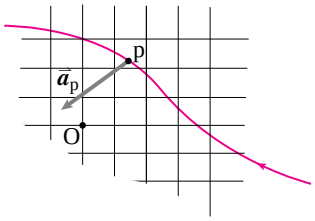


Figure 5.63: A particle moving on a curve. (a) shows the position vector is an arrow from the origin to the point on the curve. On the position curve the particle is shown at two times: t and $t+\Delta t$. The velocity at time t is roughly parallel to the difference between these two positions. The velocity is then shown at these two times in (b). The acceleration is roughly parallel to the difference between these two velocities. In (c) the acceleration is drawn on the path roughly parallel to the difference in velocities.

(Filename:tfigure2.2)

or

$$\vec{r}(t) = r_x(t)\hat{\imath} + r_y(t)\hat{\jmath} + r_z(t)\hat{k}.$$

Now we apply the definition of derivative and find

$$\begin{aligned}
\dot{\vec{r}}(t) &= \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} \\
&= \frac{\overbrace{\left(r_x(t+\Delta t)\hat{\imath}+r_y(t+\Delta t)\hat{\jmath}+r_z(t+\Delta t)\hat{k}\right)}^{\vec{r}(t+\Delta t)} - \overbrace{\left(r_x(t)\hat{\imath}+r_y(t)\hat{\jmath}+r_z(t)\hat{k}\right)}^{\vec{r}(t)}}{\Delta t} \\
&= \frac{r_x(t+\Delta t)-r_x(t)}{\Delta t}\hat{\imath} + \frac{r_y(t+\Delta t)-r_y(t)}{\Delta t}\hat{\jmath} + \frac{r_z(t+\Delta t)-r_z(t)}{\Delta t}\hat{k} \\
&= \dot{r}_x(t)\hat{\imath} + \dot{r}_y(t)\hat{\jmath} + \dot{r}_z(t)\hat{k}.
\end{aligned}$$

We have found the palatable result that the components of the velocity vector are the time derivatives of the components of the position vector ①. Vector differentiation is done to find the velocity and acceleration of particles or parts of bodies. The curve in figure 5.63 shows a particle P's path, that is, its position at a sequence of times. The position vector $\vec{r}_{P/O}$ is the arrow from the origin to a point on the curve, a different point on the curve at each instant of time. The velocity $\vec{v}$ at time t is the rate of change of position at that time, $\vec{v} \equiv \dot{\vec{r}}$.

Example: **Given position as a function of time, find the velocity.**

Given that the position of a point is:

$$\vec{r}(t) = C_1\cos(\omega\, t)\hat{\imath} + C_2\sin(\omega\, t)\hat{\jmath}$$

with $C_1 = 4\,\text{m}$, $C_2 = 2\,\text{m}$ and $\omega = 10\,\text{rad/s}$. What is the velocity (a vector) at $t = 3\,\text{s}$?

First we note that the components of $\vec{r}(t)$ have been given implicitly as

$$r_x(t) = C_1\cos(\omega\, t) \qquad \text{and} \qquad r_y(t) = C_2\sin(\omega\, t).$$

Then we find the velocity by differentiating each of the components with respect to time and re-assembling as a vector to get

$$\vec{v}(t) = \dot{\vec{r}} = -C_1\omega\sin(\omega\, t)\hat{\imath} + C_2\omega\cos(\omega\, t)\hat{\jmath}$$

Now we evaluate this expression with the given values of $C_1 = 4\,\text{m}$, $C_2 = 2\,\text{m}$, $\omega = 10\,\text{rad/s}$ and $t = 3\,\text{s}$ to get the velocity at 3 s as:

$$\begin{aligned}
\vec{v}(3\,\text{s}) &= -(4\,\text{m})(10/\text{s})\sin((10/\text{s})(3\,\text{s}))\hat{\imath} + \\
&\qquad (2\,\text{m})(10/\text{s})\cos((10/\text{s})(3\,\text{s}))\hat{\jmath} && (5.72)\\
&= \left(-40\sin(30)\hat{\imath} + 20\cos(30)\hat{\jmath}\right)\,\text{m/s} && (5.73)\\
&= \left(39.5\hat{\imath} + 3.09\hat{\jmath}\right)\,\text{m/s} && (5.74)
\end{aligned}$$

Note that the last line is calculated using the angle as measured in radians, not degrees. □

Product rule

We know three ways to multiply vectors. You can multiply a vector by a scalar, take the dot product of two vectors, and take the cross product of two vectors. Because these forms all show up in dynamics we nee to know a method for differentiating. The method is simple. All three kinds of vector multiplication obey the product rule of differentiation that you learned in freshmen calculus.

$$\begin{aligned}\frac{d}{dt}(a\vec{\boldsymbol{A}}) &= \dot{a}\vec{\boldsymbol{A}} + a\dot{\vec{\boldsymbol{A}}}\\ \frac{d}{dt}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) &= \dot{\vec{\boldsymbol{A}}}\cdot\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}\cdot\dot{\vec{\boldsymbol{B}}}\\ \frac{d}{dt}(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) &= \dot{\vec{\boldsymbol{A}}}\times\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}\times\dot{\vec{\boldsymbol{B}}}.\end{aligned}$$

The proofs of these identities is nearly an exact copy of the proof used for scalar multiplication.

Example: **Derivative of a vector of constant length.**

Assume

$$\frac{d}{dt}|\vec{\boldsymbol{C}}| = 0$$

so

$$\frac{d}{dt}|\vec{\boldsymbol{C}}|^2 = \frac{d}{dt}(\vec{\boldsymbol{C}}\cdot\vec{\boldsymbol{C}}) = 0.$$

Using the product rule above, we get

$$\frac{d}{dt}(\vec{\boldsymbol{C}}\cdot\vec{\boldsymbol{C}}) = \vec{\boldsymbol{C}}\cdot\dot{\vec{\boldsymbol{C}}} + \dot{\vec{\boldsymbol{C}}}\cdot\vec{\boldsymbol{C}} = 2\vec{\boldsymbol{C}}\cdot\dot{\vec{\boldsymbol{C}}} = 0$$

so

$$\vec{\boldsymbol{C}}\cdot\dot{\vec{\boldsymbol{C}}} = 0 \quad\Rightarrow\quad \vec{\boldsymbol{C}}\perp\dot{\vec{\boldsymbol{C}}}.$$

The rate of change of a vector of constant length is perpendicular to that vector. This observation is a useful fact to remember about time varying unit vectors, a special case of time varying constant length vectors. □

The motion quantities

The various quantities that show up in the equations of dynamics are defined on the inside cover. To calculate any of them you must multiply some combination of position, velocity and acceleration by mass.

Rate of change of a vector depends on frame

We just explained that the time derivative of a vector can be found by differentiating each of its components. This calculation depended on having a reference frame, an imaginary piece of big graph paper, and a corresponding set of base (or basis) vectors, say $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. But there can be more than one piece of imaginary graph paper. You could be holding one, Jo another, and Tanya a third. Each could be moving their graph paper around and on each paper the same given vector would change in a different way.

The rate of change of a given vector is different if calculated in different reference frames.

When applying the laws of mechanics, we must be sure that when we differentiate vectors we do so with respect to a Newtonian frame.

Because most often we use the "fixed" ground under us as a practical approximation of a Newtonian frame, we label a Newtonian frame with a curly script $\mathcal{F}$, for fixed. So, when being careful with notation we will write the velocity of point B as

$$\overset{\mathcal{F}}{\dot{\boldsymbol{r}}}_{\mathrm{B/O}}$$

Non-Newtonian frames

It is useful to understand frames that accelerate and rotate with respect to each other and with reference to Newtonian frames. These non-Newtonian frames will be discussed in chapter 9. Even though the laws of mechanics are not valid in non-Newtonian frames, non-Newtonian frames are useful help with the understanding of the motion and forces of systems composed of objects with complex relative motion.

SAMPLE 5.27 Find $\vec{L}, \dot{\vec{L}}, \vec{H}_{\rm C}, \dot{\vec{H}}_{\rm C}, E_K, \dot{E}_K$ for a given particle P with mass $m_P = 1\,\text{kg}$, given position, velocity, acceleration, and a point C. Specifically, we are given $\vec{r}_P = (\hat{\imath} + \hat{\jmath} + \hat{k})\,\text{m}$, $\vec{v}_P = 3\,\text{m/s}(\hat{\imath} + \hat{\jmath})$, $\vec{a}_P = 2\,\text{m/s}^2(\hat{\imath} - \hat{\jmath} - \hat{k})$, and $\vec{r}_C = (2\hat{\imath} + \hat{k})\,\text{m}$.

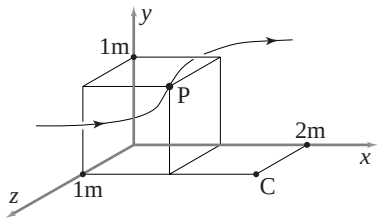


Figure 5.64: (Filename:sfig1.1.DH1)

Solution Since $\vec{r}_P = (\hat{\imath} + \hat{\jmath} + \hat{k})\,\text{m}$ and $\vec{r}_C = (2\hat{\imath} + \hat{k})\,\text{m}$,

$$\vec{r}_{P/C} = \vec{r}_P - \vec{r}_C = (-\hat{\imath} + \hat{\jmath})\,\text{m}.$$

So we have the motion quantities

$$\begin{aligned}
\vec{L} &= m\vec{v}_P \\
&= (1\,\text{kg})\cdot[(3\,\text{m/s})(\hat{\imath} + \hat{\jmath})] \\
&= 3(\hat{\imath} + \hat{\jmath})\frac{\text{kg}\cdot\text{m}}{\text{s}} \\
&= 3\,\text{N}\cdot\text{s}(\hat{\imath} + \hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{L}} &= m\vec{a}_P \\
&= (1\,\text{kg})[(2\,\text{m/s}^2)(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= 2(\hat{\imath} - \hat{\jmath} - \hat{k})\frac{\text{kg}\cdot\text{m}}{\text{s}^2} \\
&= 2\,\text{N}(\hat{\imath} - \hat{\jmath} - \hat{k})
\end{aligned}$$

$$\begin{aligned}
\vec{H}_{\rm C} &= \vec{r}_{P/C} \times m\vec{v}_P \\
&= [(-\hat{\imath} + \hat{\jmath})\,\text{m}] \times [(1\,\text{kg})3\,\text{m/s}(\hat{\imath} + \hat{\jmath})] \\
&= -6\frac{\text{kg}\cdot\text{m}^2}{\text{s}}\hat{k} \qquad (5.75)
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{H}}_{\rm C} &= \vec{r}_{P/C} \times m\vec{a} \\
&= [(-\hat{\imath} + \hat{\jmath})m] \times [(1\,\text{kg})2\,\text{m/s}^2(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= -2\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}(\hat{\imath} + \hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
E_{\rm K} &= \frac{1}{2}m|\vec{v}_P|^2 \\
&= \frac{1}{2}(1\,\text{kg})(3\sqrt{2}\,\text{m/s})^2 \\
&= 9\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2} \\
&= 9\,\text{N}\cdot\text{m}
\end{aligned}$$

$$\begin{aligned}
\dot{E}_{\rm K} &= \frac{d}{dt}(\frac{m}{2}\vec{v}_P\cdot\vec{v}_P) \\
&= \frac{m}{2}[\vec{v}_P\cdot\dot{\vec{v}}_P + \dot{\vec{v}}_P\cdot\vec{v}_P] \\
&= m\vec{v}_P\cdot\vec{a}_P \\
&= 1\,\text{kg}[(3\,\text{m/s})(\hat{\imath} + \hat{\jmath})]\cdot[(2\,\text{m/s}^2)(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= 0.
\end{aligned}$$

Note: $\frac{d}{dt}(\frac{1}{2}v^2) \neq |\vec{v}||\vec{a}|$.

SAMPLE 5.28 *Linear momentum: direct application of formula.* A 2 kg block is moving with a velocity $\vec{v}(t) = u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}$, where $u_0 = 5\,\text{m/s}$, $v_0 = 10\,\text{m/s}$, and $c = 0.5/\,\text{s}$.

(a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ at $t = 5\,\text{s}$.

(b) What is the net change in linear momentum of the block from $t = 0\,\text{s}$ to $t = 5\,\text{s}$?

Solution Since $\vec{L} = m\vec{v}$ and $\dot{\vec{L}} = \frac{d\vec{L}}{dt}$; for the given block we have

(a)

$$\begin{aligned}\vec{L}(t) &= m(u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath})\\ \dot{\vec{L}}(t) &= m(-u_0 c e^{-ct}\hat{\imath}).\end{aligned}$$

Substituting the given values, $m = 2\,\text{kg}$, $u_0 = 5\,\text{m/s}$, $v_0 = 10\,\text{m/s}$, $c = 0.5/\,\text{s}$ and $t = 5\,\text{s}$, we get

$$\begin{aligned}\vec{L}(5\,\text{s}) &= 2\,\text{kg}(5\,\text{m/s}\cdot e^{-2.5}\hat{\imath} + 10\,\text{m/s}\hat{\jmath})\\ &= (0.82\hat{\imath} + 20\hat{\jmath})\,\text{kg}\cdot\,\text{m/s}\\ \dot{\vec{L}}(5\,\text{s}) &= 2\,\text{kg}(-5\,\text{m/s}\cdot 0.5/\,\text{s}\cdot e^{-2.5}\hat{\imath})\\ &= -0.41\,\text{kg}\cdot\,\text{m/s}^2\hat{\imath} = -0.41\,\text{N}\hat{\imath}.\end{aligned}$$

$$\boxed{\vec{L} = (0.82\hat{\imath} + 20\hat{\jmath})\,\text{kg}\cdot\,\text{m/s},\ \dot{\vec{L}} = -0.41\,\text{N}\hat{\imath}}$$

(b)

$$\begin{aligned}\vec{L}(0\,\text{s}) &= 2\,\text{kg}(5\,\text{m/s}\cdot e^{0}\hat{\imath} + 10\,\text{m/s}\hat{\jmath})\\ &= (10\hat{\imath} + 20\hat{\jmath})\,\text{kg}\cdot\,\text{m/s}.\end{aligned}$$

Therefore, the net change in the linear momentum in $t = 0\,\text{s}$ to $t = 5\,\text{s}$ is,

$$\begin{aligned}\Delta\vec{L} &= \vec{L}(5\,\text{s}) - \vec{L}(0\,\text{s})\\ &= (0.82\hat{\imath} + 20\hat{\jmath})\,\text{kg}\cdot\,\text{m/s} - (10\hat{\imath} + 20\hat{\jmath})\,\text{kg}\cdot\,\text{m/s}\\ &= -9.18\,\text{kg}\cdot\,\text{m/s}\hat{\imath}.\end{aligned}$$

Note that the net change is only in x-direction. This result makes sense because the y-component of $\vec{L}$ is constant and therefore, y-component of $\dot{\vec{L}}$ is zero.

$$\boxed{\Delta\vec{L} = -9.18\,\text{kg}\cdot\,\text{m/s}\hat{\imath}}$$

SAMPLE 5.29 *Angular momentum: direct application of the formula.* The position of a particle of mass $m = 0.5\,\text{kg}$ is $\vec{r}(t) = \ell\,\sin(\omega t)\hat{\imath} + h\hat{\jmath}$; where $\omega = 2\,\text{rad/s}$, $h = 2\,\text{m}$, $\ell = 2\,\text{m}$, and $\vec{r}$ is measured from the origin.

(a) Find the angular momentum $\vec{H}_O$ of the particle about the origin at $t = 0\,\text{s}$ and $t = 5\,\text{s}$.

(b) Find the rate of change of angular momentum $\dot{\vec{H}}$ about the origin at $t = 0\,\text{s}$ and $t = 5\,\text{s}$.

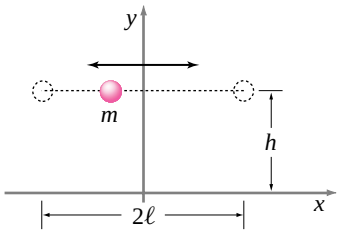


Figure 5.65: (Filename:sfig3.2.direct.appl)

Solution since $\vec{H}_O = \vec{r}_{/O} \times m\vec{v}$ and $\dot{\vec{H}}_O = \vec{r}_{/O} \times m\vec{a}$, we need to find $\vec{r}$, $\vec{v}$ and $\vec{a}$ to compute $\vec{H}_O$ and $\dot{\vec{H}}_O$. Now,

$$\begin{aligned} & \vec{r}(t) &=& \ell\,\sin(\omega t)\hat{\imath} + h\hat{\jmath} \\ \Rightarrow\quad & \vec{v}(t) &=& \dot{\vec{r}}(t) = \ell\,\omega\,\cos(\omega t)\hat{\imath} + 0\hat{\jmath} \\ \Rightarrow\quad & \vec{a}(t) &=& \ddot{\vec{r}}(t) = -\ell\,\omega^2\,\sin(\omega t)\hat{\imath} \end{aligned}$$

(a) Since the position is measured from the origin,

$$\vec{r}_{/O} = \ell\,\sin(\omega t)\hat{\imath} + h\hat{\jmath}.$$

Therefore,

$$\begin{aligned} \vec{H}_O &= \vec{r}_{/O} \times m\vec{v} = (\ell\,\sin(\omega t)\hat{\imath} + h\hat{\jmath}) \times m(\ell\,\omega\,\cos(\omega t)\hat{\imath}) \\ &= m\ell^2\omega\sin(\omega t)\cos(\omega t)(\hat{\imath}\times\hat{\imath}) + m\ell\omega h\cos(\omega t)(\hat{\jmath}\times\hat{\imath}) \\ &= -m\ell\omega h\cos(\omega t)\hat{k}. \end{aligned}$$

Now we can substitute the desired values:

$$\begin{aligned} \vec{H}_O(0\,\text{s}) &= -(0.5\,\text{kg})\cdot(2\,\text{rad/s})\cdot(2\,\text{m})\cdot(2\,\text{m})\cdot\cos(0)\hat{k} \\ &= -4\,\text{kg}\cdot\text{m}^2/\text{s}\hat{k} \\ \vec{H}_O(5\,\text{s}) &= -(4\,\text{kg}\cdot\text{m}^2/\text{s})\cdot\cos(2\,\text{rad/s}\cdot 5\,\text{s}) = 3.36\,\text{kg}\cdot\text{m}^2/\text{s}\hat{k}. \end{aligned}$$

$$\boxed{\vec{H}_O(0\,\text{s}) = -4\,\text{kg}\cdot\text{m}^2/\text{s}\hat{k}, \quad \vec{H}_O(5\,\text{s}) = 3.36\,\text{kg}\cdot\text{m}^2/\text{s}\hat{k}}$$

(b)

$$\begin{aligned} \dot{\vec{H}}_O &= \vec{r}_{/O} \times m\vec{a} \\ &= (\ell\,\sin(\omega t)\hat{\imath} + h\hat{\jmath}) \times m(-\ell\,\omega^2\sin(\omega t)\hat{\imath}) \\ &= ml\omega^2 h\sin(\omega t)\hat{k} \end{aligned}$$

Substituting the values of constants and the time, we get $\dot{\vec{H}}_O(0\,\text{s}) = \vec{0}$, and

$$\begin{aligned} \dot{\vec{H}}_O(5\,\text{s}) &= (0.5\,\text{kg})\cdot(2\,\text{m})\cdot(2\,\text{rad/s})^2\cdot(2\,\text{m})\cdot\sin(2\,\text{rad/s}\cdot 5\,\text{s})\hat{k} \\ &= -4.35(\,\text{kg}\cdot\text{m/s}^2)\cdot\text{m}\hat{k} = -4.35\,\text{N}\cdot\text{m}\hat{k}. \end{aligned}$$

$$\boxed{\dot{\vec{H}}_O(0\,\text{s}) = \vec{0}, \quad \dot{\vec{H}}_O(5\,\text{s}) = -4.35\,\text{N}\cdot\text{m}\hat{k}}$$

Comments: Note that both $\vec{H}$ and $\dot{\vec{H}}$ point out of the plane, in the $\hat{k}$ direction. $\vec{H}$ and $\dot{\vec{H}}$ are always in the $\hat{k}$ direction for all motions in the xy-plane for all masses in the xy-plane (provided, of course, that the reference point about which $\vec{H}$ and $\dot{\vec{H}}$ are calculated also lies in the xy plane).

5.8 Spatial dynamics of a particle

One of Isaac Newton's interests was the motion of the planets around the sun. By applying his equation $\vec{F} = m\vec{a}$, his law of gravitation, his calculus, and his inimitable geometric reasoning he learned much about the motions of celestial bodies. After learning the material in this section you will know enough to reproduce many of Newton's calculations. You don't need to be a Newton-like genius to solve Newton's differential equations. You can solve them on a computer. And you can use the same equations to find motions that Newton could never find, say the trajectory of projectile with a realistic model of air friction. In this chapter, the main approach we take to celestial mechanics and related topics is as follows: ①

① Eventually you may develop analytic skills which will allow you to shortcut this brute-force numerical approach, at least for some simple problems. For hard problems, even the greatest analytic geniuses resort to methods like those prescribed here.

(a) draw a free body diagram of each particle, n free body diagrams if there are n particles,
(b) find the forces on each particle in terms of their positions and velocities and any other external forces (for example, these forces could involve spring, dashpot, gravity, or air friction terms),
(c) write the linear momentum balance equations for each particle, that is write $\vec{F} = m\vec{a}$ once for each particle. That is, write n vector equations.
(d) break each vector equation into components to make 2 or 3 scalar equations for each vector equation, in 2 or 3 dimensions, respectively.
(e) write the $2n$ or $3n$ equations in first order form. You now have $4n$ or $6n$ first order ordinary differential equations in 2 or 3 dimensions, respectively.
(f) write these first order equations in standard form, with all the time derivatives on the left hand side.
(g) feed these equations to the computer, substituting values for the various parameters and appropriate initial conditions.
(h) plot some aspect(s) of the solution and
    (i) use the solution to help you find errors in your formulation, and
    (ii) interpret the solution so that it makes sense to you and increases your understanding of the system of study.

We can use this approach if the forces on all of the point masses composing the system can be found in terms of their positions, velocities, and the present time. In this section we will just look at the motion of a single particle with forces coming, say, from gravity, springs, dashpots and air drag.

Some problems are of the *instantaneous dynamics* type. That is, they use the equations of dynamics but do not involve tracking motion in time.

Example: **Knowing the forces find the acceleration.**

Say you know the forces on a particle at some instant in time, say $\vec{F}_1$ and $\vec{F}_2$, and you just want to know the acceleration at that instant. The answer is given directly by linear momentum balance as

$$\sum \vec{F}_i = m\vec{a} \quad \Rightarrow \quad \vec{a} = \frac{\vec{F}_1 + \vec{F}_2}{m}$$

□

Even some problems involving motion are simple and you can determine most all you want to know with pencil and paper.

Example: **Parabolic trajectory of a projectile**

If we assume a constant gravitational field, neglect air drag, and take the y direction as up the only force acting on a projectile is $\vec{\boldsymbol{F}} == mg\hat{\boldsymbol{\jmath}}$. Thus the "equations of motion" (linear momentum balance) are

$$-mg\hat{\boldsymbol{\jmath}} = m\vec{\boldsymbol{a}}.$$

If we take the dot product of this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ (take x and y components) we get the following two differential equations,

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

which are decoupled and have the general solution

$$\vec{\boldsymbol{r}} = (A + Bt)\hat{\boldsymbol{\imath}} + (C + Dt - gt^2/2)\hat{\boldsymbol{\jmath}}$$

which is a parametric description of all possible trajectories. By making plots or simple algebra you might convince yourself that these trajectories are parabolas for all possible A, B, C, and D. That is, neglecting air drag, the predicted trajectory of a thrown ball is a parabola. □

Some problems are hard and necessitate computer solution.

Example: **Trajectory with quadratic air drag.**

For motions of things you can see with your bare eyes moving in air, the drag force is roughly proportional to the speed squared and opposes the motion. Thus the total force on a particle is $\vec{\boldsymbol{F}} = -mg\hat{\boldsymbol{\jmath}} - Cv^2(\vec{\boldsymbol{v}}/v)$, where $\vec{\boldsymbol{v}}/v$ is a unit vector in the direction of motion. So linear momentum balance gives

$$-mg\hat{\boldsymbol{\jmath}} - Cv\vec{\boldsymbol{v}} = m\vec{\boldsymbol{a}}.$$

If we dot this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ we get

$$\ddot{x} = -(C/m)\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)\dot{x} \quad \text{and} \quad \ddot{y} = -(C/m)\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)\dot{y} - g.$$

These are two coupled second order equations that are probably not solvable with pencil and paper. But they are easily put in the form of a set of four first order equations and can be solved numerically. □

Some special problems turn out to be easy, though you might not realize it at first glance.

Example: **Zero-length spring**

Imagine a massless spring whose unstretched length is zero (see chapter 2 for a discussion of zero length springs). Assume one end is connected to a pivot at the origin and the other to a particle. Neglect gravity and air drag. The force on the mass is thus proportional to its distance from the pivot and the spring constant and pointed towards the origin: $\vec{\boldsymbol{F}} = -k\vec{\boldsymbol{r}}$. Thus linear momentum balance yields

$$-k\vec{\boldsymbol{r}} = m\vec{\boldsymbol{a}}.$$

Breaking into components we get

$$\ddot{x} = (-k/m)x \quad \text{and} \quad \ddot{y} = (-k/m)y.$$

Thus the motion can be thought of as two independent harmonic oscillators, one in the x direction and one in the y direction. The general solution is

$$\vec{r} = \left(A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t \right)\hat{\imath} + \left(C\cos\sqrt{\frac{k}{m}}\,t + D\sin\sqrt{\frac{k}{m}}\,t \right)\hat{\jmath}$$

which is always an ellipse (special cases of which are a circle and a straight line). □

Some problems are within the reach of advanced analytic methods, but can also be solved with a computer.

Example: **Path of the earth around the sun.**

Assume the sun is big and unmovable with mass M and the earth has mass m. Take the origin to be at the sun. The force on the earth is $\vec{F} = -(mMG/r^2)(\vec{r}/r)$ where $\vec{r}/r$ is a unit vector pointing from the sun to the earth. So linear momentum balance gives

$$\frac{mMG\vec{r}}{r^3} = m\vec{a}.$$

This equation *can* be solved with pencil and paper, Newton did it. But the solution is beyond this course. On the other hand the equation of motion is easily broken into components and then into a set of 4 ODEs which can be easily solved on the computer. Either by pencil and paper, or by investigation of numerical solutions, you will find that all solutions are conic sections (straight lines, parabolas, hyperbolas, and ellipses). The special case of circular motion is not far from what the earth does. □

The work-energy equation

Energy balance is one of the basic governing equations. For a single particle with no stored internal energy, the energy balance equation is

$$P = \frac{d}{dt}E_{\mathrm{K}} \tag{IIId}$$

Before getting into the technical definitions of the terms, let's first summarize the most basic of the energy equations in words.

The power P of *all* the external forces acting on a particle is the rate of change of its kinetic energy $\dot{E}_{\mathrm{K}}$.

From other physics texts and courses, you know energy principles help you solve a variety of simple problems, both in mechanics and other parts of physics. In many engineering applications, one can determine useful things about the motion of a machine or object by thinking about its energy and change of energy.

For particles and rigid bodies that interact in the simple ways we consider in this book, the energy equations can be derived from the momentum balance equations. They follow logically.

However, in practice, one uses the various work-energy relations as if they were independent. Sometimes energy equations can be used in place of, or as a check of, momentum balance equations.

Neglecting the right hand side

The right hand side of the energy equations is the rate of change of kinetic energy. This term is not zero if the speeds of the various mass points change non-negligibly. But, for negligible motion, we neglect all terms that involve motion, in this case v_i and $\dot{v}_i$. Thus, we assume that

$$\dot{E}_{\mathrm{K}} = 0.$$

Thus, for better or worse, equation 5.8 reduces to

$$P = 0. \tag{5.76}$$

The net power into the system is zero. Equation 5.76 is useful for system that can be modeled as having constant (or zero) kinetic energy.

The power into a system P

In mechanics, the sources of power are applied forces. The power of an applied force $\vec{\boldsymbol{F}}$ acting on a particle is

$$P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}},$$

where $\vec{\boldsymbol{v}}$ is the velocity of the point of the material body being acted on by the force.

If many forces are applied, then

$$P = \sum \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{v}}.$$

The work of a force $\vec{F}$: W_{12}

Previously in Physics, and more recently in one dimensional mechanics, you learned that

Work is force times distance.

This is actually a special case of the formula

$$P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}.$$

How is that? If $\vec{\boldsymbol{F}}$ is constant and parallel to the displacement $\Delta\vec{\boldsymbol{x}}$, then

$$\begin{aligned} \text{Work} &= \int \dot{W}\, dt = \int P dt = \int \vec{\boldsymbol{F}} \cdot \underbrace{\vec{\boldsymbol{v}}\, dt}_{d\vec{\boldsymbol{x}}} = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} = \vec{\boldsymbol{F}} \cdot \int d\vec{\boldsymbol{x}} \\ &= \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{x}} = F\Delta x = \text{Force} \cdot \text{distance}. \end{aligned}$$

Or,

$$dW = \dot{W}\, dt = P dt = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\, dt = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} \text{ (or } \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}).$$

Being a little more precise about notation, we can write that the work of a force acting on a particle or body in moving from state 1 to state 2 is

$$W_{12} \equiv \int_{\vec{\boldsymbol{r}}_1}^{\vec{\boldsymbol{r}}_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \tag{5.77}$$

where the path of integration is the path of the material point at which the force is applied.

Potential energy of a force

Some forces have the property that the work they do is independent of the path followed by the material point (or pair of points between which the force acts). If the work of a force is path independent in this way, then a potential energy can be defined so that the work done by the force is the decrease in the Potential Energy E_{P}: $W_{12} = E_{\mathrm{P}1} - E_{\mathrm{P}2}$. The common examples are listed below:

- **linear spring:** $E_{\mathrm{P}} = (1/2)k(\text{stretch})^2$.
- **gravity near earth's surface:** $E_{\mathrm{P}} = mgh$
- **gravity between spheres or points:** $E_{\mathrm{P}} = -MmG/r$
- **constant force $\vec{F}$ acting on a point:** $E_{\mathrm{P}} = -\vec{F} \cdot \vec{r}$

In the cases of the spring and gravity between spheres, the change in potential energy is the net work done by the spring or gravity on the pair of objects between which the force acts. If both ends of a spring are moving, the net work of the spring on the two objects to which it is connected is the decrease in potential energy of the spring.

There is a possible source of confusion in our using the same symbol E_{P} to represent the potential work of an external force and for internal potential energy. In practice, however, they are used identically, so we use the same symbol for both. The potential energy in a stretched spring is the same whether it is the cause of force on a system or it is internal to the system.

Forces that do no work

Fortunately for the evaluation of power and work one often encounters forces that do no work or forces that come in pairs where the pair of forces does no net work.

The net work done by the interaction force between body A and body B is zero if the force on body A dotted with the relative velocity of A and B is zero. Examples are:

- frictionless sliding,
- the force caused by a magnetic field on a moving charged particle.

Summary

To find the motion of a particle you draw a free body diagram, write the linear momentum balance equation and then solve it, most often on the computer. The power and energy equations can sometimes be used to check your solution and to determine special features of the solution, and in special case.

5.7 *THEORY*
Angular momentum and energy of a point mass

For a point-mass particle, we can derive the angular momentum equation (II) and the energy equation (III) from linear momentum balance in a snap.

For a single particle we have $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. Taking the cross product of both sides with the position relative to a point C gives:

$$\vec{\boldsymbol{r}}_{/C} \times \vec{\boldsymbol{F}} = \vec{\boldsymbol{r}}_{/C} \times \left(m\vec{\boldsymbol{a}}\right).$$

For a single point-mass particle the angular momentum equation is a direct un-refutable consequence of the linear momentum balance equation.

The power equation is found with a shade more difficulty. We take the equation $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and dot both sides with the velocity $\vec{\boldsymbol{v}}$ of the particle:

$$\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = m\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}}. \tag{5.78}$$

Evaluating $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}}$ is most easily done with the benefit of hindsight. So we cheat and look at the time derivative of the speed squared:

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}v^2\right) &= \frac{1}{2}\frac{d}{dt}(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}) \\ &= \frac{1}{2}\left(\dot{\vec{\boldsymbol{v}}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} \cdot \dot{\vec{\boldsymbol{v}}}\right) \\ &= \vec{\boldsymbol{v}} \cdot \dot{\vec{\boldsymbol{v}}} \\ &= \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}} \end{aligned}$$

Applying this result to equation 5.78 we get

$$\underbrace{\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}}_{P} = \frac{d}{dt}\underbrace{\left(\frac{1}{2}mv^2\right)}_{E_{\mathrm{K}}},$$

the energy (or power balance) equation for a particle.

So for one particle angular momentum balance and power balance (eqns. II and III on the inside cover) follow directly from $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.

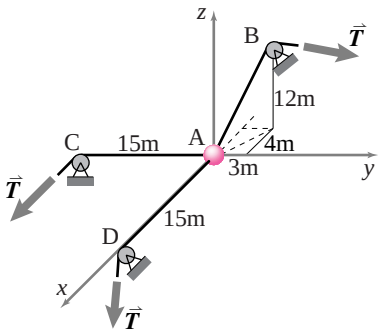


Figure 5.66: A ball in 3-D

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SAMPLE 5.30 *Acceleration of a point mass in 3-D.* A ball of mass $m = 13\,\text{kg}$ is being pulled by three strings as shown in Fig. 5.66. The tension in each string is $T = 13\,\text{N}$. Find the acceleration of the ball.

Solution The forces acting on the body are shown in the free body diagram in Fig.5.67. From geometry:

$$\begin{aligned}\hat{\boldsymbol{\lambda}} &= \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{\sqrt{4^2+3^2+12^2}} \\ &= \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}.\end{aligned}$$

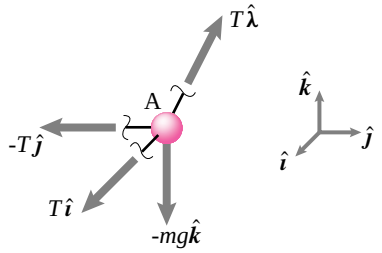


Figure 5.67: FBD of the ball

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Balance of linear momentum for the ball:

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \tag{5.79}$$

$$\begin{aligned}\sum \vec{\boldsymbol{F}} &= T\hat{\boldsymbol{\imath}} - T\hat{\boldsymbol{\jmath}} + T\hat{\boldsymbol{\lambda}} - mg\hat{\boldsymbol{k}} \\ &= T\left(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}\right) - mg\hat{\boldsymbol{k}} \\ &= \frac{T}{13}(9\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - mg\hat{\boldsymbol{k}}.\end{aligned}$$

Substituting $\sum \vec{\boldsymbol{F}}$ in eqn. (5.79):

$$\vec{\boldsymbol{a}} = \frac{T}{13m}(9\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - g\hat{\boldsymbol{k}}.$$

Now plugging in the given values: $T = 13\,\text{N}$, $m = 13\,\text{kg}$, and $g = 10\,\text{m/s}^2$, we get

$$\begin{aligned}\vec{\boldsymbol{a}} &= \frac{1\!\not{3}\,\text{N}}{1\!\not{3}\cdot 13\,\text{kg}}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - 10\,\text{m/s}^2\hat{\boldsymbol{k}} \\ &= (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2}$$

SAMPLE 5.31 Assume the expression for velocity $\vec{\boldsymbol{v}}(=\frac{d\vec{\boldsymbol{r}}}{dt})$ of a particle is given: $\vec{\boldsymbol{v}} = v_0\hat{\boldsymbol{\imath}} - gt\,\hat{\boldsymbol{\jmath}}$. Find the expressions for the x and y coordinates of the particle at a general time t, if the initial coordinates at $t = 0$ are (x_0, y_0).

Solution The position vector of the particle at any time t is

$$\vec{\boldsymbol{r}}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}.$$

We are given that

$$\vec{\boldsymbol{r}}(t=0) = x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}}.$$

Now

$$\begin{aligned} \vec{\boldsymbol{v}} \equiv \frac{d\vec{\boldsymbol{r}}}{dt} &= v_0\hat{\boldsymbol{\imath}} - gt\,\hat{\boldsymbol{\jmath}} \\ \text{or} \qquad dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} &= (v_0\hat{\boldsymbol{\imath}} - gt\,\hat{\boldsymbol{\jmath}})\,dt. \end{aligned}$$

Dotting both sides of this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, we get

$$\begin{aligned} dx &= v_0\,dt \\ \Rightarrow \quad \int_{x_0}^{x} dx &= v_0\int_0^t dt \\ \Rightarrow \quad x &= x_0 + v_0 t, \end{aligned}$$

and

$$\begin{aligned} dy &= -gt\,dt \\ \Rightarrow \quad \int_{y_0}^{y} dy &= -g\int_0^t t\,dt \\ \Rightarrow \quad y &= y_0 - \frac{1}{2}gt^2. \end{aligned}$$

Therefore,

$$\vec{\boldsymbol{r}}(t) = (x_0 + v_0 t)\hat{\boldsymbol{\imath}} + (y_0 - \frac{1}{2}gt^2)\hat{\boldsymbol{\jmath}}$$

and the (x, y) coordinates are

$$\begin{aligned} x(t) &= x_0 + v_0 t \\ y(t) &= y_0 - \frac{1}{2}gt^2. \end{aligned}$$

$$\boxed{(x_0 + v_0 t,\ y_0 - \tfrac{1}{2}gt^2)}$$

SAMPLE 5.32 *The path of a particle.* A particle moves in the xy plane such that its coordinates are given by $x(t) = at$ and $y(t) = bt^2$, where $a = 2\,\text{m/s}$ and $b = 0.5\,\text{m/s}^2$.

(a) Find the velocity and acceleration of the particle at $t = 3\,\text{s}$.

(b) Show that the path of the particle is neither a straight-line nor a circle.

Solution

(a) This problem is straightforward. We are given the position of the particle as a function of time t. We can find the velocity and acceleration by differentiating the position with respect to time:

$$\begin{aligned}
\vec{r}(t) &= \overbrace{at}^{x}\hat{\imath} + \overbrace{bt^2}^{y}\hat{\jmath} \\
\vec{v} &= \frac{d\vec{r}}{dt} = a\hat{\imath} + 2bt\hat{\jmath} \\
&= 2\,\text{m/s}\hat{\imath} + 2\cdot 0.5\,\text{m/s}^2\cdot 3\,\text{s}\hat{\jmath} \\
&= (2\hat{\imath} + 3\hat{\jmath})\,\text{m/s} \\
\vec{a} &= \frac{d\vec{v}}{dt} = 2b\hat{\jmath} \\
&= 1\,\text{m/s}^2\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{v} = (2\hat{\imath} + 3\hat{\jmath})\,\text{m/s},\ \vec{a} = 1\,\text{m/s}^2\hat{\jmath}}$$

(b) There are many ways to show that the path of the particle is neither a straight line nor a circle. One of the easiest ways is graphical. Calculate the position of the particle at various times and plot a curve through the positions. This curve is the path of the particle. Using a computer, for example, we can plot the path as follows:

```
a = 2,   b = 0.5          % specify constants
t = [0 4 8 ... 20]        % take 6 points from 0-20 sec.
x = a*t,   y = b*t^2      % calculate coordinates
plot x vs y               % plot the particle path
```

A plot so generated is shown in Fig. 5.68. Clearly, the path is neither a straight line nor a circle.

Another way to find the path of the particle is to find an explicit equation of the path. We find this equation by eliminating t from the expressions for x and y and thus relating x to y:

$$x = at \quad \Rightarrow \quad t = \frac{x}{a}.$$

Substituting x/a for t in $y = bt^2$, we get

$$y = \frac{b}{a^2}x^2$$

which is clearly not the equation of a straight line or a circle. In fact, it is the equation of a parabola, i.e., the path of the particle is parabolic.

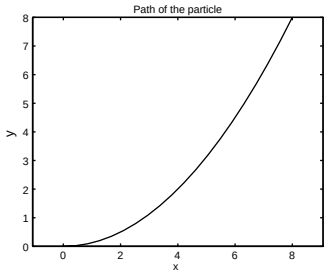


Figure 5.68: The path of the particle generated by computing its coordinates at various points in time.

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SAMPLE 5.33 *Trajectory of a food-bag.* In a flood hit area relief supplies are dropped in a 20 kg bag from a helicopter. The helicopter is flying parallel to the ground at 200 km/h and is 80 m above the ground when the package is dropped. How much horizontal distance does the bag travel before it hits the ground? Take the value of g, the gravitational acceleration, to be $10\,\mathrm{m/s^2}$.

Solution You must have solved such problems in elementary physics courses. Usually, in all projectile motion problems the equations of motion are written separately in the x and y directions, realizing that there is no force in the x direction, and then the equations are solved. Here we show you how to write and keep the equations in vector form all the way through.

The free body diagram of the bag during its free flight is shown in Fig. 5.69. The only force acting on the bag is its weight. Therefore, from the linear momentum balance for the bag we get

$$m\vec{\boldsymbol{a}} = -mg\hat{\boldsymbol{\jmath}}.$$

Let us choose the origin of our coordinate system on the ground exactly below the point at which the bag is dropped from the helicopter. Then, the initial position of the bag $\vec{\boldsymbol{r}}(0) = h\hat{\boldsymbol{\jmath}} = 80\,\mathrm{m}\,\hat{\boldsymbol{\jmath}}$. The fact that the bag is dropped from a helicopter flying horizontally gives us the initial velocity of the bag:

$$\vec{\boldsymbol{v}}(0) \equiv \dot{\vec{\boldsymbol{r}}}(0) = v_x\hat{\boldsymbol{\imath}} = 200\,\mathrm{km/h}\hat{\boldsymbol{\imath}}.$$

So now we have a 2nd order differential equation (from the linear momentum balance):

$$\ddot{\vec{\boldsymbol{r}}} = -g\hat{\boldsymbol{\jmath}}$$

with two initial conditions:

$$\vec{\boldsymbol{r}}(0) = h\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = v_x\hat{\boldsymbol{\imath}}$$

which we can solve to get the position vector of the bag at any time. Since the basis vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ do not change with time, solving the differential equation is a matter of simple integration:

$$\begin{aligned} \ddot{\vec{\boldsymbol{r}}} &\equiv \frac{d\dot{\vec{\boldsymbol{r}}}}{dt} = -g\hat{\boldsymbol{\jmath}} \\ \int d\dot{\vec{\boldsymbol{r}}} &= -\hat{\boldsymbol{\jmath}}\int g\,dt \\ \text{or} \qquad \dot{\vec{\boldsymbol{r}}} &= -gt\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1 \end{aligned} \tag{5.80}$$

and integrating once again, we get

$$\begin{aligned} \vec{\boldsymbol{r}} &= \int(-gt\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1)\,dt \\ &= -\frac{1}{2}gt^2\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1 t + \vec{\boldsymbol{c}}_2 \end{aligned} \tag{5.81}$$

where $\vec{\boldsymbol{c}}_1$ and $\vec{\boldsymbol{c}}_2$ are constants of integration and are vector quantities. Now substituting the initial conditions in eq6.1.2.1 and eq6.1.2.2 we get

$$\begin{aligned} \dot{\vec{\boldsymbol{r}}}(0) &= v_x\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{c}}_1, \quad \text{and} \\ \vec{\boldsymbol{r}}(0) &= h\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{c}}_2. \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} \vec{\boldsymbol{r}}(t) &= -\frac{1}{2}gt^2\hat{\boldsymbol{\jmath}} + v_x t\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{\jmath}} \\ &= v_x t\hat{\boldsymbol{\imath}} + (h - \frac{1}{2}gt^2)\hat{\boldsymbol{\jmath}}. \end{aligned}$$



FBD of the bag

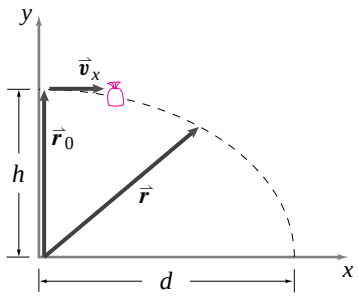


Figure 5.69: Free body diagram of the bag and the geometry of its motion.

(Filename:sfig6.1.2a)

So how do we find the horizontal distance traveled by the bag from our solution? The distance we are interested in is the x-component of $\vec{r}$, *i.e.*, $v_x t$. But we do not know t. However, when the bag hits the ground, its position vector has no y-component, *i.e.*, we can write $\vec{r} = d\hat{\imath} + 0\hat{\jmath}$ where d is the distance we are interested in. Now equating the components of $\vec{r}$ with the obtained solution, we get

$$d = v_x t \quad \text{and} \quad 0 = h - \frac{1}{2}gt^2.$$

Solving for t from the second equation and substituting in the first equation we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

$$\boxed{d = 222\,\text{m}}$$

Comments: Here we have tried to show you that solving for position from the given acceleration in vector form is not really any different than solving in scalar form provided the unit vectors involved are fixed in time. As long as the right hand side of the differential equation is integrable, the solution can be obtained. If the method shown above seems too "*mathy*" or intimidating to you then follow the usual scalar way of doing this problem.

The scalar method:

From the linear momentum balance, $-mg\hat{\jmath} = m\vec{a}$, writing the acceleration as $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$ and equating the x and y components from both sides, we get

$$a_x = 0 \quad \text{and} \quad a_y = -g.$$

Now using the formula for distance under uniform acceleration from Chapter 3, $x = x_0 + v_0 t + \frac{1}{2}at^2$, in both x and y directions, we get

$$\begin{aligned} d &= \overbrace{x_0}^{0} + v_x t + \frac{1}{2}\overbrace{a_x}^{0} t^2 \\ &= v_x t \\ 0 &= \overbrace{y_0}^{h} + \overbrace{v_y}^{0} t + \frac{1}{2}\overbrace{a_y}^{-g} t^2 \\ &= h - \frac{1}{2}gt^2 \\ \Rightarrow \quad t &= \sqrt{\frac{2h}{g}}. \end{aligned}$$

Substituting for t in the equation for d we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

as above.

SAMPLE 5.34 *Projectile motion with air drag.* A projectile is fired into the air at an initial angle θ_0 and with initial speed v_0. The air resistance to the motion is proportional to the square of the speed of the projectile. Take the constant of proportionality to be k. Find the equations of motion of the projectile in the horizontal and vertical directions assuming the air resistance to be in the opposite direction of the velocity.

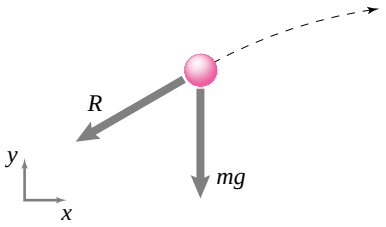


Figure 5.70: FBD of the projectile.
(Filename:sfig6.4.DH1)

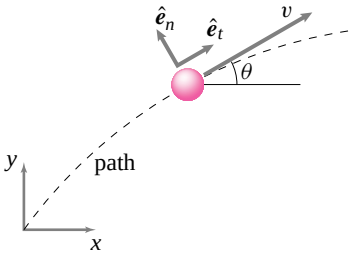


Figure 5.71: (Filename:sfig6.4.DH2)

Solution The free body diagram of the projectile is shown in the figure at some constant t during motion. At the instant shown, let the velocity of the projectile be $\vec{v} = v\hat{e}_t$ where

$$\hat{e}_t = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}.$$

Then the force due to air resistance is

$$\vec{R} = -kv^2\hat{e}_t.$$

Now applying the linear momentum balance on the projectile, we get

$$\begin{aligned} \vec{R} + m\vec{g} &= m\vec{a} \\ \text{or} \qquad -kv^2\hat{e}_t - mg\hat{\jmath} &= m\overbrace{(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})}^{\vec{a}} \end{aligned} \tag{5.82}$$

Noting that $v = |\vec{v}| = |\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}| = \sqrt{\dot{x}^2 + \dot{y}^2}$, and dotting both sides of equation 5.82 with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\begin{aligned} -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\imath}) &= m\ddot{x} \\ -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\jmath}) - mg &= m\ddot{y} \end{aligned}$$

Rearranging terms and carrying out the dot products, we get

$$\begin{aligned} \ddot{x} &= -\frac{k}{m}(\dot{x}^2 + \dot{y}^2)\cos\theta \\ \ddot{y} &= -g - \frac{k}{m}(\dot{x}^2 + \dot{y}^2)\sin\theta \end{aligned}$$

Note that θ changes with time. We can express θ in terms of $\dot{x}$ and $\dot{y}$ because θ is the slope of the trajectory:

$$\theta = \tan^{-1}\frac{dy}{dx} = \tan^{-1}\frac{dy/dt}{dx/dt} = \tan^{-1}\frac{\dot{x}}{\dot{y}} \qquad (\text{i.e., } \tan\theta = \frac{\dot{y}}{\dot{x}})$$

$$\Rightarrow \qquad \cos\theta = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \quad \text{and} \quad \sin\theta = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}.$$

Substituting these expression in to the equations for $\ddot{x}$ and $\ddot{y}$ we get

$$\boxed{\ddot{x} = -\tfrac{k}{m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \quad \ddot{y} = -\tfrac{k}{m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g}$$

SAMPLE 5.35 *Cartoon mechanics: The cannon.* It is sometimes claimed that students have trouble with dynamics because they built their intuition by watching cartoons. This claim could be rebutted on many grounds.

1) Students don't have trouble with dynamics! They love the subject.
2) Nowadays many cartoons are made using 'correct' mechanics, and
3) the cartoons are sometimes more accurate than the pedagogues anyway.

Problem: What is the path of a cannon ball? In the cartoon world the cannon ball goes in a straight line out the cannon then comes to a stop and then starts falling. Of course a good physicist knows the path is a parabola. Or is it?

Solution The drag force on a cannon ball moving through air is approximately proportional to the speed squared and resists motion. Gravity is approximately constant. Then

$$\begin{aligned}\vec{F}_{\text{drag}} &= cv^2 \cdot (\text{unit vector opposing motion}) \\ &= cv^2 \cdot \left(\frac{-\vec{v}}{|\vec{v}|}\right) \\ &= -c|\vec{v}|\vec{v} \\ &= -c\sqrt{\dot{x}^2+\dot{y}^2}\,(\dot{x}\hat{\imath}+\dot{y}\hat{\jmath})\end{aligned}$$

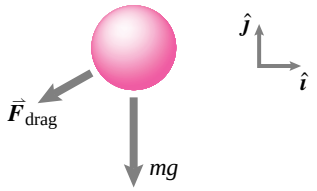


Figure 5.72: (Filename:sfig3.5.cart.cannon)

So LMB gives

$$\sum \vec{F} = \dot{\vec{L}}$$
$$\left\{-mg\hat{\jmath} - c\sqrt{\dot{x}^2+\dot{y}^2}(\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}) = m(\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath})\right\}$$

$$\{\}\cdot\hat{\imath} \quad\Rightarrow\quad \ddot{x} = \left[-c\sqrt{\dot{x}^2+\dot{y}^2}\dot{x}/m\right]$$
$$\ddot{y} = -c\sqrt{\dot{x}^2+\dot{y}^2}\dot{y}/m - g$$

Solving these equations numerically with reasonable values ① of $\dot{x}_0$, $\dot{y}_0$, m and c gives

① To be precise, if the launch speed is much faster than the 'terminal velocity' of the falling ball.

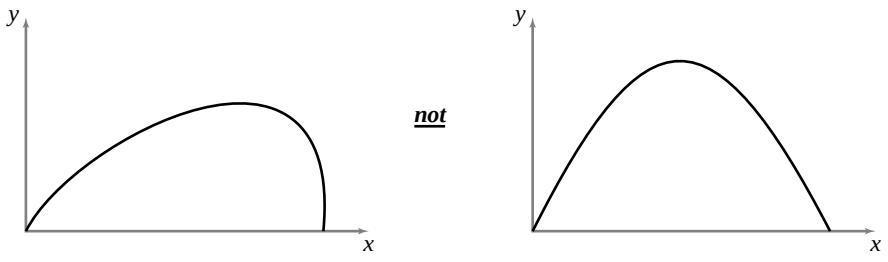


Figure 5.73: (Filename:sfig3.5.cart.cannon.graph)

which is closer to a cartoon's triangle than to a naive physicist's parabola.

5.9 Central-force motion and celestial mechanics

One of Isaac Newton's greatest achievements was the explanation of Kepler's laws of planetary motion. Kepler, using the meticulous observations of Tycho Brahe characterized the orbits of the planets about the sun with his 3 famous laws:

- Each planet travels on an ellipse with the sun at one focus.
- Each planet goes faster when it is close to the sun and slower when it is further. It speeds and slows so that the line segment connecting the planet to the sun sweeps out area at a constant rate.
- Planets that are further from the sun take longer to go around. More exactly, the periods are proportional to the lengths of the ellipses to the 3/2 power.

Newton, using his equation $\vec{F} = m\vec{a}$ and his law of universal gravitational attraction, was able to formulate a differential equation governing planetary motion. He was also able to solve this equation and found that it exactly predicts all three of Kepler's laws.

The Newtonian description of planetary motion is the most historically significant example of *central-force motion* where,

- the only force acting on a particle is directed towards the origin of a given coordinate system, and
- the magnitude of the force depends only on radius.

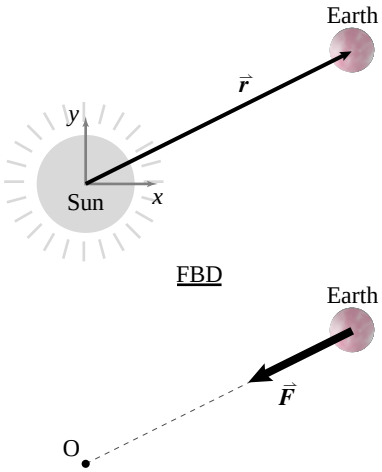


Figure 5.74: The earth moving around a fixed sun. The attraction force $\vec{F}$ is directed "centrally" towards the sun and has magnitude proportional to both masses and inversely proportional to the distance squared.

(Filename:tfigure.earthfixedsun)

① Soon after Newton, Cavendish found G in his lab by delicately measuring the *small* attractive force between two balls. The gravitational attraction between two 1 kg balls a meter apart is about a ten-millionth of a billionth of a Newton (a Newton is about a fifth of a pound).

If we define the position of the particle as $\vec{r}$ with magnitude r, linear momentum balance for central-force motion is

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ \Rightarrow \quad \vec{F} &= m\vec{a} \\ \Rightarrow \quad F(r)\left(\frac{-\vec{r}}{r}\right) &= m\ddot{\vec{r}}, \end{aligned} \tag{5.83}$$

where $-\vec{r}/r$ is a unit vector pointed toward the origin and $F(r)$ is the magnitude of the origin-attracting force.

For the rest of this section we consider some of the consequences of eqn. (5.83). We start with the most historically important example.

Motion of the earth around a fixed sun

For simplicity let's assume that the sun does not move and that the motion of the earth lies in a plane. Newton's law of gravitation says that the attractive force of the sun on the earth is proportional to the masses of the sun and earth and inversely proportional to the distance between them squared (Fig. 5.74). Thus we have ①

$$F = \frac{Gm_e m_s}{r^2}$$

where m_e and m_s are the masses of the earth and sun, r is the distance between the earth and sun. 'Big G' is a universal constant $G \approx 6.67 \cdot 10^{-17}\,\mathrm{N\,m^2/kg^2}$. What

is the vector-valued force on the earth? It is its magnitude times a unit vector in the appropriate direction.

$$\begin{aligned} \vec{F} &= \left(\frac{Gm_em_s}{r^2}\right)\left(\frac{-\vec{r}}{|\vec{r}|}\right) \\ \Rightarrow \quad \vec{F} &= -Gm_em_s\left(\frac{\vec{r}}{r^3}\right) \\ \Rightarrow \quad \vec{F} &= -Gm_em_s\left(\frac{x\hat{\imath}+y\hat{\jmath}}{(x^2+y^2)^{3/2}}\right) \end{aligned} \tag{5.84}$$

where we have used that $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, $r = |\vec{r}| = \sqrt{x^2+y^2}$, and $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$. Now we can write the linear momentum balance equation for the earth in great detail.

$$\vec{F} = m\vec{a} \quad \Rightarrow \quad -Gm_em_s\left(\frac{x\hat{\imath}+y\hat{\jmath}}{(x^2+y^2)^{3/2}}\right) = m_e\left(\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath}\right) \tag{5.85}$$

Taking the dot product of equation 5.85 with $\hat{\imath}$ and $\hat{\jmath}$ successively (*i.e.*, taking x and y components) gives two scalar second order ordinary differential equations:

$$\ddot{x} = \frac{-Gm_sx}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-Gm_sy}{(x^2+y^2)^{3/2}}. \tag{5.86}$$

This pair of coupled second order differential equations describes the motion of the earth.① Pencil and paper solution is possible, Newton did it, but is a little too hard for this book. So we resort to computer solution. To set this up we put equations eqn. (5.86) in the form of a set of coupled first order ordinary differential equations. If we define $z_1 = x$, $z_2 = \dot{x}$, $z_3 = y$, and $z_4 = \dot{y}$. We can now write equations 5.86 as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -Gm_sz_1/(z_1^2+z_3^2)^{3/2} \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -Gm_sz_3/(z_1^2+z_3^2)^{3/2}. \end{aligned} \tag{5.87}$$

To actually solve these numerically we need a value for Gm_s and initial conditions. The solutions of these equations on the computer are all, within numerical error, consistent with Kepler's laws.

Without a full solution, there are some things we can figure out relatively easily.

① Note that G appears in the product Gm_s. Newton didn't know the value of big G, but he could do a lot of figuring without it. All he needed was the product Gm_s which he could find from the period and radius of the earth's orbit. The entanglement of G with the mass of the sun is why some people call Cavendish's measurement of big G, "weighing the sun". From Newton's calculation of Gm_s and Cavendish's measurement of G you can find m_s. Naturally, the real history is a bit more complicated. Cavendish presented his result as weighing the earth.

Circular orbits

We generally think of the motions of the planets as being roughly circular orbits. In fact, for any attractive central force one of the possible motions is a circular orbit. Rather than trying to derive this, let's assume a circular solution and see if it solves the equations of motion. A constant speed circular orbit with angular frequency ω and radius r_o obeys the parametric equation

$$\begin{aligned} \vec{r} &= r_o\left(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}\right) \\ \text{differentiating twice} \quad \Rightarrow \quad \ddot{\vec{r}} &= -\omega^2 r_o\left(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}\right) \\ &= -\omega^2\vec{r}. \end{aligned} \tag{5.88}$$

Comparing eqn. (5.88) with eqn. (5.83) we see we have an identity (a solution to the equation) if

$$\omega^2 = \frac{F(r)}{mr}.$$

In the case of gravitational attraction where $m = m_e$ we have $F(r) = Gm_s m_e/r^2$ so we get circular motion with

$$\omega^2 = \frac{Gm_s}{r^3} \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{}{Gm_s}} \quad r^{\frac{3}{2}} \tag{5.89}$$

because angular frequency is inversely proportional to the period ($\omega = 2\pi/T$). We have, for the special case of circular orbits, derived Kepler's third law. The orbital period is proportional to the orbital size to the 3/2 power.

Conservation of energy

Any force of the form

$$\vec{\boldsymbol{F}} = -F(r)\frac{\vec{\boldsymbol{r}}}{r}$$

is conservative and is associated with a potential energy given by the indefinite integral

$$E_{\mathrm{P}} = \int F(r)dr.$$

For the case of gravitational attraction, the potential energy is

$$E_{\mathrm{P}} = \frac{-Gm_s m_e}{r}$$

where we could add an arbitrary constant. Thus, one of the features of planetary motion is that for a given orbit the energy is constant in time:

$$\begin{aligned} \text{Constant} &= E_{\mathrm{K}} + E_{\mathrm{P}} \\ &= = \frac{1}{2}mv^2 + \frac{-Gm_s m_e}{r} \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{-Gm_s m_e}{\sqrt{x^2+y^2}}. \end{aligned} \tag{5.90}$$

If that constant is bigger than zero than the orbit has enough energy to have positive kinetic energy even when infinitely far from the sun. Such orbits are said to have more than "escape velocity" and they do indeed have open hyperbola-shaped orbits, and only pass close to the sun at most once.

Motion of rockets and artificial satellites

Rockets and the like move around the earth much like planets, comets and asteroids move around the sun. All of the equations for planetary motion apply. But you need to substitute the mass of the earth for m_s and the mass of the satellite for m_e. Thus we can write the governing equation eqn. (5.85) as

$$-GMm\left(\frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) = m\left(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}\right) \tag{5.91}$$

where now M is the mass of the earth and m is the mass of the satellite. At the surface of the earth $r = R$, the earth's radius, and $GM/R^2 = g$ so we can rewrite the governing equation for rockets and the like as

$$-gR^2\left(\frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) = \left(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}\right). \tag{5.92}$$

Another central-force example: force proportional to radius

A less famous, but also useful, example of central force is where the attraction force is proportional to the radius. In this case the governing equations are:

$$\begin{aligned} \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ -k\vec{\boldsymbol{r}} &= m\ddot{\vec{\boldsymbol{r}}} \\ -k(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}) &= m\left(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}\right). \end{aligned} \tag{5.93}$$

Dotting both sides with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ we get two uncoupled linear homogeneous constant coefficient differential equations:

$$\ddot{x} + \frac{k}{m}x = 0 \quad \text{and} \quad \ddot{y} + \frac{k}{m}y = 0.$$

These you recognize as the harmonic oscillator equations so we can pick off the general solutions immediately as:

$$x = A\cos(\lambda t) + B\sin(\lambda t) \quad \text{and} \quad y = C\cos(\lambda t) + D\sin(\lambda t) \tag{5.94}$$

where A, B, C, and D are arbitrary constants which are determined by initial conditions. For all A, B, C, and D eqn. (5.94) describes an ellipse (or a special case of an ellipse, like a circle or a straight line). In the case of planetary motion we also had ellipses. In this case, however, the center of attraction is at the center of the ellipse and not at one of the foci.

Conservation of angular momentum and Kepler's second law

If we take the linear momentum balance equation eqn. (5.83) and take the cross product of both sides with $\vec{\boldsymbol{r}}$ we get the following.

$$\begin{aligned} \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ \Rightarrow \quad F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right) &= m\ddot{\vec{\boldsymbol{r}}} \\ \Rightarrow \quad \vec{\boldsymbol{r}} \times \left(F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right)\right) &= \vec{\boldsymbol{r}} \times \left(m\ddot{\vec{\boldsymbol{r}}}\right) \\ \Rightarrow \quad \vec{\boldsymbol{0}} &= \frac{d}{dt}\left(m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}\right) \qquad (\text{because} \dot{\vec{\boldsymbol{r}}} \times \dot{\vec{\boldsymbol{r}}} = \vec{\boldsymbol{0}}) \\ \Rightarrow \quad \text{constant} &= m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}. \end{aligned} \tag{5.95}$$

But this last quantity is exactly the rate at which area is swept out by a moving particle. Thus Kepler's third law has been derived for all central-force motions (not just inverse square attractions). The last quantity is also the angular momentum of the particle. Thus for a particle in central force motion we have derived conservation of angular momentum from $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.

SAMPLE 5.36 *Circular orbits of planets:* Refer to eqn. (5.86) in the text that governs the motion of planets around a fixed sun.

(a) Let $x = A\cos(\lambda t)$ and $y = A\sin(\lambda t)$. Show that x and y satisfy the equations of planetary motion and that they describe a circular orbit.

(b) Show that the solution assumed in (a) satisfies Kepler's third law by showing that the orbital period $T = 2\pi/\lambda$ is proportional to the 3/2 power of the size of the orbit (which can be characterized by its radius).

Solution

(a) The governing equation of planetary motion can be written as

$$\begin{aligned} \frac{\ddot{x}}{x} = \frac{-Gm_s}{(x^2+y^2)^{3/2}} &= \frac{\ddot{y}}{y} \\ \Rightarrow \quad \ddot{x}y - \ddot{y}x &= 0 \end{aligned} \tag{5.96}$$

Now,

$$\begin{aligned} x &= A\cos(\lambda t) \quad \Rightarrow \quad \ddot{x} = -\lambda^2 A\cos(\lambda t) \\ y &= A\sin(\lambda t) \quad \Rightarrow \quad \ddot{y} = -\lambda^2 A\cos(\lambda t) \end{aligned}$$

Substituting these values in eqn. (5.96), we get

$$-\lambda^2 A^2\cos(\lambda t)\cdot\sin(\lambda t) + \lambda^2 A\sin(\lambda t)\cdot\cos(\lambda t) \overset{\surd}{=} 0$$

Thus the assumed form of x and y satisfy the governing equations of planetary motion, *i.e.*, $x(t) = A\cos(\lambda t)$ and $y(t) = A\sin(\lambda t)$ form a solution of planetary motion. Now, it is easy to show that

$$x^2 + y^2 = \cos^2(\lambda t) + \sin^2(\lambda t) = 1,$$

i.e., x and y satisfy the equation of a circle of radius A. Thus, the assumed solution gives a circular orbit.

(b) Substituting $x = A\cos(\lambda t)$ in eqn. (5.86), and noting that square of the radius of the orbit is $r^2 = x^2 + y^2 = A^2$, we get

$$\begin{aligned} -\lambda^2 A\cos(\lambda t) &= -Gm_s\frac{A\cos(\lambda t)}{r^3} \\ \Rightarrow \quad \lambda^2 &= \frac{Gm_s}{A^3} \\ \text{or} \qquad \left(\frac{2\pi}{T}\right)^2 &= \frac{Gm_s}{A^3} \\ \Rightarrow \quad T^2 &= \frac{4\pi^2}{Gm_s}A^3 \\ \text{or} \qquad T &= KA^{3/2} \end{aligned}$$

where $K = 2\pi/\sqrt{Gm_s}$ is a constant. Thus the orbital period T is proportional to the 3/2 power of the radius, or the size, of the circular orbit.

Of, course, the same holds true for elliptic orbits too, but it is harder to show that analytically using cartesian coordinates, x and y.

◁

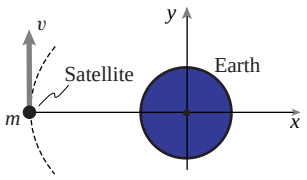


Figure 5.75: (Filename:sfig5.9.satorbit)

SAMPLE 5.37 *Numerical computation of satellite orbits:* The following data is known for an earth satellite: mass = 2000 kg, the distance to the closest point (perigee) on its orbit from the earth's surface = 1100 km, and its velocity at perigee, which is purely tangential, is 9500 km/s. The radius of the earth is 6400 km and the acceleration due to gravity $g = 9.8\,\text{m/s}^2$.

(a) Solve the equations of motion of the satellite numerically with the given data and show that the orbit of the satellite is elliptical. Find the apogee of the orbit and the speed of the satellite at the apogee.

(b) From the data at apogee and perigee show that the angular momentum and the energy of the satellite are conserved.

(c) Find the orbital period of the satellite and show that it satisfies Kepler's third law (in equality form).

Solution

(a) The equations of motion of a satellite around a fixed earth are

$$\ddot{x} = \frac{-gR^2x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-gR^2y}{(x^2+y^2)^{3/2}}.$$

where g is the acceleration due to gravity and R is the radius of the earth (see eqn. (5.91) in text). From the given data at perigee, the initial conditions are

$$x(0) = -7500\,\text{km}, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 9500\,\text{m/s}.$$

In order to solve the equations of motion by numerical integration, we first rewrite these equations as four first order equations:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -gR^2z_1/(z_1^2+z_3^2)^{3/2} \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -gR^2z_3/(z_1^2+z_3^2)^{3/2}. \end{aligned}$$

Now the given initial conditions in terms of the new variables are

$$z1(0) = -7.5\times10^6\,\text{m}, \quad z2(0) = 0, \quad z3(0) = 0, \quad z4(0) = 9500\,\text{m/s}.$$

We are now ready to go to a computer. We implement the following pseudocode on the computer to solve the problem.

```
ODEs ={z1dot=z2,  z2dot=-g*R^2*z_1/(z_1^2+z_3^2)^{3/2},
       z3dot=z4,  z4dot=-g*R^2*z_3/(z_1^2+z_3^2)^{3/2}}
    IC   ={z1(0)=-7.5E06, z2(0)=0, z3(0)=0, z4(0)=9500}
    Set  g = 9.81, R = 6.4E06
    Solve ODEs with IC for t=0 to t=4E04
    Plot z1 vs z3
```

Results obtained from implementing the code above with a Runge-Kutta method based integrator is shown in Fig. 5.75 where we have also plotted the earth centered at the origin to put the orbit in perspective. The orbit is clearly elliptic. From the computer output, we find the following data for the apogee.

$$x = 4.0049\times10^7\,\text{m}, \quad \dot{x} = 0, \quad y = 0, \quad \dot{y} = -1.7791\times10^3\,\text{m/s}$$

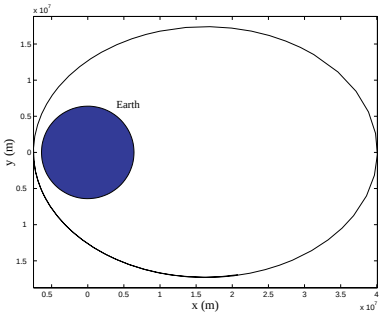


Figure 5.76: The elliptical orbit of the satellite, obtained from numerical integration of the equations of motion.

(Filename:sfig5.9.satorbit.a)

(b) The expressions for energy E and angular momentum H for a satellite are,

$$\begin{aligned} E &= E_{\rm K} + E_{\rm P} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{GMm}{r} \\ \vec{\boldsymbol{H}}_0 &= \vec{\boldsymbol{r}} \times m\vec{\boldsymbol{v}} = (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}) \times m(\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}) = m(x\dot{y} - y\dot{x})\hat{\boldsymbol{k}} \end{aligned}$$

At both apogee and perigee, $y = 0$ and the velocity (which is tangential) is in the y direction, *i.e.*, $\dot{x} = 0$. Therefore, the expressions for energy and angular momentum become simpler:

$$E = \frac{1}{2}m\dot{y}^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{y}^2 - \frac{gR^2m}{|x|} \quad \text{and} \quad H = mx\dot{y}$$

Let E_1 and H_1 be the energy and the angular momentum of the satellite at the perigee, respectively, and E_2 and H_2 be the respective quantities at the apogee. Then, from the given data,

$$\begin{aligned} E_1 = \frac{1}{2}m\dot{y}_1^2 - \frac{gR^2m}{|x_1|} &= \frac{1}{2}2000\,\text{kg}\cdot(9500\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}{7.5\times10^6\,\text{m}} \\ &= -1.6901\times10^{10}\ \text{Joules} \\ H_1 = mx_1\dot{y}_1 &= 2000\,\text{kg}\cdot(-7.5\times10^6\,\text{m})\cdot(9500\,\text{m/s}) \\ &= -1.4250\times10^{14}\,\text{N·m}\cdot\,\text{s} \\ E_2 = \frac{1}{2}m\dot{y}_2^2 - \frac{gR^2m}{|x_2|} &= \frac{1}{2}2000\,\text{kg}\cdot(-1779\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}{4.0049\times10^7\,\text{m}} \\ &= -1.6901\times10^{10}\text{Joules} \\ H_2 = mx_2\dot{y}_2 &= 2000\,\text{kg}\cdot(4.0049\times10^7\,\text{m})\cdot(-1779\,\text{m/s}) \\ &= -1.4250\times10^{14}\,\text{N·m}\cdot\,\text{s} \end{aligned}$$

Clearly, the energy and the angular momentum are conserved.

(c) From the computer output, we find the time at which the satellite returns to the perigee for the first time. This is the orbital period. From the output data, we get the orbital period to be 3.6335×10^4 s $= 10.09$ hrs. Now let us compare this result with the analytical value of the orbital period.

Let A be the semimajor axis of the elliptic orbit. Then the square of the orbital time period T is given by

$$T^2 = \frac{4\pi^2A^3}{gR^2}.$$

For the orbit we have obtained by numerical integration,

$$\begin{aligned} 2A &= |x_1| + |x_2| = 7.5\times10^6\,\text{m} + 4.0049\times10^7\,\text{m} \\ &= 4.7549\times10^7\,\text{m} \\ \Rightarrow\quad A &= 2.3774\times10^7\,\text{m} \end{aligned}$$

Hence,

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2\cdot(2.3774\times10^7\,\text{m})^3}{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}} \\ &= 3.6335\times10^4\,\text{s}. \end{aligned}$$

which is the same value as obtained from numerical solution.

$$\boxed{T = 3.6335\times10^4\,\text{s} = 10.09\ \text{hrs}}$$

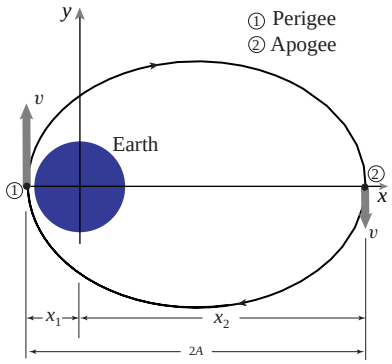


Figure 5.77: The elliptical orbit of the satellite. The perigee and apogee are marked as points 1 and 2 on the orbit.

(Filename:sfig5.9.satorbit.b)

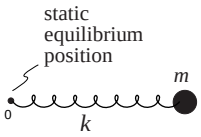


Figure 5.78: (Filename:sfig5.9.zerospring)

① No spring can have zero relaxed length, however, a spring can be configured in various ways to make it behave as if it has zero relaxed length. For example, let a spring be fixed to the ground and let its free end pass through a hole in a horizontal table. Let the relaxed length of the spring be exactly up to the top of the hole. Now, if the spring is pulled further and tied to a mass that is constrained to move on the horizontal table, then the spring behaves like a zero-length spring for the planar motion of the mass around the hole in the table. This is because the length of the position vector of the mass is exactly the stretch in the spring.

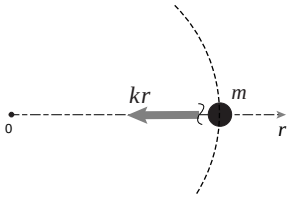


Figure 5.79: Free body diagram of the mass.

(Filename:sfig5.9.zerospring.a)

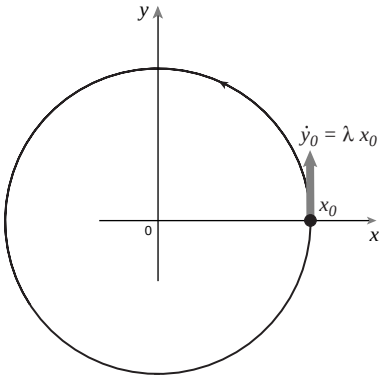


Figure 5.80: Circular trajectory of the mass.

(Filename:sfig5.9.zerospring.b)

SAMPLE 5.38 *Zero-length spring and central force motion:* A zero-length spring ①(the relaxed length is zero) is tied to a mass $m = 1\,\text{kg}$ on one end and fixed on the other end. The spring stiffness is $k = 1\,\text{N}/m$.

(a) Find appropriate initial conditions for the mass so that its trajectory is a straight line along the y-axis.
(b) Find appropriate initial conditions for the mass so that its trajectory is a circle.
(c) Can you find any condition on initial conditions that guarantees elliptic orbits of the mass?
(d) Let $\vec{r}(0) = 0.5\,\text{m}\hat{\imath}$ and $\dot{\vec{r}}(0) = (0.5\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s}$. Describe the motion of the mass by plotting its trajectory for 12 s.

Solution Let the position of the mass be $\vec{r}$ at some instant t. Since the relaxed length of the spring is zero, the stretch in the spring is $|\vec{r}|$ and the spring force on the mass is $-k\vec{r}$. Then the equation of motion of the mass is

$$\begin{aligned} -k\vec{r} &= m\ddot{\vec{r}} \\ -k(x\hat{\imath} + y\hat{\jmath}) &= m\left(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}\right) \\ \Rightarrow \quad \ddot{x} + \frac{k}{m}x &= 0 \quad \text{and} \quad \ddot{y} + \frac{k}{m}y = 0. \end{aligned}$$

Thus the equations of motion are decoupled in the x and y directions. The solutions, as discussed in the text (see eqn. (5.94)), are

$$x = A\cos(\lambda t) + B\sin(\lambda t) \quad \text{and} \quad y = C\cos(\lambda t) + D\sin(\lambda t) \tag{5.97}$$

where the constants A, B, C, D are determined from initial conditions. Let us take the most general initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, $y(0) = y_0$, and $\dot{y}(0) = \dot{y}_0$. By substituting these values in x and y equations above and their derivatives, we get

$$A = x_0, \quad B = \dot{x}_0/\lambda, \quad C = y_0, \quad D = \dot{y}_0/\lambda.$$

Substituting these values we get

$$x = x_0\cos(\lambda t) + \dot{x}_0/\lambda\sin(\lambda t) \quad \text{and} \quad y = y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t) \tag{5.98}$$

(a) For a straight line motion along the y-axis, we should have the x-component of motion identically zero. We can, therefore, set $x_0 = 0$, $\dot{x}_0 = 0$ and take any value for y_0 and $\dot{y}_0$ to give

$$x(t) = 0 \quad \text{and} \quad y(t) = y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t).$$

(b) For a circular trajectory, we must pick initial conditions such that we get $x^2 + y^2 = (\text{a constant})^2$. We can easily achieve this by choosing, say, $x(0) = x_0, \dot{x}(0) = 0, y(0) = 0$, and $\dot{y}(0) = x_0\lambda$. Substituting these values in eqn. (5.98), we get

$$x^2 + y^2 = x_0^2\cos^2(\lambda t) + \left(\frac{x_0\lambda}{\lambda}\right)^2\sin^2(\lambda t) = x_0^2$$

which is a circular orbit of radius x_0. Note that the initial position of the mass for this orbit is $\vec{r}(0) = x_0\hat{\imath}$), and the initial velocity is ($\vec{v}(0) = x_0\lambda\hat{\jmath}$), *i.e.*, the velocity is normal to the position vector ($\vec{r}\cdot\vec{v} = 0$), and the magnitude of the velocity is dependent on the magnitude of the position vector, in fact , it must be exactly equal to the product of the distance from the center and the orbital frequency λ.

(c) In order to have elliptic orbits, the initial conditions should be selected such that x and y satisfy the equation of an ellipse. By examining the solutions in eqn. (5.98), we see that if we set $\dot{x}_0 = 0$ and $y_0 = 0$ and let the other two initial conditions have any value, x_0 and $\dot{y}_0$, we get

$$\begin{aligned} & x(t) = x_0 \cos(\lambda t) \qquad \text{and} \qquad y(t) = (\dot{y}_0/\lambda)\sin(\lambda t) \\ \Rightarrow \quad & \frac{x^2}{x_0^2} + \frac{y^2}{(\dot{y}/\lambda)^2} \quad = \quad \cos^2(\lambda t) + \sin^2(\lambda t) = 1 \end{aligned}$$

which is the equation of an ellipse with semimajor axis x_0 and semiminor axis $\dot{y}/\lambda$. Of course, the symmetry of the equations implies that we could also get elliptic orbits by setting $x_0 = 0$ and $\dot{y}_0 = 0$, and letting the other two initial conditions be arbitrary. Thus the condition for elliptic orbits is to have the initial velocity normal to the position vector (either $\vec{\boldsymbol{r}}(0) = x_0\hat{\boldsymbol{\imath}}$, $\dot{\vec{\boldsymbol{r}}}(0) = \dot{y}_0\hat{\boldsymbol{\jmath}}$ or $\vec{\boldsymbol{r}}(0) = y_0\hat{\boldsymbol{\jmath}}$, $\dot{\vec{\boldsymbol{r}}}(0) = \dot{x}_0\hat{\boldsymbol{\imath}}$, or more generally, $\vec{\boldsymbol{r}}(0) = r_0\hat{\boldsymbol{\lambda}}$, $\dot{\vec{\boldsymbol{r}}}(0) = v\hat{\boldsymbol{n}}$ where $\hat{\boldsymbol{\lambda}}$ is a unit vector along the position vector of the mass and $\hat{\boldsymbol{n}}$ is normal to $\hat{\boldsymbol{\lambda}}$.

Note that the condition obtained in (b) for circular orbits is just a special case of the condition for elliptic orbits (well, a circle is just a special case of an ellipse). Therefore, if we keep x_0 fixed and vary $\dot{y}_0$ we can get different elliptic orbits, including a circular one, based on the same major axis. Taking $x_0 = 1\,\text{m}$, we show different orbits obtained for the mass by varying $\dot{y}_0$ in Fig. 5.81

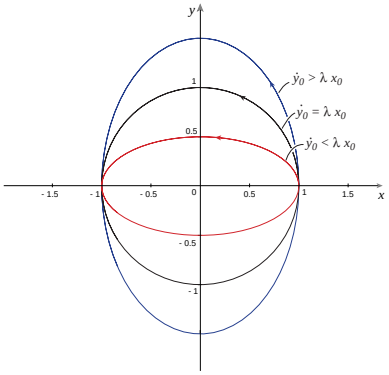


Figure 5.81: Elliptic orbits of the mass obtained from the initial conditions $x_0 = 1\,\text{m}$, $\dot{x}_0 = 0$, $y_0 = 0$, and various values of $\dot{y}_0$.

(Filename:sfig5.9.zerospring.c)

(d) By substituting the given initial values $x_0 = 0.5\,\text{m}$, $\dot{x}(0) = 0.5\,\text{m/s}$, $y(0) = 0$ and $\dot{y} = 0.6\,\text{m/s}$ in eqn. (5.98) and and noting that $\lambda \equiv \sqrt{k/m} = \sqrt{(1N/m)/(1\,\text{kg})} = (1/s)$, we get

$$\begin{aligned} x(t) &= (0.5\,\text{m}) \cdot \cos\left(\frac{1}{s}\cdot t\right) + \left(\frac{0.5\,\text{m/s}}{s}\right) \cdot \sin\left(\frac{1}{s}\cdot t\right) \\ y(t) &= \left(\frac{0.6\,\text{m/s}}{s}\right) \cdot \sin\left(\frac{1}{s}\cdot t\right) \end{aligned}$$

The functions $x(t)$ and $y(t)$ do not seem to describe any simple geometric path immediately. We could, perhaps, do some mathematical manipulations and try to get a relationship between x and y that we can recognize. In stead, let us plot the orbit on a computer to see the path that the mass takes during its motion with these initial conditions. To plot this orbit, we evaluate x and y at, say, 100 values of t between 0 and 10 s and then plot x vs y.

```
t = [0 0.1 0.2 ... 9.9 10]
x = 0.5 * cos(t) + 0.6 * sin(t)
y = 0.6 * sin(t)
plot x vs y
```

The plot obtained by performing these operations on a computer is shown in Fig. 5.82.

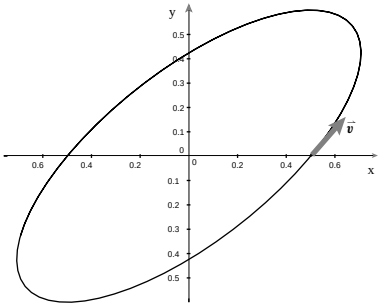


Figure 5.82: The orbit of the mass obtained from the initial conditions $x_0 = 0.5\,\text{m}$, $\dot{x}_0 = 0.5\,\text{m/s}$, $y_0 = 0$, and $\dot{y}_0 = 0.6\,\text{m/s}$.

(Filename:sfig5.9.zerospring.d)

◁

5.10 Coupled motions of particles in space

In the previous two sections you have seen that once you know the forces on a particle, or on how those forces vary with position, velocity and time, you can easily set up the equations of motion. That is, the linear momentum balance equation for a particle

$$\vec{F} = m\vec{a}$$

with initial conditions gives a well defined mathematical problem whose solution is the motion of the particle. The solution may be hard or impossible to find with pencil and paper, but can generally be found in a straightforward way with numerical integration.

Now we generalize this simple point of view to two, three or more particles. Assume you know enough about a system so that you know the forces on each particle if someone tells you the time and the positions and velocities of all the particles. Then that means we can write the governing equations for the system of particles like this:

$$\begin{aligned} \vec{a}_1 &= \frac{1}{m_1}\vec{F}_1 \\ \vec{a}_2 &= \frac{1}{m_2}\vec{F}_2 \\ \vec{a}_3 &= \frac{1}{m_3}\vec{F}_3 \\ &\text{etc.} \end{aligned} \tag{5.99}$$

where $\vec{F}_1$, $\vec{F}_2$ *etc*.are the total of the forces on the corresponding particles. If the force on each particle comes from well-understood air-friction, from springs are dashpots connected here and there, or from gravity interactions with other particles, *etc*., then all the forces on all the particles are known given the positions and velocities of the particles. Thus eqn. (5.99) can be written as a system of first order differential equations in standard form ready for computer simulation. Given accurate initial conditions and a good computer and the motions of all the particles can be found accurately.

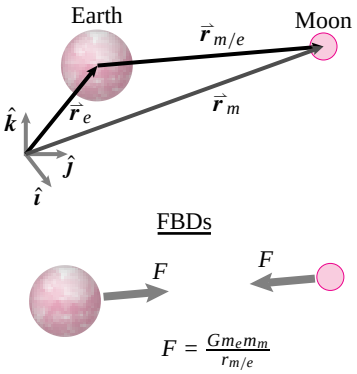


Figure 5.83: The earth and moon. Position is measured relative to some inertial point C.

(Filename:tfigure.earthmoon)

Example: **Coupled motion of the earth and moon in three dimensions.**

Let's neglect the sun and just look at the coupled motions of the earth and moon. They attract each other by the same law of gravity that we used for the sun and earth. The difference between this problem and a central-force problem is that we now need to look at the 'absolute' positions of the sun and moon ($\vec{r}_e$ and $\vec{r}_m$), and the 'relative' position, say $\vec{r}_{m/e} \equiv \vec{r}_m - \vec{r}_e$ (Fig. 5.83).

The linear momentum balance equations are now

$$m_e \ddot{\vec{r}}_e = \frac{-Gm_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3} \quad \text{and} \tag{5.100}$$

$$m_m \ddot{\vec{r}}_m = \frac{+Gm_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3}, \tag{5.101}$$

which, when broken into x, y, and z components give 6 second order ordinary differential equations. These equations can be written as 12 first order equations by defining a list of 12 z variables: $z_1 = x_e$, $z_2 = \dot{x}_e$, $z_3 = \dot{y}_e$, $z_4 = y_e$, *etc*.

After one finds solutions with appropriate initial conditions one can see if the computer finds such truths (that is, features of the exact solution of the differential equations) as:

(a) that the line between the earth and moon always lies on one fixed plane,
(b) the center of mass moves at constant speed on a straight line,
(c) relative to the center of mass both the earth and moon travel on paths that are conic sections
(d) the energy of the system is constant,
(e) and that the angular momentum of the system about the center of mass is a constant.

□

If we could think of all materials as made of atoms, and of all the atoms moving in deterministic ways governed by Newton's laws and known force laws, and we knew the initial positions and velocities accurately enough, then we could accurately predict the motions of all things for all time.

To put it in other words, given a simple atomic view of the world and a big computer, we could end a course on dynamics here. You know how to use $\vec{F} = m\vec{a}$ for each atom, so you can simulate anything made of atoms. Now there are some serious catches here, so before proceeding we name some of them:

- there are no computers big enough to accurately integrate Newton's laws for the 10^{23} or so atoms needed to describe macroscopic objects;
- the laws of interaction between atoms are not simple and are not that well known;
- the state of the world (the positions and velocities of all the atoms is not that well known);
- the solutions of the equations are often unstable in that a very small error in the initial conditions propagates into a large error in the calculations;
- the world is not deterministic, quantum mechanics says that you *cannot* know the state of the world perfectly; and
- massive simulations, even if accurate, are not always the best way to understand how things work.

Despite these limitations on the ultimate utility of the approach, in this section we look at the nature of systems of interacting particles. In particular we look at the momentum, angular momentum, and energy of a system of particles.

Linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$

One of our three basic dynamics equations is linear momentum balance:

$$\sum \vec{F} = \dot{\vec{L}}.$$

The first quantity of interest in this section is the linear momentum $\vec{L}$①, whose derivative, $\dot{\vec{L}}$, with respect to a Newtonian frame is so important. Linear momentum is a measure of the translational motion of a system.

$$\underbrace{\vec{L}}_{\text{linear momentum}} \equiv \underbrace{\sum m_i \vec{v}_i}_{\substack{\text{summed over} \\ \text{all the mass particles}}} = m_{\text{tot}} \vec{v}_{cm} \tag{5.102}$$

① In Isaac Newton's language: *'The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly'*. In other words, Newton's dynamics equations for a particle were based on the product of $\vec{v}$ and m. This quantity, $m\vec{v}$, is now called $\vec{L}$, the linear momentum of a particle.

Example: **Center of Mass position, velocity, and acceleration**

A particle of mass $m_A = 3\,\text{kg}$ and another point of mass $m_B = 2\,\text{kg}$ have positions, respectively,

$$\vec{r}_A(t) = \left[3\hat{\imath} + 5\left(\frac{t}{\text{s}}\right)\hat{\jmath}\right]\text{m}, \text{ and } \vec{r}_B(t) = \left[6\left(\frac{t^2}{\text{s}^2}\right)\hat{\imath} - 4\hat{\jmath}\right]\text{m}$$

due to forces that we do not discuss here. The position of the the center of mass of the system of particles, according to equation 2.12 on page 62, is

$$\begin{aligned}\vec{r}_{cm}(t) &= \frac{\overbrace{m_A\vec{r}_A(t) + m_B\vec{r}_B(t)}^{\sum m_i\vec{r}_i}}{\underbrace{(m_A + m_B)}_{m_{\text{tot}}=5\,\text{kg}}} \\ \vec{r}_{cm}(t) &= \left[\left(\frac{9}{5} + \frac{12}{5}\left(\frac{t^2}{\text{s}^2}\right)\right)\hat{\imath} + \left(3\left(\frac{t}{\text{s}}\right) - \frac{8}{5}\right)\hat{\jmath}\right]\text{m}.\end{aligned}$$

To get the velocity and acceleration of the center of mass, we differentiate the position of the center of mass once and twice, respectively, to get①

$$\vec{v}_{cm}(t) = \dot{\vec{r}}_{cm}(t) = \left[\frac{24}{5}\left(\frac{t}{\text{s}^2}\right)\hat{\imath} + \frac{3}{\text{s}}\hat{\jmath}\right]\text{m} = \left[\frac{24}{5}\left(\frac{t}{\text{s}}\right)\hat{\imath} + 3\hat{\jmath}\right]\text{m/s}$$

and

$$\vec{a}_{cm}(t) = \dot{\vec{v}}_{cm}(t) = \ddot{\vec{r}}_{cm}(t) = \left[\frac{24}{5}\left(\frac{1}{\text{s}^2}\right)\hat{\imath}\right]\text{m} = \left(\frac{24}{5}\right)\text{m/s}^2\hat{\imath}.$$

In this example, the center of mass turns out to have constant acceleration in the x-direction. □

① That is, particle A travels on the line $x = 3\,\text{m}$ with constant speed $\dot{r}_{A_y} = 5\,\text{m/s}$ and particle B travels on the line $y = -4\,\text{m}$ at changing speed $\dot{r}_{B_x} = 12t\,(\text{m/s}^2)$.

The second part of equation 5.102 follows from the definition of the center of mass (see box 5.8 on page 314).② The total linear momentum of a system is the same as that of a particle that is located at the center of mass and which has mass equal to that of the whole system. The linear momentum is also given by

$$\vec{L} = \frac{d}{dt}\left(m_{\text{tot}}\vec{r}_{cm}\right).$$

② Some books use the symbol $\vec{P}$ for linear momentum. Because $\vec{P}$ is often used to mean force or impulse and P for power we use $\vec{L}$ for linear momentum.

5.8 Velocity and acceleration of the center of mass of a system of particles

The average position of mass in a system is at a point called the center of mass. The position of the center of mass is

$$\vec{r}_{cm} = \frac{\sum \vec{r}_i\, m_i}{m_{\text{tot}}}.$$

Multiplying through by m_{tot}, we get

$$\vec{r}_{cm}m_{\text{tot}} = \sum \vec{r}_i\, m_i.$$

By taking the time derivatives of the equation above, we get

$$\begin{aligned}\vec{v}_{cm}m_{\text{tot}} &= \sum \vec{v}_i\, m_i \quad \text{and} \\ \vec{a}_{cm}m_{\text{tot}} &= \sum \vec{a}_i\, \text{m}_i \quad .\end{aligned}$$

for the velocity and acceleration of the center of mass. The results above are useful for simplifying various momenta and energy expressions. Note, for example, that

$$\begin{aligned}\vec{L} = \sum \vec{v}_i\, \text{m}_i &= \vec{r}_{cm}m_{\text{tot}} \\ \dot{\vec{L}} = \sum \vec{a}_i\, \text{m}_i &= \vec{a}_{cm}m_{\text{tot}}.\end{aligned}$$

We only consider systems of fixed mass, $\frac{d}{dt}(m_{\text{tot}}) = 0$. Thus, for a fixed mass system, the linear momentum of the system is equal to the total mass of the system times the derivative of the center of mass position.

Finally, since the sum defining linear momentum can be grouped any which way (the associative rule of addition) the linear momentum can be found by dividing the system into parts and using the mass of those parts and the center of mass motion of those parts. That is, the sum $\sum m_i \vec{v}_i$ can be interpreted as the sum over the center of mass velocities and masses of the various subsystems, say the parts of a machine.

Example: **System Momentum**

See figure 5.84 for a schematic example of the total momentum of system being made of the sum of the momenta of its two parts. □

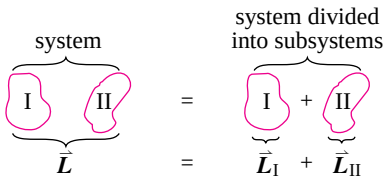


Figure 5.84: System composed of two parts. The momentum of the whole is the sum of the momentums of the two parts.

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The reasoning for this allowed subdivision is similar to that for the center of mass in box 2.9 on page 67.

The quantity $\dot{\vec{L}}$ figures a little more directly in our presentation of dynamics than just plain $\vec{L}$ ①. The rate of change of linear momentum, $\dot{\vec{L}}$, is

$$\begin{aligned} \dot{\vec{L}} &= \frac{d}{dt}\vec{L} \\ &= \frac{d}{dt}\sum m_i \vec{v}_i \\ &= m_{\text{tot}}\frac{d\,\vec{v}_{\text{cm}}}{dt} \\ \dot{\vec{L}} &= m_{\text{tot}}\vec{a}_{\text{cm}} \end{aligned}$$

The last three equations could be thought of as the *definition* of $\dot{\vec{L}}$. That $\dot{\vec{L}}$ turns out to be $\frac{d}{dt}(\vec{L})$ is, then, a derived result. Again, using the definition of center of mass,

> the total rate of change of linear momentum is the same as that of a particle that is located at the center of mass which has mass equal to that of the whole system.

The rate of change of linear momentum is also given by

$$\dot{\vec{L}} = \frac{d}{dt}(m_{\text{tot}}\vec{v}_{\text{cm}}) = \frac{d^2}{dt^2}(m_{\text{tot}}\vec{r}_{\text{cm}}).$$

The momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ can be expressed in terms of the total mass of a system and the motion of the center of mass. This simplification holds for any system, however complex, and any motion, however contorted and wild.

Angular momentum $\vec{H}$ and its rate of change $\dot{\vec{H}}$

After linear momentum balance, the second basic mechanics principle is angular momentum balance:

$$\sum \vec{M}_{\text{C}} = \dot{\vec{H}}_{\text{C}},$$

where C is any point, preferably one that is fixed in a Newtonian frame.If you choose your point C to be a moving point you may have the confusing problem that the quantity we would like to call $\dot{\vec{H}}_{\text{C}}$ is not the time derivative of $\vec{H}_{\text{C}}$. The first quantity

① No slight of Sir Isaac is intended.

of interest in this sub-section is the angular momentum with respect to some point C, $\vec{\boldsymbol{H}}_{\mathrm{C}}$, whose rate of change $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} = d\vec{\boldsymbol{H}}_{\mathrm{C}}/dt$ is so important.

$$\underbrace{\vec{\boldsymbol{H}}_{\mathrm{C}}}_{\text{angular momentum.}} \equiv \underbrace{\sum \vec{r}_{i/C} \times m_i \vec{v}_i}_{\substack{\text{summed over all} \\ \text{the mass particles}}}$$

A useful theorem about angular momentum is the following (see box 5.9 on page 317), applicable to all systems

angular momentum due to center of mass motion

angular momentum relative to the center of mass

$$\vec{\boldsymbol{H}}_{\mathrm{C}} = \overbrace{\vec{r}_{cm/C} \times \vec{v}_{cm} m_{\mathrm{tot}}}^{} + \overbrace{\sum \underbrace{\vec{r}_{i/cm}}_{} \times \underbrace{\vec{v}_{i/cm}}_{} m_i}^{} \,. \tag{5.103}$$

position of m_i relative to the center of mass $\vec{r}_{i/cm} \equiv \vec{r}_i - \vec{r}_{cm}$

velocity of m_i relative to the center of mass $\vec{v}_{i/cm} \equiv \vec{v}_i - \vec{v}_{cm}$

A system of particles is shown in figure 5.85. The angular momentum of any system is the same as that of a particle at its center of mass *plus* the angular momentum associated with motion relative to the center of mass.

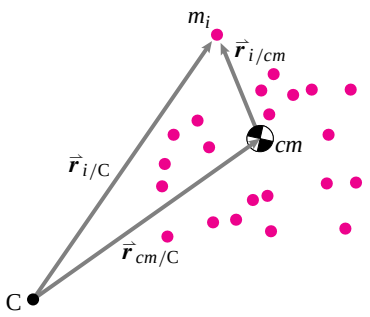


Figure 5.85: A system of particles showing its center of mass and the i_{th} particle of mass m_i. The i_{th} particle has position $\vec{r}_{i/cm}$ with respect to the center of mass. The center of mass has position $\vec{r}_{cm/C}$ with respect to the point C

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The angular momentum about point C is a measure of the average rotation rate of the system about point C. Angular momentum is not so intuitive as linear momentum for a number of reasons:

- First, recall that linear momentum is the derivative of the total mass times the center of mass position. Unfortunately, in general,

  angular momentum is not the derivative of anything.

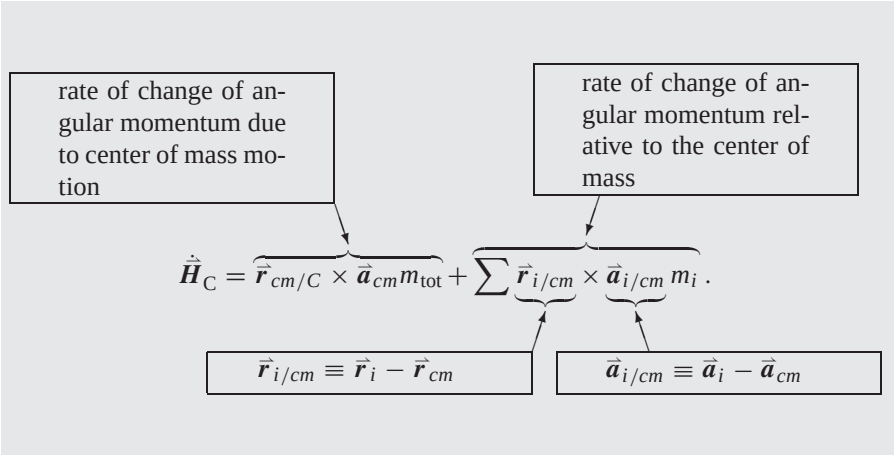
- Second, the angular momentum of a given system at a given time depends on the reference point C. So there is not one single quantity that is *the* angular momentum. For different points C_1, C_2, etc., the same system has different angular momentums.
- Finally, calculation of angular momentum involves a vector cross product and many beginning dynamics students are intimidated by vector cross products.

Despite these confusions, the concept of angular momentum allows the solution of many practical problems and eventually becomes somewhat intuitive.

Actually, it is $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$ which is the more fundamental quantity. $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$ is what you use in the equation of motion. You can find $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$ from $\vec{\boldsymbol{H}}_{\mathrm{C}}$ as shown in the box on page 318. But, in general,

$$\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} \equiv \sum \vec{r}_{i/\mathrm{C}} \times (\mathrm{m}_i \vec{a}_i).$$

A useful theorem about rate of change of angular momentum is the following (see box 5.9 on page 317), applicable to all systems:



This expression is completely analogous to equation 5.103 on page 316 and is derived in a manner nearly identical to that shown in box 5.9 on page 317. The rate of change of angular momentum of any system is the same as that of a particle at its center of mass *plus* the rate of change of angular momentum associated with motion relative

5.9 *THEORY*
Simplifying $\vec{H}_C$ using the center of mass

The definition of angular momentum relative to a point C is

$$\vec{H}_C = \sum \vec{r}_{i/C} \times m_i \vec{v}_i .$$

If we rewrite $\vec{v}_i$ as

$$\vec{v}_i = (\vec{v}_i - \vec{v}_{cm}) + \vec{v}_{cm} = \vec{v}_{i/cm} + \vec{v}_{cm}$$

and

$$\vec{r}_i = (\vec{r}_i - \vec{r}_{cm}) + \vec{r}_{cm} = \vec{r}_{i/cm} + \vec{r}_{cm}$$

then

$$\begin{aligned}
\vec{H}_C &= \sum \left(\vec{r}_{cm} + \vec{r}_{i/cm}\right) \times \left[\vec{v}_{cm} + \vec{v}_{i/cm}\right] m_i . \\
&= \sum \vec{r}_{cm} \times \vec{v}_{cm} m_i + \sum \vec{r}_{i/cm} \times \vec{v}_{i/cm} m_i \\
&\quad + \sum \vec{r}_{cm} \times \vec{v}_{i/cm} m_i + \sum \vec{r}_{i/cm} \times \vec{v}_{cm} m_i \\
&= \vec{r}_{cm} \times \vec{v}_{cm} m_{tot} + \sum \vec{r}_{i/cm} \times \vec{v}_{i/cm}\, m_i \\
&\quad + \vec{r}_{cm} \times \underbrace{\left[\sum \vec{v}_{i/cm}\, m_i\right]}_{\vec{0}} + \underbrace{\left[\sum \vec{r}_{i/cm}\, m_i\right]}_{\vec{0}} \times \vec{v}_{cm}
\end{aligned}$$

So,

$$\vec{H}_C = \underbrace{\vec{r}_{cm} \times \vec{v}_{cm} m_{tot}}_{\text{contribution of center of mass motion}} + \underbrace{\sum \vec{r}_{i/cm} \times \vec{v}_{i/cm}\, m_i}_{\text{contribution of motion relative to center of mass}} .$$

The reason $\sum \vec{r}_{i/cm}\, m_i = \vec{0}$ is somewhat intuitive. It is what you would calculate if you were looking for the center of mass relative to the center of mass. More formally,

$$\begin{aligned}
\sum \vec{r}_{i/cm}\, m_i &= \sum (\vec{r}_i - \vec{r}_{cm})\, m_i \\
&= \underbrace{\sum \vec{r}_i\, m_i}_{m_{tot}\vec{r}_{cm}} - m_{tot}\vec{r}_{cm} \\
&= \vec{0} .
\end{aligned}$$

Similarly, $\sum \vec{v}_{i/cm}\, m_i = \vec{0}$ because it is what you would calculate if you were looking for the velocity of the center of mass relative to the center of mass.

The central result of this box is that

> angular momentum of any system is that due to motion *of* the center of mass *plus* motion *relative to* the center of mass.

to the center of mass. A special point for any system is, as we have mentioned, the center of mass. In the above equations for angular momentum we could take C to be a fixed point in space that happens to coincide with the center of mass. In this case we would most naturally define $\vec{\boldsymbol{H}}_{\text{cm}} = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}\, dm$ with $\vec{\boldsymbol{v}}$ being the absolute velocity. But we have the following theorem:

$$\vec{\boldsymbol{H}}_{\text{cm}} = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}\, dm = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}_{/cm}\, dm$$

where $\vec{\boldsymbol{r}}_{/cm} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{cm}$ and $\vec{\boldsymbol{v}}_{/cm} = \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_{cm}$. Similarly,

$$\dot{\vec{\boldsymbol{H}}}_{\text{cm}} = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{a}}\, dm = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{a}}_{/cm}\, dm.$$

with $\vec{\boldsymbol{a}}_{/cm} = \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_{cm}$. That is,

> *the angular momentum and rate of change of angular momentum relative to the center of mass, defined in terms of the velocity and acceleration* **relative** *to the center of mass, are the same as the angular momentum and the rate of change of angular momentum defined in terms of a fixed point in space that coincides with the center of mass.*

The angular momentum relative to the center of mass $\vec{\boldsymbol{H}}_{\text{cm}}$ can be calculated with all positions and velocities calculated relative to the center of mass. Similarly, the rate of change of angular momentum relative to the center of mass $\dot{\vec{\boldsymbol{H}}}_{\text{cm}}$ can be calculated with all positions and *accelerations* calculated relative to the center of mass.

Combining the results above we get the often used result:

$$\sum \vec{\boldsymbol{M}}_{i/cm} = \dot{\vec{\boldsymbol{H}}}_{\text{cm}} \tag{5.104}$$

This formula is the version of angular momentum balance that many people think of as being basic. In this equation, $\dot{\vec{\boldsymbol{H}}}_{\text{cm}}$ can be found using either the absolute acceleration

5.10 Relation between $\frac{d}{dt}\vec{\boldsymbol{H}}_{\text{C}}$ and $\vec{\boldsymbol{H}}_{\text{C}}$

The expression for $\dot{\vec{\boldsymbol{H}}}_{\text{C}}$ follows from that for $\vec{\boldsymbol{H}}_{\text{C}}$ but requires a few steps of algebra to show. Like the rate of change of linear momentum, $\dot{\vec{\boldsymbol{L}}}$, the derivative of $\vec{\boldsymbol{L}}$, the derivative of angular momentum must be taken with respect to a Newtonian frame in order to be useful in momentum balance equations. Note that since we assumed that C is a point fixed in a Newtonian frame that $\frac{d}{dt}\vec{\boldsymbol{r}}_{i/\text{C}} = \vec{\boldsymbol{v}}_{i/\text{C}} = \vec{\boldsymbol{v}}_i$. Starting with the definition of $\dot{\vec{\boldsymbol{H}}}_{\text{C}}$, we can calculate as follows:

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{\text{C}} &= \frac{d}{dt}\vec{\boldsymbol{H}}_{\text{C}} \\
&= \frac{d}{dt}\sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{v}}_i) \\
&= \sum \frac{d}{dt}\vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{v}}_i) + \vec{\boldsymbol{r}}_{i/C} \times (m_i \frac{d}{dt}\vec{\boldsymbol{v}}_i) \\
&= \sum \overbrace{\underbrace{\vec{\boldsymbol{v}}_i}_{\frac{d}{dt}\vec{\boldsymbol{r}}_{i/C}} \times (m_i \vec{\boldsymbol{v}}_i)}^{\vec{\boldsymbol{0}}} + \vec{\boldsymbol{r}}_{i/C} \times (m_i \frac{d}{dt}\vec{\boldsymbol{v}}_i)
\end{aligned}$$

$$\dot{\vec{\boldsymbol{H}}}_{\text{C}} = \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{a}}_i),$$

We have used the fact that the product rule of differentiation works for cross products between vector-valued functions of time. This final formula, $\dot{\vec{\boldsymbol{H}}}_{\text{C}} = \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{a}}_i)$, or its integral form, $\dot{\vec{\boldsymbol{H}}}_{\text{C}} = \int \vec{\boldsymbol{r}}_{i/C} \times \vec{\boldsymbol{a}}_i\, dm$ are always applicable. They can be simplified in many special cases which we will discuss in this chapter and those that follow.

$\vec{a}$ or the acceleration relative to the center of mass, $\vec{a}_{/cm}$. The same $\dot{\vec{H}}_{\text{cm}}$ is found both ways. In this book, we do not give equation 5.104 quite such central status as equations III where the reference point can be any point C not just the center of mass.

Kinetic energy E_{K}

The equation of mechanical energy balance (III) is:

$$P = \dot{E}_{\text{K}} + \dot{E}_{\text{P}} + \dot{E}_{int}.$$

For discrete systems, the kinetic energy is calculated as

$$\frac{1}{2}\sum m_i\, v_i^2$$

and its rate of change as

$$\frac{d}{dt}\left[\frac{1}{2}\sum m_i\, v_i^2\right].$$

There is also a general result about the kinetic energy that takes advantage of the center of mass. The kinetic energy for any system in any motion can be decomposed into the sum of two terms. One is associated with the motion of the center of mass of the system and the other is associated with motion relative to the center of mass. Namely,

$$E_{\text{K}} = \underbrace{\frac{1}{2}m_{\text{tot}}v_{cm}^2}_{\text{kinetic energy due to center of mass motion}} + \underbrace{\frac{1}{2}\sum m_i v_{i/cm}^2}_{\text{kinetic energy relative to the center of mass}},$$

$$= \frac{1}{2}m_{tot}v_{cm}^2 + E_{\text{K}/cm}$$

5.11 Using $\vec{H}_{\text{O}}$ and $\dot{\vec{H}}_{\text{O}}$ to find $\vec{H}_{\text{C}}$ and $\dot{\vec{H}}_{\text{C}}$

You can find the angular momentum $\vec{H}_{\text{C}}$ relative to a fixed point C if you know the angular momentum $\vec{H}_{\text{O}}$ relative to some other fixed point O and also know the linear momentum of the system $\vec{L}$ (which does not depend on the reference point). The result is:

$$\vec{H}_{\text{C}} = \vec{H}_{\text{O}} + \vec{r}_{O/C} \times \vec{L}.$$

The formula is similar to the formula for the effective moment of a system of forces that you learned in statics: $\vec{M}_{\text{C}} = \vec{M}_{\text{O}} + \vec{r}_{O/C} \times \vec{F}_{tot}$. Similarly, for the rate of change of angular momentum we have:

$$\dot{\vec{H}}_{\text{C}} = \dot{\vec{H}}_{\text{O}} + \vec{r}_{O/C} \times \dot{\vec{L}}$$

So once you have found $\dot{\vec{L}}$ and also $\dot{\vec{H}}_{\text{O}}$ with respect to some point O you can easily calculate the right hand sides of the momentum balance equations using any point C that you like.

where

$$\begin{aligned} E_{\mathrm{K}/cm} &= \frac{1}{2}\sum m_i v_{i/cm}^2 && \text{for discrete systems, and} \\ &= \frac{1}{2}\int (v_{/cm})^2\, dm && \text{for continuous systems.} \end{aligned}$$

The results above can be verified by direct expansion of the basic definitions of E_{K} and the center of mass. To repeat,

> *the kinetic energy of a system is the same as the kinetic energy of a particle with the system's mass at the center of mass plus kinetic energy due to motion relative to the center of mass.*

In this chapter, all particles in the system are assumed to have the same velocity so that they all have the same velocity as the center of mass. Thus, $\vec{v}_{i/cm} = \vec{\mathbf{0}}$ for all particles, and for straight line motion,

$$E_{\mathrm{K}} = \frac{1}{2} m_{\mathrm{tot}} v_{cm}^2.$$

Summary on general results about $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{\mathrm{C}}$, $\dot{\vec{H}}_{\mathrm{C}}$, E_{K}, and center of mass

$$\begin{aligned} m_{\mathrm{tot}}\vec{r}_{cm} &= \sum \vec{r}_i\, m_i && \text{for all systems} \\ m_{\mathrm{tot}}\vec{v}_{cm} &= \sum \vec{v}_i\, m_i && \text{for all systems} \\ m_{\mathrm{tot}}\vec{a}_{cm} &= \sum \vec{a}_i\, m_i && \text{for all systems} \end{aligned}$$

$$\begin{aligned} \vec{L} &= \sum m_i\ \vec{v}_i = m_{tot}\vec{v}_{cm} && \text{for all systems} \\ \dot{\vec{L}} &= \sum m_i\ \vec{a}_i = m_{tot}\vec{a}_{cm} && \text{for all systems} \end{aligned}$$

$$\begin{aligned} \vec{H}_{\mathrm{C}} &= \sum \vec{r}_{i/C} \times (m_i\ \vec{v}_i) && \text{for all systems} \\ &= \vec{r}_{cm/C} \times m_{\mathrm{tot}}\vec{v}_{cm} + \sum \vec{r}_{i/cm} \times (\vec{v}_{i/cm} m_i) && \text{for all systems} \\ &= \vec{H}_{\mathrm{O}} + \vec{r}_{O/C} \times \vec{L} && \text{for all systems} \\ \dot{\vec{H}}_{\mathrm{C}} &= \sum \vec{r}_{i/C} \times (m_i\ \vec{a}_i) && \text{for all systems} \\ &= \vec{r}_{cm/C} \times m_{\mathrm{tot}}\vec{a}_{cm} + \sum \vec{r}_{i/cm} \times (\vec{a}_{i/cm} m_i) && \text{for all systems} \\ &= \dot{\vec{H}}_{\mathrm{O}} + \vec{r}_{O/C} \times \dot{\vec{L}} && \text{for all systems} \\ E_{\mathrm{K}} &= \frac{1}{2}\sum m_i\ v_i^2 && \text{for all systems} \\ &= \frac{1}{2} m_{tot}\ v_{cm}^2 + \frac{1}{2}\sum m_i\ v_{i/cm}^2 && \text{for all systems} \\ \dot{E}_{\mathrm{K}} &= \sum m_i\ v_i\ \dot{v}_i && \text{for all systems} \\ &= m_{tot}\ v_{cm}\ \dot{v}_{cm} + \sum m_i\ v_{i/cm}\ \dot{v}_{i/cm} && \text{for all systems} \end{aligned}$$

5.12 *THEORY*

Deriving system momentum balance from the particle equations.

In the front cover you see that we have linear and angular momentum balance equations that apply to arbitrary systems. Another approach to mechanics is to use the equation

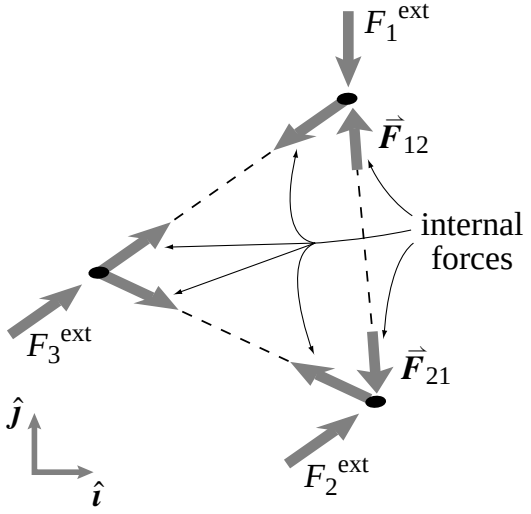
$$\vec{F} = m\vec{a}$$

for every particle in the system and then *derive* the system linear and angular momentum balance equations. This derivation depends on the following assumptions

(a) All bodies and systems are composed of point masses.

(b) These point masses interact in a pair-wise manner. For every pair of point masses A and B the interaction force is equal and opposite and along the line connecting the point masses.

We then look at any system, which we now assume is a system of point masses, and apply $\vec{F} = m\vec{a}$ to every point mass and add the equations for all point masses in the system. For each point mass we can break the total force into two parts: 1) the interaction forces between the point mass and other point masses in the system, these forces are 'internal' forces ($\vec{F}^{int}$), and 2) the forces acting on the system from the outside, the 'external' forces. The situation is shown for a three particle system below.



System linear momentum balance

Now lets take the equation $\sum \vec{F} = m\vec{a}$ for each particle and add over all the particles.

$$\underbrace{\sum_{\text{all particles}} \left[\sum_{\text{each particle}} \vec{F} \right]}_{\text{The sum of all forces on the system, internal and external}} = \sum_{\text{all particles}} m_i \vec{a}_i$$

Since all the internal forces come in cancelling pairs we can rewrite this equation as:

$$\underbrace{\sum_{\text{all external forces}} \vec{F}^{ext}}_{\text{Only the external forces, the ones acting on the system from the outside.}} = \sum_{\text{all particles}} m_i \vec{a}_i$$

That is, we have derived equation I in the front cover from $\vec{F} = m\vec{a}$ for a point mass by assuming the system is composed of point masses with pair-wise equal and opposite forces.

System angular momentum balance

For any particle we can take the equation

$$\sum_{\text{forces on particle i}} \vec{F} = m_i \vec{a}_i$$

and take the cross product of both sides with the position of the particle relative to some point C:

$$\vec{r}_{i/C} \times \left[\sum_{\text{forces on particle i}} \vec{F} \right] = \vec{r}_{i/C} \times \left[m_i \vec{a}_i \right].$$

Now we can add this equation up over all the particles to get

$$\underbrace{\sum_{particles} \left\{ \vec{r}_{i/C} \times \left[\sum_{on\ particle\ i} \vec{F} \right] \right\}}_{r/C \times \vec{F} \text{ added up for all forces on the system, internal and external}} = \sum_{particles} \left\{ \vec{r}_{i/C} \times \left[m_i \vec{a}_i \right] \right\}.$$

But, by our pair-wise assumption, for every internal force there is an equal and opposite force with the same line of action. So all the internal forces drop out of this sum and we have:

$$\underbrace{\sum_{\text{all external forces}} \vec{r}_{i/C} \times \vec{F}_i^{ext}}_{\text{Only the external forces, the ones acting on the system from the outside.}} = \sum_{\text{all particles}} \vec{r}_{i/C} \times m_i \vec{a}_i.$$

This equation is equation II, the system angular momentum balance equation (assuming we do not allow the application of any pure moments).

The derivations above are classic and are found in essentially all mechanics books. However, some people feel it is fine to take the system linear momentum balance and angular momentum balance equations as postulates and not make the subject of mechanics depend on the unrealistic view of so-simply interacting point masses.

5.13 A preview of rigid body simplifications and advanced kinematics

We have formulas for the motion quantities $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{\rm C}$, and $\dot{\vec{H}}_{\rm C}$ and $E_{\rm K}$ in terms of the positions, velocities, and accelerations of all of the mass bits in a system. Most often in this book we deal with the mechanics of *rigid bodies*, objects with negligible deformation. This assumed simplification means that the relative motions of the 10^{23} or so atoms in a body are highly restricted. In fact, if one knows these five vectors:

- $\vec{r}_{\rm cm}$, the position of the center of mass,
- $\vec{v}_{\rm cm}$, the velocity of the center of mass,
- $\vec{a}_{\rm cm}$, the acceleration of the center of mass
- $\vec{\omega}$, the *angular velocity* of the body, and
- $\vec{\alpha}$, the *angular acceleration* of the body,

then one can find the position, velocity, and acceleration of every point on the body in terms of its position relative to the center of mass, $\vec{r}_{/\rm cm} = \vec{r} - \vec{r}_{\rm cm}$.

We will save the derivations for later since we have not yet discussed the concepts of angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$.

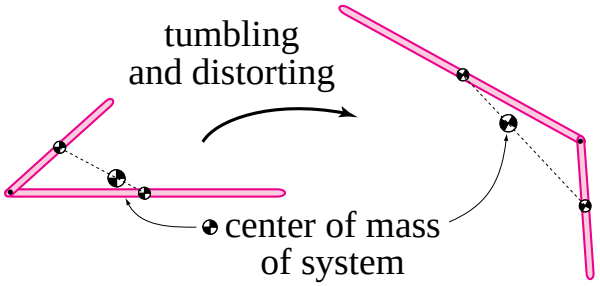
We will also use a new quantity $[\boldsymbol{I}^{\rm cm}]$, the *moment of inertia matrix*. For 2-D problems, $[\boldsymbol{I}^{\rm cm}]$ is just a number. For 3-D problems, $[\boldsymbol{I}^{\rm cm}]$ is a matrix; hence, the square brackets [ ], our notation for a matrix.

As intimidating as these new concepts may appear now, they lead to a *vast* simplification over the alternative — summing over 10^{23} particles or so.

Note that the formulas for linear momentum $\vec{L}$ and rate of change of linear momentum $\dot{\vec{L}}$ do not really look any simpler for a rigid body than the general case.

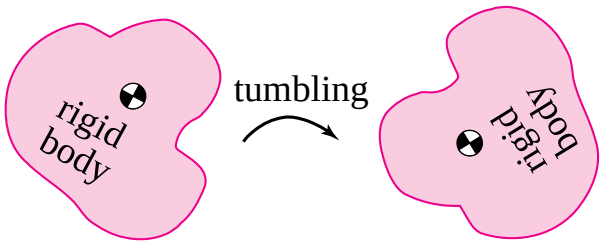
$$\begin{aligned} \vec{L} &= m_{\rm tot}\vec{v}_{\rm cm} \\ \dot{\vec{L}} &= m_{\rm tot}\vec{a}_{\rm cm} \end{aligned}$$

But, they are actually simpler in the following sense. For a general system, when we write $\vec{v}_{\rm cm}$, we are talking about an abstract point that moves in a different way than any point on the system. For example, consider the linked arms below, tumbling in space.

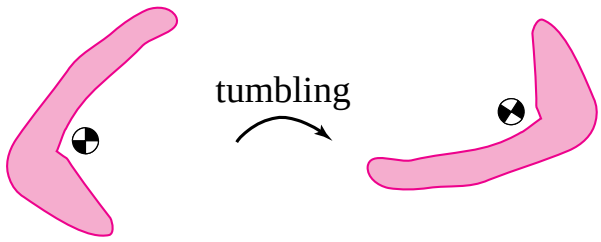


The center of mass is not even on any point in the system and, although it represents the average position in the system, it does not move with any point on the system.

On the other hand, for a *rigid body*, the center of mass is fixed relative to the body as the body moves,



even if the center of mass is not on the body, such as for this 'L-shaped' object.



In this case, the center of mass is not literally *on* the body. It is fixed with respect to the body, however. If you were rigidly attached to the body and fixed your gaze on the location of the center of mass, it would not waver in your view as the body, with you attached, tumbled wildly. In this sense the center of mass is fixed "on" a rigid body even if not on the body at all.

SAMPLE 5.39 *Location of the center of mass.* A structure is made up of three point masses, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$. At the moment of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\text{m/s}\hat{\boldsymbol{\imath}}$, $2\,\text{m/s}(\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}})$ and $1\,\text{m/s}\hat{\boldsymbol{\jmath}}$, respectively.

(a) Find the coordinates of the center of mass of the structure.

(b) Find the velocity of the center of mass.

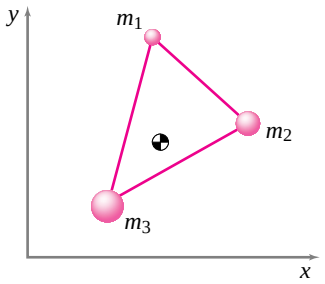


Figure 5.86: (Filename:sfig2.4.2)

Solution

(a) Let $(\bar{x}, \bar{y})$ be the coordinates of the mass-center. Then from the definition of mass-center

$$\begin{aligned}\bar{x} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 1.25\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.75\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{7.25\,\cancel{\text{kg}}\cdot\,\text{m}}{6\,\cancel{\text{kg}}} = 1.25\,\text{m}.\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{y} &= \frac{\sum m_i y_i}{\sum m_i} \\ &= \frac{1\,\text{kg}\cdot 3\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.5\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{8.55\,\cancel{\text{kg}}\cdot\,\text{m}}{6\,\cancel{\text{kg}}} = 1.42\,\text{m}.\end{aligned}$$

Thus the center of mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\text{m},\ 1.42\,\text{m})}$$

(b) For a system of particles, the linear momentum

$$\begin{aligned}\vec{\boldsymbol{L}} &= \sum m_i\vec{\boldsymbol{v}}_i = m_{\text{tot}}\vec{\boldsymbol{v}}_{cm} \\ \Rightarrow\quad \vec{\boldsymbol{v}}_{cm} &= \frac{\sum m_i\vec{\boldsymbol{v}}_i}{m_{\text{tot}}} \\ &= \frac{1\,\text{kg}\cdot(2\,\text{m/s}\hat{\boldsymbol{\imath}}) + 2\,\text{kg}\cdot(2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}})\,\text{m/s} + 3\,\text{kg}\cdot(1\,\text{m/s}\hat{\boldsymbol{\jmath}})}{6\,\text{kg}} \\ &= \frac{(6\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}})\,\cancel{\text{kg}}\cdot\,\text{m/s}}{6\,\cancel{\text{kg}}} \\ &= 1\,\text{m/s}\hat{\boldsymbol{\imath}} + 0.5\,\text{m/s}\hat{\boldsymbol{\jmath}}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_{cm} = 1\,\text{m/s}\hat{\boldsymbol{\imath}} + 0.5\,\text{m/s}\hat{\boldsymbol{\jmath}}}$$

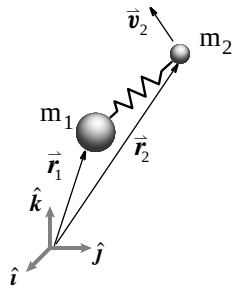


Figure 5.87: (Filename:sfig5.10.hopper)

SAMPLE 5.40 *A spring-mass system in space.* A spring-mass system consists of two masses, $m_1 = 10\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a weak spring with stiffness $k = 1\,\text{N/m}$. The spring has zero relaxed length. The system is in 3-D space where there is no gravity. At the moment f observaion, *i.e.*, at $t = 0$, $\vec{r}_1 = \vec{0}$, $\vec{r}_2 = 1\,\text{m}(\hat{\imath} + \hat{\jmath} + \hat{k})$, $\dot{\vec{r}}_1 = \vec{0}$, and $\dot{\vec{r}}_2 = \sqrt{6}\,\text{m/s}(-\hat{\imath} + \hat{\jmath})$. Track the motion of the system for the next 20 seconds. In particular,

(a) Plot the trajectory of the two masses in space.
(b) Plot the trajectory of the center of mass of the system.
(c) Plot the trajectory of the two masses as seen by an observer sitting at the center of mass.
(d) Compute and plot the total energy of the system and show that it remains constant during the entire motion.

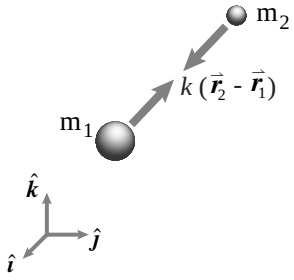


Figure 5.88: Free body diagram of the two masses.

(Filename:sfig5.10.hopper.a)

Solution The free body diagrams of the two masses are shown in Fig. 5.88. The only force acting on each mass is the force due to the spring which is directed along the line joining the two masses. Thus, the system represents a central force problem. From the linear momentum balance of the two masses, we can write the equations of motion as follows.

$$\begin{aligned} m_1\ddot{\vec{r}}_1 &= k(\vec{r}_2 - \vec{r}_1) \\ m_2\ddot{\vec{r}}_2 &= -k(\vec{r}_2 - \vec{r}_1) \end{aligned}$$

Let $\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$. Substituting above and dotting the two equations with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, we get

$$\begin{aligned} \ddot{x}_1 &= \frac{k}{m_1}(x_2 - x_1); & \ddot{x}_2 &= -\frac{k}{m_2}(x_2 - x_1) \\ \ddot{y}_1 &= \frac{k}{m_1}(y_2 - y_1); & \ddot{y}_2 &= -\frac{k}{m_2}(y_2 - y_1) \\ \ddot{z}_1 &= \frac{k}{m_1}(z_2 - z_1); & \ddot{z}_2 &= -\frac{k}{m_2}(z_2 - z_1) \end{aligned}$$

Thus we get six second order coupled linear ODEs as equations of motion.

(a) To plot the trajectory of the two masses, we need to solve for $\vec{r}_1(t)$ and $\vec{r}_2(t)$, *i.e.*, for $x_1(t), y_1(t), z_1(t)$, and $x_2(t), y_2(t), z_2(t)$. We can do this by first writing the six second order equations as a set of 12 first order equations and then solving them using a numerical ODE solver. Here is a pseudocode to accomplish this task.

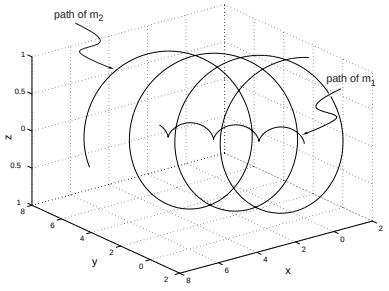


Figure 5.89: 3-D trajectory of m_1 and m_2 plotted from numerical solution of the equations of motion.

(Filename:sfig5.10.hopper.b)

```
ODEs = {x1dot = u1,
        u1dot = k/m1*(x2-x1),
        y1dot = v1,
        v1dot = k/m1*(y2-y1),
        z1dot = w1,
        w1dot = k/m1*(z2-z1),
        x2dot = u2,
        u2dot = -k/m2*(x2-x1),
        y2dot = v2,
        v2dot = -k/m2*(y2-y1),
        z2dot = w2,
        w2dot = -k/m2*(z2-z1) }

IC   = {x1(0)=0, y1(0)=0, z1(0)=0,
```

```
        u1(0)=0, v1(0)=0, w1(0)=0,
        x2(0)=1, y2(0)=1, z2(0)=1,
        u2(0)=-sqrt(6), v2(0)=sqrt(6), w2(0)=0}
Set  k=1, m1=10, m2=1
Solve ODEs with IC for t=0 to t=20
Plot {x1,y1,z1} and {x2,y2,z2}
```

The 3-D plot showing the trajectory of the two masses obtained from the numerical solution is shown in Fig. 5.89. From the plot, it seems like the smaller mass goes around the bigger mass as the bigger mass moves on its trajectory.

(b) We can find the trajectory of the center of mass using the following relationships.

$$x_{\rm cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \quad y_{\rm cm} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}, \quad z_{\rm cm} = \frac{m_1z_1 + m_2z_2}{m_1 + m_2}.$$

Since there is no external force on the system if we consider the two masses and the spring together, the center of mass of the system has zero acceleration. Therefore, we expect the center of mass to move on a straight path with constant velocity. The center of mass coordinates $x_{\rm cm}$, $y_{\rm cm}$, and $z_{\rm cm}$ are plotted against time in Fig. 5.90 which show that the center of mass moves on a straight line in a plane parallel to the xy-plane (z is constant). This is expected since the initial velocity of the center of has no z-component:

$$\begin{aligned}\vec{\boldsymbol{v}}_{\rm cm} &= \frac{m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2}{m_1 + m_2}\\ &= \frac{m_1\cdot\vec{\boldsymbol{0}} + 1\,\text{kg}\cdot\sqrt{(}6)\,\text{m/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}{10\,\text{kg} + 1\,\text{kg}}\\ &= 0.22\,\text{m/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).\end{aligned}$$

(c) The trajectory of the two masses with respect to the center of mass can be easily obtained by the following relationships.

$$\begin{aligned}x_{1/cm} &= x_1 - x_{\rm cm}, \quad & y_{1/cm} &= y_1 - y_{\rm cm}, \quad & z_{1/cm} &= z_1 - z_{\rm cm}\\ x_{2/cm} &= x_2 - x_{\rm cm}, \quad & y_{2/cm} &= y_2 - y_{\rm cm}, \quad & z_{2/cm} &= z_2 - z_{\rm cm}\end{aligned}$$

The trajectories thus obtained are shown in Fig. 5.90. It is clear that the two masses have closed orbits with respect to the center of mass. These closed orbits are actually conic sections as we would expect in a central force problem.

(d) We can calculate the kinetic energy of the two masses and the potential energy of the spring at each instant during the motion and add them up to find the total energy.

$$\begin{aligned}(E_k)_{m_1} &= \frac{1}{2}m_1(u_1^2 + v_1^2 + w_1^2)\\ (E_k)_{m_2} &= \frac{1}{2}m_2(u_2^2 + v_2^2 + w_2^2)\\ E_p &= \frac{1}{2}k[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]\\ E_{\rm total} &= (E_k)_{m_1} + (E_k)_{m_2} + E_p\end{aligned}$$

The energies so calculated are plotted in Fig. 5.91. It is clear from the plot that the total energy remains constant during the entire motion.

□

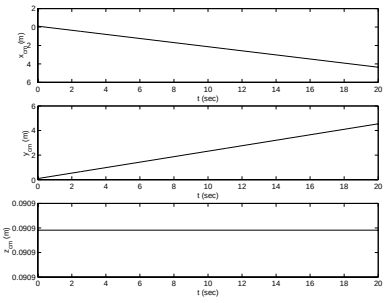


Figure 5.90: The center of mass coordinates $x_{\rm cm}(t)$, $y_{\rm cm}(t)$, and $z_{\rm cm}(t)$. The center of mass moves on a straight line in a plane parallel to the xy-plane.

(Filename:sfig5.10.hopper.c)

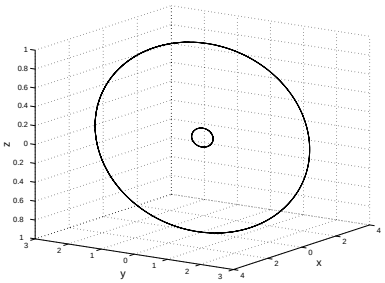


Figure 5.91: The paths of m_1 and m_2 as seen from the center of mass. The two masses are on closed orbits with respect to the center of mass.

(Filename:sfig5.10.hopper.d)

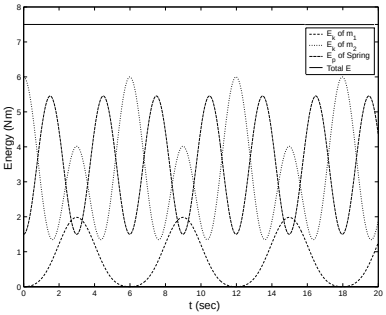


Figure 5.92: The kinetic energy of the two masses and the potential energy of the spring sum up to the constant total energy of the system.

(Filename:sfig5.10.hopper.e)

6 Constrained straight line motion

In the previous chapter you learned that it is basically straightforward to write the equations of motion for a particle, or for a collection of a few particles, if you had a model for the forces on the particles in terms of their positions and velocities and time. After writing $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each particle, finding unknown forces and accelerations is then a matter of solving linear algebraic equations, and finding the motions is a matter of solving the resulting differential equations.

These methods run into difficulty when positions and velocities are not sufficient for calculating the interaction forces. This situation arrises when we model connections between particles or bodies as inextensible or undeformable. When the connections are undeformable the motions are coupled in a way that is described geometrically rather than with forces. Such geometric coupling is called *kinematic constraint*.

The utility of free body diagrams, the principle of action and reaction, the linear and angular momentum balance equations, and the balance of energy apply to all systems, no matter how they move. But, it is easiest to learn how to use the principles if we start with systems that move in simple ways. In this short chapter, we will discuss the mechanics of 'bodies' where every point in the body moves in a straight line and where every point has the same velocity and acceleration as every other point.



Figure 6.1: A train running on straight level track is in straight-line motion, neglecting, of course, the wheel rotation, the bouncing, the moving engine parts, and the wandering eyes of the passengers.

(Filename:tfigure3.0.train)

Example: **Train on Straight Level Tracks**

Consider a train on straight level tracks. If we focus on the body of the train, we can approximate the motion as straight line motion. All parts move the same amount in the same fixed direction. □

In the next section we maintain a particle view and just consider the forces and motions in one direction.

6.1 1-D constrained motion and pulleys

The kinematic constraints we consider here are those imposed by connection with bars or ropes. Consider a car towing another with a strong light chain. We may not want to consider the elasticity of the chain but instead idealize the chain as an inextensible connection. This idealization of zero deformation is a simplification. But it is a simplification that requires special treatment. It is the simplest example of a kinematic constraint.

Figure 6.2 shows a schematic of one car pulling another. The force F is the force transmitted from the road to the front car through the tires. The tension T is the tension in the connecting chain. One-dimensional free body diagrams are also shown. From linear momentum balance for each of the objects (modeled as particles):

$$T = m_1\ddot{x}_1 \quad \text{and} \quad F - T = m_2\ddot{x}_2. \tag{6.1}$$

But these equations are the same as we would have if the cars were connected by a spring, a dashpot, or any idealized-as-massless connector. We need to somehow indicate using equations that the two particles are not allowed to move independently. We need something to replace the constitutive law that we would have used for a spring or dashpot.

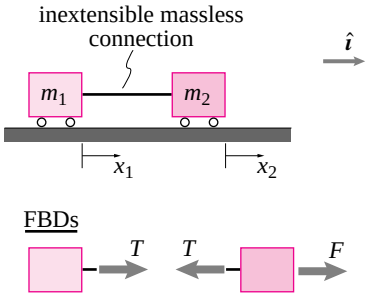


Figure 6.2: A schematic of one car pulling another, or of a boat pulling a barge.

(Filename:tfigure.boatpullsbarge)

Kinematic constraint

The geometric (or kinematic) restriction that two masses must move in lock-step is as

$$x_1 = x_2 + \text{Constant}.$$

We can differentiate the kinematic constraint twice to get

$$\ddot{x}_1 = \ddot{x}_2. \tag{6.2}$$

If we take F and the two masses as given, equations 6.1 and 6.2 are three equations for the unknowns $\ddot{x}_1$, $\ddot{x}_2$, and T. In matrix form,we have:

$$\begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.$$

We can solve these equations to find $\ddot{x}_1$, $\ddot{x}_2$, and T in terms of F.

If all we are interested in is the acceleration of the cars it would be nice to avoid even having to think about the constraint force. One way to avoid dealing with the constraint force is to draw a free body diagram of the entire system as in figure 6.3. If we just call the acceleration of the system $\ddot{x}$ we have, from linear momentum balance, that

$$F = (m_1 + m_2)\ddot{x},$$

which is one equation in one unknown.

This simplest possible example shows two ways of dealing with kinematic constraints:

(a) Use separate free body diagrams and equations of motion for each particle and then add extra kinematic constraint equations, or

(b) do something clever to try to avoid having to find constraint forces.

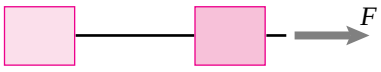


Figure 6.3: A free body diagram of the whole system. Note that the unknown tension (constraint) force does not show.

(Filename:tfigure.twocarstogether)

A generalization of this constraint is the rigid-body constraint where not just two, but many particles are assumed to keep constant distance from one another and in two or three dimensions. Another important constraint is an ideal hinge connection between two objects. Much of the theory of mechanics after Newton has been motivated by a desire to deal easily with these and other kinematic constraints. In fact, one way of characterizing the primary difficulty of dynamics is as the difficulty of dealing with kinematic constraints.

Pulleys

Pulleys are used to redirect force to amplify or attenuate force and to amplify or attenuate motion. Like a lever, a pulley system is an example of a mechanical transmission. Problems with pulleys are solved by using two facts about idealized string. First, ideal string is inextensible so the sum of the string lengths, over the different inter-pulley sections, adds to a constant.

$$\ell_1 + \ell_2 + \ell_3 + \ell_4 + ... = \text{constant}$$

Second, for round pulleys of negligible mass and no bearing friction, tension is constant along the length of the string. The tension on one side of a pulley is the same as the tension on the other side. And this can carry on if a rope is wrapped around several pulleys.

$$T_1 = T_2 = T_3 \ldots$$

See figure 4.4 on page 112 and the related text which shows why $T_1 = T_2$ for one pulley idealized as frictionless and massless.

We use the trivial pulley example in figure 6.4 to show how to analyze the relative motion of various points in a pulley system.

Example: **Length of string calculation**

Starting from point A, we add up the lengths of string

$$\ell_{tot} = x_A + \pi r + x_B \equiv constant. \tag{6.3}$$

The portion of string wrapped around the pulley contacts half of the pulley so that it's length is half the pulley circumference, πr. Even if x_A and x_B change in time and different portions of string wrap around the pulley, the length of string touching the pulley is always πr.

We can now formally deduce the intuitively obvious relations between the velocities and accelerations of points A and B. Differentiating equation 6.3 with respect to time once and then again, we get

$$\begin{aligned} \dot{\ell}_{tot} = 0 &= \dot{x}_A + 0 + \dot{x}_B \\ \Rightarrow \dot{x}_A &= -\dot{x}_B \\ \Rightarrow \ddot{x}_A &= -\ddot{x}_B \end{aligned} \tag{6.4}$$

When point A is displaced to the right by an amount Δx_A, the point B is displaced exactly the same amount but to the left; that is, $\Delta x_A = -\Delta x_B$. Note that in order to derive the kinematic relations 6.4 for the pulley system, we never need to know the total length of the string, only that it is constant in time. The constant quantities, the pulley half-circumference and the string length, get 'killed' in the process of differentiation. □

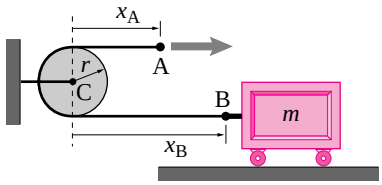


Figure 6.4: One mass, one pulley, and one string

(Filename:tfigure3.pulleyex)

Commonly we think of pulleys as small and thus never account for the pulley-contacting string length. Luckily this approximation generally leads to no error because we most often are interested in displacements, velocities, and accelerations. In these cases the pulley contact length drops out of the equations anyway.

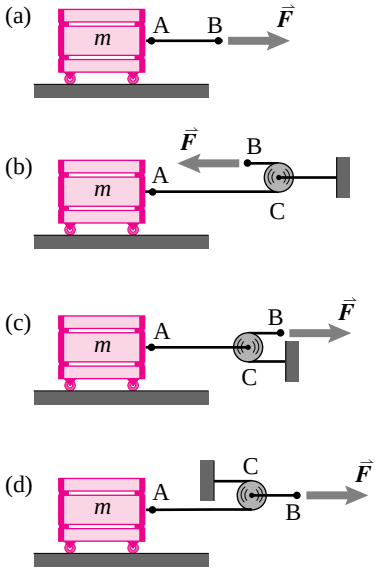


Figure 6.5: The four classic cases: (a) no pulley, (b) a pulley system with no mechanical advantage, (c) a pulley system that attenuates force and amplifies motion, and (d) a pulley system that multiplies force and attenuates motion.

(Filename:tfigure.pulley1)

The classic simple uses of pulleys

First imagine trying to move a load with no pulley as in Fig. 6.5a. The force you apply goes right to the mass. This is like direct drive with no transmission.

Now you would like to use pulleys to help you move the mass. In the cases we consider here the mass is on a frictionless support and we are trying to accelerate it. But the concepts are the same if there are also resisting forces on the mass. What can we do with one pulley? Three possibilities are shown in Fig. 6.5b-d which might, at a blinking glance, look roughly the same. But they are quite different. Here we discuss each design qualitatively. The details of the calculations are a homework problem.

In Fig. 6.5b we pull one direction and the force accelerates the other way. This illustrates one use of a pulley, to redirect an applied force. The force on the mass has magnitude $|\vec{F}|$ and there is no mechanical advantage.

Fig. 6.5c shows the most classic use of a pulley. A free body diagram of the pulley at C will show you that the tension in rope AC is $2|\vec{F}|$ and we have thus doubled the force acting on the mass. However, counting string length and displacement you will see that point A moves only half the distance that point B moves. Thus the force at B is multiplied by two to give the force at A and the displacement at B is divided by two to give the displacement at A. This is a general property of ideal transmissions, from levers to pulleys to gear boxes:

> If force is amplified then motion is equally attenuated.

This result is most solidly understood using energy balance. The power of the force at B goes entirely into the mass. On the other hand if we cut the string AB, the same amount of power must be applied to the mass (it gains the same energy). Thus the product of the tension and velocity at A must equal the product of the tension and velocity at B.

$$T_A v_A = T_B v_B$$

Fig. 6.5d shows the opposite use of a pulley. A free body diagram of the pulley shows that the tension in AB is $\frac{1}{2}|\vec{F}|$. Thus the force is attenuated by a factor of 2. A kinematic analysis reveals that the motion of A is twice that of B. Thus, as expected from energy considerations, the motion is amplified when the force is attenuated.

Effective mass

Of concern for design of machines that people work with is the feel of the machine. One aspect of feel is the effective mass. The *effective mass* is defined by the response of a point when a force is applied.

$$m_{\text{eff}} = \frac{|\vec{F}_{\text{B}}|}{|\vec{a}_{\text{B}}|}.$$

For the case of Fig. 6.5a and Fig. 6.5b the effective mass of point B is the mass of the block, m. For the case of Fig. 6.5c the block has twice the twice $|\vec{F}|$ acting on it and point B has twice the acceleration of point A, so the effective mass of point B is $m/4$. For the case of Fig. 6.5d, the mass only has half $|\vec{F}|$ acting on it and point B only has half the acceleration of point A, so the effective mass is $4m$.

These special cases exemplify the general rule:

> The effective mass of one end of a transmission is the mass of the other end multiplied by the square of the motion amplification ratio.

In terms of the effective mass, the systems Fig. 6.5c and Fig. 6.5d which might look so similar, actually differ by a factor of $2^2 \cdot 2^2 = 16$.

SAMPLE 6.1 *Find the motion of two cars.* One car is towing another of equal mass on level ground. The thrust of the wheels of the first car is F. The second car rolls frictionlessly. Find the acceleration of the system two ways:

(a) using separate free body diagrams,
(b) using a system free body diagram.

Solution

(a) From linear momentum balance of the two cars, we get

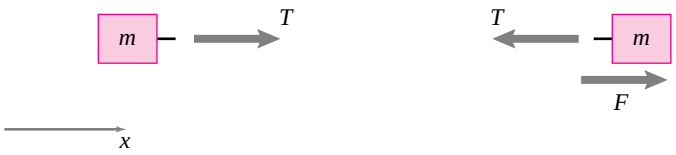


Figure 6.6: (Filename:sfig4.1.twocars.fbda)

$$m\ddot{x}_1 = T \tag{6.5}$$
$$F - T = m\ddot{x}_2 \tag{6.6}$$

The kinematic constraint of towing (the cars move together, *i.e.*, no relative displacement between the cars) gives

$$\ddot{x}_1 - \ddot{x}_2 = 0 \tag{6.7}$$

Solving eqns. (6.5), (6.6), and (6.7) simultaneously, we get

$$\ddot{x}_1 = \ddot{x}_2 = \frac{F}{2m} \quad (T = \frac{F}{2})$$

(b) From linear momentum balance of the two cars as one system, we get



Figure 6.7: (Filename:sfig4.1.twocars.fbdb)

$$m\ddot{x} + m\ddot{x} = F$$
$$\ddot{x} = F/2m$$

$$\boxed{\ddot{x} = \ddot{x}_1 = \ddot{x}_2 = F/2m}$$

SAMPLE 6.2 *Pulley kinematics.* For the masses and ideal-massless pulleys shown in figure 6.8, find the acceleration of mass A in terms of the acceleration of mass B. Pulley C is fixed to the ceiling and pulley D is free to move vertically. All strings are inextensible.

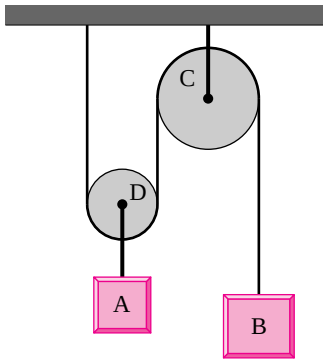


Figure 6.8: (Filename:sfig3.3.DH1)

Solution Let us measure the position of the two masses from a fixed point, say the center of pulley C. (Since C is fixed, its center is fixed too.) Let y_A and y_B be the vertical distances of masses A and B, respectively, from the chosen reference (C). Then the position vectors of A and B are:

$$\vec{r}_A = y_A\hat{j} \quad \text{and} \quad \vec{r}_B = y_B\hat{j}.$$

Therefore, the velocities and accelerations of the two masses are

$$\begin{aligned} \vec{v}_A &= \dot{y}_A\hat{j}, \qquad \vec{v}_B = \dot{y}_B\hat{j}, \\ \vec{a}_A &= \ddot{y}_A\hat{j}, \qquad \vec{a}_B = \ddot{y}_B\hat{j}. \end{aligned}$$

Since all quantities are in the same direction ($\hat{j}$), we can drop $\hat{j}$ from our calculations and just do scalar calculations. We are asked to relate $\ddot{y}_A$ to $\ddot{y}_B$.

In all pulley problems, the trick in doing kinematic calculations is to relate the variable positions to the fixed length of the string. Here, the length of the string ℓ_{tot} is: ①

$$\begin{aligned} \ell_{tot} &= \text{ab} + \text{bc} + \text{cd} + \text{de} + \text{ef} = \text{constant} \\ \text{where} \quad \text{ab} &= \underbrace{\text{aa}'}_{\text{constant}} + \underbrace{\text{a}'\text{b}}_{(=\text{cd}=y_D)} \\ \text{bc} &= \text{string over the pulley D} = \text{constant} \\ \text{de} &= \text{string over the pulley C} = \text{constant} \\ \text{ef} &= y_B \\ \text{thus} \quad \ell_{tot} &= 2y_D + y_B + \overbrace{\text{constant}}^{(\text{aa}'+\text{bc}+\text{de})}. \end{aligned}$$

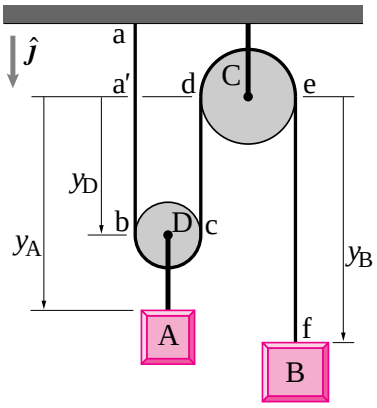


Figure 6.9: (Filename:sfig3.3.DH2)

① We have done an elaborate calculation of ℓ_{tot} here. Usually, the constant lengths over the pulleys and some constant segments such as aa′ are ignored in calculating ℓ_{tot}. These constant length segments can be ignored because they drop out of the equation when we take time derivatives to relate velocities and accelerations of different points, such as B and D here.

Taking the time derivative on both sides, we get

$\overbrace{\frac{d}{dt}(\ell_{tot})}$ (0 because ℓ_{tot} does not change with time) $= 2\dot{y}_D + \dot{y}_B \quad \Rightarrow \quad \dot{y}_D = -\frac{1}{2}\dot{y}_B$ (6.8)

$$\Rightarrow \quad \ddot{y}_D = -\frac{1}{2}\ddot{y}_B. \qquad (6.9)$$

But $y_D = y_A - \text{AD}$ and AD = constant

$$\Rightarrow \quad \dot{y}_D = \dot{y}_A \quad \text{and} \quad \ddot{y}_D = \ddot{y}_A.$$

Thus, substituting $\dot{y}_A$ and $\ddot{y}_A$ for $\dot{y}_D$ and $\ddot{y}_D$ in (6.8) and (6.9) we get

$$\dot{y}_A = -\frac{1}{2}\dot{y}_B \quad \text{and} \quad \ddot{y}_A = -\frac{1}{2}\ddot{y}_B$$

$$\boxed{\ddot{y}_A = -\tfrac{1}{2}\ddot{y}_B}$$

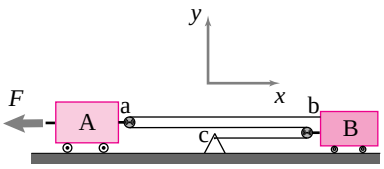


Figure 6.10: A two-mass pulley system.

(Filename:sfig3.3.1)

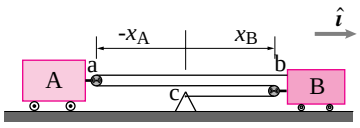


Figure 6.11: Pulley kinematics. Note that the distance from c to a is minus the x coordinate of a.

(Filename:sfig3.3.1b)

SAMPLE 6.3 *A two-mass pulley system.* The two masses shown in Fig. 6.10 have frictionless bases and round frictionless pulleys. The inextensible cord connecting them is always taut. Given that $F = 130\,\text{N}$, $m_A = m_B = m = 40\,\text{kg}$, find the acceleration of the two blocks using:

(a) linear momentum balance (LMB) and
(b) energy balance.

Solution

(a) **Using Linear Momentum Balance:** The free body diagrams of the two masses A and B are shown in Fig. 6.12 above. Linear momentum balance for mass A gives (assuming $\vec{a}_A = a_A\hat{\imath}$ and $\vec{a}_B = a_B\hat{\imath}$):

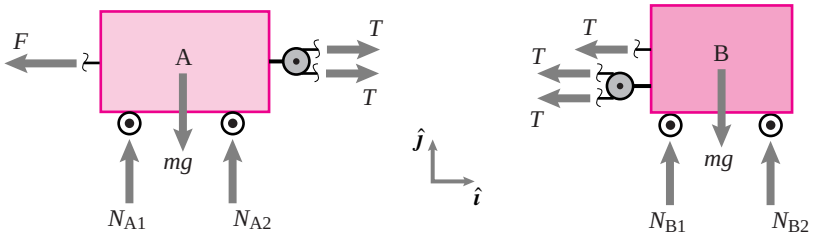


Figure 6.12: (Filename:sfig3.3.1a)

$$\begin{aligned} (2T - F)\hat{\imath} + (N_{A1} + N_{A2} - mg)\hat{\jmath} &= m\vec{a}_A = -ma_A\hat{\imath} \\ \text{(dotting with } \hat{\jmath}) \quad \Rightarrow \qquad 2N_A &= mg \\ \text{and} \qquad 2T - F &= ma_A \end{aligned} \tag{6.10}$$

Similarly, linear momentum balance for mass B gives:

$$\begin{aligned} -3T\hat{\imath} + (N_{B1} + N_{B2} - mg)\hat{\jmath} &= m\vec{a}_B = ma_B\hat{\imath} \\ \Rightarrow \qquad 2N_B &= mg \\ \text{and} \qquad -3T &= ma_B. \end{aligned} \tag{6.11}$$

From (6.10) and (6.11) we have three unknowns: T, a_A, a_B, but only 2 equations!. We need an extra equation to solve for the three unknowns.①
We can get the extra equation from kinematics. Since A and B are connected by a string of fixed length, their accelerations must be related. For simplicity, and since these terms drop out anyway, we neglect the radius of the pulleys and the lengths of the little connecting cords. Using the fixed point C as the origin of our xy coordinate system we can write

$$\begin{aligned} \ell_{tot} &\equiv \text{length of the string connecting A and B} \\ &= 3x_B + 2(-x_A) \\ \Rightarrow \qquad \overbrace{\dot{\ell}_{tot}}^{0} &= 3\dot{x}_B + 2(-\dot{x}_A) \\ \Rightarrow \qquad \dot{x}_B &= -\frac{2}{3}(-\dot{x}_A) \quad \Rightarrow \quad \ddot{x}_B = -\frac{2}{3}(-\ddot{x}_A) \end{aligned} \tag{6.12}$$

Since

$$\vec{v}_A = v_A\hat{\imath} = -(-\dot{x}_A)\hat{\imath},$$

① You may be tempted to use angular momentum balance (AMB) to get an extra equation. In this case AMB could help determine the vertical reactions, but offers no help in finding the rope tension or the accelerations.

$$\begin{aligned}\vec{\boldsymbol{a}}_A &= a_A\hat{\boldsymbol{\imath}} = \ddot{x}_A\hat{\boldsymbol{\imath}},\\ \vec{\boldsymbol{v}}_B &= v_B\hat{\boldsymbol{\imath}} = \dot{x}_B\hat{\boldsymbol{\imath}}, \text{ and}\\ \vec{\boldsymbol{a}}_B &= a_B\hat{\boldsymbol{\imath}} = \ddot{x}_B\hat{\boldsymbol{\imath}},\end{aligned}$$

we get

$$a_B = \frac{2}{3}a_A. \tag{6.13}$$

Substituting (6.13) into (6.11), we get

$$9T = -2m_B a_A. \tag{6.14}$$

Now solving (6.10) and (6.14) for T, we get

$$T = \frac{2F}{13} = \frac{2\cdot 130\,\text{N}}{13} = 20\,\text{N}.$$

Therefore,

$$\begin{aligned}a_A &= -\frac{9T}{2m} = -\frac{9\cdot 20\,\text{N}}{2\cdot 40\,\text{kg}} = -2.25\,\text{m/s}^2\\ a_B &= \frac{2}{3}a_A = -1.5\,\text{m/s}^2\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_A = -2.25\,\text{m/s}^2\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{a}}_B = -1.5\,\text{m/s}^2\hat{\boldsymbol{\imath}}.}$$

(b) **Using Power Balance (III):** We have,

$$P = \dot{E_{\text{K}}}.$$

The power balance equation becomes

$$\sum \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}} = m\,a_A\,v_A + m_B\,a_B\,v_B.$$

Because the force at A is the only force that does work on the system, when we apply power balance to the whole system (using $\vec{\boldsymbol{F}} = -F\hat{\boldsymbol{\imath}}$), we get

$$\begin{aligned}-Fv_A &= m_A v_A a_A + m v_B a_B\\ \text{or}\qquad F &= -ma_A - m\frac{v_B}{v_A}a_B\\ &= -a_A(m + m\frac{v_B}{v_A}\frac{a_B}{a_A}).\end{aligned}$$

Substituting $a_B = 2/3a_A$ and $v_B = 2/3v_A$ from Eqn. (6.13),

$$a_A = \frac{-F}{m + \frac{4}{9}m} = \frac{-130\,\text{N}}{40\,\text{kg}(1+\frac{4}{9})} = -2.25\,\text{m/s}^2,$$

and since $a_B = 2/3a_A$,

$$a_B = -1.5\,\text{m/s}^2,$$

which are the same accelerations as found before.

$$\boxed{a_A = -2.25\,\text{m/s}^2\hat{\boldsymbol{\imath}},\ a_B = -1.5\,\text{m/s}^2\hat{\boldsymbol{\imath}}}$$

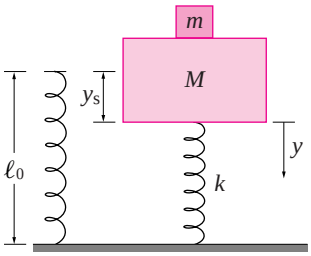


Figure 6.13: (Filename:sfig10.1.5)

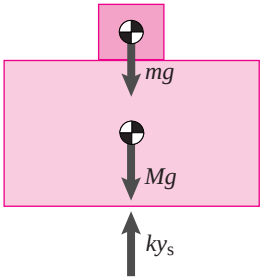


Figure 6.14: Free body diagram of the two masses as one system when in static equilibrium (this special case could be skipped as it follows from the free body diagram below).

(Filename:sfig10.1.5a)

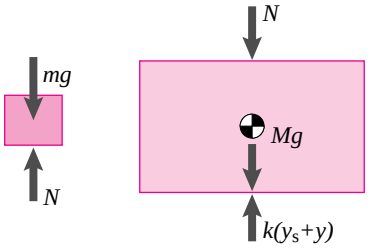


Figure 6.15: Free body diagrams of the individual masses.

(Filename:sfig10.1.5b)

SAMPLE 6.4 In static equilibrium the spring in Fig. 6.13 is compressed by y_s from its unstretched length ℓ_0. Now, the spring is compressed by an additional amount y_0 and released with no initial velocity.

(a) Find the force on the top mass m exerted by the lower mass M.
(b) When does this force become minimum? Can this force become zero?
(c) Can the force on m due to M ever be negative?

Solution

(a) The free body diagram of the two masses is shown in Figure 6.14 when the system is in static equilibrium. From linear momentum balance we have

$$\sum \vec{F} = \vec{0} \qquad \Rightarrow \qquad ky_s = (m+M)g. \tag{6.15}$$

The free body diagrams of the two masses at an arbitrary position y during motion are given in Figure 6.15. Since the two masses oscillate together, they have the same acceleration. From linear momentum balance for mass m we get

$$mg - N = m\ddot{y}. \tag{6.16}$$

We are interested in finding the normal force N. Clearly, we need to find $\ddot{y}$ to calculate N. Now, from linear momentum balance for mass M we get

$$Mg + N - k(y + y_s) = M\ddot{y}. \tag{6.17}$$

Adding Eqn (6.16) with Eqn (6.17) we get

$$(m+M)g - ky - ky_s = (m+M)\ddot{y}.$$

But $ky_s = (m+M)g$ from Eqn 6.15. Therefore, the equation of motion of the system is

$$-ky = (m+M)\ddot{y}$$
$$\text{or} \quad \ddot{y} + \frac{k}{(m+M)}y = 0. \tag{6.18}$$

As you recall from your study of the harmonic oscillator, the general solution of this differential equation is

$$y(t) = A\sin\lambda t + B\cos\lambda t \tag{6.19}$$

where $\lambda = \sqrt{\frac{k}{m+M}}$ and the constants A and B are to be determined from the initial conditions. From Eqn (6.19) we obtain

$$\dot{y}(t) = A\lambda\cos\lambda t - B\lambda\sin\lambda t. \tag{6.20}$$

Substituting the given initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$ in Eqns (6.19) and (6.20), respectively, we get

$$\begin{aligned} y(0) &= y_0 = B \\ \dot{y}(0) &= 0 = A\lambda \qquad \Rightarrow \qquad A = 0. \end{aligned}$$

Thus,

$$y(t) = y_0\cos\lambda t. \tag{6.21}$$

Now we can find the acceleration by differentiating Eqn (6.21) twice :

$$\ddot{y} = -y_0\lambda^2\cos\lambda t.$$

Substituting this expression in Eqn (6.16) we get the force applied by mass M on the smaller mass m:

$$\begin{aligned} mg - N &= m\overbrace{(-y_0\lambda^2\cos\lambda t)}^{\ddot{y}} \\ \Rightarrow \quad N &= mg + my_0\lambda^2\cos\lambda t \\ &= m(g + y_0\lambda^2\cos\lambda t) \end{aligned} \tag{6.22}$$

$$\boxed{N = m(g + y_0\lambda^2\cos\lambda t)}$$

(b) Since $\cos\lambda t$ varies between ± 1, the value of the force N varies between $mg \pm y_0\lambda^2$. Clearly, N attains its minimum value when $\cos\lambda t = -1$, *i.e.*, when $\lambda t = \pi$. This condition is met when the spring is fully stretched and the mass is at its highest vertical position. At this point,

$$N \equiv N_{min} = m(g - y_0\lambda^2)$$

If y_0, the initial displacement from the static equilibrium position, is chosen such that $y_0 = \frac{g}{\lambda^2}$, then $N = 0$ when $\cos\lambda t = -1$, *i.e.*, at the topmost point in the vertical motion. This condition means that the two masses momentarily lose contact with each other when they are about to begin their downward motion. ◁

(c) From Eqn (6.22) we can get a negative value of N when $\cos\lambda t = -1$ and $y_0\lambda^2 > g$. However, a negative value for N is nonsense unless the blocks are glued. Without glue the bigger mass M cannot apply a negative compression on m, *i.e.*, it cannot "suck" m. When $y_0\lambda^2 > g$ then N becomes zero before $\cos\lambda t$ decreases to -1. That is, assuming no bonding, the two masses lose contact on their way to the highest vertical position but before reaching the highest point. Beyond that point, the equations of motion derived above are no longer valid for unglued blocks because the equations assume contact between m and M. Eqn (6.22) is inapplicable when $N \leq 0$. ◁

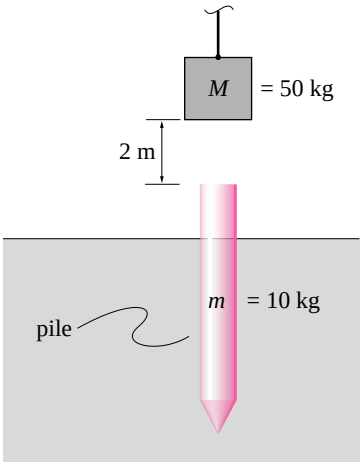


Figure 6.16: (Filename:sfig3.5.DH1)

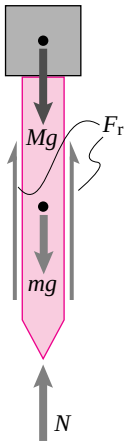


Figure 6.17: Free body diagram of the hammer and pile system. F_r is the total resistance of the ground.

(Filename:sfig3.5.DH2)

SAMPLE 6.5 *Driving a pile into the ground.* A cylindrical wooden pile of mass 10 kg and cross-sectional diameter 20 cm is driven into the ground with the blows of a hammer. The hammer is a block of steel with mass 50 kg which is dropped from a height of 2 m to deliver the blow. At the nth blow the pile is driven into the ground by 5 cm. Assuming the impact between the hammer and the pile to be totally inelastic (*i.e.*, the two stick together), find the average resistance of the soil to penetration of the pile.

Solution Let F_r be the average (constant over the period of driving the pile by 5 cm) resistance of the soil. From the free body diagram of the pile and hammer system, we have

$$\sum \vec{\boldsymbol{F}} = -mg\hat{\boldsymbol{j}} - Mg\hat{\boldsymbol{j}} + N\hat{\boldsymbol{j}} + F_r\hat{\boldsymbol{j}}.$$

But N is the normal reaction of the ground, which from static equilibrium, must be equal to $mg + Mg$. Thus,

$$\sum \vec{\boldsymbol{F}} = Fr\,\hat{\boldsymbol{j}}.$$

Therefore, from linear momentum balance ($\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$),

$$\vec{\boldsymbol{a}} = \frac{F_r}{M+m}\hat{\boldsymbol{j}}.$$

Now we need to find the acceleration from given conditions. Let v be the speed of the hammer just before impact and V be the combined speed of the hammer and the pile immediately after impact. Then, treating the hammer and the pile as one system, we can ignore all other forces *during* the impact (none of the external forces: gravity, soil resistance, ground reaction, is comparable to the impulsive impact force, see page 87). The impact force is internal to the system. Therefore, during impact, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$ which implies that linear momentum is conserved. Thus

$$\begin{aligned} -mv\hat{\boldsymbol{j}} &= -(m+M)V\hat{\boldsymbol{j}} \\ \Rightarrow \qquad V &= \left(\frac{m}{m+M}\right)v = \frac{10\,\text{kg}}{60\,\text{kg}}v = \frac{1}{6}v. \end{aligned}$$

The hammer speed v can be easily calculated, since it is the free fall speed from a height of 2 m:

$$v = \sqrt{2gh} = 2\sqrt{9.81\text{m}^2/\text{s}^2} \quad or \quad v = 6.26\,\text{m/s} \quad \Rightarrow \quad V = \frac{1}{6}v = 1.04\,\text{m/s}.$$

The pile and the hammer travel a distance of $s = 5\,\text{cm}$ under the deceleration a. The initial speed $V = 1.04\,\text{m/s}$ and the final speed = 0. Plugging these quantities into the one-dimensional kinematic formula

$$v^2 = v_0^2 + 2as,$$

we get,

$$\begin{aligned} 0 &= V^2 - 2as \quad \text{(Note that } a \text{ is negative)} \\ \Rightarrow \quad a &= \frac{V^2}{2s} = \frac{(1.04\,\text{m/s})^2}{2\times 0.05\,\text{m}} = 10.82\,\text{m/s}^2. \end{aligned}$$

Thus $\vec{\boldsymbol{a}} = 10.82\,\text{m/s}^2\hat{\boldsymbol{j}}$. Therefore,

$$F_r = (m+M)a = (60\,\text{kg})\cdot(10.82\,\text{m/s}^2) = 649.2\,\text{N}$$

$$\boxed{F_r \approx 650\,\text{N}}$$

6.2 2-D and 3-D forces even though the motion is straight

Even if all the motion is in a single direction, an engineer may still have to consider two- or three-dimensional forces.

Example: **Piston in a cylinder.**

Consider a piston sliding vertically in a cylinder. For now let us neglect the spatial extent of the cylinder. Let's assume a coefficient of friction μ between the piston and the cylinder wall. Let's assume that the connecting rod has negligible mass so it can be treated as a two-force member as discussed in section 4.1. That is, the force transmitted to the piston from the connecting rod is along the connecting rod. The free body diagram of the system is shown in figure 6.18. We have also assumed that the piston is moving up so the friction force is directed down, resisting the motion. We can set up the linear momentum balance equation as follows:

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ -N\hat{\imath} - \mu N\hat{\jmath} + T\hat{\boldsymbol{\lambda}}_{rod} &= m_{piston}\, a\hat{\jmath}. \end{aligned}$$

If we assume that the acceleration $a\hat{\jmath}$ of the piston is known, as is its mass m_{piston}, the coefficient of friction μ, and the orientation of the connecting rod $\hat{\boldsymbol{\lambda}}_{rod}$, then we can solve for the rod tension T and the normal reaction N. Note that even though the piston moves in one direction, the momentum balance equation is a two-dimensional vector equation. □

The kinematically simple motions we assume in this chapter simplify the evaluation of the right hand sides of the vector momentum balance equations. But they are still vector equations.

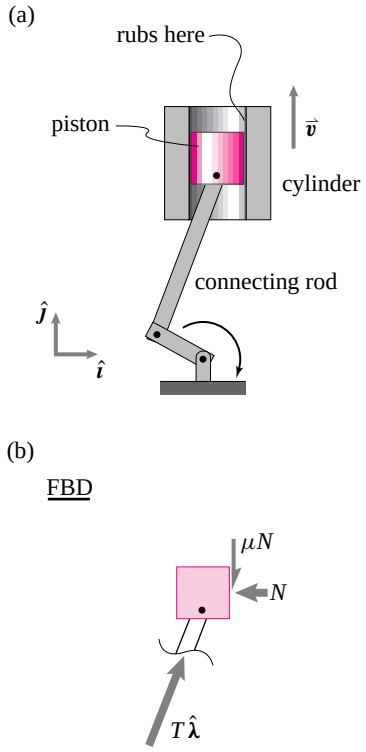


Figure 6.18: (a) shows a piston in a cylinder. (b) shows a free body diagram of the piston. To draw this FBD, we have assumed: (1) a coefficient of friction μ between the piston and cylinder wall, and (2) negligible mass for the connecting rod, and (3) ignored the spatial extent of the cylinder.

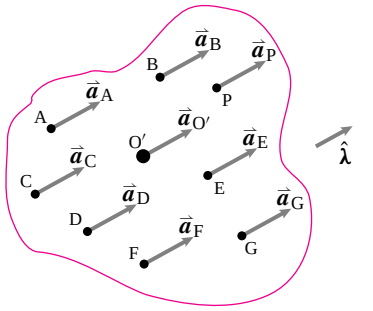
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Highly constrained bodies

This chapter is about rigid bodies that do not rotate. Most objects will not agree to be the topic of such discussion without being forced into doing so. In general, one expects bodies to rotate or move along a curved path. To keep an object that is subject to various forces from rotating or curving takes some constraint. The object needs to be rigid and held by wires, rods, rails, hinges, welds, etc. that keep it from spinning, keeping it in parallel motion. We do not mean to imply that the presence of constraint always is associated with disallowance of rotation — constraints could even cause rotation. But to keep a rigid object in straight-line motion usually does require some kind of constraint.

Of common interest for constrained structures is making sure that static and dynamic loads do not cause failure of the constraints. For example, suppose a truck hauls a very heavy load that is held down by chains tightened by come-alongs. When the truck accelerates, what is the tension in the chains, and will it exceed the strength limit of the chains so that they might break?

In this chapter, we assume all points of a system or body are moving in a straight line with the same velocity and acceleration. Let's consider a set of points in the system of interest. Let's call them A to G, or generically, P. Say a reference point on the system is O', just a point that we distinguish for convenience. O' may be the



$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}_D = \vec{a}_E = \vec{a}_F = \vec{a}_G = \vec{a}_{O'}$

Figure 6.19: Parallel motion: all points on the body have the same acceleration $\vec{a} = a\hat{\boldsymbol{\lambda}}$. For straight-line motion: $\hat{\boldsymbol{\lambda}}(t)$=constant in time and $\vec{v} = v\hat{\boldsymbol{\lambda}}$.

(Filename:tfigure3.1a)

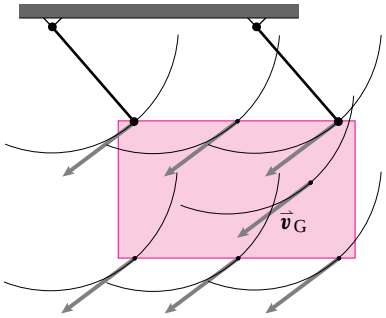


Figure 6.20: A swing showing instantaneous parallel motion which is *curvilinear*. At every instant, each point has the same velocity as the others, but the motion is not in a straight line.

(Filename:tfigure3.swing)

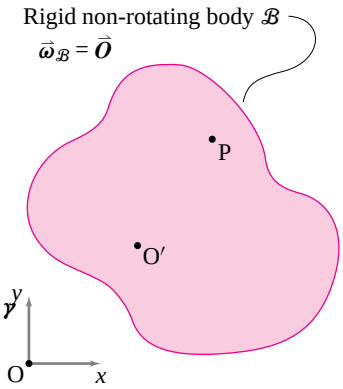


Figure 6.21: A non-rotating body $\mathcal{B}$ with points O' and P.

(Filename:tfigure3.2.1)

center of mass, the origin of a local coordinate system, or a fleck of dirt that serves as a marker. By *parallel motion*, we mean that the system happens to move in such a way that $\vec{a}_P = \vec{a}_{O'}$, and $\vec{v}_P = \vec{v}_{O'}$ (Fig. 6.19). That is,

$$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}_D = \vec{a}_E = \vec{a}_F = \vec{a}_G = \vec{a}_P = \vec{a}_{O'}$$

at every instant in time. We also assume that $\vec{v}_A = \ldots = \vec{v}_P = \vec{v}_{O'}$.

A special case of parallel motion is straight-line motion.

a system moves with straight-line motion if it moves like a non-rotating rigid body, in a straight line.

For straight-line motion, the velocity of the body is in a fixed unchanging direction. If we call a unit vector in that direction $\hat{\boldsymbol{\lambda}}$, then we have

$$\vec{v}(t) = v(t)\hat{\boldsymbol{\lambda}}, \qquad \vec{a}(t) = a(t)\hat{\boldsymbol{\lambda}} \qquad \text{and} \qquad \vec{r}(t) = \vec{r}_0 + s(t)\hat{\boldsymbol{\lambda}}$$

for every point in the system. $\vec{r}_0$ is the position of a point at time 0 and s is the distance the point moves in the $\hat{\boldsymbol{\lambda}}$ direction. Every point in the system has the same s, v, a, and $\hat{\boldsymbol{\lambda}}$ as the other points. There are a variety of problems of practical interest that can be idealized as fitting into this class, notably, the motions of things constrained to move on belts, roads, and rails, like the train in figure 6.1.

Example: **Parallel swing is not straight-line motion**

The swing shown does not rotate — all points on the swing have the same velocity. The motion of all particles are parallel but, since paths are curved, this motion is not straight-line motion. Such curvilinear parallel motion will be discussed in Chapter 7. □

A special way of analyzing straight-line motion is with one-dimensional mechanics as we did in the previous section. For one-dimensional mechanics, we assume that, in addition to the restricted kinematics, everything of interest mechanically happens in the $\hat{\boldsymbol{\lambda}}$ direction, often taken to be the x direction. That is, we ignore *all* torques and angular momenta, and only consider the $\hat{\boldsymbol{\lambda}}$ components of the forces (*i.e.*, $\vec{F}\cdot\hat{\boldsymbol{\lambda}}$) and linear momentum ($\vec{L}\cdot\hat{\boldsymbol{\lambda}}$). For example, if $\hat{\boldsymbol{\lambda}}$ is in the $\hat{\boldsymbol{\imath}}$ direction, the components would be F_x and L_x.

Before we proceed with discussion of the details of the mechanics of straight-line motion we present some ideas that are also more generally applicable. That is, the concept of the center of mass allows some useful simplifications of the general expressions for $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_C$, $\dot{\vec{H}}_C$ and E_K.

Velocity of a point

The velocity of any point P on a non-rotating rigid body (such as for straight-line motion) is the same as that of any reference point on the body (see Fig. 6.21).

$$\vec{v}_P = \vec{v}_{O'}$$

A more general case, which you will learn in later chapters, is shown as 5b in Table II at the back of the book. This formula concerns rotational rate which we will measure with the vector $\vec{\omega}$. For now all you need to know is that $\vec{\omega} = \vec{0}$ when something is not rotating. In 5b in Table II, if you set $\vec{\omega}_{\mathcal{B}} = 0$ and $\vec{v}_{P/\mathcal{B}} = \vec{0}$ it says that $\vec{v}_P = \dot{\vec{r}}_{O'/O}$ or in shorthand, $\vec{v}_P = \vec{v}_{O'}$, as we have written above.

Acceleration of a point

Similarly, the acceleration of every point on a non-rotating rigid body is the same as every other point. The more general case, not needed in this chapter, is shown as entry 5c in Table II at the back of the book.

Angular momentum and its rate of change, $\vec{H}_C$ and $\dot{\vec{H}}_C$ for straight-line motion

For the motions in this chapter, where $\vec{a}_i = \vec{a}_{cm}$ and thus $\vec{a}_{i/cm} = \vec{0}$, angular momentum considerations are simplified, as explained in Box 6.2 on 341. For a rigid body $\mathcal{B}$ in more general motion, the calculation of rate of change of angular momentum involves the angular velocity $\vec{\omega}_{\mathcal{B}}$, its rate of change $\dot{\vec{\omega}}_{\mathcal{B}}$, and the moment of inertia matrix $[\boldsymbol{I}^{\rm cm}]$. ①You will learn about these things in later chapters. If you look at Table I, entry 6d, you will see a general formula that reduces to the formula above if you assume no rotation and thus use $\vec{\omega} = \vec{0}$ and $\dot{\vec{\omega}} = \vec{0}$. But for straight-line motion (and for parallel motion), the calculations turn out to be the same as we would get if we put a single point mass at the center of mass:②

$$\begin{aligned}\vec{H}_{\rm C} &\equiv \sum(\vec{r}_{i/{\rm C}} \times m_i \vec{v}_i) &&= \vec{r}_{cm/C} \times (m_{\rm total}\vec{v}_{\rm cm}),\\ \dot{\vec{H}}_{\rm C} &\equiv \sum(\vec{r}_{i/{\rm C}} \times m_i \vec{a}_i) &&= \vec{r}_{cm/C} \times (m_{\rm total}\vec{a}_{cm}).\end{aligned}$$

① Calculating rate of change of angular momentum will get more difficult as the book progresses. But rate of change of linear momentum is simple, at least in concept.

$$\dot{\vec{L}} = m_{tot}\vec{a}_{cm}$$

in this chapter, as well as in the rest of this book!

②**Caution:** Unfortunately, the special motions in this chapter are almost the only cases where the angular momentum and its rate of change are so easy to calculate.

Approach

To study such systems, we:

- draw a free body diagram, showing the appropriate forces and couples at places where connections are 'cut',
- state reasonable kinematic assumptions based on the motions that the constraints allow,
- write linear and/or angular momentum balance equations and/or energy balance, and
- solve for quantities of interest.

Angular momentum balance about a judiciously chosen axis is a particularly useful tool for reducing the number of equations that need to be solved.

Example: **Plate on a cart**

A uniform rectangular plate $ABCD$ of mass m is supported by a light rigid rod DE and a hinge joint at point B. The dimensions are as shown. The cart has acceleration $a_x\hat{\imath}$ due to a force $F\hat{\imath}$ and the constraints of the wheels. Referring to the free body diagram in figure 6.22 and writing

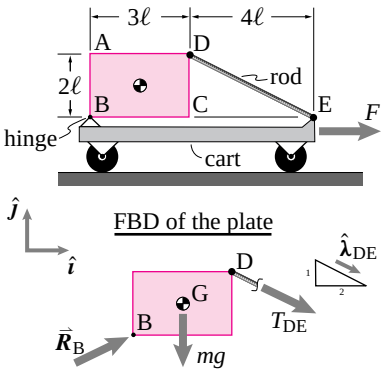


Figure 6.22: Uniform plate supported by a hinge and a rod on an accelerating cart.

(Filename:tfigure3.2D.guyed)

6.1 *THEORY*
Calculation of $\vec{H}_{\rm C}$ and $\dot{\vec{H}}_{\rm C}$ for straight-line motion

For straight-line motion, and parallel motion in general, we can derive the simplification in the calculation of $\vec{H}_{\rm C}$ as follows:

$$\begin{aligned}\vec{H}_{\rm C} &\equiv \sum \vec{r}_{i/{\rm C}} \times {\rm m}_i \vec{v}_i \,(\text{ definition})\\ &= \sum \vec{r}_{i/{\rm C}} \times {\rm m}_i \vec{v}_{\rm cm} \quad (\text{since, } \vec{v}_i = \vec{v}_{\rm cm})\\ &= \left(\sum \vec{r}_{i/{\rm C}}\,{\rm m}_i\right) \times \vec{v}_{\rm cm},\\ &= \vec{r}_{\rm cm/C} \times (m_{\rm tot}\vec{v}_{\rm cm}),\\ &\quad (\text{since, } \textstyle\sum \vec{r}_{i/{\rm C}}\,{\rm m}_i \equiv m_{\rm tot}\vec{r}_{\rm cm/C}).\end{aligned}$$

The derivation that $\dot{\vec{H}}_{\rm C} = \vec{r}_{\rm cm/C} \times (m\vec{a}_{\rm cm})$ follows exactly from $\dot{\vec{H}}_{\rm C} \equiv \sum \vec{r}_{i/{\rm C}} \times {\rm m}_i \vec{a}_i$ by exactly the same reasoning.

angular momentum balance for the plate about point B, we can get an equation for the tension in the rod T_{DE} in terms of m and a_x:

$$\begin{aligned} \sum \vec{M}_{/B} &= \dot{\vec{H}}_{/B} \\ \left\{ \vec{r}_{D/B} \times (T_{DE}\hat{\lambda}_{DE}) + \vec{r}_{G/B} \times (-mg\hat{j}) \right. &= \left. \vec{r}_{G/B} \times (ma_x\hat{i}) \right\} \\ \{\} \cdot \hat{k} \Rightarrow \qquad T_{DE} &= \frac{\sqrt{5}}{7} m(a_x - \frac{3}{2}g). \end{aligned}$$

□

Sliding and pseudo-sliding objects

A car coming to a stop can be roughly modeled as a rigid body that translates and does not rotate. That is, at least for a first approximation, the rotation of the car, as the suspension deforms, can be neglected. The free body diagram will show various forces with lines of action that do not all act through a single point so that angular momentum balance must be used to analyze the system (even though there is no rotation!). Similarly, a bicycle which is braking or a box that is skidding (if not tipping) may be analyzed by assuming straight-line motion.

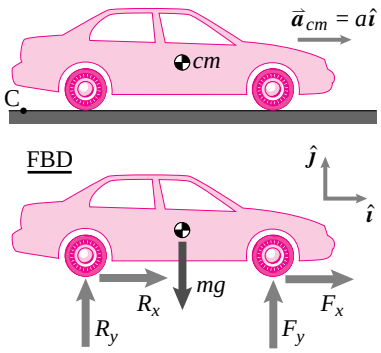


Figure 6.23: A four-wheel drive car accelerating but not tipping. See fig. 6.24 on page 342 for more about FBDs involving wheel contact.

(Filename:tfigure3.4wd.car)

Example: **Car skidding**

Consider the accelerating four-wheel drive car in figure 6.23. The motion quantities for the car are $\dot{\vec{L}} = m_{car}\vec{a}_{car}$ and $\dot{\vec{H}}_C = \vec{r}_{cm/C} \times \vec{a}_{car} m_{car}$. We could calculate angular momentum balance relative to the car's center of mass in which case $\sum \vec{M}_{\text{cm}} = \dot{\vec{H}}_{\text{cm}} = \vec{0}$ (because the position of the center of mass relative to the center of mass is $\vec{0}$). □

As mentioned, it is often useful to calculate angular momentum balance of sliding objects about points of contact (such as where tires contact the road) or about points that lie on lines of action of applied forces when writing angular momentum balance to solve for forces or accelerations. To do so usually eliminates some unknown reactions from the equations to be solved. For example, the angular momentum balance equation about the rear-wheel contact of a car does not contain the rear-wheel contact forces.

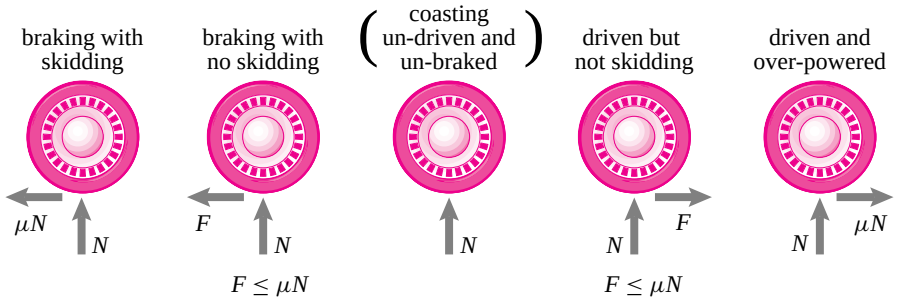


Figure 6.24: Partial free body diagrams of wheel in a braking or accelerating car that is pointed and moving to the right. The force of the ground on the tire is shown. But the forces of the axle, gravity, and brakes on the wheel are not shown. An ideal point-contact wheel is assumed.

(Filename:tfigure3.2.car.breaking)

Wheels

The function of wheels is to allow easy sliding-like motion between objects. On the other hand, wheels do sometimes slip due to:

- being overpowered (as in a screeching accelerating car),
- being braked hard, or
- having very bad bearings (like a rusty toy car).

How wheels are treated when analyzing cars, bikes, and the like depends on both the application and on the level of detail one requires. In *this chapter*, we will always assume that wheels have negligible mass. Thus, when we treat the special case of un-driven and un-braked wheels our free body diagrams will be as in figure 3.20 on page 92 and *not* like the one in figure **??** on page **??**. All of the various cases for a car traveling to the right are shown with partial free body diagrams of a wheel in figure 6.24. For the purposes of actually solving problems, we have accepted Coulomb's law of friction as a model for contacting interaction (see pages 88-90).

3-D forces in straight-line motion

The ideas we have discussed apply as well in three dimensions as in two. As you learned from doing statics problems, working out the details is more involved. In particular, vector methods must be used carefully. Also, as for statics, three dimensional problems often yield simple results and simple intuitions by considering angular momentum balance about an axis.

Angular momentum balance about an axis

The simplest way to think of angular momentum balance about an axis is to look at angular momentum balance about a point and then take a dot product with a unit vector along an axis:

$$\hat{\boldsymbol{\lambda}} \cdot \left\{ \sum \vec{\boldsymbol{M}}_{/\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \right\}.$$

Note that the axis need not correspond to any mechanical device in any way resembling and axle. The equation above applies for any point C and any vector $\hat{\boldsymbol{\lambda}}$. If you choose C and $\hat{\boldsymbol{\lambda}}$ judiciously many terms in your equations may drop out.

SAMPLE 6.6 *Force in braking.* A front-wheel-drive car of mass $m = 1200\,\text{kg}$ is cruising at $v = 60\,\text{mph}$ on a straight road when the driver slams on the brake. The car slows down to 20 mph in 4 s while maintaining its straight path. What is the average force (average in time) applied on the car during braking?

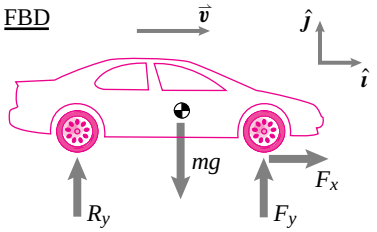


Figure 6.25: Free body diagram of a front-wheel-drive car during braking. Note that we have (arbitrarily) pointed F_x to the right. The algebra in this problem will tell us that $F_x < 0$.

(Filename:sfig2.6.3a)

Solution Let us assume that we have an xy coordinate system in which the car is traveling along the x-axis during the entire time under consideration. Then, the velocity of the car before braking $\vec{v}_1$ and after braking $\vec{v}_2$ are

$$\vec{v}_1 = v_1\hat{\imath} = 60\,\text{mph}\hat{\imath} \quad \text{and} \quad \vec{v}_2 = v_2\hat{\imath} = 20\,\text{mph}\hat{\imath}.$$

The linear impulse during braking is $\vec{F}_{ave}\Delta t$ where $\vec{F} \equiv F_x\hat{\imath}$ (see free body diagram of the car). Now, from the impulse-momentum relationship,

$$\vec{F}\Delta t = \vec{L}_2 - \vec{L}_1,$$

where $\vec{L}_1$ and $\vec{L}_2$ are linear momenta of the car before and after braking, respectively, and $\vec{F}$ is the average applied force. Therefore,

$$\begin{aligned}
\vec{F} &= \frac{1}{\Delta t}(\vec{L}_2 - \vec{L}_1) \\
&= \frac{m}{\Delta t}(\vec{v}_2 - \vec{v}_1) \\
&= \frac{1200\,\text{kg}}{4\,\text{s}}(20 - 60)\,\text{mph}\hat{\imath} \\
&= -12000\frac{\text{kg}}{\text{s}} \cdot \frac{\cancel{\text{mi}}}{\cancel{\text{hr}}} \cdot \frac{1600\ \text{m}}{1\ \cancel{\text{mi}}} \cdot \frac{1\ \cancel{\text{hr}}}{3600\,\text{s}}\hat{\imath} \\
&= -\frac{16,000}{3}\,\text{kg}\cdot\,\text{m/s}^2\hat{\imath} = -5.33\,\text{kN}\hat{\imath}.
\end{aligned}$$

Thus

$$\begin{aligned}
F_x\hat{\imath} &= -5.33\,\text{kN}\hat{\imath} \\
\Rightarrow \quad F_x &= -5.33\,\text{kN}.
\end{aligned}$$

$$\boxed{F_x = -5.33\,\text{kN}}$$

SAMPLE 6.7 *Sliding to a stop.* A block of mass $m = 2.5\,\text{kg}$ slides down a frictionless incline from a 5 m height. The block encounters a frictional bed AB of length 1 m on the ground. If the speed of the block is 9 m/s at point B, find the coefficient of friction between the block and the frictional surface AB.

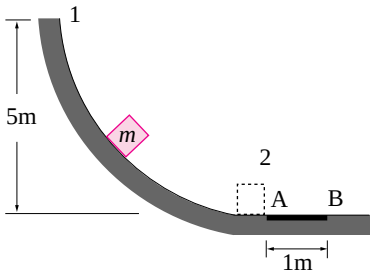


Figure 6.26: (Filename:sfig2.9.1a)

Solution We divide the problem in two parts: We first find the speed of the block as it reaches point A using conservation of energy for its motion on the inclined surface, and then use the work-energy principle to find the speed at B. Let the ground level be the datum for potential energy and let v be the speed at A. For the motion on the incline;

$$\begin{aligned}(E_{\text{K}})_1 + (E_{\text{P}})_1 &= (E_{\text{K}})_2 + (E_{\text{P}})_2\\ 0 + mgh &= \frac{1}{2}mv^2 + 0\\ \Rightarrow \quad v &= \sqrt{2gh}\\ &= \sqrt{2 \cdot 9.81\,\text{m/s}^2 \cdot 5\,\text{m}}\\ &= 9.90\,\text{m/s}.\end{aligned}$$

Now, as the block slides on the surface AB, a force of friction $= \mu\,\text{N} = \mu mg$ (since $\text{N} = mg$, from linear momentum balance in the vertical direction) opposes the motion (see Fig. 9.33). Work done by this force on the block is

$$\begin{aligned}W &= \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{r}}\\ &= -\mu mg\hat{\boldsymbol{\imath}} \cdot (1\,\text{m})\hat{\boldsymbol{\imath}}\\ &= -\mu mg(1\,\text{m}).\end{aligned}$$

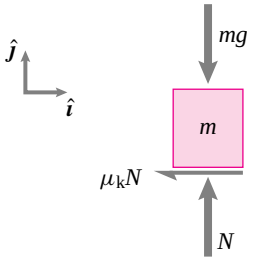


Figure 6.27: Free body diagram of the block on the frictional surface.
(Filename:sfig2.9.1b)

From the work-energy relationship (*e.g.*, see the inside cover) we have,

$$\begin{aligned}W &= \Delta E_{\text{K}} = (E_{\text{K}})_2 - (E_{\text{K}})_1\\ \Rightarrow \quad (E_{\text{K}})_2 &= (E_{\text{K}})_1 + W\\ \frac{1}{2}mv_B^2 &= \frac{1}{2}mv^2 - \mu mg(1\,\text{m})\\ -\mu mg(1\,\text{m}) &= \frac{1}{2}m(v_B^2 - v^2)\\ \Rightarrow \quad \mu &= \frac{1}{2g(1\,\text{m})}(v_B^2 - v^2)\\ &= \frac{(9.90\,\text{m/s})^2 - (9\,\text{m/s})^2}{2 \cdot 9.81\,\text{m/s}^2 \cdot 1\,\text{m}}\\ &= 0.87\end{aligned}$$

$$\boxed{\mu = 0.87}$$

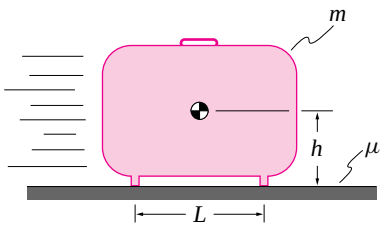


Figure 6.28: A suitcase in motion.

(Filename:sfig3.5.1)

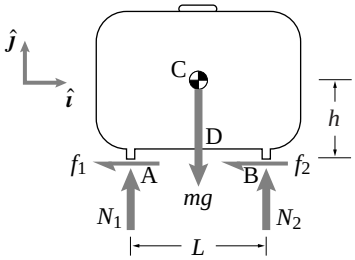


Figure 6.29: FBD of the suitcase.

(Filename:sfig3.5.1a)

SAMPLE 6.8 *A suitcase skidding on frictional ground.* A suitcase of mass m is pushed and sent sliding on a horizontal surface. The suitcase slides without any rotation. A and B are the only contact points of the suitcase with the ground. If the coefficient of friction between the suitcase and the ground is μ, find all the forces applied by the ground on the suitcase. Discuss the results obtained for normal forces.

Solution As usual, the first thing we do is draw a free body diagram of the suitcase. The FBD is shown in Fig. 6.29. Assuming Coulomb's law of friction holds, we can write

$$\vec{F}_1 = -\mu N_1 \hat{\imath} \quad \text{and} \quad \vec{F}_2 = -\mu N_2 \hat{\imath}. \tag{6.23}$$

Now we write the balance of linear momentum for the suitcase:

$$\sum \vec{F} = m\vec{a}$$

$$\Rightarrow \qquad -(F_1 + F_2)\hat{\imath} + (N_1 + N_2 - mg)\hat{\jmath} = ma\hat{\imath} \tag{6.24}$$

where $\vec{a} = a\hat{\imath}$ is the unknown acceleration. Dotting eqn. (6.24) with $\hat{\imath}$ and $\hat{\jmath}$ and substituting for F_1 and F_2 from eqn. (6.23) we get

$$-\mu(N_1 + N_2) = ma \tag{6.25}$$

$$N_1 + N_2 = mg \tag{6.26}$$

Equations (6.25) and (6.26) represent 2 scalar equations in three unknowns N_1, N_2 and a. Obviously, we need another equation to solve for these unknowns.

We can write the balance of angular momentum about any point. Points A or B are good choices because they each eliminate some reaction components. Let us write the balance of angular momentum about point A:

$$\sum \vec{M}_{\mathrm{A}} = \dot{\vec{H}}_{\mathrm{A}}$$

$$\begin{aligned} \sum \vec{M}_{\mathrm{A}} &= \vec{r}_{B/A} \times N_2\hat{\jmath} + \vec{r}_{D/A} \times (-mg)\hat{\jmath} \\ &= L\hat{\imath} \times N_2\hat{\jmath} + \frac{L}{2}\hat{\imath} \times (-mg)\hat{\jmath} \\ &= (LN_2 - mg\frac{L}{2})\hat{k} \end{aligned} \tag{6.27}$$

and

$$\dot{\vec{H}}_{\mathrm{A}} = \vec{r}_{cm/A} \times m\vec{a} \tag{6.28}$$

$$= (\frac{L}{2}\hat{\imath} + h\hat{\jmath}) \times ma\hat{\imath}$$

$$= -mah\hat{k} \tag{6.29}$$

Equating (6.27) and (6.29) and dotting both sides with $\hat{k}$ we get the following third scalar equation:

$$LN_2 - mg\frac{L}{2} = -mah. \tag{6.30}$$

Solving eqns. (6.25) and (6.26) for a we get

$$a = -\mu g$$

and substituting this value of a in eqn. (6.30) we get

$$\begin{aligned} N_2 &= \frac{m\mu gh + mgL/2}{L} \\ &= mg\left(\frac{1}{2} + \frac{h}{L}\mu\right). \end{aligned}$$

Substituting the value of N_2 in either of the equations (6.25) or (6.26) we get

$$N_1 = mg\left(\frac{1}{2} - \frac{h}{L}\mu\right).$$

$$\boxed{N_1 = mg(\tfrac{1}{2} - \tfrac{h}{L}\mu),\ N_2 = mg(\tfrac{1}{2} + \tfrac{h}{L}\mu),\ f_1 = \mu N_1,\ f_2 = \mu N_2.}$$

Discussion: From the expressions for N_1 and N_2 we see that

(a) $N_1 = N_2 = \frac{1}{2}mg$ if $\mu = 0$ because without friction there is no deceleration. The problem becomes equivalent to a statics problem.

(b) $N_1 = N_2 \approx \frac{1}{2}mg$ if $L >> h$. In this case, the moment produced by the friction forces is too small to cause a significant difference in the magnitudes of the normal forces. For example, take $L = 20h$ and calculate moment about the center of mass to convince yourself.

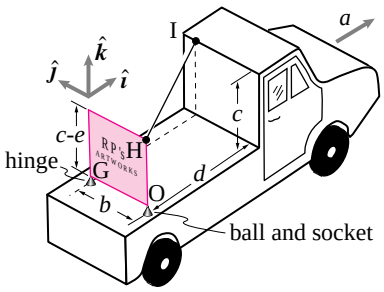


Figure 6.30: An accelerating board in 3-D

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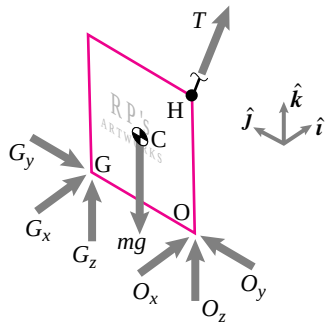


Figure 6.31: FBD of the board

(Filename:sfig3.5.2a)

SAMPLE 6.9 *Uniform acceleration of a board in 3-D.* A uniform sign-board of mass $m = 20\,\text{kg}$ sits in the back of an accelerating flatbed truck. The board is supported with ball-and-socket joint at O and a hinge at G. A light rod from H to I keeps the board from falling over. The truck is on level ground and has forward acceleration $\vec{a} = 0.6\,\text{m/s}^2\hat{\imath}$. The relevant dimensions are $b = 1.5\,\text{m}$, $c = 1.5\,\text{m}$, $d = 3\,\text{m}$, $e = 0.5\,\text{m}$. There is gravity ($g = 10\,\text{m/s}^2$).

(a) Draw a free body diagram of the board.
(b) Set up equations to solve for all the unknown forces shown on the FBD.
(c) Use the balance of angular momentum about an axis to find the tension in the rod.

Solution

(a) The free body diagram of the board is shown in Fig. 6.31.
(b) Linear momentum balance for the board:

$$\sum \vec{F} = m\vec{a}, \quad \text{or}$$

$$(G_x + O_x)\hat{\imath} + (G_y + O_y)\hat{\jmath} + (G_z + O_z - mg)\hat{k} + T\hat{\boldsymbol{\lambda}}_{HI} = ma\hat{\imath} \qquad (6.31)$$

where

$$\hat{\boldsymbol{\lambda}}_{HI} = \frac{d\hat{\imath} + b\hat{\jmath} + e\hat{k}}{\sqrt{d^2 + b^2 + e^2}} = \frac{d\hat{\imath} + b\hat{\jmath} + e\hat{k}}{\ell},$$

and where ℓ is the length of the rod HI.
Dotting eqn. (6.31) with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ we get the following three scalar equations:

$$G_x + O_x + T\frac{d}{\ell} = ma \qquad (6.32)$$

$$G_y + O_y + T\frac{b}{\ell} = 0 \qquad (6.33)$$

$$G_z + O_z + T\frac{e}{\ell} = mg \qquad (6.34)$$

Angular momentum balance about point G:

$$\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$$

$$\begin{aligned}
\sum \vec{M}_{/G} &= \vec{r}_{C/G} \times (-mg\hat{k}) + \vec{r}_{O/G} \times (O_x\hat{\imath} + O_z\hat{k}) + \vec{r}_{H/G} \times T\hat{\boldsymbol{\lambda}}_{HI} \\
&= (-\frac{b}{2}\hat{\jmath} + \frac{c-e}{2}\hat{k}) \times (-mg\hat{k}) - b\hat{\jmath} \times (O_x\hat{\imath} + O_z\hat{k}) \\
&\quad + [-b\hat{\jmath} + (c-e)\hat{k}] \times \frac{T}{\ell}(d\hat{\imath} + b\hat{\jmath} + e\hat{k}) \\
&= \left(\frac{b}{2}mg - bO_y - be\frac{T}{\ell} - (c-e)b\frac{T}{\ell}\right)\hat{\imath} \\
&\quad + (c-e)d\frac{T}{\ell}\hat{\jmath} + \left(bO_x + bd\frac{T}{\ell}\right)\hat{k} \qquad (6.35)
\end{aligned}$$

and

$$\begin{aligned}
\dot{\vec{H}}_{/G} &= \vec{r}_{C/G} \times ma\hat{\imath} \\
&= (-\frac{b}{2}\hat{\jmath} + \frac{c-e}{2}\hat{k}) \times ma\hat{\imath} \\
&= \frac{b}{2}ma\hat{k} + \frac{c-e}{2}ma\hat{\jmath}. \qquad (6.36)
\end{aligned}$$

Equating (6.35) and (6.36) and dotting both sides with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ we get the following three additional scalar equations:

$$G_y + O_y + \frac{c}{\ell}T = \frac{1}{2}mg \tag{6.37}$$

$$\frac{d}{\ell}T = \frac{1}{2}ma \tag{6.38}$$

$$O_x + \frac{d}{\ell}T = \frac{1}{2}ma \tag{6.39}$$

Now we have seven scalar equations for six unknowns — O_x, $G_y + O_y$, O_z, G_x, G_z, and T. Note, however, that G_y and O_y appear as the sum $G_y + O_y$. That is, they cannot be found independently. This mathematical problem corresponds to the physical circumstance that the supports at points O and G could be squeezing the plate along the line OG with, say, $O_y = 1000$ lbf and $G_y = -1000$ lbf. To make problems like this tractable, people often make assumptions like, 'Assume $G_y = 0$'. Fortunately, no one asked us to find O_y or G_y and we can find the tension in the wire HI without adding assumptions about the pre-stress in the structure.

(c) Balance of angular momentum about axis OG gives:

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG} \cdot \sum \vec{\boldsymbol{M}}_{/G} &= \hat{\boldsymbol{\lambda}}_{OG} \cdot \dot{\vec{\boldsymbol{H}}}_{/G} \\ &= \hat{\boldsymbol{\lambda}}_{OG} \cdot (\vec{\boldsymbol{r}}_{C/G} \times ma\hat{\boldsymbol{\imath}}).\end{aligned} \tag{6.40}$$

Since all reaction forces and the weight go through axis OG, they do not produce any moment about this axis (convince yourself that the forces from the reactions have no torque about the axis by calculation or geometry). Therefore,

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG} \cdot \sum \vec{\boldsymbol{M}}_{/G} &= \hat{\boldsymbol{\jmath}} \cdot (\vec{\boldsymbol{r}}_{H/G} \times T\hat{\boldsymbol{\lambda}}_{HI}) \\ &= T\frac{d(c-e)}{\ell}.\end{aligned} \tag{6.41}$$

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG} \cdot (\vec{\boldsymbol{r}}_{C/G} \times ma\hat{\boldsymbol{\imath}}) &= \hat{\boldsymbol{\jmath}} \cdot \left[(\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}) \times ma\hat{\boldsymbol{\imath}} \right] \\ &= ma\frac{(c-e)}{2}.\end{aligned} \tag{6.42}$$

Equating (6.41) and (6.42), we get

$$\begin{aligned}T &= \frac{ma\ell}{2d} \\ &= \frac{20\,\text{kg} \cdot 0.5\,\text{m/s}^2 \cdot 3.39\,\text{m}}{2 \cdot 3\,\text{m}} \\ &= 6.78\,\text{N}.\end{aligned}$$

$$\boxed{T_{HI} = 6.78\,\text{N}}$$

SAMPLE 6.10 *Computer solution of algebraic equations.* In the previous sample problem (Sample 6.9), six equations were obtained to solve for the six unknown forces (assuming $G_y = 0$). (i) Set up the six equations in matrix form and (ii) solve the matrix equation on a computer. Check the solution by substituting the values obtained in one or two equations.

Solution

(a) The six scalar equations — (6.32), (6.33), (6.34), (6.37), (6.38), and (6.39) are amenable to hand calculations. We, however, set up these equations in matrix form and solve the matrix equation on the computer. The matrix form of the equations is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{d}{\ell} \\ 0 & 1 & 0 & 0 & 0 & \frac{b}{\ell} \\ 0 & 0 & 1 & 0 & 1 & \frac{e}{\ell} \\ 0 & 1 & 0 & 0 & 0 & \frac{c}{\ell} \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \\ 1 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \end{bmatrix} \begin{Bmatrix} O_x \\ O_y \\ O_z \\ G_x \\ G_z \\ T \end{Bmatrix} = \begin{Bmatrix} ma \\ 0 \\ mg \\ mg/2 \\ ma/2 \\ ma/2 \end{Bmatrix}. \tag{6.43}$$

The above equation can be written, in matrix notation, as

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

where **A** is the coefficient matrix, **x** is the vector of the unknown forces, and **b** is the vector on the right hand side of the equation. Now we are ready to solve the system on the computer.

① Be careful with units. Most computer programs will not take care of your units. They only deal with numerical input and output. You should, therefore, make sure that your variables have proper units for the required calculations. Either do dimensionless calculations or use consistent units for all quantities.

(b) The following is pseudo-code to solve the above equation. ①

```
m = 20, a = 0.6,
b = 1.5, c = 1.5, d = 3, e = 0.5, g = 10,
l = sqrt(b^2 + d^2 + e^2),
A = [1 0 0 1 0 d/l
     0 1 0 0 0 b/l
     0 0 1 0 1 e/l
     0 1 0 0 0 c/l
     0 0 0 0 0 d/l
     1 0 0 0 0 d/l]
b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'
{Solve A x = b for x}

x =                    % this is the computer output
         0
   -3.0000
   97.0000
    6.0000
  102.0000
    6.7823
```

Figure 6.32: (Filename:sfig3.5.matlab)

The solution obtained from the computer means:

$$\boxed{O_x = 0,\ O_y = -3\,\text{N},\ O_z = 97\,\text{N},\ G_x = 6\,\text{N},\ G_z = 102\,\text{N},\ T = 6.78\,\text{N}.}$$

We now hand-check the solution by substituting the values obtained in, say, Eqns. (6.33) and (6.38). Before we substitute the values of forces, we need to calculate the length ℓ.

$$\begin{aligned} \ell &= \sqrt{d^2 + b^2 + e^2} \\ &= 3.3912\,\text{m}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Eqn. (6.33):} \qquad O_y + T\frac{b}{\ell} &= -3\,\text{N} + 6.78\,\text{N} \cdot \frac{1.5\,\text{m}}{3.3912\,\text{m}} \\ &\overset{\surd}{=} 0, \\ \text{Eqn. (6.38):} \qquad \frac{d}{\ell}T - \frac{1}{2}ma &= \frac{3\,\text{m}}{3.3912\,\text{m}}6.78\,\text{N} - \frac{1}{2}20\,\text{kg}\,0.6\,\text{m/s}^2 \\ &\overset{\surd}{=} 0. \end{aligned}$$

Thus, the computer solution agrees with our equations.

7 Circular motion

When a rigid object moves, it translates and rotates. In general, the points on the body move on complicated paths. When considering the unconstrained motions of particles in chapter 5, such as the motion of a thrown ball, we observed such curved paths. Things which are constrained to translate in straight lines were covered in the previous chapter. Now we would like to consider motion that may be constrained to a curved path. More specifically, this chapter concerns mechanics when particles move on the archetypal curved path, a circle. Circular motion deserves special attention because

- the most common connection between moving parts on a machine is with a bearing (or hinge or axle) (Fig. 7.1), if the axle is fixed then all points on a part so-connected move in circles;
- circular motion is the simplest case of curved-path motion;
- circular motion provides a simple way to introduce time-varying base vectors;
- in some sense, that you will appreciate with hind-sight, circular motion includes all of the conceptual ingredients of more general curved motions;
- at least in 2 dimensions, the only way two particles on one rigid body can move relative to each other is by circular motion (no matter how the body is moving); and
- circular motion is the simplest case where we can introduce two important rigid body concepts:
  - angular velocity, and
  - moment of inertia.

Useful calculations can be made for many things by approximating their motion as one for which all particles are going in circles. For example, a jet engine's turbine

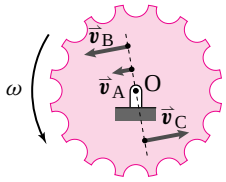


Figure 7.1: All the points on a gear move in circles, assuming the axle is stationary.

(Filename:tfigure4.1a)

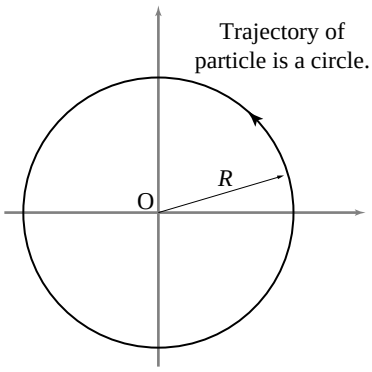


Figure 7.2: Trajectory of particle for circular motion.

(Filename:tfigure1.i)

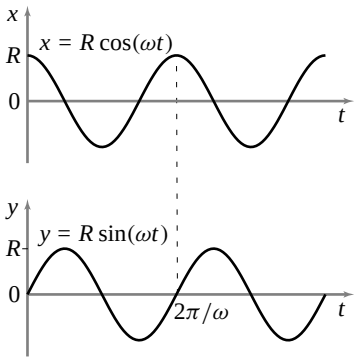


Figure 7.3: Plots of x versus t and y versus t for a particle going in a circle of radius R at constant rate. Both x and y vary as sinusoidal functions of time: $x = R\cos(\omega t)$ and $y = R\sin(\omega t)$.

(Filename:tfigure1.h)

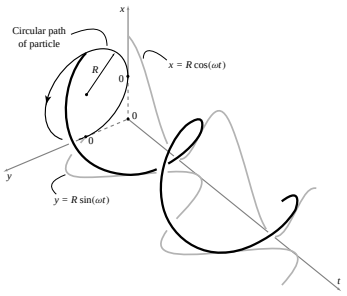


Figure 7.4: Plot of x and y versus time for a particle going in circles. x versus t is the cosine curve, y versus t is the sine curve. Together they make up the helical curve in three-dimensional space.

(Filename:tfigure1.j)

blade, a car engine's crank shaft, a car's wheel, a windmill's propeller, the earth spinning about its axis, points on a clock pendulum, a bicycle's approximately circular path when going around a corner, and a spinning satellite might all be reasonably approximated by the assumption of circular motion.

This chapter concerns only motion in two dimensions. The next chapter discusses circular motion in a three-dimensional context. The first two sections consider the kinematics and mechanics of a single particle going in circles. The later sections concern the kinematics and mechanics of rigid bodies.

For the systems in this chapter, we have, as always,

$$\text{linear momentum balance,} \qquad \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}}$$

$$\text{angular momentum balance,} \qquad \sum \vec{\boldsymbol{M}}_{i/C} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}},$$

$$\text{and power balance:} \qquad P = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} + \dot{E}_{\mathrm{int}}.$$

The left hand sides of the momentum equations are found using the forces and couples shown on the free body diagram of the system of interest; the right sides are evaluated in terms of motion of the system. Because you already know how to work with forces and moments, the primary new skill in this chapter is to learning to evaluate $\dot{\vec{\boldsymbol{L}}}$, $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$, and $\dot{E}_{\mathrm{K}}$ for a rotating particle or rigid body. Mostly towards this end, you need to know the velocity and accelerations of points on a rigid body.

7.1 Kinematics of a particle in planar circular motion

Consider a particle on the xy plane going in circles around the origin at a constant rate. One way of representing this situation is with the equation:

$$\vec{\boldsymbol{r}} = R\cos(\omega t)\hat{\boldsymbol{\imath}} + R\sin(\omega t)\hat{\boldsymbol{\jmath}},$$

with R and ω constants. Another way is with the pair of equations:

$$x = R\cos(\omega t) \qquad \text{and} \qquad y = R\sin(\omega t).$$

How do we represent this motion graphically? One way is to plot the particle trajectory, that is, the path of the particle. Figure 7.2 shows a circle of radius R drawn on the xy plane. Note that this plot doesn't show the speed the particle moves in circles. That is, a particle moving in circles slowly and another moving quickly would both would have the same plotted trajectory.

Another approach is to plot the functions $x(t)$ and $y(t)$ as in Fig. 7.3. This figure shows how x and y vary in time but does not directly convey that the particle is going in circles. How do you make these plots? Using a calculator or computer you can evaluate x and$y$ for a range of values of t. Then, using pencil and paper, a plotting calculator, or a computer, plot x *vs* t, y *vs* t, and y *vs* x.

If one wishes to see both the trajectory and the time history of both variables one can make a 3-D plot of xy position versus time (Fig. 7.3). The shadows of this curve (a helix) on the three coordinate planes are the three graphs just discussed. How you make such a graph with a computer depends on the software you use.

Finally, rather than representing time as a spatial coordinate, one can use time directly by making an animated movie on a computer screen showing a particle on the xy plane as it moves. Move your finger around in circles on the table. That's it. These days, the solutions of complex dynamics problems are often presented with computer animations.

The velocity and acceleration of a point going in circles: polar coordinates

Let's redraw Fig. 7.3 but introduce unit base vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the direction of the position vector $\vec{\boldsymbol{R}}$ and perpendicular to $\vec{\boldsymbol{R}}$. At any instant in time, the radial unit vector $\hat{\boldsymbol{e}}_R$ is directed from the center of the circle towards the point of interest and the transverse vector $\hat{\boldsymbol{e}}_\theta$, perpendicular to $\hat{\boldsymbol{e}}_R$, is tangent to the circle at the point of interest. Note that, as the particle goes around, its $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ unit vectors change. Note also, that every particle has its own $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ vectors. We will make frequent use the polar coordinate unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$.

Here is one of many possible derivations of the polar coordinate velocity and acceleration formulas.

First, observe that the position of the particle is (see figure 7.5)

$$\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R.$$

That is, the position vector is the distance from the origin times a unit vector in the direction of the particle's position. Given the position, it is just a matter of *careful* differentiation to find velocity and acceleration. First, velocity is the time derivative of position, so

$$\vec{\boldsymbol{v}} = \frac{d}{dt}\vec{\boldsymbol{R}} = \frac{d}{dt}(R\hat{\boldsymbol{e}}_R) = \underbrace{\dot{R}}_{0}\hat{\boldsymbol{e}}_R + R\dot{\hat{\boldsymbol{e}}}_R.$$

Since a circle has constant radius, R is constant and $\dot{R}$ is zero. But what is the rate of change of $\hat{\boldsymbol{e}}_R$ with respect to time, $\dot{\hat{\boldsymbol{e}}}_R$?

We could find $\dot{\hat{\boldsymbol{e}}}_R$ using the geometry of figure 7.6. $\Delta\hat{\boldsymbol{e}}_R$ is evidently in the direction $\hat{\boldsymbol{e}}_\theta$ and has magnitude $\Delta\theta$ so $\Delta\hat{\boldsymbol{e}}_R \approx (\Delta\theta)\hat{\boldsymbol{e}}_\theta$. Dividing by Δt, we have $\Delta\hat{\boldsymbol{e}}_R/\Delta t \approx (\Delta\theta/\Delta t)\hat{\boldsymbol{e}}_\theta$. So, using this sloppy calculus, we get $\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$. Similarly, we could get $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$.

Alternatively, we can be a little less geometric and a little more algebraic, and use the decomposition of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ into cartesian coordinates. These decompositions are found by looking at the projections of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the x and y-directions (see figure 7.7).

$$\begin{aligned} \hat{\boldsymbol{e}}_R &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{e}}_\theta &= -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}} \end{aligned} \tag{7.1}$$

So to find $\dot{\hat{\boldsymbol{e}}}_R$ we just differentiate, taking into account that θ is changing with time but that the unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are fixed (so they don't change with time).

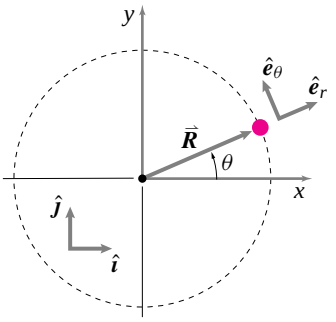


Figure 7.5: A particle going in circles. The position vector of the particle relative to the center of the circle is $\vec{\boldsymbol{R}}$. It makes an angle θ measured counter-clockwise from the positive x-axis. The unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ are shown in the radial and tangential directions, the directions of increasing R and increasing θ.

(Filename:tfigure4.4)

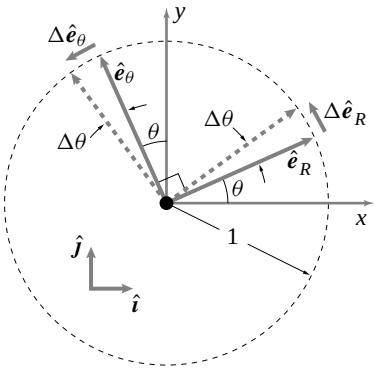


Figure 7.6: A close up view of the unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$. They make an angle θ with the positive x and y-axis, respectively. As the particle advances an amount $\Delta\theta$ both $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ change. In particular, for small $\Delta\theta$, $\Delta\hat{\boldsymbol{e}}_R$ is approximately in the $\hat{\boldsymbol{e}}_\theta$ direction and $\Delta\hat{\boldsymbol{e}}_\theta$ is approximately in the $-\hat{\boldsymbol{e}}_R$ direction.

(Filename:tfigure4.5)

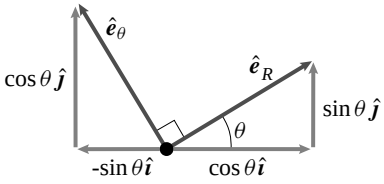


Figure 7.7: Projections of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the x and y directions

(Filename:tfigure4.5a)

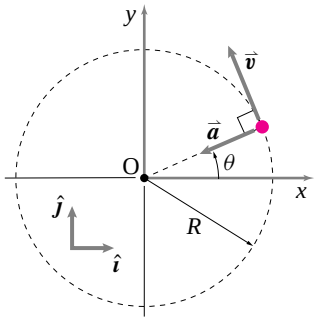


Figure 7.8: The directions of velocity $\vec{v}$ and acceleration $\vec{a}$ are shown for a particle going in circles at *constant* rate. The velocity is tangent to the circle and the acceleration is directed towards the center of the circle.

(Filename:tfigure4.6)

$$\begin{aligned}\dot{\hat{e}}_R &= \frac{d}{dt}(\cos\theta\hat{i} + \sin\theta\hat{j}) = -\dot{\theta}\sin\theta\hat{i} + \dot{\theta}\cos\theta\hat{j} = \dot{\theta}\hat{e}_\theta \\ \dot{\hat{e}}_\theta &= \frac{d}{dt}(-\sin\theta\hat{i} + \cos\theta\hat{j}) = -\dot{\theta}\hat{e}_R\end{aligned}$$

We had to use the chain rule, that is

$$\frac{d\sin\theta(t)}{dt} = \frac{d\sin\theta}{d\theta}\frac{d\theta(t)}{dt} = \dot{\theta}\cos\theta.$$

Now, we know $\dot{\hat{e}}_R$ so we can find $\vec{v}$,

$$\vec{v} = \dot{\vec{R}} = R\dot{\hat{e}}_R = R(\dot{\theta}\hat{e}_\theta) = R\dot{\theta}\hat{e}_\theta. \tag{7.2}$$

Similarly we can find $\ddot{\vec{R}}$ by differentiating once again,

$$\vec{a} = \ddot{\vec{R}} = \dot{\vec{v}} = \frac{d}{dt}(R\dot{\theta}\hat{e}_\theta) = \underbrace{\dot{R}\dot{\theta}\hat{e}_\theta}_{\vec{0}} + R\ddot{\theta}\hat{e}_\theta + R\dot{\theta}\dot{\hat{e}}_\theta \tag{7.3}$$

The first term on the right hand side is zero because $\dot{R}$ is 0 for circular motion. The third term is evaluated using the formula we just found for the rate of change of $\hat{e}_\theta$: $\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_R$. So,

$$\vec{a} = R\dot{\theta}(-\dot{\theta}\hat{e}_R) + R\ddot{\theta}\hat{e}_\theta = -(\dot{\theta})^2 R\hat{e}_R + R\ddot{\theta}\hat{e}_\theta = -\dot{\theta}^2\vec{R} + R\ddot{\theta}\hat{e}_\theta. \tag{7.4}$$

The velocity $\vec{v}$ and acceleration $\vec{a}$ are shown for a particle going in circles at constant rate in figure 7.8.

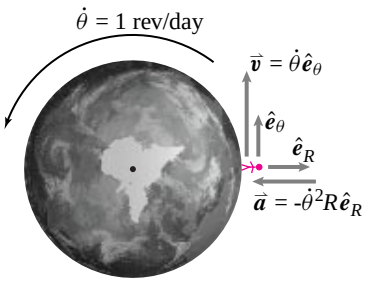


Figure 7.9: (Filename:tfigure4.1.example1)

Example: **A person standing on the earth's equator**

A person standing on the equator has velocity

$$\begin{aligned}\vec{v} = \dot{\theta}R\hat{e}_\theta &\approx \left(\frac{2\pi\text{ rad}}{24\text{ hr}}\right)4000\text{ mi}\hat{e}_\theta \\ &\approx 1050\text{ mph}\hat{e}_\theta \approx 1535\text{ ft/s}\hat{e}_\theta\end{aligned}$$

and acceleration

$$\begin{aligned}\vec{a} = -\dot{\theta}^2 R\hat{e}_R &\approx -\left(\frac{2\pi\text{ rad}}{24\text{ hr}}\right)^2 4000\text{ mi}\hat{e}_R \\ &\approx -274\text{ mi/ hr}^2\hat{e}_R \approx -0.11\text{ ft/s}^2\hat{e}_R.\end{aligned}$$

The velocity of a person standing on the equator, due to the earth's rotation, is about 1000 mph tangent to the earth. Her acceleration is about $0.11\,\mathrm{ft/s^2}$ towards the center of the earth. This acceleration is about $1/300$ of g, the inwards acceleration of a person in frictionless free-fall. □

Another derivation of the velocity and acceleration formulas

We now repeat the derivation for velocity and acceleration, but more concisely. The position of the particle is $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R$. Recall that the rates of change of the polar base vectors are $\dot{\hat{\boldsymbol{e}}}_R = \dot\theta\hat{\boldsymbol{e}}_\theta$ and $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot\theta\hat{\boldsymbol{e}}_R$. We find the velocity by differentiating the position with respect to time, keeping R constant.

$$\begin{aligned}\vec{\boldsymbol{v}} &= \frac{d}{dt}\vec{\boldsymbol{R}} = \frac{d}{dt}(R\hat{\boldsymbol{e}}_R) = \underbrace{\dot R}_{0}\hat{\boldsymbol{e}}_R + R\dot{\hat{\boldsymbol{e}}}_R \\ &= \dot{\vec{\boldsymbol{R}}} = R\dot{\hat{\boldsymbol{e}}}_R = R\dot\theta\hat{\boldsymbol{e}}_\theta\end{aligned}$$

We can find the acceleration $\vec{\boldsymbol{a}}$ by differentiating again,

$$\begin{aligned}\vec{\boldsymbol{a}} &= \ddot{\vec{\boldsymbol{R}}} = \dot{\vec{\boldsymbol{v}}} = \frac{d}{dt}(R\dot\theta\hat{\boldsymbol{e}}_\theta) \\ &= (\dot R\dot\theta\hat{\boldsymbol{e}}_\theta) + (R\ddot\theta\hat{\boldsymbol{e}}_\theta) + (R\dot\theta\,\overbrace{\dot{\hat{\boldsymbol{e}}}_\theta}^{-\dot\theta\hat{\boldsymbol{e}}_R}) \\ &= -(\dot\theta)^2R\hat{\boldsymbol{e}}_R + \ddot\theta R\hat{\boldsymbol{e}}_\theta = -(v^2/R)\hat{\boldsymbol{e}}_R + \dot v\hat{\boldsymbol{e}}_\theta.\end{aligned}$$

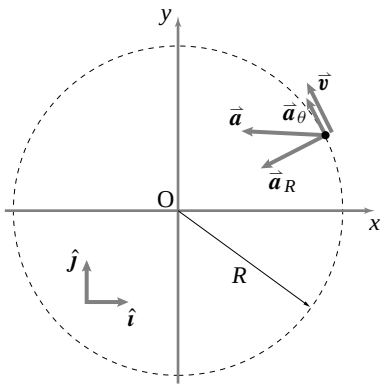


Figure 7.10: The directions of velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ are shown for a particle going in circles at variable rate. The velocity is tangent to the circle and the acceleration is the sum of two components: one directed towards the center of the circle and one tangent to the circle.

(Filename:tfigure5.6)

Thus, the formulas for velocity and acceleration of a point undergoing variable rate circular motion in 2-D are:

$$\begin{aligned}\vec{\boldsymbol{v}} &= R\dot\theta\hat{\boldsymbol{e}}_\theta \\ \vec{\boldsymbol{a}} &= -\frac{v^2}{R}\hat{\boldsymbol{e}}_R + \dot v\hat{\boldsymbol{e}}_\theta,\end{aligned}$$

where $\dot v$ is the rate of change of tangential speed①.

The rotation θ can vary with t arbitrarily, depending on the problem at hand.

For uniform rotational acceleration, $\frac{d}{dt}\omega = \alpha =$ constant, the following formulas are useful for some elementary problems:

$$\omega(t) = \omega_0 + \alpha t, \text{ and} \tag{7.5}$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2. \tag{7.6}$$

You can also write the above formulas in terms of $\dot\theta$, $\ddot\theta$, etc., by simply substituting $\dot\theta$ for ω and $\ddot\theta$ for α (see samples).

①**Caution:** Note that the rate of change of speed is not the magnitude of the acceleration: $\dot v \neq |\vec{\boldsymbol{a}}|$ or in other words: $\frac{d}{dt}|\vec{\boldsymbol{v}}| \neq |\frac{d}{dt}\vec{\boldsymbol{v}}|$. Consider the case of a car driving in circles at constant rate. Its rate of change of speed is zero, yet it has an acceleration.

The motion quantities

We can use our results for velocity and acceleration to better evaluate the momenta and energy quantities. These results will allow us to do mechanics problems associated with circular motion. For one particle in constant rate circular motion.

$$\begin{array}{lcccll}
\vec{\boldsymbol{L}} & = & \vec{\boldsymbol{v}}m & = & R\dot{\theta}\hat{\boldsymbol{e}}_\theta m, & \\
\dot{\vec{\boldsymbol{L}}} & = & \vec{\boldsymbol{a}}m & = & (-\dot{\theta}^2\vec{\boldsymbol{R}} + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta)m, & \\
\vec{\boldsymbol{H}}_{\mathrm{O}} & = & \vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{v}}m & = & R^2\dot{\theta}m\hat{\boldsymbol{k}}, & \\
\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} & = & \vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{a}}m & = & R^2\ddot{\theta}m\hat{\boldsymbol{k}}, & \\
E_{\mathrm{K}} & = & \frac{1}{2}v^2m & = & \frac{1}{2}R^2\dot{\theta}^2m, & \text{and} \\
\dot{E}_{\mathrm{K}} & = & \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}}\, m & = & mR^2\dot{\theta}\ddot{\theta} &
\end{array}$$

We have used the fact that $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}}$ which can be verified with the right hand rule definition of the cross product or using the Cartesian representation of the polar base vectors.

SAMPLE 7.1 *The velocity vector.* A particle executes circular motion in the xy plane with constant speed $v = 5\,\text{m/s}$. At $t = 0$ the particle is at $\theta = 0$. Given that the radius of the circular orbit is 2.5 m, find the velocity of the particle at $t = 2\,\text{sec}$.

Solution

$$\begin{aligned} \text{It is given that} \quad & R = 2.5\,\text{m} \\ & v = \text{constant} = 5\,\text{m/s} \\ & \theta_{(t=0)} = 0. \end{aligned}$$

The velocity of a particle in constant-rate circular motion is:

$$\begin{aligned} \vec{\boldsymbol{v}} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta \\ \text{where} \quad \hat{\boldsymbol{e}}_\theta &= -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}. \end{aligned}$$

Since R is a constant and $v = |\vec{\boldsymbol{v}}| = R\dot{\theta}$ is a constant, $\dot{\theta} = \dfrac{v}{R} = \dfrac{5\,\text{m/s}}{2.5\,\text{m}} = 2\dfrac{\text{rad}}{\text{s}}$ is also constant.

$$\text{Thus} \qquad \vec{\boldsymbol{v}}_{(t=2\,\text{s})} = \underbrace{R\dot{\theta}}_{v}\,\hat{\boldsymbol{e}}_\theta\,|_{t=2\,\text{s}} = 5\,\text{m/s}\;\hat{\boldsymbol{e}}_{\theta(t=2\,\text{s})}.$$

Clearly, we need to find $\hat{\boldsymbol{e}}_\theta$ at $t = 2\,\text{sec}$.

$$\begin{aligned} \text{Now} \qquad \dot{\theta} \equiv \tfrac{d\theta}{dt} &= 2\,\text{rad/s} \\ \Rightarrow \qquad \textstyle\int_0^\theta d\theta &= \displaystyle\int_0^{2\,\text{s}} 2\,\text{rad/s}\,dt \\ \Rightarrow \qquad \theta &= (2\,\text{rad/s})\;t\;|_0^{2\,\text{s}} \\ &= 2\,\text{rad/s}\cdot 2\,\text{s} \\ &= 4\,\text{rad}. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{\boldsymbol{e}}_\theta &= -\sin 4\hat{\boldsymbol{\imath}} + \cos 4\hat{\boldsymbol{\jmath}} \\ &= 0.76\hat{\boldsymbol{\imath}} - 0.65\hat{\boldsymbol{\jmath}}, \end{aligned}$$

and

$$\begin{aligned} \vec{\boldsymbol{v}}_{(2\,\text{s})} &= 5\,\text{m/s}(0.76\hat{\boldsymbol{\imath}} - 0.65\hat{\boldsymbol{\jmath}}) \\ &= (3.78\hat{\boldsymbol{\imath}} - 3.27\hat{\boldsymbol{\jmath}})\,\text{m/s}. \end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}} = (3.78\hat{\boldsymbol{\imath}} - 3.27\hat{\boldsymbol{\jmath}})\,\text{m/s}}$$

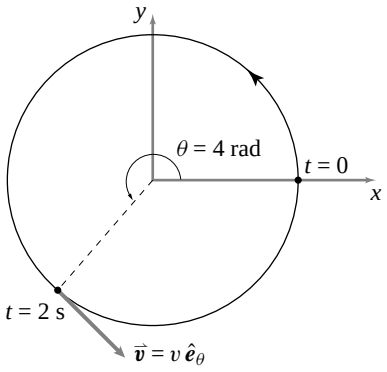


Figure 7.11: (Filename:sfig4.1.DH)

SAMPLE 7.2 *Basic kinematics:* A point mass executes circular motion with angular acceleration $\ddot{\theta} = 5\,\text{rad/s}^2$. The radius of the circular path is 0.25 m. If the mass starts from rest at $\theta = 0^o$, find and draw

(a) the velocity of the mass at $\theta = 0^o,\ 30^o,\ 90^o$, and 210^o,

(b) the acceleration of the mass at $\theta = 0^o,\ 30^o,\ 90^o$, and 210^o.

Solution We are given, $\ddot{\theta} = 5\,\text{rad/s}^2$, and $R = 0.25\,\text{m}$.

(a) The velocity $\vec{\boldsymbol{v}}$ in circular (constant or non-constant rate) motion is given by:

$$\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta.$$

So, to find the velocity at different positions we need $\dot{\theta}$ at those positions. Here the angular acceleration is constant, i.e., $\ddot{\theta} = 5\,\text{rad/s}^2$. Therefore, we can use the formula ①

$$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\ddot{\theta}\theta$$

to find the angular speed $\dot{\theta}$ at various θ's. But $\dot{\theta}_0 = 0$ (mass starts from rest), therefore $\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$. Now we make a table for computing the velocities at different positions:

Position (θ)	θ in radians	$\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$	$\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta$
0^o	0	0 rad/s	$\vec{\boldsymbol{0}}$
30^o	$\pi/6$	$\sqrt{10\pi/6} = 2.29\,\text{rad/s}$	$0.57\,\text{m/s}\hat{\boldsymbol{e}}_\theta$
90^o	$\pi/3$	$\sqrt{10\pi/3} = 3.34\,\text{rad/s}$	$0.81\,\text{m/s}\hat{\boldsymbol{e}}_\theta$
210^o	$7\pi/6$	$\sqrt{70\pi/6} = 2.29\,\text{rad/s}$	$1.51\,\text{m/s}\hat{\boldsymbol{e}}_\theta$

The computed velocities are shown in Fig. 7.12.

(b) The acceleration of the mass is given by

$$\begin{aligned}\vec{\boldsymbol{a}} &= \overbrace{a_R\hat{\boldsymbol{e}}_R}^{\text{radial}} + \overbrace{a_\theta\hat{\boldsymbol{e}}_\theta}^{\text{tangential}} \\ &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta.\end{aligned}$$

Since $\ddot{\theta}$ is constant, the tangential component of the acceleration is constant at all positions. We have already calculated $\dot{\theta}$ at various positions, so we can easily calculate the radial (also called the normal) component of the acceleration. Thus we can find the acceleration. For example, at $\theta = 30^o$,

$$\begin{aligned}\vec{\boldsymbol{a}} &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta \\ &= -0.25\,\text{m}\cdot\frac{10\pi}{6}\frac{1}{\text{s}^2}\hat{\boldsymbol{e}}_R + 0.25\,\text{m}\cdot 5\frac{1}{\text{s}^2}\hat{\boldsymbol{e}}_\theta \\ &= -1.31\,\text{m/s}^2\hat{\boldsymbol{e}}_R + 1.25\,\text{m/s}^2\hat{\boldsymbol{e}}_\theta.\end{aligned}$$

Similarly, we find the acceleration of the mass at other positions by substituting the values of R, $\ddot{\theta}$ and $\dot{\theta}$ in the formula and tabulate the results in the table below.

Position (θ)	$a_r = -R\dot{\theta}^2$	$a_\theta = R\ddot{\theta}$	$\vec{\boldsymbol{a}} = a_r\hat{\boldsymbol{e}}_R + a_\theta\hat{\boldsymbol{e}}_\theta$
0^o	0	$1.25\,\text{m/s}^2$	$1.25\,\text{m/s}^2\hat{\boldsymbol{e}}_\theta$
30^o	$-1.31\,\text{m/s}^2$	$1.25\,\text{m/s}^2$	$(-1.31\hat{\boldsymbol{e}}_R + 1.25\hat{\boldsymbol{e}}_\theta)\,\text{m/s}^2$
90^o	$-2.62\,\text{m/s}^2$	$1.25\,\text{m/s}^2$	$(-2.62\hat{\boldsymbol{e}}_R + 1.25\hat{\boldsymbol{e}}_\theta)\,\text{m/s}^2$
210^o	$-9.16\,\text{m/s}^2$	$1.25\,\text{m/s}^2$	$(-9.16\hat{\boldsymbol{e}}_R + 1.25\hat{\boldsymbol{e}}_\theta)\,\text{m/s}^2$

① We use this formula because we need $\dot{\theta}$ at different values of θ. In elementary physics books, the same formula is usually written as

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

where α is the constant angular acceleration.

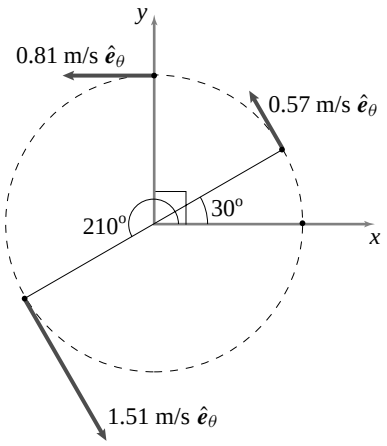


Figure 7.12: Velocity of the mass at $\theta = 0^o,\ 30^o,\ 90^o$, and 210^o.

(Filename:sfig5.1.1a)

The accelerations computed are shown in Fig. 7.13. The acceleration vector as well as its tangential and radial components are shown in the figure at each position.

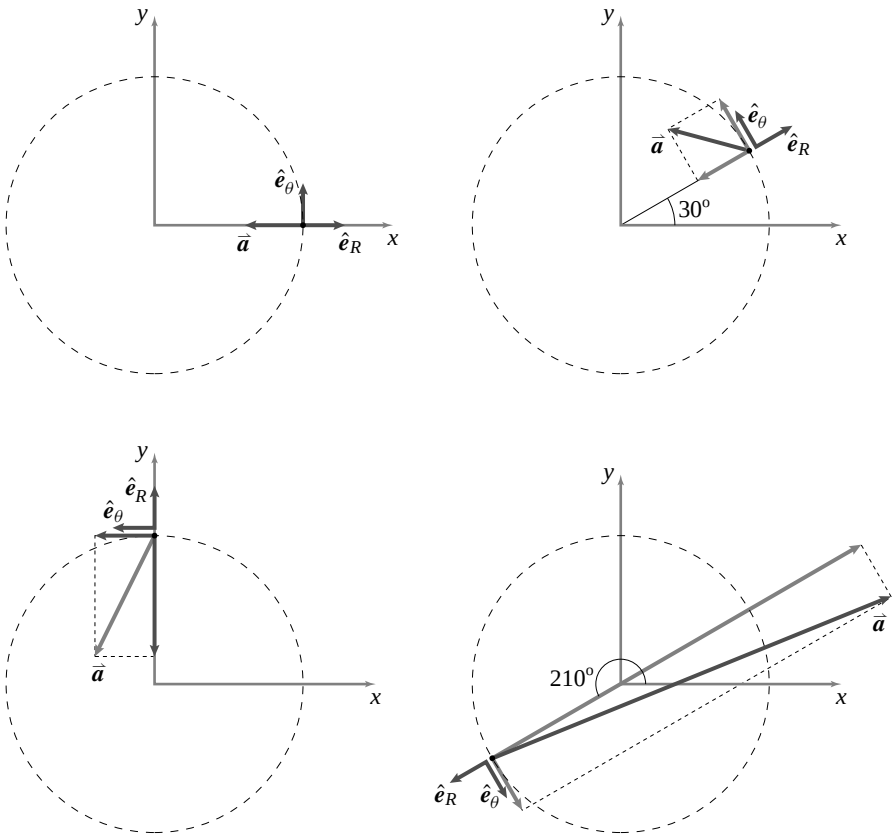


Figure 7.13: Acceleration of the mass at $\theta = 0^o,\ 30^o,\ 90^o$, and 210^o. The radial and tangential components are shown with grey arrows. As the angular velocity increases, the radial component of the acceleration increases; therefore, the total acceleration vector leans more and more towards the radial direction.

(Filename:sfig5.1.1b)

SAMPLE 7.3 In an experiment, the magnitude of angular deceleration of a spinning ball is found to be proportional to its angular speed ω (ie., $\dot{\omega} \propto -\omega$). Assume that the proportionality constant is k and find an expression for ω as a function of t, given that $\omega(t = 0) = \omega_0$.

Solution The equation given is:

$$\dot{\omega} = \frac{d\omega}{dt} = -k\omega. \tag{7.7}$$

Let us guess a solution of the exponential form with arbitrary constants and plug into Eqn. (7.7) to check if our solution works. Let $\omega(t) = C_1 e^{C_2 t}$. Substituting in Eqn. (7.7), we get

$$\begin{aligned} C_1 C_2 e^{C_2 t} &= -k C_1 e^{C_2 t} \\ \Rightarrow \quad C_2 &= -k, \\ \text{also,} \quad \omega(0) &= \omega_0 = C_1 e^{C_2 \cdot 0} \\ \Rightarrow \quad C_1 &= \omega_0. \end{aligned}$$

Therefore,

$$\omega(t) = \omega_0 e^{-kt}. \tag{7.8}$$

Alternatively,

$$\begin{aligned} \frac{d\omega}{\omega} &= -k\,dt \\ \text{or} \quad \int_{\omega_0}^{\omega(t)} \frac{d\omega}{\omega} &= -\int_0^t k\,dt \\ \Rightarrow \quad \ln\omega|_{\omega_0}^{\omega(t)} &= -kt \\ \Rightarrow \quad \ln\omega(t) - \ln\omega_0 &= -kt \\ \Rightarrow \quad \ln\left(\frac{\omega(t)}{\omega_0}\right) &= -kt \\ \Rightarrow \quad \frac{\omega(t)}{\omega_0} &= e^{-kt}. \end{aligned}$$

Therefore,

$$\omega(t) = \omega_0\, e^{-kt},$$

which is the same solution as equation (7.8).

$$\boxed{\omega(t) = \omega_0\, e^{-kt}}$$

SAMPLE 7.4 *Using kinematic formulae:* The spinning wheel of a stationary exercise bike is brought to rest from 100 rpm by applying brakes over a period of 5 seconds.

(a) Find the average angular deceleration of the wheel.
(b) Find the number of revolutions it makes during the braking.

Solution We are given,

$$\dot{\theta}_0 = 100\,\text{rpm}, \quad \dot{\theta}_{\text{final}} = 0, \quad \text{and} \quad t = 5\,\text{s}.$$

(a) Let α be the average (constant) deceleration. Then

$$\dot{\theta}_{\text{final}} = \dot{\theta}_0 - \alpha t.$$

Therefore,

$$\begin{aligned} \alpha &= \frac{\dot{\theta}_0 - \dot{\theta}_{\text{final}}}{t} \\ &= \frac{100\,\text{rpm} - 0\,\text{rpm}}{5\,\text{s}} \\ &= \frac{100\,\text{rev}}{60\,\text{s}} \cdot \frac{1}{5\,\text{s}} \\ &= 0.33 \frac{\text{rev}}{\text{s}^2}. \end{aligned}$$

$$\boxed{\alpha = 0.33 \tfrac{\text{rev}}{\text{s}^2}}$$

(b) To find the number of revolutions made during the braking period, we use the formula

$$\theta(t) = \underbrace{\theta_0}_{0} + \dot{\theta}_0 t + \frac{1}{2}(-\alpha)t^2 = \dot{\theta}_0 t - \frac{1}{2}\alpha t^2.$$

Substituting the known values, we get

$$\begin{aligned} \theta &= \frac{100\,\text{rev}}{60\,\text{s}} \cdot 5\,\text{s} - \frac{1}{2}\, 0.33 \frac{\text{rev}}{\text{s}^2} \cdot 25\,\text{s}^2 \\ &= 8.33\,\text{rev} - 4.12\,\text{rev} \\ &= 4.21\,\text{rev}. \end{aligned}$$

$$\boxed{\theta = 4.21\,\text{rev}}$$

Comments:

- Note the negative sign used in both the formulae above. Since α is deceleration, that is, a negative acceleration, we have used negative sign with α in the formulae.
- Note that it is not always necessary to convert rpm in rad/s. Here we changed rpm to rev/ s because time was given in seconds.

SAMPLE 7.5 *Non-constant acceleration:* A particle of mass 500 grams executes circular motion with radius $R = 100\,\text{cm}$ and angular acceleration $\ddot{\theta}(t) = c\sin\beta t$, where $c = 2\,\text{rad/s}^2$ and $\beta = 2\,\text{rad/s}$.

(a) Find the position of the particle after 10 seconds if the particle starts from rest, that is, $\theta(0) = 0$.

(b) How much kinetic energy does the particle have at the position found above?

Solution

(a) We are given $\ddot{\theta}(t) = c\sin\beta t$, $\dot{\theta}(0) = 0$ and $\theta(0) = 0$. We have to find $\theta(10\,\text{s})$. Basically, we have to solve a second order differential equation with given initial conditions.

$$\begin{aligned} \ddot{\theta} \equiv \frac{d}{dt}(\dot{\theta}) &= c\sin\beta t \\ \Rightarrow \int_{\dot{\theta}_0=0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t c\sin\beta\tau\, d\tau \\ \dot{\theta}(t) &= -\frac{c}{\beta}\cos\beta\tau\Big|_0^t = \frac{c}{\beta}(1-\cos\beta t). \end{aligned}$$

Thus, we get the expression for the angular speed $\dot{\theta}(t)$. We can solve for the position $\theta(t)$ by integrating once more:

$$\begin{aligned} \dot{\theta} \equiv \frac{d}{dt}(\theta) &= \frac{c}{\beta}(1-\cos\beta t) \\ \Rightarrow \int_{\theta_0=0}^{\theta(t)} d\theta &= \int_0^t \frac{c}{\beta}(1-\cos\beta\tau) \\ \theta(t) &= \frac{c}{\beta}\left[\tau - \frac{\sin\beta\tau}{\beta}\right]_0^t \\ &= \frac{c}{\beta^2}(\beta t - \sin\beta t). \end{aligned}$$

Now substituting $t = 10\,\text{s}$ in the last expression along with the values of other constants, we get

$$\begin{aligned} \theta(10\,\text{s}) &= \frac{2\,\text{rad/s}^2}{(2\,\text{rad/s})^2}[2\,\text{rad/s}\cdot 10\,\text{s} - \sin(2\,\text{rad/s}\cdot 10\,\text{s})] \\ &= 9.54\,\text{rad}. \end{aligned}$$

$$\boxed{\theta = 9.54\,\text{rad}}$$

(b) The kinetic energy of the particle is given by

$$\begin{aligned} E_\text{K} &= \frac{1}{2}mv^2 = \frac{1}{2}m(R\dot{\theta})^2 \\ &= \frac{1}{2}mR^2[\underbrace{\frac{c}{\beta}(1-\cos\beta t)}_{\dot{\theta}(t)}]^2 \\ &= \frac{1}{2}0.5\,\text{kg}\cdot 1\,\text{m}^2\cdot\left[\frac{2\,\text{rad/s}^2}{2\,\text{rad/s}}\cdot(1-\cos(20))\right]^2 \\ &= 0.086\,\text{kg}\cdot\,\text{m}^2\cdot\,\text{s}^2 = 0.086\text{Joule}. \end{aligned}$$

$$\boxed{E_\text{K} = 0.086\text{J}}$$

7.2 Dynamics of a particle in circular motion

The simplest examples of circular motion concern the motion of a particle constrained by a massless connection to be a fixed distance from a support point.

Example: **Rock spinning on a string**

Neglecting gravity, we can now deal with the familiar problem of a point mass being held in constant circular-rate motion by a massless string or rod. Linear momentum balance for the mass gives:

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ \Rightarrow -T\hat{e}_R &= m\vec{a} \\ \left\{-T\hat{e}_R\right. &= \left. m(-\dot{\theta}^2\ell\hat{e}_R)\right\} \\ \{\}\cdot\hat{e}_R \Rightarrow \qquad & T = \dot{\theta}^2\ell m = (v^2/\ell)m \end{aligned}$$

The force required to keep a mass in constant rate circular motion is mv^2/ℓ (sometimes remembered as mv^2/R). □

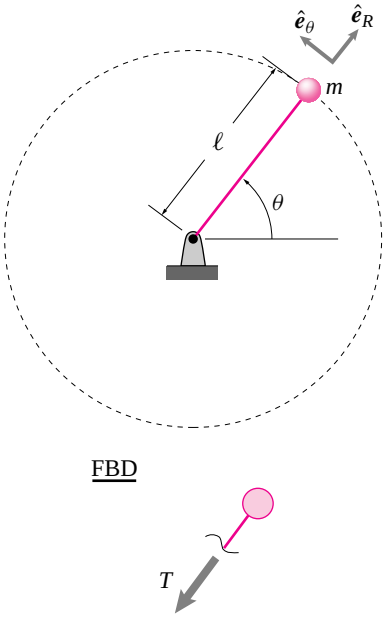


Figure 7.14: Point mass spinning in circles. Sketch of system and a free body diagram.

(Filename:tfigure4.1.rockandstring)

The simple pendulum

As a child's swing, the inside of a grandfather clock, a hypnotist's device, or a gallows, the motion of a simple pendulum is a clear image to all of us. Galileo studied the simple pendulum and it is a topic in freshman physics. It turns out, forcing a pendulum in an appropriate manner can lead to chaos. Pendula are also useful as models of many phenomena from the swing of leg joints in walking to the tipping of a chimney in an earthquake.

For starters, we consider a 2-D pendulum of fixed length with no forcing other than gravity. All mass is concentrated at a point. The tension in the pendulum rod acts along the length since it is a massless two-force body. Of primary interest is the motion of the pendulum. First we find governing differential equations. Here are two approaches to getting the equation of motion.

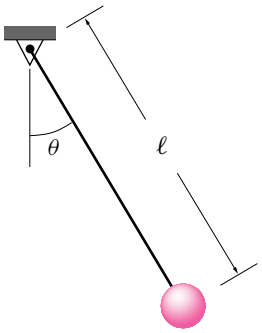


Figure 7.15: The simple pendulum.

(Filename:tfigure5.spend)

Method One

The equation of linear momentum balance is

$$\sum \vec{F} = \overbrace{\dot{\vec{L}}}^{m\vec{a}}$$

Evaluating the left and right hand sides, we get

$$-T\hat{e}_R + (-mg)\hat{j} = m[\ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R] \tag{7.9}$$

From the picture we see that $\hat{e}_R = \sin\theta\hat{i} - \cos\theta\hat{j}$ and $\hat{e}_\theta = \cos\theta\hat{i} + \sin\theta\hat{j}$. So, upon substitution into the equation above, we get

$$-T(\sin\theta\hat{i} - \cos\theta\hat{j}) + (-mg)\hat{j} = m[\ell\ddot{\theta}(\cos\theta\hat{i} + \sin\theta\hat{j}) - \ell\dot{\theta}^2(\sin\theta\hat{i} - \cos\theta\hat{j})]$$

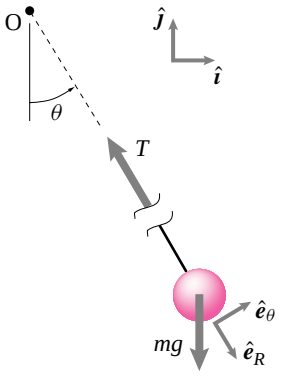


Figure 7.16: Free body diagram of the simple pendulum.

(Filename:tfigure5.spend.fbd)

Breaking this equation into its x and y components (by dotting with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$) gives

$$-T\sin\theta = m[\ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta] \text{ and} \tag{7.10}$$

$$T\cos\theta - mg = m[\ell\ddot{\theta}\sin\theta + \ell\dot{\theta}^2\cos\theta] \tag{7.11}$$

which are two simultaneous equations that we can solve for the two unknowns T and $\ddot{\theta}$ to get

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \tag{7.12}$$

$$T = m[\ell\dot{\theta}^2 + g\cos\theta]. \tag{7.13}$$

A little quicker way to get the equation of motion would be to take equation 7.9 and dot it with $\hat{\boldsymbol{e}}_\theta$.

$$-T\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{e}}_\theta}_{0} + -mg\underbrace{\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{e}}_\theta}_{\sin\theta} = m\ell\ddot{\theta}\underbrace{\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{e}}_\theta}_{1} - \ell\dot{\theta}^2\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{e}}_\theta}_{0}$$

$$\Rightarrow \quad -mg\sin\theta = m\ell\ddot{\theta}$$

$$\Rightarrow \quad \ddot{\theta} = -\tfrac{g}{\ell}\sin\theta.$$

Method Two

Using angular momentum balance, we can 'kill' the tension term at the start. Taking angular momentum balance about the point O, we get

$$\sum \vec{\boldsymbol{M}}_O = \dot{\vec{\boldsymbol{H}}}_O$$

$$-mg\ell\sin\theta\hat{\boldsymbol{k}} = \vec{\boldsymbol{r}}_{/O}\times\vec{\boldsymbol{a}}\,m$$

$\ell\hat{\boldsymbol{e}}_R$ $\qquad$ $\ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R$

$$-mg\ell\sin\theta\hat{\boldsymbol{k}} = m\ell^2\ddot{\theta}\hat{\boldsymbol{k}}$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

since $\hat{\boldsymbol{e}}_R\times\hat{\boldsymbol{e}}_R = 0$ and $\hat{\boldsymbol{e}}_R\times\hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}}$. So, the governing equation for a simple pendulum is

$$\boxed{\ddot{\theta} = -\tfrac{g}{\ell}\sin\theta}$$

For small angles, $\sin\theta \approx \theta$, so we have

$$\ddot{\theta} = -\frac{g}{\ell}\theta$$

for small oscillations. This equation describes a harmonic oscillator with $\frac{g}{\ell}$ replacing the $\sqrt{\frac{k}{m}}$ coefficient in a spring-mass system.

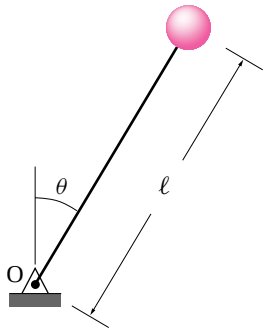


Figure 7.17: The inverted pendulum

(Filename:tfigure5.spend.inv)

The inverted pendulum

A pendulum with the mass-end up is called an inverted pendulum. By methods just like we used for the regular pendulum, we find the equation of motion to be

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta$$

which, for small θ, is well approximated by

$$\ddot{\theta} = \frac{g}{\ell}\theta.$$

As opposed to the simple pendulum, which has oscillatory solutions, this differential equation has exponential solutions, indicating the inherent *instability* of the inverted pendulum; that is, its tendency to fall over when slightly disturbed from the vertical position.

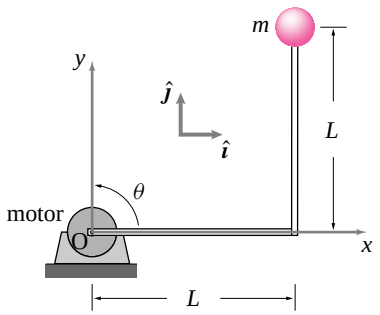


Figure 7.18: The motor rotates the structure at a constant angular speed in the counterclockwise direction.

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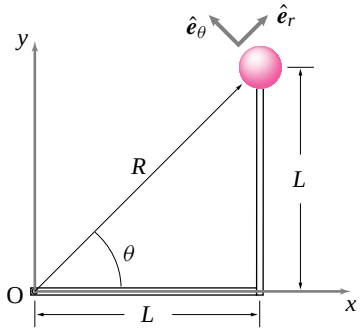


Figure 7.19: The ball follows a circular path of radius R. The position, velocity, and acceleration of the ball can be expressed in terms of the polar basis vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$.

(Filename:sfig4.1.1a)

SAMPLE 7.6 *Circular motion in 2-D.* Two bars, each of negligible mass and length $L = 3\,\text{ft}$, are welded together at right angles to form an 'L' shaped structure. The structure supports a $3.2\,\text{lbf}$ $(= mg)$ ball at one end and is connected to a motor on the other end (see Fig. 7.18). The motor rotates the structure in the vertical plane at a constant rate $\dot{\theta} = 10\,\text{rad/s}$ in the counter-clockwise direction. Take $g = 32\,\text{ft/s}^2$. At the instant shown in Fig. 7.18, find

(a) the velocity of the ball,
(b) the acceleration of the ball, and
(c) the net force and moment applied by the motor and the support at O on the structure.

Solution The motor rotates the structure at a constant rate. Therefore, the ball is going in circles with angular velocity $\vec{\boldsymbol{\omega}} = \dot{\theta}\hat{\boldsymbol{k}} = 10\,\text{rad/s}\hat{\boldsymbol{k}}$. The radius of the circle is $R = \sqrt{L^2 + L^2} = L\sqrt{2}$. Since the motion is in the xy plane, we use the following formulae to find the velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$.

$$\vec{\boldsymbol{v}} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta$$

$$\vec{\boldsymbol{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta,$$

where $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ are the polar basis vectors shown in Fig. 7.19. In Fig. 7.19, we note that $\theta = 45^\text{o}$. Therefore,

$$\begin{aligned}
\hat{\boldsymbol{e}}_R &= \cos\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{j}} \\
&= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}), \\
\hat{\boldsymbol{e}}_\theta &= -\sin\theta\hat{\boldsymbol{i}} + \cos\theta\hat{\boldsymbol{j}} \\
&= \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}).
\end{aligned}$$

Since $R = L\sqrt{2} = 3\sqrt{2}\,\text{ft}$ is constant, $\dot{R} = 0$ and $\ddot{R} = 0$. Thus,

(a) the velocity of the ball is

$$\begin{aligned}
\vec{\boldsymbol{v}} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta \\
&= 3\sqrt{2}\,\text{ft} \cdot 10\,\text{rad/s}\hat{\boldsymbol{e}}_\theta \\
&= 30\sqrt{2}\,\text{ft/s} \cdot \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) \\
&= 30\,\text{ft/s}(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}).
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}} = 30\,\text{ft/s}(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})}$$

(b) The acceleration of the ball is

$$\begin{aligned}
\vec{\boldsymbol{a}} &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R \\
&= -3\sqrt{2}\,\text{ft} \cdot (10\,\text{rad/s})^2\hat{\boldsymbol{e}}_R \\
&= -300\sqrt{2}\,\text{ft/s}^2 \cdot \frac{1}{\sqrt{2}}(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) \\
&= -300\,\text{ft/s}^2(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}).
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = -300\,\text{ft/s}^2(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})}$$

(c) Let the net force and the moment applied by the motor-support system be $\vec{F}$ and $\vec{M}$ as shown in Fig. 7.20. From the linear momentum balance for the structure,

$$\begin{aligned}
\sum \vec{F} &= m\vec{a} \\
\vec{F} - mg\hat{j} &= m\vec{a} \\
\Rightarrow \quad \vec{F} &= m\vec{a} + mg\hat{j} \\
&= \frac{\overbrace{3.2\,\text{lbf}}^{m}}{32\,\text{ft/s}^2}(-300\sqrt{2}\,\text{ft/s}^2)\hat{e}_R + \overbrace{3.2\,\text{lbf}}^{mg}\hat{j} \\
&= -30\sqrt{2}\,\text{lbf}\hat{e}_R + 3.2\,\text{lbf}\hat{j}. \\
&= -30\sqrt{2}\,\text{lbf}\frac{1}{\sqrt{2}}(\hat{i}+\hat{j}) + 3.2\,\text{lbf}\hat{j} \\
&= -30\,\text{lbf}\hat{i} - 26.8\,\text{lbf}\hat{j}.
\end{aligned}$$

Similarly, from the angular momentum balance for the structure,

$$\begin{aligned}
\sum \vec{M}_O &= \dot{\vec{H}}_O, \\
\text{where} \quad \sum \vec{M}_O &= \vec{M} + \vec{r}_{/O} \times mg(-\hat{j}) \\
&= \vec{M} + \underbrace{R\hat{e}_R}_{L(\hat{i}+\hat{j})} \times mg(-\hat{j}) \\
&= \vec{M} - mgL\hat{k}, \\
\text{and} \quad \dot{\vec{H}}_O &= \vec{r}_{/O} \times m\vec{a} \\
&= R\hat{e}_R \times m(-R\dot{\theta}^2\hat{e}_R) \\
&= -mR^2\dot{\theta}^2 \underbrace{(\hat{e}_R \times \hat{e}_R)}_{\vec{0}} \\
&= \vec{0}.
\end{aligned}$$

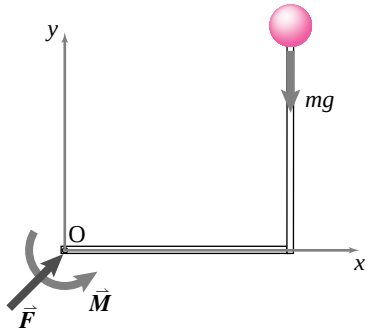


Figure 7.20: Free body diagram of the structure.

(Filename:sfig4.1.1b)

Therefore,

$$\begin{aligned}
\vec{M} &= mgL\hat{k} \\
&= \underbrace{3.2\,\text{lbf}}_{mg} \cdot \underbrace{3\,\text{ft}}_{L}\hat{k} = 9.6\,\text{lbf}\cdot\text{ft}\hat{k}.
\end{aligned}$$

$$\boxed{\vec{F} = -30\,\text{lbf}\hat{i} - 26.8\,\text{lbf}\hat{j}, \quad \vec{M} = 9.6\,\text{lbf}\cdot\text{ft}\hat{k}}$$

Note: If there was no gravity, the moment applied by the motor would be zero.

SAMPLE 7.7 A 50 gm point mass executes circular motion with angular acceleration $\ddot{\theta} = 2\,\text{rad/s}^2$. The radius of the circular path is 200 cm. If the mass starts from rest at $t = 0$, find

(a) Its angular momentum $\vec{\boldsymbol{H}}$ about the center at $t = 5\,\text{s}$.

(b) Its rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}$ about the center.

Solution

(a) From the definition of angular momentum,

$$\begin{aligned}\vec{\boldsymbol{H}}_{\text{O}} &= \vec{\boldsymbol{r}}_{/0} \times m\vec{\boldsymbol{v}} \\ &= R\hat{\boldsymbol{e}}_R \times m\dot{\theta} R\hat{\boldsymbol{e}}_\theta \\ &= mR^2\dot{\theta}(\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta) \\ &= mR^2\dot{\theta}\hat{\boldsymbol{k}}\end{aligned}$$

On the right hand side of this equation, the only unknown is $\dot{\theta}$. Thus to find $\vec{\boldsymbol{H}}_{\text{O}}$ at $t = 5\,\text{s}$, we need to find $\dot{\theta}$ at $t = 5\,\text{s}$. Now,

$$\begin{aligned}\ddot{\theta} &= \frac{d\dot{\theta}}{dt} \\ d\dot{\theta} &= \ddot{\theta}\,dt \\ \int_{\dot{\theta}_0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t \ddot{\theta}\,dt \\ \dot{\theta}(t) - \dot{\theta}_0 &= \ddot{\theta}(t_t - t_0) \\ \dot{\theta} &= \dot{\theta}_0 + \ddot{\theta}(t - t_0)\end{aligned}$$

Writing α for $\ddot{\theta}$ and substituting $t_0 = 0$ in the above expression, we get $\dot{\theta}(t) = \dot{\theta}_0 + \alpha t$, which is the angular speed version of the linear speed formula $v(t) = v_0 + at$. ① Substituting $t = 5\,\text{s}$, $\dot{\theta}_0 = 0$, and $\alpha = 2\,\text{rad/s}^2$ we get $\dot{\theta} = 2\,\text{rad/s}^2 \cdot 5\,\text{s} = 10\,\text{rad/s}$. Therefore,

① Be warned that these formulae are valid only for constant acceleration.

$$\begin{aligned}\vec{\boldsymbol{H}}_{\text{O}} &= 0.05\,\text{kg} \cdot (0.2\,\text{m})^2 \cdot 10\,\text{rad/s}\hat{\boldsymbol{k}} \\ &= 0.02\,\text{kg.m}^2/\,\text{s} = 0.02\,\text{N·m}\cdot\,\text{s}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}}_{\text{O}} = 0.02\,\text{N·m}\cdot\,\text{s}.}$$

(b) Similarly, we can calculate the rate of change of angular momentum:

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{\text{O}} &= \vec{\boldsymbol{r}}_{/0} \times m\vec{\boldsymbol{a}} \\ &= R\hat{\boldsymbol{e}}_R \times m(R\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \dot{\theta}^2 R\hat{\boldsymbol{e}}_R) \\ &= mR^2\ddot{\theta}(\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta) \\ &= mR^2\ddot{\theta}\hat{\boldsymbol{k}} \\ &= 0.02\,\text{kg} \cdot (0.2m)^2 \cdot 2\,\text{rad/s}^2\hat{\boldsymbol{k}} \\ &= 0.004\,\text{kg}\cdot\,\text{m}^2/\,\text{s}^2 = 0.004\,\text{N·m}\end{aligned}$$

$$\boxed{\dot{\vec{\boldsymbol{H}}}_{\text{O}} = 0.004\,\text{N·m}}$$

SAMPLE 7.8 *The simple pendulum.* A simple pendulum swings about its vertical equilibrium position (2-D motion) with maximum amplitude $\theta_{max} = 10^o$. Find

(a) the magnitude of the maximum angular acceleration,
(b) the maximum tension in the string.

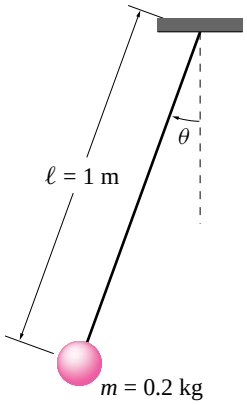


Figure 7.21: (Filename:sfig5.5.DH1)

Solution

(a) The equation of motion of the pendulum is given by (see equation 7.12 of text):

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

We are given that $|\theta| \leq \theta_{max}$. For $\theta_{max} = 10^o = 0.1745\,\text{rad}$, $\sin\theta_{max} = 0.1736$. Thus we see that $\sin\theta \approx \theta$ even when θ is maximum. Therefore, we can safely use linear approximation (although we could solve this problem without it); i.e.,

$$\ddot{\theta} = -\frac{g}{\ell}\theta.$$

Clearly, $|\ddot{\theta}|$ is maximum when θ is maximum. Thus,

$$|\ddot{\theta}|_{max} = \frac{g}{\ell}\theta_{max} = \frac{9.81\,\text{m/s}^2}{1\,\text{m}}\cdot(0.1745\,\text{rad}) = 1.71\,\text{rad/s}^2.$$

$$\boxed{|\ddot{\theta}|_{\text{max}} = 1.71\,\text{rad/s}^2}$$

(b) The tension in the string is given by (see equation 7.13 of text):

$$T = m(\ell\dot{\theta}^2 + g\cos\theta).$$

This time, we will not make the small angle assumption. We can find T_{max} and where it is maximum as follows using conservation of energy. Let the position of maximum amplitude be position 1. and the position at any θ be position 2. At the its maximum amplitude, the mass comes to rest and switches directions; thus, its angular velocity and, hence, its kinetic energy is zero there. Using conservation of energy, we have

$$\begin{aligned} E_{\text{K1}} + E_{\text{P1}} &= E_{\text{K2}} + E_{\text{P2}} \\ 0 + mg\ell(1-\cos\theta_{max}) &= \frac{1}{2}m(\ell\dot{\theta})^2 + mg\ell(1-\cos\theta). \qquad (7.14) \end{aligned}$$

(7.15)

and solving for $\dot{\theta}$,

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}(\cos\theta - \cos(\theta_{max}))}.$$

Therefore, the tension at any θ is

$$T = m(\ell\dot{\theta}^2 + g\cos\theta) = mg(3\cos\theta - 2\cos(\theta_{max})).$$

To find the maximum value of the tension T, we set its derivative with respect to θ equal to zero and find that, for $0 \leq \theta \leq \theta_{max}$, T is maximum when $\theta = 0$, or

$$T_{max} = mg(3\cos(0) - 2\cos(\theta_{max})) = 0.2\,\text{kg}\cdot 9.81\,\text{m/s}^2[3 - 1.97] = 2.02\,\text{N}.$$

The maximum tension corresponds to maximum speed which occurs at the bottom of the swing where all of the potential energy is converted to kinetic energy.

$$\boxed{T_{max} = 2.02\,\text{N}}$$

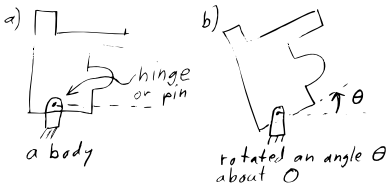


Figure 7.22: A body has rotated an angle θ counterclockwise about 0.

(Filename:tfigure.circmot2D)

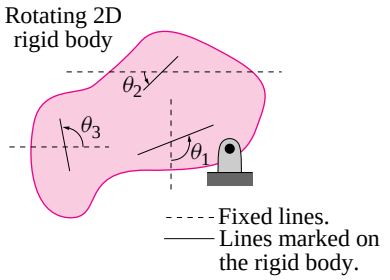


Figure 7.23: **Rotation of lines on a rotating rigid body.** Some real or imagined lines marked on the rigid body are shown. They make the angles θ_1, θ_2, θ_3, ... with respect to various fixed lines which do not rotate. As the body rotates, each of these angles increases by the same amount.

(Filename:tfigure4.2Domega)

7.3 Kinematics of a rigid body in planar circular motion

The most common form of flexible attachment in machine design is a hinge or pin connection (Fig. 7.22). For simplicity, in this chapter we study the situation where a machine part is hinged to a structure which does not move. If we take the hinge axis to be the z axis fixed at O, then the hinge's job is to keep the part's only possible motion to be rotation about O. As usual in this book, we think of the part as rigid. Thus to study dynamics of a hinged part we need to understand the position, velocity and acceleration of points on a rigid body which rotates. This section discusses the geometry and algebra of rotation, of the rotation rate which we will call the angular velocity, and the rate of change of the angular velocity.

The rest of the book rests heavily on the material in this section.

Rotation of a rigid body counterclockwise by θ

We start by imagining the object in some configuration which we call the *reference configuration* or *reference state*. Often the reference state is one where prominent features of the object are aligned with the vertical or horizontal direction or with prominent features of another nearby part. The reference state may or may not be the start of the motion of interest. We measure an object's rotation relative to the reference state, as in Fig. 7.22 where a body is shown and shown again, rotated. For definiteness, rotation is the change, relative to the reference state, in the counterclockwise angle θ of a reference line marked in the body relative to a fixed line outside. Which reference line? Fortunately,

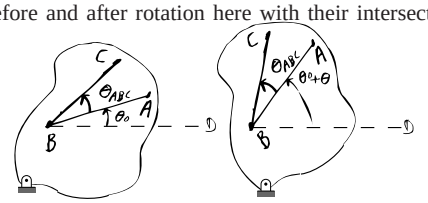
All real or imagined lines marked on a rotating rigid body rotate by the same angle.

7.1 *THEORY*

Rotation is uniquely defined for a rigid body (2D)

We show here the intuitively clear result that, starting at a reference orientation, all lines marked on a rigid body rotate by the same angle θ.

Because the body is rigid, the act of rotation preserves all distances between pairs of points. That's a geometric definition of the word *rigid*. Thus, by the "similar triangle theorem" (side-side-side) of elementary geometry, all relative angles between marked line segments are preserved by the rotation. Consider a pair of line segments with each segment defined by two points on the body. First, extend the segments to their point of intersection. Such a pair of lines is shown before and after rotation here with their intersection at B.



Initially BA makes and angle θ_0 with a horizontal reference line. BC then makes an angle of $\theta_{\rm ABC} + \theta_0$. After rotation we measure the angle to the line BD (displaced in a parallel manner). BA now makes an angle of $\theta_0 + \theta$ where θ is the angle of rotation of the body. By the addition of angles in the rotated configuration line BC now makes an angle of $\theta_{\rm ABC} + \theta_0 + \theta$ which is, because angle $\theta_{\rm ABC}$ is unchanged, also an increase by θ in the angle made by BC with the horizontal reference line. Both lines rotate by the same angle θ.

We could use one of these lines and compare with an arbitrary third line and show that those have equal rotation also, and so on for any lines of interest in the body. So all lines rotate by the same angle θ. The demonstration for a pair of parallel lines is easy, they stay parallel so always make a common angle with any reference line.

Thus, all lines marked on a rigid body rotate by the same angle θ so the concept of a body's rotation from a given reference state is uniquely defined.

(See box 7.1). Thus, once we have decided on a reference configuration, we can measure the rotation of the body, and of all lines marked on the body, with a single number, the *rotation angle* θ①.

① In three dimensions the situation is more complicated. Rotation of a rigid body is also well defined, but its representation is more complicated than a single number θ.

Rotated coordinates and base vectors $\hat{\imath}'$ and $\hat{\jmath}'$

Often it is convenient two pick two orthogonal lines on a body and give them distinguished status as *body fixed* rotating coordinate axes x' and y'. The algebra we will develop is most simple if these axes are chosen to be parallel with a fixed x and y axes when $\theta = 0$ in the reference configuration.

We will follow a point P at $\vec{r}_{\rm P}$. With this rotating coordinate axes x' and y' are associated rotating base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ (Fig. 8.9). The position coordinates of P, in the rotating coordinates, are $[\vec{r}]_{x'y'} = [x', y']$, which we sometimes write as $[\vec{r}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

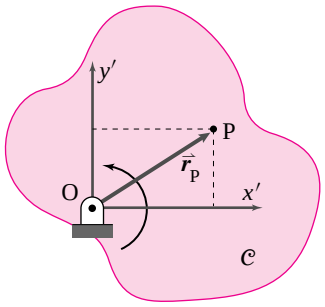


Figure 7.24: A rotating rigid body $\mathcal{C}$ with rotating coordinates $x'y'$ rigidly attached.

(Filename:tfigure4.intro.rot.frames)

Example: **A particle on the x' axis**

If a particle of interest is fixed on the x'-axis at position $x' = 3\,\text{cm}$, then we have.

$$\vec{r}_{\rm P} = 3\,\text{cm}\hat{\imath}'$$

for all time, even as the body rotates. □

For a general point P fixed to a body rotating about O it is always true that

$$\vec{r}_{\rm P} = x'\hat{\imath}' + y'\hat{\jmath}', \tag{7.16}$$

(7.17)

with x' and y' not changing as θ increases. Obviously point P moves, but its coordinates x' and y' do not change. The change in motion is expressed in eqn. (7.18) by the base vectors changing as the body rotates. Thus we could write more explicitly that.

$$\vec{r}_{\rm P} = x'\hat{\imath}'(\theta) + y'\hat{\jmath}'(\theta) \tag{7.18}$$

In particular, just like for polar base vectors (see eqn. (7.2) on page 355) we can express the rotating base vectors in terms of the fixed base vectors and θ.

$$\begin{aligned} \hat{\imath}' &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}, \\ \hat{\jmath}' &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}. \end{aligned} \tag{7.19}$$

One also sometimes wants to know the fixed basis vectors in terms of the rotating vectors,

$$\begin{aligned} \hat{\imath} &= \cos\theta\hat{\imath}' - \sin\theta\hat{\jmath}' \\ \hat{\jmath} &= \sin\theta\hat{\imath}' \quad \cos\theta\hat{\jmath}'. \end{aligned} \tag{7.20}$$

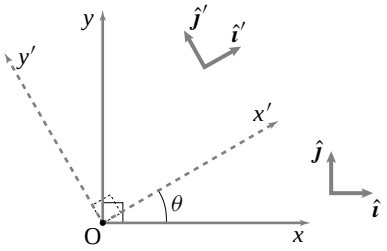


Figure 7.25: Fixed coordinate axes and rotating coordinate axes.

(Filename:tfigure4.1.rot.coord)

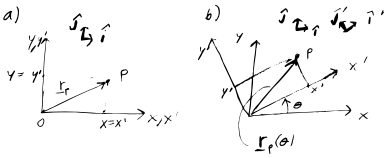


Figure 7.26: The x' and y' coordinates of a point fixed on a rotating body stay constant while the base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ change while they rotate with the body.

(Filename:tfigure.rotatepbytheta)

You should review the material in section 2.2 to see how these formulae can be derived with dot products.

We will use the phrase *reference frame* or just *frame* to mean "a coordinate system attached to a rigid body". One could imagine that the coordinate grid is like a metal framework that rotates with the body. We would refer to a calculation based on the rotating coordinates in Fig. 8.9 as "in the frame $\mathcal{C}$" or "using the $x'y'$ frame" or "in the $\hat{\boldsymbol{\imath}}'\hat{\boldsymbol{\jmath}}'$ frame.

(Advanced aside. Sometimes a *reference frame* is defined as the set of all coordinate systems that could be attached to a rigid body. Two coordinate systems, even if rotated with respect to each other, then represent the same frame so long as they rotate together at the time of interest. Some of the results we will develop only depend on this definition of frame, that the coordinates are glued to the body, and not on their orientation on the body.)

In computer calculations it is often best to manipulate lists and arrays of numbers and not geometric vectors. Thus we like to keep track of coordinates of vectors. Lets look at a point whose coordinates we know in the reference configuration: $\left[\vec{\boldsymbol{r}}_{\rm P}{}^{\rm ref}\right]_{xy}$. Taking the body axes and fixed axes coincide in the reference configuration, the body coordinates of a point $\left[\vec{\boldsymbol{r}}_{\rm P}\right]_{x'y'}$ are equal to the space fixed coordinates of the point in the reference configuration $\left[\vec{\boldsymbol{r}}_{\rm P}{}^{\rm ref}\right]_{xy}$. We can think of the point as equivalently defined either way.

Coordinate representation of rotations using $[R]$

Here is the key question: What are the fixed basis coordinates of a point with coordinates $[\vec{\boldsymbol{r}}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$? Here is one way to find the answer:

$$\begin{aligned} \vec{\boldsymbol{r}}_{\rm P} &= x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' \\ &= x'(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + y'(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) \\ &= \underbrace{\big((\cos\theta)x' - (\sin\theta)y'\big)}_{x}\hat{\boldsymbol{\imath}} + \underbrace{\big((\sin\theta)x' + (\cos\theta)y'\big)}_{y}\hat{\boldsymbol{\jmath}} \end{aligned} \tag{7.21}$$

so we can pull out the x and y coordinates compactly as,

$$[\vec{\boldsymbol{r}}_{\rm P}]_{xy} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta\, x' + \sin\theta(-y') \\ \sin\theta\, x' + \cos\theta(y') \end{bmatrix}. \tag{7.22}$$

But this can, in turn be written in matrix notation as

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \text{or} \\ \left[\vec{\boldsymbol{r}}_{\rm P}\right]_{xy} &= [R]\left[\vec{\boldsymbol{r}}_{\rm P}\right]_{x'y'}. \quad \text{or} \\ \left[\vec{\boldsymbol{r}}_{\rm P}\right]_{xy} &= [R]\left[\vec{\boldsymbol{r}}_{\rm P}{}^{\rm ref}\right]_{xy}, \end{aligned} \tag{7.23}$$

The matrix $[R]$ or $[R(\theta)]$ is the *rotation matrix* for counterclockwise rotations by θ. If you know the coordinates of a point on a body before rotation, you can find its coordinates after rotation by multiplying the coordinate column vector by the matrix

$[R]$. A feature of eqn. (7.23) is that the same matrix $[R]$ prescribes the coordinate change for every different point on the body. Thus for points called 1, 2 and 3 we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [R] \begin{bmatrix} x_1' \\ y_1' \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = [R] \begin{bmatrix} x_2' \\ y_2' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = [R] \begin{bmatrix} x_3' \\ y_3' \end{bmatrix}.$$

A more compact way to write a matrix times a list of column vectors is to arrange the column vectors one next to the other in a matrix. By multiplying this matrix by $[R]$ we get a new matrix whose columns are the new coordinates of various points. For example,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = [R] \begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{bmatrix}. \tag{7.24}$$

Eqn. 7.24 is useful for computer animation of rotating things in video games (and in dynamics simulations too).

Example: **Rotate a picture**

If a simple picture of a house is drawn by connecting the six points (Fig. 7.27a) with the first point at $(x, y) = (1, 2)$, the second at $(x, y) = (3, 2)$, etc., and the sixth point on top of the first, we have,

$$[xy \text{ points BEFORE}] \equiv \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix}.$$

After a 30^o counter-clockwise rotation about O, the coordinates of the house, in a coordinate system that rotates with the house, are unchanged (Fig. 7.27b). But in the fixed (non-rotating, Newtonian) coordinate system the new coordinates of the rotated house points are,

$$\begin{aligned} [xy \text{ points AFTER}] &= [R][xy \text{ points BEFORE}] \\ &= \begin{bmatrix} \sqrt{3}/2 & -.5 \\ .5 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix} \\ &\approx \begin{bmatrix} -0.1 & 1.6 & 0.6 & -0.8 & -1.1 & -0.1 \\ 2.2 & 3.2 & 5.0 & 5.3 & 4.0 & 2.2 \end{bmatrix} \end{aligned}$$

as shown in Fig. 7.27c. □

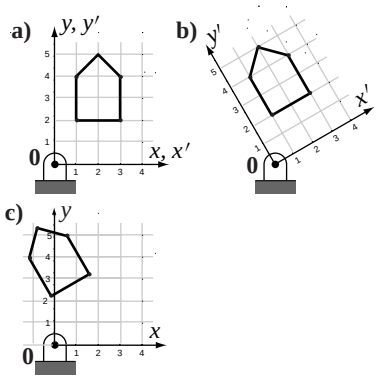


Figure 7.27: a) A house is drawn by connecting lines between 6 points, b) the house and coordinate system are rotated, thus its coordinates in the rotating system do not change c) But the coordinates in the original system do change

(Filename:tfigure.rotatedhouse)

Angular velocity of a rigid body: $\vec{\omega}$

Thus far we have talked about rotation, but not how it varies in time. Dynamics is about motion, velocities and accelerations, so we need to think about rotation rates and their rate of change.

In 2D, a rigid body's net rotation is most simply measured by the change that a line marked on the body (any line) makes with a fixed line (any fixed line). We have called this net change of angle θ. Thus, the simplest measure of rotation rate is $\dot{\theta} \equiv \frac{d\theta}{dt}$. Because all marked lines rotate the same amount they all have the same rates of change, so $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = etc$. So the concept of rotation rate of a rigid body, just like the concept of rotation, transcends the concept of rotation rate of this or that line. So we give it a special symbol ω (omega),

For all lines marked on a rigid body,

$$\omega \equiv \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \ldots = \dot{\theta}. \tag{7.25}$$

For calculation purposes in 2D, and necessarily in 3D, we think of angular velocity as a vector. Its direction is the axis of the rotation which is $\hat{\boldsymbol{k}}$ for bodies in the xy plane. Its scalar part is ω. So, the *angular velocity vector* is

$$\vec{\boldsymbol{\omega}} \equiv \omega \hat{\boldsymbol{k}} \tag{7.26}$$

with ω as defined in eqn. (7.25).

Rate of change of $\hat{\imath}'$, $\hat{\jmath}'$

Our first use of the angular velocity vector $\vec{\boldsymbol{\omega}}$ is to calculate the rate of change of rotating unit base vectors. We can find the rate of change of, say, $\hat{\boldsymbol{\imath}}'$, by taking the time derivate of the first of eqn. (7.19), and using the chain rule while recognizing that $\theta = \theta(t)$. We can also make an analogy with polar coordinates (page 355), where we think of $\hat{\boldsymbol{e}}_R$ as like $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{e}}_\theta$ as like $\hat{\boldsymbol{\jmath}}'$. We found there that $\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$ and $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$. Either way,

$$\begin{aligned} \dot{\hat{\boldsymbol{\imath}}}' &= \dot{\theta}\hat{\boldsymbol{\jmath}}' \quad \text{or} \quad \dot{\hat{\boldsymbol{\imath}}}' = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}' \quad \text{and} \\ \dot{\hat{\boldsymbol{\jmath}}}' &= -\dot{\theta}\hat{\boldsymbol{\imath}}' \quad \text{or} \quad \dot{\hat{\boldsymbol{\jmath}}}' = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\jmath}}' \end{aligned} \tag{7.27}$$

because $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\imath}}' = -\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\jmath}}'$. Depending on the tastes of your lecturer, you may find eqn. (7.27) one of the most used equations from this point onward①

① Eqn. 7.27 is sometimes considered the definition of $\vec{\boldsymbol{\omega}}$. In this view, $\vec{\boldsymbol{\omega}}$ is the vector that determines $\dot{\hat{\boldsymbol{\imath}}}'$ and $\dot{\hat{\boldsymbol{\jmath}}}'$ by the formulas $\dot{\hat{\boldsymbol{\imath}}}' = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}'$ and $\dot{\hat{\boldsymbol{\jmath}}}' = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\jmath}}'$. In that approach one then shows that such a vector exists and that it is $\vec{\boldsymbol{\omega}} = \hat{\boldsymbol{\imath}}' \times \dot{\hat{\boldsymbol{\imath}}}'$ which happens to be the same as our $\vec{\boldsymbol{\omega}} = \dot{\theta}\hat{\boldsymbol{k}}$.

The fixed Newtonian reference frame $\mathcal{F}$

All of mechanics depends on the laws of mechanics. A frame in which Newton's laws are accurate is called a *Newtonian frame*. In engineering practice the frames we use as approximations of a Newtonian frame often seem, loosely speaking, somehow still. So we sometimes call such a frame *the fixed frame* and label it with a script capital $\mathcal{F}$. When we talk about velocity and acceleration of mass points, for use in the equations of mechanics, we are always talking about the velocity and acceleration relative to a fixed Newtonian frame.

Assume x and y are the coordinates of a vector, say $\vec{\boldsymbol{r}}_{\mathrm{P}}$ is a fixed frame with fixed axis (with associated constant base vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$). When we write $\dot{\vec{\boldsymbol{r}}}_{\mathrm{P}}$ we mean $\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$. But we could be more explicit (and notationally ornate) and write

$$\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P}}}{dt} \equiv {}^{\mathcal{F}}\dot{\vec{\boldsymbol{r}}}_{\mathrm{P}} \quad \text{by which we mean} \quad \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}.$$

The $\mathcal{F}$ in front of the time derivative (or in front of the dot) means that when we calculate a derivative we hold the base vectors of $\mathcal{F}$ constant. This is no surprise, because for $\mathcal{F}$ the base vectors *are* constant. In general, however, when taking a derivative in a given frame you

- write vectors in terms of base vectors stuck to the frame, and
- only differentiate the components.

We will avoid the ornate notation of labeling frames when can. If you don't see any script capital letters floating around in front of derivatives, you can assume that we are taking derivatives relative to a fixed Newtonian frame.

Velocity of a point fixed on a rigid body

Lets call some rotating body $\mathcal{B}$ (script capital B) to which is glued a coordinate system $x'y'$ with base vectors $\hat{\imath}'$ and $\hat{\jmath}'$. Consider a point P at $\vec{r}_{\rm P}$ that is glued to the body. That is, the x' and y' coordinates of $\vec{r}_{\rm P}$ do not change in time. Using the new frame notation we can write

$$\frac{{}^{\mathcal{B}}d\vec{r}_{\rm P}}{dt} \equiv {}^{\mathcal{B}}\dot{\vec{r}}_{\rm P} = \quad \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' = \vec{0}.$$

That is, relative to a moving frame, the velocity of a point glued to the frame is zero. (Big surprise!)

We would like to know the velocity of such a point in the fixed frame. We just take the derivative, using the differentiation rules we have developed.

$$\begin{aligned} \vec{r}_{\rm P} &= x'\hat{\imath}' + y'\hat{\jmath}' \\ \Rightarrow \quad \vec{v}_{\rm P} = \dot{\vec{r}}_{\rm P} &= \frac{d}{dt}\left(x'\hat{\imath}' + y'\hat{\jmath}'\right) = x'\dot{\hat{\imath}}' + y'\dot{\hat{\jmath}}' = x'(\vec{\omega}\times\hat{\imath}') + y'(\vec{\omega}\times\hat{\jmath}') \\ &= \vec{\omega}\times(x'\hat{\imath}' + y'\hat{\jmath}') \end{aligned}$$

where $\vec{r}_{\rm P}$ is the simple way to write $\frac{{}^{\mathcal{F}}d\vec{r}_{\rm P}}{dt}$. Thus,

$$\vec{v}_{\rm P} = \vec{\omega}\times\vec{r}_{\rm P} \tag{7.28}$$

We can rewrite eqn. (7.28) in a minimalist or elaborate notation as

$$\begin{aligned} \vec{v} &= \vec{\omega}\times\vec{r} \qquad \text{or} \\ \frac{{}^{\mathcal{F}}d\vec{r}_{\rm P}}{dt} &= \vec{\omega}_{\mathcal{B}/\mathcal{F}}\times\vec{r}_{\rm P/O}. \end{aligned}$$

In the first case you have to use common sense to know what point you are talking about, that you are interested in the velocity of the same point and that it is on a body rotating with absolute angular velocity $\vec{\omega}$. In the second case everything is laid out clearly (which is why it looks so confusing!). On the left side of the equation it says that we are interested in how point P moves relative to, not just any frame, but the fixed frame $\mathcal{F}$. On the right side we make clear that the rotation rate we are looking at is that of body $\mathcal{B}$ relative to $\mathcal{F}$ and not some other relative rotation. We further make clear that the formula only makes sense if the position of the point P is measured relative to a point which doesn't move, namely 0.

What we have just found largely duplicates what we already learned in section 7.1 for points moving in circles. The slight generalization is that the same angular velocity $\vec{\omega}$ can be used to calculate the velocities of multiple points on one rigid body. The key idea remains: the velocity of a point going in circles is tangent to the circle it is going around and with magnitude proportional both to distance from the center and the angular rate of rotation (Fig. **??**a).

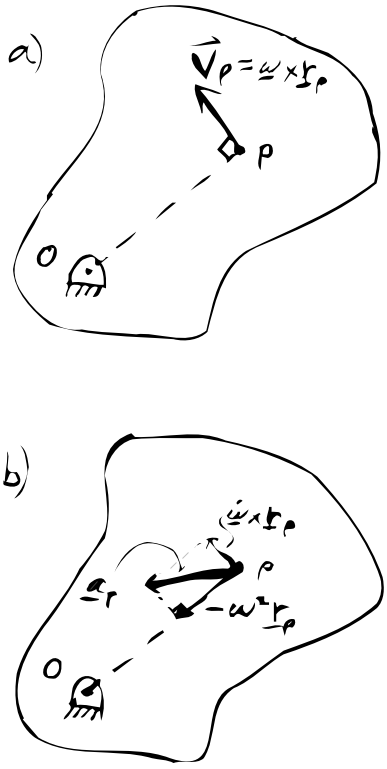


Figure 7.28: Velocity and acceleration of two points on a rigid body rotating about 0.

(Filename:tfigure.velandaccelofp)

Acceleration of a point on a rotating rigid body

Let's again consider a point with position

$$\vec{r}_{\rm P} = x'\hat{\imath}' + y'\hat{\jmath}'.$$

Relative to the frame $\mathcal{B}$ to which a point is attached, its acceleration is zero. (Surprise again!) But what is its acceleration in the fixed frame? We find this by writing the position vector and then differentiating twice, repeatedly using the product rule and eqn. (7.27).

Leaving off the ornate pre-super-script $\mathcal{F}$ for simplicity, we have

$$\begin{aligned}\vec{a}_{\rm P} = \dot{\vec{v}}_{\rm P} &= \frac{d}{dt}\left(\frac{d}{dt}\left(x'\hat{\imath}' + y'\hat{\jmath}'\right)\right) \\ &= \frac{d}{dt}\left(x'(\vec{\omega}\times\hat{\imath}') + y'(\vec{\omega}\times\hat{\jmath}')\right). \end{aligned} \tag{7.29}$$

To continue we need to use the product rule of differentiation for the cross product of two time dependent vectors like this:

$$\begin{aligned}\tfrac{d}{dt}\left(\vec{\omega}\times\hat{\imath}'\right) &= \dot{\vec{\omega}}\times\hat{\imath}' + \vec{\omega}\times\dot{\hat{\imath}}' &= \dot{\vec{\omega}}\times\hat{\imath}' + \vec{\omega}\times(\vec{\omega}\times\hat{\imath}'), \\ \tfrac{d}{dt}\left(\vec{\omega}\times\hat{\jmath}'\right) &= \dot{\vec{\omega}}\times\hat{\jmath}' + \vec{\omega}\times\dot{\hat{\jmath}}' &= \dot{\vec{\omega}}\times\hat{\jmath}' + \vec{\omega}\times(\vec{\omega}\times\hat{\jmath}'). \end{aligned} \tag{7.30}$$

Substituting back into eqn. (7.29) we get

$$\begin{aligned}\vec{a}_{\rm P} &= \left(x'(\dot{\vec{\omega}}\times\hat{\imath}' + \vec{\omega}\times(\vec{\omega}\times\hat{\imath}')) + y'(\dot{\vec{\omega}}\times\hat{\jmath}' + \vec{\omega}\times(\vec{\omega}\times\hat{\jmath}'))\right) \\ &= \dot{\vec{\omega}}\times(x'\hat{\imath}' + y'\hat{\imath}') + \vec{\omega}\times\left(\vec{\omega}\times\left(x'\hat{\imath}' + y'\hat{\imath}'\right)\right) \\ &= \dot{\vec{\omega}}\times\vec{r}_{\rm P} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{\rm P}\right) \end{aligned} \tag{7.31}$$

which is hardly intuitive at a glance[①] Recalling that in 2D $\vec{\omega} = \omega\hat{k}$ we can use either the right hand rule or manipulation of unit vectors to rewrite eqn. (7.31) as

$$\vec{a}_{\rm P} = \dot{\omega}\hat{k}\times\vec{r}_{\rm P} - \omega^2\vec{r}_{\rm P} \tag{7.32}$$

where $\omega = \dot{\theta}$ and $\dot{\omega} = \ddot{\theta}$ and theta is the counterclockwise rotation of any line marked on the body relative to any fixed line.

Thus, as we found in section 7.1 for a particle going in circles, the acceleration can be written as the sum of two terms, a tangential acceleration $\dot{\omega}\hat{k}\times\vec{r}_{\rm P}$ due to increasing tangential speed, and a centrally directed (centripetal) acceleration $-\omega^2\vec{r}_{\rm P}$ due to the direction of the velocity continuously changing towards the center (see Fig. 7.28b). The generalization we have made in this section is that the same $\vec{\omega}$ can be used to calculate the acceleration for all the different points on one rotating body. A second brief derivation of the acceleration eqn. (7.32) goes like this (using minimalist notation):

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(\vec{\omega}\times\vec{r}) = \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\dot{\vec{r}} = \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) = \dot{\vec{\omega}}\times\vec{r} - \omega^2\vec{r}.$$

① Although the form eqn. (7.31) is not of much immediate use, if you are going to continue on to the mechanics of mechanisms or three dimensional mechanics, you should follow the derivation of eqn. (7.31) carefully.

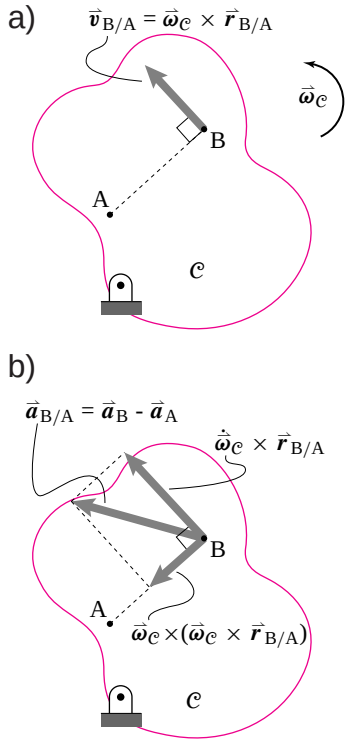


Figure 7.29: Relative velocity and acceleration of two points A and B on the same body $\mathcal{C}$.

(Filename:tfigure5.vel.accel.rel)

Relative motion of points on a rigid body

As you well know by now, the position of point B relative to point A is $\vec{r}_{B/A} \equiv \vec{r}_B - \vec{r}_A$. Similarly the relative velocity and acceleration of two points A and B is defined to be

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A \quad \text{and} \quad \vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A \tag{7.33}$$

So, the relative velocity (as calculated relative to a fixed frame) of two points glued to one spinning rigid body $\mathcal{B}$ is given by

$$\begin{aligned} \vec{v}_{B/A} &\equiv \vec{v}_B - \vec{v}_A && (7.34)\\ &= \vec{\omega} \times \vec{r}_{B/O} - \vec{\omega} \times \vec{r}_{A/O} && (7.35)\\ &= \vec{\omega} \times (\vec{r}_{B/O} - \vec{r}_{A/O}) && (7.36)\\ &= \vec{\omega} \times \vec{r}_{B/A}, && (7.37)\\ & && (7.38) \end{aligned}$$

where point O is the point in the Newtonian frame on the fixed axis of rotation and $\vec{\omega} = \vec{\omega}_{\mathcal{B}}$ is the angular velocity of $\mathcal{B}$. Clearly, since points A and B are fixed on $\mathcal{B}$ their velocities and hence their relative velocity as observed in a reference frame fixed to $\mathcal{B}$ is $\vec{0}$. But, point A has some absolute velocity that is different from the absolute velocity of point B. So they have a relative velocity as seen in the fixed frame. And it is what you would expect if B was just going in circles around A. Similarly, the relative acceleration of two points glued to one rigid body spinning at constant rate is

$$\vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A = -\omega^2 \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}). \tag{7.39}$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point O fixed on the axis. These kinematics results, 8.14 and 8.15, are useful for calculating angular momentum relative to the center of mass. They are also sometimes useful for the understanding of the motions of machines with moving connected parts.

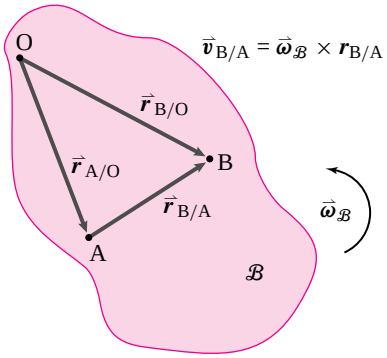


Figure 7.30: The acceleration of B relative to A if they are both on the same rotating rigid body.

(Filename:tfigure4.vel.accel.rel)

Another definition of $\vec{\omega}$

For two points on one rigid body we have that

$$\dot{\vec{r}}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}. \tag{7.40}$$

This last equation (8.16) is perhaps the most fundamental equation for those desiring a deeper understanding of rotation. In three dimensions, unless one uses matrix representations of rotation, equation (8.16) is *the* defining equation for the angular velocity $\vec{\omega}$ of a rigid body.

Calculating relative velocity directly, using rotating frames

A coordinate system $x'y'$ to a rotating rigid body $\mathcal{C}$, defines a reference frame $\mathcal{C}$ (Fig. 8.9). Recall, the base vectors in this frame change in time by

$$\frac{d}{dt}\hat{\imath}' = \vec{\omega}_{\mathcal{C}} \times \hat{\imath}' \quad \text{and} \quad \frac{d}{dt}\hat{\jmath}' = \vec{\omega}_{\mathcal{C}} \times \hat{\jmath}'.$$

If we now write the relative position of B to A in terms of $\hat{\imath}'$ and $\hat{\jmath}'$, we have

$$\vec{r}_{B/A} = x'\hat{\imath}' + y'\hat{\jmath}'.$$

Since the coordinates x' and y' rotate with the body to which A and B are attached, they are constant with respect to that body,

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

So

$$\begin{aligned}
\frac{d}{dt}(\vec{r}_{B/A}) &= \frac{d}{dt}\left(x'\hat{\imath}' + y'\hat{\jmath}'\right) \\
&= \underbrace{\dot{x}'}_{0}\hat{\imath}' + x'\frac{d}{dt}\hat{\imath}' + \underbrace{\dot{y}'}_{0}\hat{\jmath}' + y'\frac{d}{dt}\hat{\jmath}' \\
&= x'(\vec{\omega}_{\mathcal{C}} \times \hat{\imath}') + y'(\vec{\omega}_{\mathcal{C}} \times \hat{\jmath}') \\
&= \vec{\omega}_{\mathcal{C}} \times \underbrace{(x'\hat{\imath}' + y'\hat{\jmath}')}_{\vec{r}_{B/A}} \\
&= \vec{\omega}_{\mathcal{C}} \times \vec{r}_{B/A}.
\end{aligned}$$

We could similarly calculate $\vec{a}_{\mathrm{B/A}}$ by taking another derivative to get

$$\vec{a}_{\mathrm{B/A}} = \vec{\omega}_{\mathcal{C}} \times \left(\vec{\omega}_{\mathcal{C}} \times \vec{r}_{B/A}\right) + \dot{\vec{\omega}}_{\mathcal{C}} \times \vec{r}_{B/A}.$$

The concept of measuring velocities and accelerations relative to a rotating frame will be of central interest chapters 10 and 11.

7.2 Plato's discussion of spinning in circles

In discussion of an object maintaining contradictory attributes simultaneously . . .

" Socrates: Now let's have a more precise agreement so that we won't have any grounds for dispute as we proceed. If someone were to say of a human being standing still, but moving his hands and head, that the same man at the same time stands still and moves, I don't suppose we'd claim that it should be said like that, but rather that one part of him stands still and another moves. Isn't that so?

Glaucon: Yes it is.

Socrates: Then if the man who says this should become still more charming and make the subtle point that tops as wholes stand still and move at the same time when the peg is fixed in the same place and they spin, or that anything else going around in a circle on the same spot does this too, we wouldn't accept it because it's not with respect to the same part of themselves that such things are at the same time both at rest and in motion. But we'd say that they have in them both a straight and a circumference; and with respect to the straight they stand still since they don't lean in any direction –while with respect to the circumference they move in a circle; and when the straight inclines to the right the left, forward, or backward at the same time that it's spinning, then in no way does it stand still.
Glaucon: And we'd be right."

This chapter is about things that are still with respect to their own parts (they do not distort) but in which the points do move in circles.

SAMPLE 7.9 A uniform bar AB of length $\ell = 50\,\text{cm}$ rotates counterclockwise about point A with constant angular speed ω. At the instant shown in Fig 7.31 the linear speed v_C of the center of mass C is $7.5\,\text{cm/s}$.

(a) What is the angular speed of the bar?
(b) What is the angular velocity of the bar?
(c) What is the linear velocity of end B?
(d) By what angles do the angular positions of points C and B change in 2 seconds?

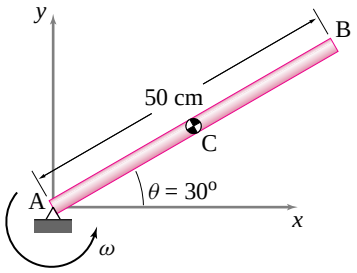


Figure 7.31: (Filename:sfig4.4.1)

Solution Let the angular velocity of the bar be $\vec{\omega} = \dot{\theta}\hat{k}$.

(a) Angular speed of the bar = $\dot{\theta}$. The linear speed of point C is $v_C = 7.5\,\text{cm/s}$. Now,

$$\begin{aligned} v_C &= \dot{\theta}\, r_C \\ \Rightarrow \quad \dot{\theta} &= \frac{v_C}{r_C} = \frac{7.5\,\text{cm/s}}{25\,\text{cm}} = 0.3\,\text{rad/s}. \end{aligned}$$

$$\boxed{\dot{\theta} = 0.3\,\text{rad/s}}$$

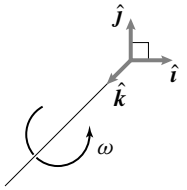


Figure 7.32: (Filename:sfig4.4.1a)

(b) The angular velocity of the bar is $\vec{\omega} = \dot{\theta}\hat{k} = 0.3\,\text{rad/s}\hat{k}$.

$$\boxed{\vec{\omega} = 0.3\,\text{rad/s}\hat{k}}$$

(c)

$$\begin{aligned} \vec{v}_B &= \vec{\omega} \times \vec{r}_B = \dot{\theta}\hat{k} \times \ell(\cos\theta\hat{i} + \sin\theta\hat{j}) \\ &= \dot{\theta}\ell(\cos\theta\hat{j} - \sin\theta\hat{i}) \\ &= 0.3\,\text{rad/s}\cdot 50\,\text{cm}(\frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i}) \\ &= 15\,\text{cm/s}(\frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i}). \end{aligned}$$

$$\boxed{\vec{v}_B = 15\,\text{cm/s}(\tfrac{\sqrt{3}}{2}\hat{j} - \tfrac{1}{2}\hat{i})}$$

We can also write $\vec{v}_B = 15\,\text{cm/s}\hat{e}_\theta$ where $\hat{e}_\theta = \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i}$.

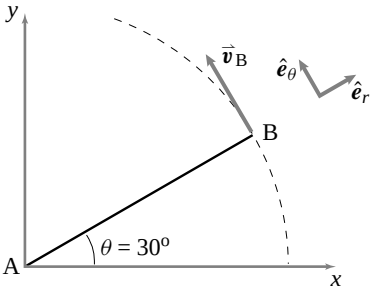


Figure 7.33:
$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$
$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$
$\vec{v}_B = |\vec{v}_B|\hat{e}_\theta$
(Filename:sfig4.4.1b)

(d) Let θ_1 be the position of point C at some time t_1 and θ_2 be the position at time t_2. We want to find $\Delta\theta = \theta_2 - \theta_1$ for $t_2 - t_1 = 2\,\text{s}$.

$$\begin{aligned} \frac{d\theta}{dt} &= \dot{\theta} = \text{constant} = 0.3\,\text{rad/s}. \\ \Rightarrow \quad d\theta &= (0.3\,\text{rad/s})dt. \\ \Rightarrow \quad \int_{\theta_1}^{\theta_2} d\theta &= \int_{t_1}^{t_2} (0.3\,\text{rad/s})dt. \\ \Rightarrow \quad \theta_2 - \theta_1 &= 0.3\,\text{rad/s}(t_2 - t_1) \\ \text{or} \quad \Delta\theta &= 0.3\frac{\text{rad}}{\cancel{s}}\cdot 2\cancel{s} = 0.6\,\text{rad}. \end{aligned}$$

The change in position of point B is the same as that of point C. In fact, all points on AB undergo the same change in angular position because AB is a rigid body.

$$\boxed{\Delta\theta_C = \Delta\theta_B = 0.6\,\text{rad}}$$

SAMPLE 7.10 A flywheel of diameter 2 ft is made of cast iron. To avoid extremely high stresses and cracks it is recommended that the peripheral speed not exceed 6000 to 7000 ft/min. What is the corresponding rpm rating for the wheel?

Solution

$$\begin{aligned} \text{Diameter of the wheel} &= 2\,\text{ft.} \\ \Rightarrow \quad \text{radius of wheel} &= 1\,\text{ft.} \end{aligned}$$

Now,

$$\begin{aligned} v &= \omega r \\ \Rightarrow \quad \omega &= \frac{v}{r} = \frac{6000\,\text{ft/min}}{1\,\text{ft}} \\ &= 6000 \frac{\text{rad}}{\text{min}} \cdot \frac{1\text{rev}}{2\pi\ \text{rad}} \\ &= 955\,\text{rpm.} \end{aligned}$$

Similarly, corresponding to $v = 7000\,\text{ft/min}$

$$\begin{aligned} \omega &= \frac{7000\,\text{ft/min}}{1\,\text{ft}} \\ &= 7000 \frac{\text{rad}}{\text{min}} \cdot \frac{1\text{rev}}{2\pi\ \text{rad}} \\ &= 1114\,\text{rpm.} \end{aligned}$$

Thus the rpm rating of the wheel should read 955 – 1114 rpm.

$$\boxed{\omega = 955 \text{ to } 1114\,\text{rpm.}}$$

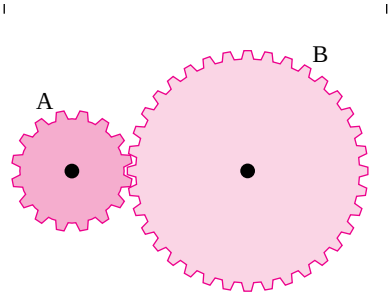


Figure 7.34: (Filename:sfig4.4.3)

SAMPLE 7.11 Two gears A and B have the diameter ratio of 1:2. Gear A drives gear B. If the output at gear B is required to be 150 rpm, what should be the angular speed of the driving gear? Assume no slip at the contact point.

Solution Let C and C′ be the points of contact on gear A and B respectively at some instant t. Since there is no relative slip between C and C′, both points must have the same linear velocity at instant t. If the velocities are the same, then the linear speeds must also be the same. Thus

$$\begin{aligned} v_C &= v_{C'} \\ \Rightarrow \quad \omega_A r_A &= \omega_B r_B \\ \Rightarrow \quad \omega_A &= \omega_B \cdot \frac{r_B}{r_A} \\ &= \omega_B \cdot \frac{2r}{r} = 2\omega_B \\ &= (2)\cdot(150\,\text{rpm}) \\ &= 300\,\text{rpm.} \end{aligned}$$

$$\boxed{\omega_A = 300\,\text{rpm}}$$

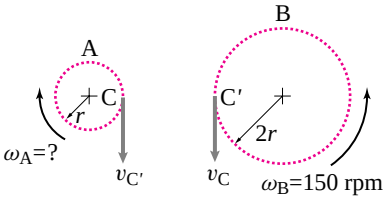


Figure 7.35: (Filename:sfig4.4.3a)

SAMPLE 7.12 A uniform rigid rod AB of length $\ell = 0.6\,\text{m}$ is connected to two rigid links OA and OB. The assembly rotates about point O at a constant rate. At the instant shown, when rod AB is vertical, the velocities of points A and B are $\vec{v}_A = -4.64\,\text{m/s}\hat{j} - 1.87\,\text{m/s}\hat{i}$, and $\vec{v}_B = 1.87\,\text{m/s}\hat{i} - 4.64\,\text{m/s}\hat{j}$. Find the angular velocity of bar AB. What is the length R of the links?

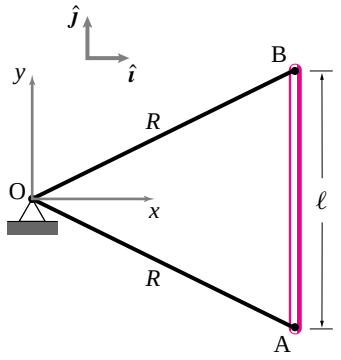


Figure 7.36: (Filename:sfig4.4.4)

Solution Let the angular velocity of the rod AB be $\vec{\omega} = \omega\hat{k}$. ① Since we are given the velocities of two points on the rod we can use the relative velocity formula to find $\vec{\omega}$:

$$\begin{aligned}
\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A} &= \vec{v}_B - \vec{v}_A \\
\text{or} \quad \underbrace{\omega\hat{k}}_{\vec{\omega}} \times \underbrace{\ell\hat{j}}_{\vec{r}_{B/A}} &= (1.87\hat{i} - 4.64\hat{j})\,\text{m/s} - (-4.64\hat{j} - 1.87\hat{i})\,\text{m/s} \\
\text{or} \quad \omega\ell(-\hat{i}) &= (1.87\hat{i} + 1.87\hat{i})\,\text{m/s} - (4.64\hat{j} - 4.64\hat{j})\,\text{m/s} \\
&= 3.74\hat{i}\,\text{m/s} \\
\Rightarrow \quad \omega &= -\frac{3.74\,\text{m/s}}{\ell} \\
&= -\frac{3.74}{0.6}\,\text{rad/s} \\
&= -6.233\,\text{rad/s} \qquad (7.41)
\end{aligned}$$

Thus, $\vec{\omega} = -6.233\,\text{rad/s}\hat{k}$.

$$\boxed{\vec{\omega} = -6.23\,\text{rad/s}\hat{k}}$$

Let θ be the angle between link OA and the horizontal axis. Now,

$$\begin{aligned}
\vec{v}_A = \vec{\omega} \times \vec{r}_A &= \omega\hat{k} \times \underbrace{R(\cos\theta\hat{i} - \sin\theta\hat{j})}_{\vec{r}_A} \\
\text{or} \quad (-4.64\hat{j} - 1.87\hat{i})\,\text{m/s} &= \omega R(\cos\theta\hat{j} + \sin\theta\hat{i})
\end{aligned}$$

Dotting both sides of the equation with $\hat{i}$ and $\hat{j}$ we get

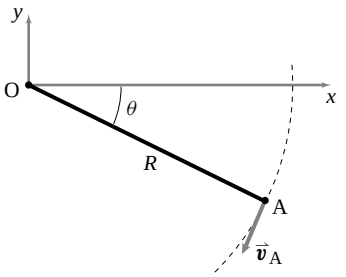
$$\begin{aligned}
-1.87\,\text{m/s} &= \omega R\sin\theta \qquad (7.42) \\
-4.64\,\text{m/s} &= \omega R\cos\theta \qquad (7.43)
\end{aligned}$$

Squaring and adding Eqns (7.42) and (7.43) together we get

$$\begin{aligned}
\omega^2 R^2 &= (-4.64\,\text{m/s})^2 + (-1.187\,\text{m/s})^2 \\
&= 25.026\,\text{m}^2/\,\text{s}^2 \\
\Rightarrow \quad R^2 &= \frac{25.026\,\text{m}^2/\,\text{s}^2}{(-6.23\,\text{rad/s})^2} = 0.645\,\text{m}^2 \\
\Rightarrow \quad R &= 0.8\,\text{m}
\end{aligned}$$

$$\boxed{R = 0.8\,\text{m}}$$

① We know that the rod rotates about the z-axis but we do not know the sense of the rotation *i.e.*, $+\hat{k}$ or $-\hat{k}$. Here we have assumed that $\vec{\omega}$ is in the positive $\hat{k}$ direction, although just by sketching $\vec{v}_A$ we can easily see that $\vec{\omega}$ must be in the $-\hat{k}$ direction.



$\vec{v}_\text{A} = \vec{\omega} \times \vec{r}_\text{A}$
and $\vec{r}_\text{A} = R\cos\theta\hat{i} - R\sin\theta\hat{j}$

Figure 7.37: (Filename:sfig4.4.4a)

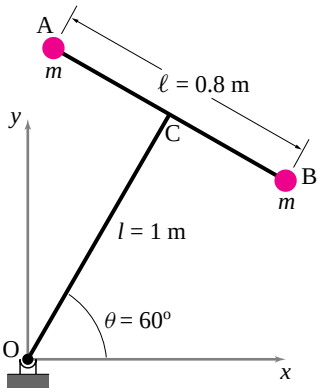


Figure 7.38: (Filename:sfig4.3.1)

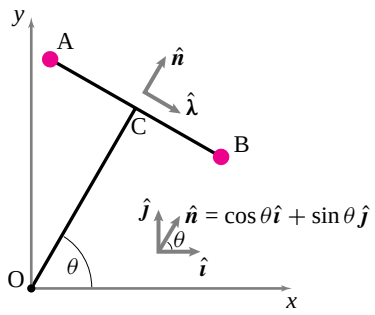


Figure 7.39: (Filename:sfig4.3.1a)

① The vector $\vec{r}_{B/A}$ may also be expressed directly in terms of unit vectors $\hat{\imath}$ and $\hat{\jmath}$, but it involves a little bit more geometry. Note how assuming $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ in the directions shown makes calculations easier and cleaner.

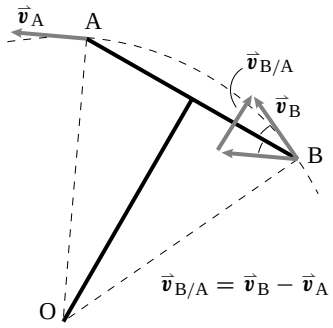


Figure 7.40: (Filename:sfig4.3.1b)

SAMPLE 7.13 A dumbbell AB, made of two equal masses and a rigid rod AB of negligible mass, is welded to a rigid arm OC such that OC is perpendicular to AB. Arm OC rotates about O at a constant angular velocity $\vec{\boldsymbol{\omega}} = 10\,\text{rad/s}\hat{\boldsymbol{k}}$. At the instant when $\theta = 60^o$, find the relative velocity of B with respect to A.

Solution Since A and B are two points on the same rigid body (AB) and the body is spinning about point O at a constant rate, we may use the relative velocity formula

$$\vec{\boldsymbol{v}}_{B/A} \equiv \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \tag{7.44}$$

to find the relative velocity of B with respect to A. We are given $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} = 10\,\text{rad/s}\hat{\boldsymbol{k}}$. Let $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ be unit vectors parallel to AB and OC respectively. Since OC$\perp$AB, we have $\hat{\boldsymbol{n}} \perp \hat{\boldsymbol{\lambda}}$. Now we may write vector $\vec{\boldsymbol{r}}_{B/A}$ as ①

$$\vec{\boldsymbol{r}}_{B/A} = \ell\hat{\boldsymbol{\lambda}}$$

Substituting $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{r}}_{B/A}$ in Eqn (7.44) we get

$$\begin{aligned}
\vec{\boldsymbol{v}}_{B/A} &= \omega\hat{\boldsymbol{k}} \times \ell\hat{\boldsymbol{\lambda}} \\
&= \omega\ell\,\underbrace{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}})}_{\hat{\boldsymbol{n}}} \\
&= \omega\ell\hat{\boldsymbol{n}} \\
&= \omega\ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\
&= 10\,\text{rad/s}(0.8\,\text{m})\cdot(\frac{1}{2}\hat{\boldsymbol{\imath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}}) \\
&= 4\,\text{m/s}(\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}})
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_{B/A} = 4\,\text{m/s}(\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}})}$$

Comments: $\vec{\boldsymbol{v}}_{B/A}$ can also be obtained by adding vectors $\vec{\boldsymbol{v}}_B$ and $-\vec{\boldsymbol{v}}_A$ geometrically. Since A and B execute circular motion with the same radius $R = OA = OB$, the magnitudes of $\vec{\boldsymbol{v}}_B$ and $\vec{\boldsymbol{v}}_A$ are the same ($= \omega R$) and since the velocity in circular motion is tangential to the circular path, $\vec{\boldsymbol{v}}_A \perp OA$ and $\vec{\boldsymbol{v}}_B \perp OB$. Then moving $\vec{\boldsymbol{v}}_A$ to point B, we can easily find $\vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A = \vec{\boldsymbol{v}}_{B/A}$. Its direction is found to be perpendicular to AB, *i.e.*, along OC.

SAMPLE 7.14 For the same problem and geometry as in Sample 7.13, find the acceleration of point B relative to point A.

Solution Since points A and B are on the same rigid body AB which is rotating at a constant rate $\omega = 10\,\text{rad/s}$, the relative acceleration of B is:

$$\begin{aligned}\vec{a}_{B/A} &= \vec{a}_B - \vec{a}_A = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \\ &= \omega\hat{k} \times (\omega\hat{k} \times \ell\hat{\lambda}) \\ &= \omega\hat{k} \times \omega\ell\hat{n} \quad (\text{since } \hat{k} \times \hat{\lambda} = \hat{n}) \\ &= \omega^2\ell(\hat{k} \times \hat{n}) \\ &= \omega^2\ell(-\hat{\lambda})\end{aligned}$$

①Now we need to express $\hat{\lambda}$ in terms of known basis vectors $\hat{\imath}$ and $\hat{\jmath}$. If you are good with geometry, then by knowing that $\hat{\lambda} \perp \hat{n}$ and $\hat{n} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ you can immediately write

$$\hat{\lambda} = \sin\theta\hat{\imath} - \cos\theta\hat{\jmath} \quad (\text{so that } \hat{\lambda}\cdot\hat{n} = 0).$$

Or you may draw a big and clear picture of $\hat{\lambda}$, $\hat{n}$, $\hat{\imath}$ and $\hat{\jmath}$ and label the angles as shown in Fig 7.41. Then, it is easy to see that

$$\hat{\lambda} = \sin\theta\hat{\imath} - \cos\theta\hat{\jmath}.$$

Substituting for $\hat{\lambda}$ in the expression for $\vec{a}_{B/A}$, we get

$$\begin{aligned}\vec{a}_{B/A} &= -\omega^2\ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \\ &= -100\frac{\text{rad}^2}{\text{s}^2}\cdot\left[0.8\,\text{m}(\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath})\right] \\ &= -40\,\text{m/s}^2(\sqrt{3}\hat{\imath} - \hat{\jmath})\end{aligned}$$

$$\boxed{\vec{a}_{B/A} = -40\,\text{m/s}^2(\sqrt{3}\hat{\imath} - \hat{\jmath})}$$

Comments: We could also find $\vec{a}_{B/A}$ using geometry and geometric addition of vectors. Since A and B are going in circles about O at constant speed, their accelerations are centripetal accelerations. Thus, $\vec{a}_A$ points along AO and $\vec{a}_B$ points along BO. Also $|\vec{a}_A| = |\vec{a}_B| = \omega^2(\text{OA})$. Now adding $-\vec{a}_A$ to $\vec{a}_B$ we get $\vec{a}_{B/A}$ which is seen to be along BA.

① If a body rotates in a plane, *i.e.*, $\vec{\omega} = \omega\hat{k}$, then $\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2\vec{r}$. Using this fact we can immediately write $\vec{a}_{B/A} = -\omega^2\vec{r}_{B/A} = -\omega^2\ell\hat{\lambda}$.

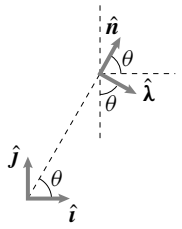


Figure 7.41: The geometry of vectors $\hat{\imath}$ and $\hat{n}$.
$\hat{\lambda} = -\cos\theta\hat{\jmath} + \sin\theta\hat{\imath}$
(Filename:sfig4.3.2)

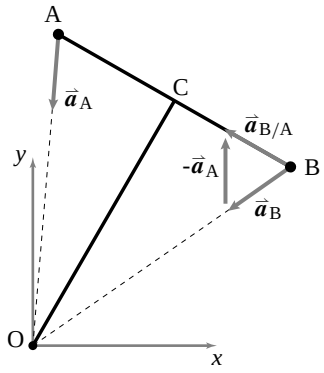


Figure 7.42: (Filename:sfig4.3.2a)

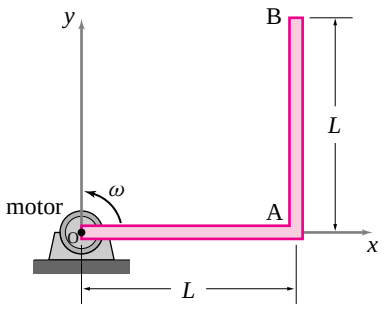


Figure 7.43: An 'L' shaped bar rotates at speed ω about point O.

(Filename:sfig5.3.1a)

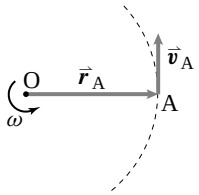


Figure 7.44: $\vec{v}_A = \vec{\omega} \times \vec{r}_A$ is tangential to the circular path of point A.

(Filename:sfig5.3.1b)

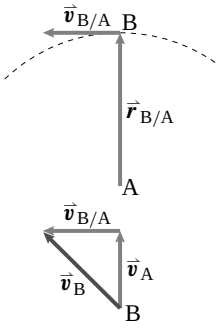


Figure 7.45: $\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$ and $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$.

(Filename:sfig5.3.1c)

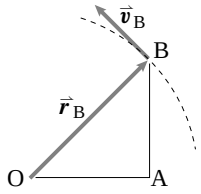


Figure 7.46: $\vec{v}_B = \vec{\omega} \times \vec{r}_B$.

(Filename:sfig5.3.1d)

SAMPLE 7.15 *Test the velocity formula on something you know.* The motor at O in Fig. 7.43 rotates the 'L' shaped bar OAB in counterclockwise direction at an angular speed which increases at $\dot{\omega} = 2.5\,\text{rad/s}^2$. At the instant shown, the angular speed $\omega = 4.5\,\text{rad/s}$. Each arm of the bar is of length L = 2 ft.

(a) Find the velocity of point A.

(b) Find the relative velocity $\vec{v}_{B/A}$ $(= \vec{\omega} \times \vec{r}_{B/A})$ and use the result to find the absolute velocity of point B $(\vec{v}_B = \vec{v}_A + \vec{v}_{B/A})$.

(c) Find the velocity of point B directly. Check the answer obtained in part (b) against the new answer.

Solution

(a) As the bar rotates, every point on the bar goes in circles centered at point O. Therefore, we can easily find the velocity of any point on the bar using circular motion formula $\vec{v} = \vec{\omega} \times \vec{r}$. Thus,

$$\begin{aligned} \vec{v}_A &= \vec{\omega} \times \vec{r}_A = \omega\hat{k} \times L\hat{\imath} = \omega L\hat{\jmath} \\ &= 4.5\,\text{rad/s} \cdot 2\,\text{ft}\hat{\jmath} = 9\,\text{ft/s}\hat{\jmath}. \end{aligned}$$

The velocity vector $\vec{v}_A$ is shown in Fig. 7.44.

$$\boxed{\vec{v}_A = 9\,\text{ft/s}\hat{\jmath}}$$

(b) Point B and A are on the same rigid body. Therefore, with respect to point A, point B goes in circles about A. Hence the relative velocity of B with respect to A is

$$\begin{aligned} \vec{v}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A} \\ &= \omega\hat{k} \times L\hat{\jmath} = -\omega L\hat{\imath} \\ &= -4.5\,\text{rad/s} \cdot 2\,\text{ft}\hat{\imath} = -9\,\text{ft/s}\hat{\imath}. \\ \text{and} \quad \vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} \\ &= 9\,\text{ft/s}(-\hat{\imath} + \hat{\jmath}). \end{aligned}$$

These velocities are shown in Fig. 7.45.

$$\boxed{\vec{v}_{B/A} = -9\,\text{ft/s}\hat{\imath},\ \vec{v}_B = 9\,\text{ft/s}(-\hat{\imath} + \hat{\jmath})}$$

(c) Since point B goes in circles of radius OB about point O, we can find its velocity directly using circular motion formula:

$$\begin{aligned} \vec{v}_B &= \vec{\omega} \times \vec{r}_B \\ &= \omega\hat{k} \times (L\hat{\imath} + L\hat{\jmath}) = \omega L(\hat{\jmath} - \hat{\imath}) \\ &= 9\,\text{ft/s}(-\hat{\imath} + \hat{\jmath}). \end{aligned}$$

The velocity vector is shown in Fig. 7.46. Of course this velocity is the same velocity as obtained in part (b) above.

$$\boxed{\vec{v}_B = 9\,\text{ft/s}(-\hat{\imath} + \hat{\jmath})}$$

Note: Nothing in this sample uses $\dot{\omega}$!

SAMPLE 7.16 *Test the acceleration formula on something you know.* Consider the 'L' shaped bar of Sample 7.15 again. At the instant shown, the bar is rotating at 4 rad/s and is slowing down at the rate of 2 rad/s^2.

(i) Find the acceleration of point A.
(ii) Find the relative acceleration $\vec{a}_{B/A}$ of point B with respect to point A and use the result to find the absolute acceleration of point B ($\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$).
(iii) Find the acceleration of point B directly and verify the result obtained in (ii).

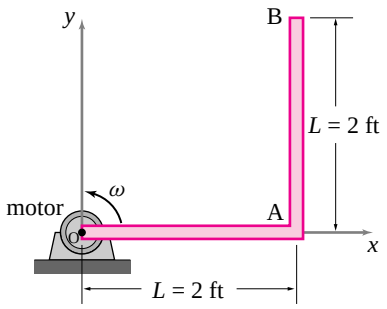


Figure 7.47: The 'L' shaped bar is rotating counterclockwise and is slowing down.
(Filename:sfig5.3.2a)

Solution We are given:

$$\vec{\omega} = \omega\hat{k} = 4\,\text{rad/s}\hat{k}, \quad \text{and} \quad \dot{\vec{\omega}} = -\dot{\omega}\hat{k} = -2\,\text{rad/s}^2\hat{k}.$$

(i) Point A is going in circles of radius L. Hence,

$$\begin{aligned}\vec{a}_A &= \dot{\vec{\omega}} \times \vec{r}_A + \vec{\omega} \times (\vec{\omega} \times \vec{r}_A) = \dot{\vec{\omega}} \times \vec{r}_A - \omega^2 \vec{r}_A \\ &= -\dot{\omega}\hat{k} \times L\hat{\imath} - \omega^2 L\hat{\imath} = -\dot{\omega}L\hat{\jmath} - \omega^2 L\hat{\imath} \\ &= -2\,\text{rad/s} \cdot 2\,\text{ft}\hat{\jmath} - (4\,\text{rad/s})^2 \cdot 2\,\text{ft}\hat{\imath} \\ &= -(4\hat{\jmath} + 32\hat{\imath})\,\text{ft/s}^2.\end{aligned}$$

$$\boxed{\vec{a}_A = -(4\hat{\jmath} + 32\hat{\imath})\,\text{ft/s}^2}$$

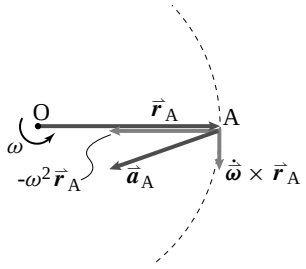


Figure 7.48: (Filename:sfig5.3.2b)

(ii) The relative acceleration of point B with respect to point A is found by considering the motion of B with respect to A. Since both the points are on the same rigid body, point B executes circular motion with respect to point A. Therefore,

$$\begin{aligned}\vec{a}_{B/A} &= \dot{\vec{\omega}} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) = \dot{\vec{\omega}} \times \vec{r}_{B/A} - \omega^2 \\ &= -\dot{\omega}\hat{k} \times L\hat{\jmath} - \omega^2 L\hat{\jmath} \\ &= \dot{\omega}L\hat{\imath} - \omega^2 L\hat{\jmath} = 2\,\text{rad/s}^2 \cdot 2\,\text{ft}\hat{\imath} - (4\,\text{rad/s})^2 \cdot 2\,\text{ft}\hat{\jmath} \\ &= (4\hat{\imath} - 32\hat{\jmath})\,\text{ft/s}^2,\end{aligned}$$

and

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = (-28\hat{\imath} - 36\hat{\jmath})\,\text{ft/s}^2.$$

$$\boxed{\vec{a}_B = -(28\hat{\imath} + 36\hat{\jmath})\,\text{ft/s}^2}$$

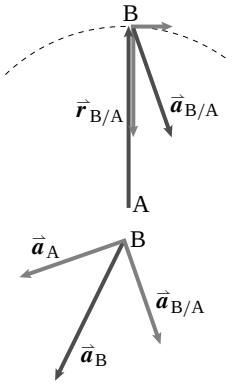


Figure 7.49: (Filename:sfig5.3.2c)

(iii) Since point B is going in circles of radius OB about point O, we can find the acceleration of B as follows.

$$\begin{aligned}\vec{a}_B &= \dot{\vec{\omega}} \times \vec{r}_B + \vec{\omega} \times (\vec{\omega} \times \vec{r}_B) \\ &= \dot{\vec{\omega}} \times \vec{r}_B - \omega^2 \vec{r}_B \\ &= -\dot{\omega}\hat{k} \times (L\hat{\imath} + L\hat{\jmath}) - \omega^2(L\hat{\imath} + L\hat{\jmath}) \\ &= (-\dot{\omega}L - \omega^2 L)\hat{\jmath} + (\dot{\omega}L - \omega^2 L)\hat{\imath} \\ &= (-4 - 32)\,\text{ft/s}^2\hat{\jmath} + (4 - 32)\,\text{ft/s}^2\hat{\imath} \\ &= (-36\hat{\jmath} - 28\hat{\imath})\,\text{ft/s}^2.\end{aligned}$$

This acceleration is, naturally again, the same acceleration as found in (ii) above.

$$\boxed{\vec{a}_B = -(28\hat{\imath} + 36\hat{\jmath})\,\text{ft/s}^2}$$

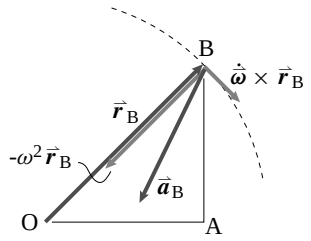


Figure 7.50: (Filename:sfig5.3.2d)

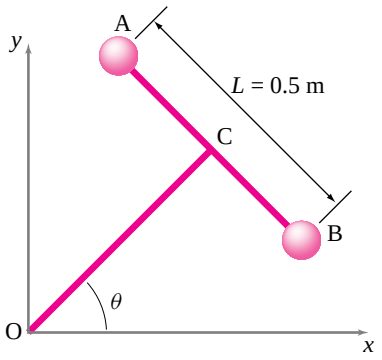


Figure 7.51: *Relative velocity and acceleration:*

(Filename:sfig5.3.3)

SAMPLE 7.17 The dumbbell AB shown in the figure rotates counterclockwise about point O with angular acceleration $3\,\text{rad/s}^2$. Bar AB is perpendicular to bar OC. At the instant of interest, $\theta = 45^\text{o}$ and the angular speed is 2 rad/s.

(a) Find the velocity of point B relative to point A. Will this relative velocity be different if the dumbbell were rotating at a *constant* rate of 2 rad/s?

(b) Without calculations, draw a vector approximately representing the acceleration of B relative to A.

(c) Find the acceleration of point B relative to A. What can you say about the direction of this vector as the motion progresses in time?

Solution

(a) **Velocity of B relative to A:**

$$\begin{aligned}\vec{\boldsymbol{v}}_{B/A} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \\ &= \dot{\theta}\hat{\boldsymbol{k}} \times L(\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) \\ &= \dot{\theta}L(\sin\theta\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\boldsymbol{\imath}}) \\ &= 2\,\text{rad/s}\cdot 0.5\,\text{m}(\sin 45^\text{o}\hat{\boldsymbol{\jmath}} + \cos 45^\text{o}\hat{\boldsymbol{\imath}}) \\ &= 0.707\,\text{m/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).\end{aligned}$$

Thus the relative velocity is perpendicular to AB, that is, parallel to OC.

No, the relative velocity will not be any different at the instant of interest if the dumbbell were rotating at constant rate. As is evident from the formula, the relative velocity only depends on $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{r}}_{B/A}$, and not on $\dot{\vec{\boldsymbol{\omega}}}$. Therefore, $\vec{\boldsymbol{v}}_{B/A}$ will be the same if at the instant of interest, $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{r}}_{B/A}$ are the same.

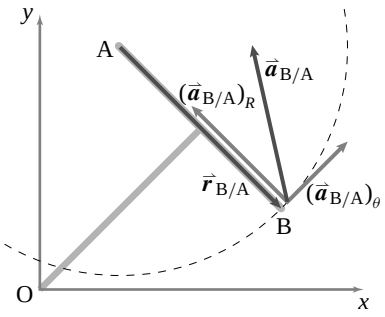


Figure 7.52: To draw the relative acceleration of B, $\vec{\boldsymbol{a}}_{B/A}$, consider point B going in circles about point A.

(Filename:sfig5.3.3a)

(b) **Relative acceleration vector:** The velocity and acceleration of some point B on a rigid body *relative* to some other point A on the same body is the same as the velocity and acceleration of B if the body is considered to rotate about point A with the same angular velocity and acceleration as given. Therefore, to find the relative velocity and acceleration of B, we take A to be the center of rotation and draw the circular path of B, and then draw the velocity and acceleration vectors of B.

Since we know that the acceleration of a point under circular motion has tangential ($\dot{\vec{\boldsymbol{\omega}}}\times\vec{\boldsymbol{r}}$ or $\ddot{\theta}\,R\hat{\boldsymbol{e}}_\theta$ in 2-D) and radial or centripetal ($\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$ or $-\dot{\theta}^2 R\hat{\boldsymbol{e}}_R$ in 2-D) components, the total acceleration being the vector sum of these components, we draw an approximate acceleration vector of point B as shown in Fig. 7.52.

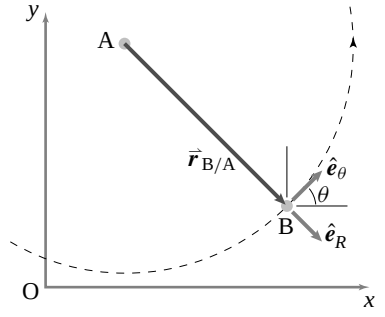


Figure 7.53: The geometry of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ for the imagined motion of B about A. $\hat{\boldsymbol{e}}_R = \sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{e}}_\theta = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$.

(Filename:sfig5.3.3b)

(c) **Acceleration of B relative to A:**

$$\begin{aligned}\vec{\boldsymbol{a}}_{B/A} &= \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}) \\ &= \ddot{\theta}\hat{\boldsymbol{k}} \times L\hat{\boldsymbol{e}}_R + \dot{\theta}\hat{\boldsymbol{k}} \times (\dot{\theta}\hat{\boldsymbol{k}} \times L\hat{\boldsymbol{e}}_R) \\ &= L\,\ddot{\theta}\hat{\boldsymbol{e}}_\theta - L\,\dot{\theta}^2\hat{\boldsymbol{e}}_R \\ &= 0.5\,\text{m}\cdot 3\,\text{rad/s}^2(\cos 45^\text{o}\hat{\boldsymbol{\imath}} + \sin 45^\text{o}\hat{\boldsymbol{\jmath}}) \\ &\quad -0.5\,\text{m}\cdot(2\,\text{rad/s})^2(\sin 45^\text{o}\hat{\boldsymbol{\imath}} - \cos 45^\text{o}\hat{\boldsymbol{\jmath}}) \\ &= 1.061\,\text{m/s}^2(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) - 1.414\,\text{m/s}^2(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}) \\ &= (-0.353\hat{\boldsymbol{\imath}} + 2.474\hat{\boldsymbol{\jmath}})\,\text{m/s}^2.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_{B/A} = (-0.353\hat{\boldsymbol{\imath}} + 2.474\hat{\boldsymbol{\jmath}})\,\text{m/s}^2}$$

7.4 Dynamics of a rigid body in planar circular motion

Our goal here is to set up and evaluate in simple terms, the momentum, angular momentum, and energy balance equations for a planar body that is rotating about one point. The most common application is the mechanical analysis of a part held in place by a hinge or bearing.

Mechanics

If we can calculate the velocity and acceleration of every point in a system, we can evaluate all the momentum and energy terms in the equations of motion (inside cover), namely: $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_C$, $\dot{\vec{H}}_C$, E_K and $\dot{E}_K$ for any reference point C of our choosing. For rotational motion these calculations are a little more complex than the special case of straight-line motion in chapter 6, where all points in a system had the same acceleration as each other.

For circular motion of a rigid body, we just well-learned in the previous section that the velocities and accelerations are

$$\begin{aligned} \vec{v} &= \vec{\omega} \times \vec{r}, \\ \vec{a} &= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}), \end{aligned}$$

where $\vec{\omega}$ is the angular velocity of the body relative to a fixed frame and $\vec{r}$ is the position of a point relative to the axis of rotation. These relations apply to every point on a rotating rigid body.

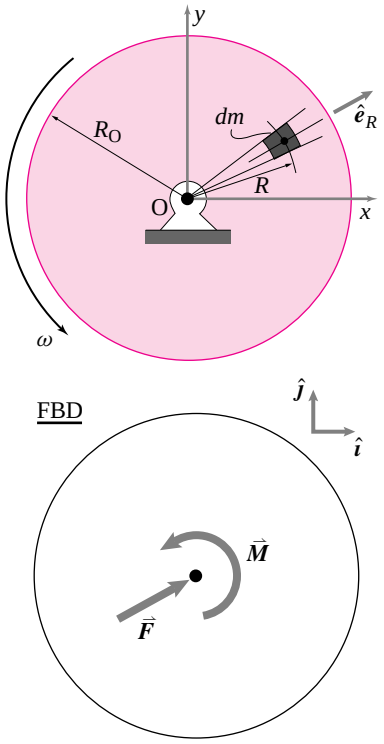


Figure 7.54: A uniform disk turned by a motor at a constant rate.

(Filename:tfigure4.3.motordisk)

Example: **Spinning disk**

The round flat uniform disk in figure 8.18 is in the xy plane spinning at the constant rate $\vec{\omega} = \omega\hat{k}$ about its center. It has mass m_{tot} and radius R_0. What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

$$\sum \vec{F}_i = \dot{\vec{L}} = m_{tot}\vec{a}_{cm} = \vec{0},$$

Which we could also have calculated by evaluating the integral $\dot{\vec{L}} \equiv \int \vec{a}\, dm$ instead of using the general result that $\dot{\vec{L}} = m_{tot}\vec{a}_{cm}$. From angular momentum balance we have:

$$\begin{aligned} \sum \vec{M}_{i/O} &= \dot{\vec{H}}_{/O} \\ \Rightarrow \quad \vec{M} &= \int \vec{r}_{/O} \times \vec{a}\, dm \\ &= \int_0^{R_0} \int_0^{2\pi} (R\hat{e}_R) \times (-R\omega^2 \hat{e}_R) \underbrace{\frac{m_{tot}}{\pi R_O^2} \overbrace{R\, d\theta\, dR}^{dA}}_{dm} \\ &= \int \int \vec{0}\, d\theta\, dR \\ &= \vec{0}. \end{aligned}$$

So the net force and moment needed are $\vec{F} = \vec{0}$ and $\vec{M} = \vec{0}$. Like that a particle moves at constant velocity with no force, it rotates at constant rate with no torque (at least in 2D). □

We'd now like to consider the most general case that the subject of the section allows, an arbitrarily shaped 2D rigid body with arbitrary ω and $\dot{\omega}$.

Linear momentum

For any system in any motion we know, as we have often used, that

$$\vec{\boldsymbol{L}} = m_{\text{tot}}\vec{\boldsymbol{v}}_{\text{cm}} \quad \text{and} \quad \dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}.$$

For a rigid body, the center of mass is a particular point G that is fixed relative to the body. So the velocity and acceleration of that point can be expressed the same way as for any other point. So, for a body in planar rotational motion about 0

$$\vec{\boldsymbol{L}} = m_{\text{tot}}\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{G}/0} \quad \text{and} \quad \dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\left(\dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{\text{G}/0} + -\omega^2\vec{\boldsymbol{r}}_{\text{G}/0})\right).$$

If the center of mass is at 0 the momentum and its rate of change are zero. But if the center of mass is off the axis of rotation, there must be a net force on the object with a component parallel to $\vec{\boldsymbol{r}}_{0/\text{G}}$ (if $\omega \neq 0$) and a component orthogonal to $\vec{\boldsymbol{r}}_{0/\text{G}}$ (if $\dot{\omega} \neq 0$). This net force need not be applied at 0 or G or any other special place on the object.

Angular momentum: $\vec{\boldsymbol{H}}_{\text{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{\text{O}}$

The angular momentum itself is easy enough to calculate,

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\text{O}} &= \int_{\text{all mass}} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\, dm && \text{(a)} \\
&= \int \vec{\boldsymbol{r}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\, dm && \text{(b)} \\
&= \omega\hat{\boldsymbol{k}} \int r^2\, dm && \text{(c)} \\
\Rightarrow \quad H_0 &= \omega \int r^2\, dm. && \text{(d)}
\end{aligned} \tag{7.45}$$

Here eqn. (7.45)c is the vector equation. But since both sides are in the $\hat{\boldsymbol{k}}$ direction we can dot both sides with $\hat{\boldsymbol{k}}$ to get the scalar moment equation eqn. (7.45)d, taking both M_{net} and ω as positive when counterclockwise.

To get the all important angular momentum balance equation for this system we could easily differentiate eqn. (7.45), taking note that the derivative is being taken relative to a fixed frame. More reliably, we use the general expression for $\dot{\vec{\boldsymbol{H}}}_{\text{O}}$ to write the angular momentum balance equation as follows.

$$\begin{aligned}
\text{Net moment}_{/0} &= \text{rate of change of angular momentum}_{/0} && \text{(a)} \\
\vec{\boldsymbol{M}}_{\text{net}} &= \dot{\vec{\boldsymbol{H}}}_{\text{O}} && \text{(b)} \\
&= \int_{\text{all mass}} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}}\, dm && \text{(c)} \\
&= \int \vec{\boldsymbol{r}} \times \left(-\omega^2\vec{\boldsymbol{r}} + \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}\right) dm && \text{(d)} \\
&= \int \vec{\boldsymbol{r}} \times \left(\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}\right) dm && \text{(e)} \\
&= \int \vec{\boldsymbol{r}} \times \left(\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}\right) dm && \text{(f)} \\
\vec{\boldsymbol{M}}_{\text{net}} &= \dot{\omega}\hat{\boldsymbol{k}} \int r^2\, dm && \text{(g)} \\
\Rightarrow \quad M_{\text{net}} &= \dot{\omega} \int r^2\, dm && \text{(h)}
\end{aligned} \tag{7.46}$$

We get from from eqn. (7.46)f to eqn. (7.46)g by noting that $\vec{\boldsymbol{r}}$ is perpendicular to $\hat{\boldsymbol{k}}$. Thus, using the right hand rule twice we get $\vec{\boldsymbol{r}} \times (\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}) = r^2\hat{\boldsymbol{k}}$.

Eqn. 7.46g and eqn. (7.46)h are the vector and scalar versions of the angular momentum balance equation for for rotation of a planar body about 0.

Power and Energy

Although we could treat distributed forces similarly, lets assume that there are a set of point forces applied. And, to be contrary, lets assume the mass is continuously distributed (the derivation for rigidly connected point masses would be similar). The power balance equation for one rotating rigid body is (discussed below):

$$\begin{aligned}
\text{Net power in} &= \text{rate of change of kinetic energy} && \text{(a)}\\
P &= \dot{E}_{\mathrm{K}} && \text{(b)}\\
\sum_{\text{all applied forces}} \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{v}}_i &= \frac{d}{dt}\int_{\text{all mass}} \frac{1}{2} v^2\, dm && \text{(c)}\\
\sum \vec{\boldsymbol{F}}_i \cdot (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_i) &= \frac{d}{dt}\int \frac{1}{2}(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) \cdot (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\, dm && \text{(d)}\\
\sum \vec{\boldsymbol{\omega}} \cdot (\vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i) &= \frac{d}{dt}\int \frac{1}{2}\omega^2 r^2\, dm && \text{(e)}\\
\vec{\boldsymbol{\omega}} \cdot \sum (\vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i) &= \frac{d}{dt}\left(\frac{1}{2}\omega^2\right)\int r^2\, dm && \text{(f)}\\
\vec{\boldsymbol{\omega}} \cdot \sum \vec{\boldsymbol{M}}_i &= \dot{\omega}\omega \int r^2\, dm && \text{(g)}\\
\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{M}}_{\text{tot}} &= \dot{\vec{\boldsymbol{\omega}}} \cdot \underbrace{\left(\vec{\boldsymbol{\omega}} \int r^2\, dm\right)}_{\vec{\boldsymbol{H}}_{/0}} && \text{(h)}
\end{aligned} \tag{7.47}$$

When not notated clearly, positions and moments are relative to the hinge at 0. Derivation 7.47 is two derivations in one. The left side about power and the right side about kinetic energy. Lets discuss one at a time.

On the left side of eqn. (7.47) we note in (c) that the power of each force is the dot product of the force with the velocity of the point it touches. In (d) we use what we know about the velocities of points on rotating rigid bodies. In (e) we use the vector identity $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}} = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{C}} \times \vec{\boldsymbol{A}}$ from chapter 2. In (f) we note that $\vec{\boldsymbol{\omega}}$ is common to all points so factors out of the sum. In (g) we note that $\vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}_i$ is the moment of the force about pt O. And in (g) we sum the moments of the forces. So we find that The power of a set of forces acting on a rigid body is the product of thier net moment(about 0) and the body angular velocity,

$$P = \vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{M}}_{\text{tot}}. \tag{7.48}$$

On the right side of eqn. (7.47) we note in (c) that the kinetic energy is the sum of the kinetic energy of the mass increments. In (d) we use what we know about the velocities of these bits of mass, given that they are on a common rotating body. In (e) we use that the magnitude of the cross product of orthogonal vectors is the product of the magnitudes ($|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = AB$) and that the dot product of a vector with itself is its magnitude squared ($\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{A}} = A^2$). In (f) we factor out ω^2 because it is common to all the mass increments and note that the remaining integral is constant in time for a rigid body. In (g) we carry out the derivative. In (h) we de-simplify the result from (g) in order to show a more general form that we will find later in 3D mechanics. Eqn. (h) follows from (g) because $\vec{\boldsymbol{\omega}}$ is parallel to $\dot{\vec{\boldsymbol{\omega}}}$ for 2D rotations.

Note that we started here with the basic power balance equation from the front inside cover. Instead, we could have derived power balance from our angular momentum balance expression (see box 7.4 on 392).

7.3 *THEORY*

The relation between angular momentum balance and power balance

For this system, angular momentum balance can be derived from power balance and visa versa. Thus neither is essentially more fundamental than the other and both are reliable. First we can derive power balance from angular momentum balance as follows:

$$\begin{aligned} \vec{\boldsymbol{M}}_{\text{net}} &= \dot{\omega}\hat{\boldsymbol{k}} \int r^2\, dm \\ \vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{M}}_{\text{net}} &= \vec{\boldsymbol{\omega}} \cdot \left(\dot{\omega}\hat{\boldsymbol{k}} \int r^2\, dm \right). \\ P &= \dot{E}_{\text{K}} \end{aligned} \tag{7.49}$$

That is, when we dot both sides of the angular momentum equation with $\vec{\boldsymbol{\omega}}$ we get on the left side a term which we recognize as the power of the forces and on the right side a term which is the rate of change of kinetic energy.

The opposite derivation starts with the power balance Fig. 7.47(g)

$$\begin{aligned} \vec{\boldsymbol{\omega}} \cdot \sum \vec{\boldsymbol{M}}_i &= \dot{\omega}\omega \int r^2\, dm \quad \text{(g)} \\ \Rightarrow \quad \omega \left(\hat{\boldsymbol{k}} \cdot \sum \vec{\boldsymbol{M}}_i \right) &= \dot{\omega}\omega \int r^2\, dm \\ \Rightarrow \quad \left(\hat{\boldsymbol{k}} \cdot \sum \vec{\boldsymbol{M}}_i \right) &= \dot{\omega} \int r^2\, dm \end{aligned} \tag{7.50}$$

and, assuming $\omega \neq 0$, divide by ω to get the angular momentum equation for planar rotational motion.

SAMPLE 7.18 *A rod going in circles at constant rate.* A uniform rod of mass m and length ℓ is connected to a motor at end O. A ball of mass m is attached to the rod at end B. The motor turns the rod in counterclockwise direction at a constant angular speed ω. There is gravity pointing in the $-\hat{\boldsymbol{\jmath}}$ direction. Find the torque applied by the motor (i) at the instant shown and (ii) when $\theta = 0^o,\ 90^o,\ 180^o$. How does the torque change if the angular speed is doubled?

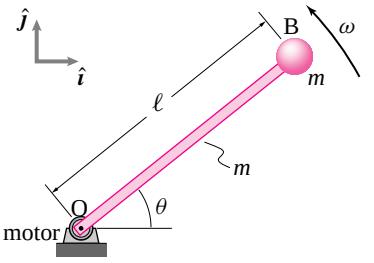
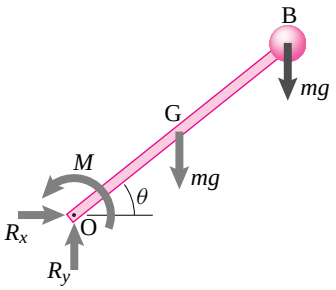


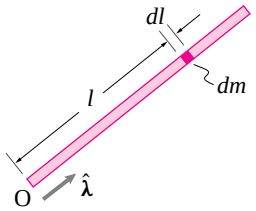
Figure 7.55: A rod goes in circles at a constant rate.

(Filename:sfig4.5.5)

Solution The FBD of the rod and ball system is shown in Fig. 7.56(a). Since the system is undergoing circular motion at a constant speed, the acceleration of the ball as well as every point on the rod is just radial (pointing towards the center of rotation O) and is given by $\vec{\boldsymbol{a}} = -\omega^2 r\hat{\boldsymbol{\lambda}}$ where r is the radial distance from the center O to the point of interest and $\hat{\boldsymbol{\lambda}}$ is a unit vector along OB pointing away from O (Fig. 7.56(b)).



(a) FBD of rod+ball system



(b) Calculation of $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ of the rod

Figure 7.56: A rod goes in circles at a constant rate.

(Filename:sfig4.5.5a)

Angular Momentum Balance about point O gives

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$$

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \vec{\boldsymbol{r}}_{G/O} \times (-mg\hat{\boldsymbol{\jmath}}) + \vec{\boldsymbol{r}}_{B/O} \times (-mg\hat{\boldsymbol{\jmath}}) + M\hat{\boldsymbol{k}} \\
&= -\frac{\ell}{2}\cos\theta mg\hat{\boldsymbol{k}} - \ell\cos\theta mg\hat{\boldsymbol{k}} + M\hat{\boldsymbol{k}} \\
&= (M - \frac{3\ell}{2}mg\cos\theta)\hat{\boldsymbol{k}} \qquad (7.51)
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} &= \overbrace{\vec{\boldsymbol{r}}_{B/O} \times m\vec{\boldsymbol{a}}_B}^{\dot{\vec{\boldsymbol{H}}}_{\mathrm{ball/O}}} + \overbrace{\int_m \vec{\boldsymbol{r}}_{dm/O} \times \vec{\boldsymbol{a}}_{dm}\, dm}^{\dot{\vec{\boldsymbol{H}}}_{\mathrm{rod/O}}} \\
&= \ell\hat{\boldsymbol{\lambda}} \times (-m\omega^2\ell\hat{\boldsymbol{\lambda}}) + \int_m \overbrace{l\hat{\boldsymbol{\lambda}}}^{\vec{\boldsymbol{r}}_{dm/O}} \times \overbrace{(-\omega^2 l\hat{\boldsymbol{\lambda}})}^{\vec{\boldsymbol{a}}_{dm}} dm \\
&= \vec{\boldsymbol{0}} \qquad (\text{since } \hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\lambda}} = \vec{\boldsymbol{0}}) \qquad (7.52)
\end{aligned}$$

(i) Equating (7.51) and (7.52) we get

$$M = \frac{3}{2}mg\ell\cos\theta.$$

$$\boxed{M = \tfrac{3}{2}mg\ell\cos\theta}$$

(ii) Substituting the given values of θ in the above expression we get

$$M(\theta = 0^o) = \frac{3}{2}mg\ell, \quad M(\theta = 90^o) = 0 \quad M(\theta = 180^o) = -\frac{3}{2}mg\ell$$

$$\boxed{M(0^o) = \tfrac{3}{2}mg\ell, \quad M(90^o) = 0 \quad M(180^o) = -\tfrac{3}{2}mg\ell}$$

The values obtained above make sense (at least qualitatively). To make the rod and the ball go up from the 0^o position, the motor has to apply some torque in the counterclockwise direction. In the 90^o position no torque is required for the *dynamic balance*. In 180^o position the system is accelerating downwards under gravity; therefore, the motor has to apply a clockwise torque to make the system maintain a uniform speed.

It is clear from the expression of the torque that it does not depend on the value of the angular speed ω! Therefore, the torque will not change if the speed is doubled. In fact, as long as the speed remains constant at any value, the only torque required to maintain the motion is the torque to counteract the moments at O due to gravity.

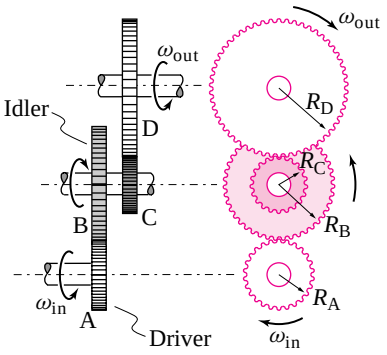


Figure 7.57: A compound gear train.
(Filename:sfig4.5.7)

SAMPLE 7.19 *A compound gear train.* When the gear of an input shaft, often called the *driver* or the *pinion*, is directly meshed in with the gear of an output shaft, the motion of the output shaft is opposite to that of the input shaft. To get the output motion in the same direction as that of the input motion, an *idler* gear is used. If the idler shaft has more than one gear in mesh, then the gear train is called a *compound gear train*.

In the gear train shown in Fig. 7.57, the input shaft is rotating at 2000 rpm and the input torque is 200 N-m. The efficiency (defined as the ratio of output power to input power) of the train is 0.96 and the various radii of the gears are: $R_A = 5\,\text{cm}$, $R_B = 8\,\text{cm}$, $R_C = 4\,\text{cm}$, and $R_D = 10\,\text{cm}$. Find

(a) the input power P_{in} and the output power P_{out},
(b) the output speed ω_{out}, and
(c) the output torque.

Solution

(a) The power:

$$\begin{aligned} P_{in} &= M_{in}\omega_{in} = 200\,\text{N·m} \cdot 2000\,\text{rpm} \\ &= 400000\,\text{N·m} \cdot \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi}{1\,\text{rev}} \cdot \frac{1\,\text{min}}{60\,\text{s}} \\ &= 41887.9\,\text{N m/s} \approx 42\,\text{kW}. \\ \Rightarrow \quad p_{out} &= \text{efficiency} \cdot P_{in} = 0.96 \cdot 42\,\text{kW} \approx 40\,\text{kW} \end{aligned}$$

$$\boxed{P_{in} = 42\,\text{kW}, \quad p_{out} = 40\,\text{kW}}$$

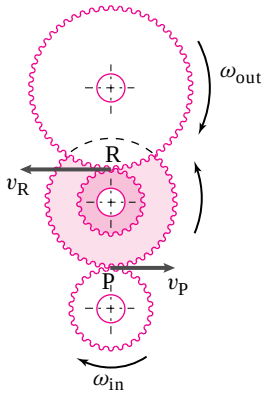


Figure 7.58: (Filename:sfig4.5.7a)

(b) The angular speed of meshing gears can be easily calculated by realizing that the linear speed of the point of contact has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus,

$$\begin{aligned} & v_P = \omega_{in} R_A = \omega_B R_B \\ \Rightarrow \quad & \omega_B = \omega_{in} \cdot \frac{R_A}{R_B} \\ \text{and} \quad & v_R = \omega_C R_C = \omega_{out} R_D \\ \Rightarrow \quad & \omega_{out} = \omega_C \cdot \frac{R_C}{R_D} \\ \text{But} \quad & \omega_C = \omega_B \\ \Rightarrow \quad & \omega_{out} = \omega_{in} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \\ & = 2000\,\text{rpm} \cdot \frac{5}{8} \cdot \frac{4}{10} = 500\,\text{rpm}. \end{aligned}$$

$$\boxed{\omega_{out} = 500\,\text{rpm}}$$

(c) The output torque,

$$\begin{aligned} M_{out} &= \frac{P_{out}}{\omega_{out}} = \frac{40\,\text{kW}}{500\,\text{rpm}} \\ &= \frac{40}{500} \cdot 1000 \frac{\text{N·m}}{\text{s}} \cdot \frac{\text{min}}{\text{rev}} \cdot \frac{1\,\text{rev}}{2\pi} \cdot \frac{60\,\text{s}}{1\,\text{min}} \\ &= 764\,\text{N·m}. \end{aligned}$$

$$\boxed{M_{out} = 764\,\text{N·m}}$$

SAMPLE 7.20 *At the onset of motion:* A $2' \times 4'$ rectangular plate of mass 20 lbm is pivoted at one of its corners as shown in the figure. The plate is released from rest in the position shown. Find the force on the support immediately after release.

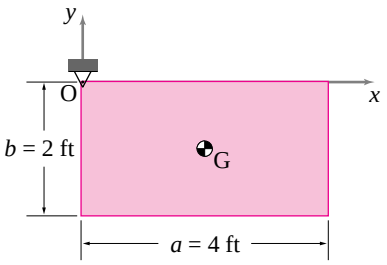


Figure 7.59: A rectangular plate is released from rest from the position shown.
(Filename:sfig5.4.3)

Solution The free body diagram of the plate is shown in Fig. 7.60. The force $\vec{F}$ applied on the plate by the support is unknown.

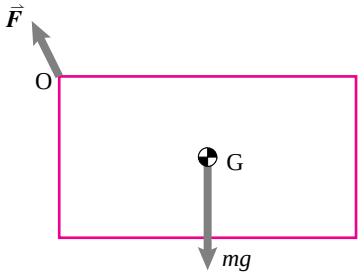
The linear momentum balance for the plate gives

$$\begin{aligned} \sum \vec{F} &= m\vec{a}_G \\ \vec{F} - mg\hat{\jmath} &= m(\ddot{\theta}\, r_{G/O}\hat{e}_\theta - \dot{\theta}^2 R\hat{e}_R) \\ &= m\,\ddot{\theta}\, r_{G/O}\hat{e}_\theta \quad (\text{since } \dot{\theta} = 0 \text{ at } t = 0) \end{aligned} \tag{7.53}$$

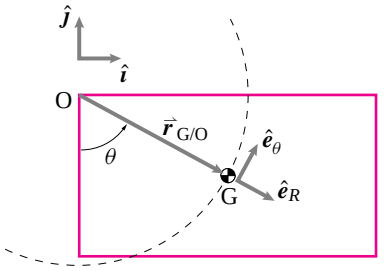
Thus to find $\vec{F}$ we need to find $\ddot{\theta}$.

The angular momentum balance for the plate about the fixed support point O gives

$$\begin{aligned} \vec{M}_O &= \dot{\vec{H}}_O \\ \vec{r}_{G/O} \times mg(-\hat{\jmath}) &= I_{zz/O}\,\ddot{\theta}\hat{k} \\ \underbrace{(\frac{a}{2}\hat{\imath} - \frac{b}{2}\hat{\jmath})}_{\vec{r}_{G/O}} \times mg(-\hat{\jmath}) &= \underbrace{\left[\frac{m(a^2+b^2)}{12} + m(\frac{a^2}{4} + \frac{b^2}{4})\right]}_{\text{parallel axis theorem}}\ddot{\theta}\hat{k} \\ -mg\frac{a}{2}\hat{k} &= \frac{m(a^2+b^2)}{3}\ddot{\theta}\hat{k} \\ \Rightarrow \quad \ddot{\theta} &= -\frac{3g\,a}{2(a^2+b^2)} \\ &= -\frac{3 \cdot 32.2\,\text{ft/s}^2 \cdot 4\,\text{ft}}{2(16+4)\,\text{ft}^2} \\ &= -9.66\,\text{rad/s}^2. \end{aligned}$$



(a) Free body diagram



(b) Geometry of motion

Figure 7.60: (a) The free body diagram of the plate. (b) The geometry of motion. From the given dimensions, $\hat{e}_R = \frac{a\hat{\imath}-b\hat{\jmath}}{(a^2+b^2)^{\frac{1}{2}}}$ and $\hat{e}_\theta = \frac{b\hat{\imath}+a\hat{\jmath}}{(a^2+b^2)^{\frac{1}{2}}}$.
(Filename:sfig5.4.3a)

Substituting this value of $\ddot{\theta}$ in eqn. (7.53), we get

$$\begin{aligned} \vec{F} &= mg\hat{\jmath} + m\,\ddot{\theta}\, r_{G/O}\hat{e}_\theta \\ &= 20\,\text{lbm} \cdot 32.2\,\text{ft/s}^2\,\hat{\jmath} + 20\,\text{lbm} \cdot (-9.66\,\text{rad/s}^2) \cdot (\underbrace{\sqrt{2^2+1^2}}_{r_{G/O}}\,\text{ft})\,\underbrace{\frac{(2\hat{\imath}+4\hat{\jmath})}{\sqrt{20}}}_{\hat{e}_\theta} \\ &= 20\,\text{lbf}\,\hat{\jmath} - \frac{20 \cdot 9.66}{32.2}(1\hat{\imath} + 2\hat{\jmath})\,\text{lbf} \qquad (\text{since } 1\,\text{lbm} \cdot \text{ ft/s}^2 = \tfrac{1}{32.2}\,\text{lbf}) \\ &= (-3\hat{\imath} + 14\hat{\jmath})\,\text{lbf}. \end{aligned}$$

$$\boxed{\vec{F} = (-3\hat{\imath} + 14\hat{\jmath})\,\text{lbf}}$$

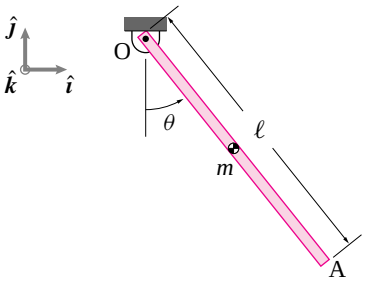


Figure 7.61: A uniform rod swings in the plane about its pinned end O.

(Filename:sfig5.4.1a)

SAMPLE 7.21 *The swinging stick.* A uniform bar of mass m and length ℓ is pinned at one of its ends O. The bar is displaced from its vertical position by an angle θ and released (Fig. 7.61).

(a) Find the equation of motion using momentum balance.
(b) Find the reaction at O as a function of $(\theta,\ \dot{\theta},\ g,\ m,\ \ell)$.

Solution First we draw a simple sketch of the given problem showing relevant geometry (Fig. 7.61(a)), and then a free body diagram of the bar (Fig. 7.61(b)).

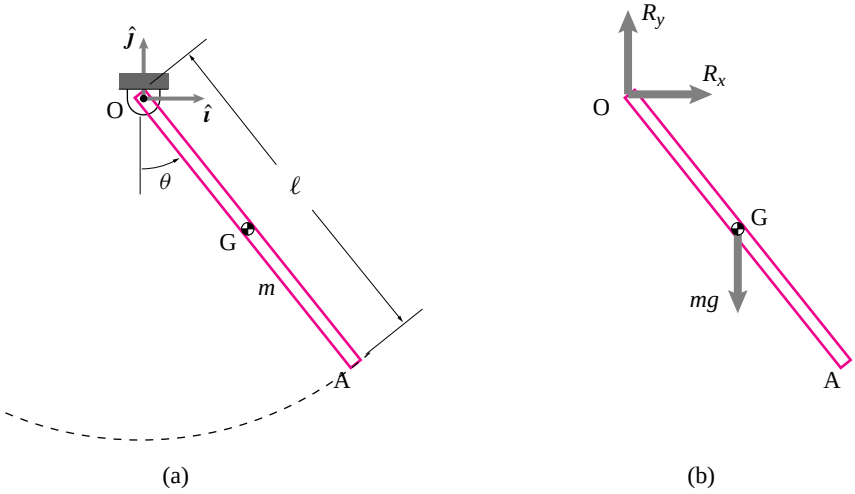


Figure 7.62: (a) A line sketch of the swinging rod and (b) free body diagram of the rod.

(Filename:sfig5.4.1b)

We should note for future reference that

$$\begin{aligned}\vec{\omega} &= \omega\hat{k} \equiv \dot{\theta}\hat{k}\\ \dot{\vec{\omega}} &= \dot{\omega}\hat{k} \equiv \ddot{\theta}\hat{k}\end{aligned}$$

(a) **Equation of motion using momentum balance:** We can write angular momentum balance about point O as

$$\sum \vec{M}_{\mathrm{O}} = \dot{\vec{H}}_{\mathrm{O}}.$$

Let us now calculate both sides of this equation:

$$\begin{aligned}\sum \vec{M}_{\mathrm{O}} &= \vec{r}_{G/O} \times mg(-\hat{j})\\ &= \frac{\ell}{2}(\sin\theta\hat{i} - \cos\theta\hat{j}) \times mg(-\hat{j})\\ &= -\frac{\ell}{2}mg\,\sin\theta\hat{k}. \qquad (7.54)\\ \dot{\vec{H}}_{\mathrm{O}} &= I_{zz/G}\,\dot{\vec{\omega}} + \vec{r}_G \times m\vec{a}_G\\ &= \frac{m\ell^2}{12}\dot{\omega}\hat{k} + \vec{r}_G \times m(\overbrace{\dot{\omega}\hat{k} \times \vec{r}_G - \omega^2\vec{r}_G}^{\vec{a}_G})\\ &= \frac{m\ell^2}{12}\dot{\omega}\hat{k} + \frac{m\ell^2}{4}\dot{\omega}\hat{k}, = \frac{m\ell^2}{3}\dot{\omega}\hat{k} \qquad (7.55)\end{aligned}$$

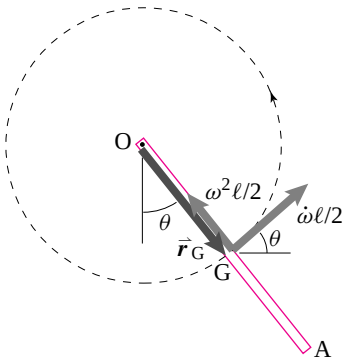


Figure 7.63: Radial and tangential components of $\vec{a}_G$. Since the radial component is parallel to $\vec{r}_G$, $\vec{r}_G \times \vec{a}_G = \frac{\ell^2}{4}\dot{\omega}\hat{k}$.

(Filename:sfig5.4.1c)

where the last step, $\vec{\boldsymbol{r}}_G \times m\vec{\boldsymbol{a}}_G = \frac{m\ell^2}{4}\dot{\omega}\hat{\boldsymbol{k}}$, should be clear from Fig. 7.63. Equating (7.54) and (7.55) we get

$$\begin{aligned} -\frac{\ell}{2}\not{m}\, g\, \sin\theta &= \not{m}\frac{\ell^2}{3}\dot{\omega} \\ \text{or} \qquad \dot{\omega} + \frac{3g}{2\ell}\sin\theta &= 0 \\ \text{or} \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. \end{aligned} \tag{7.56}$$

$$\boxed{\ddot{\theta} + \tfrac{3g}{2\ell}\sin\theta = 0}$$

(b) **Reaction at O:** Using linear momentum balance

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m\,\vec{\boldsymbol{a}}_G, \\ \text{where} \quad \sum \vec{\boldsymbol{F}} &= R_x\hat{\boldsymbol{\imath}} + (R_y - mg)\hat{\boldsymbol{\jmath}}, \\ \text{and} \quad \vec{\boldsymbol{a}}_G &= \frac{\ell}{2}\dot{\omega}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + \frac{\ell}{2}\omega^2(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) \\ &= \frac{\ell}{2}[(\dot{\omega}\cos\theta - \omega^2\sin\theta)\hat{\boldsymbol{\imath}} + (\dot{\omega}\sin\theta + \omega^2\cos\theta)\hat{\boldsymbol{\jmath}}]. \end{aligned}$$

Dotting both sides of $\sum \vec{\boldsymbol{F}} = m\,\vec{\boldsymbol{a}}_G$ with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ and rearranging, we get

$$\begin{aligned} R_x &= m\frac{\ell}{2}(\dot{\omega}\cos\theta - \omega^2\sin\theta) \\ &\equiv m\frac{\ell}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \\ R_y &= mg + m\frac{\ell}{2}(\dot{\omega}\sin\theta + \omega^2\cos\theta) \\ &\equiv mg + m\frac{\ell}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta). \end{aligned}$$

Now substituting the expression for $\ddot{\theta}$ from (7.56) in R_x and R_y, we get

$$R_x = -m\sin\theta\left(\frac{3}{4}g\cos\theta + \frac{\ell}{2}\dot{\theta}^2\right), \tag{7.57}$$

$$R_y = mg\left(1 - \frac{3}{4}\sin^2\theta\right) + m\frac{\ell}{2}\dot{\theta}^2\cos\theta. \tag{7.58}$$

$$\boxed{\vec{\boldsymbol{R}} = -m(\tfrac{3}{4}g\cos\theta + \tfrac{\ell}{2}\dot{\theta}^2)\sin\theta\hat{\boldsymbol{\imath}} + [mg(1 - \tfrac{3}{4}\sin^2\theta) + m\tfrac{\ell}{2}\dot{\theta}^2\cos\theta]\hat{\boldsymbol{\jmath}}}$$

Check: We can check the reaction force in the special case when the rod does not swing but just hangs from point O. The forces on the bar in this case have to satisfy static equilibrium. Therefore, the reaction at O must be equal to mg and directed vertically upwards. Plugging $\theta = 0$ and $\dot{\theta} = 0$ (no motion) in Eqn. (7.57) and (7.58) we get $R_x = 0$ and $R_y = mg$, the values we expect.

SAMPLE 7.22 *The swinging stick: energy balance.* Consider the same swinging stick as in Sample 7.21. The stick is, again, displaced from its vertical position by an angle θ and released (See Fig. 7.61).

(a) Find the equation of motion using energy balance.
(b) What is $\dot{\theta}$ at $\theta = 0$ if $\theta(t = 0) = \pi/2$?
(c) Find the period of small oscillations about $\theta = 0$.

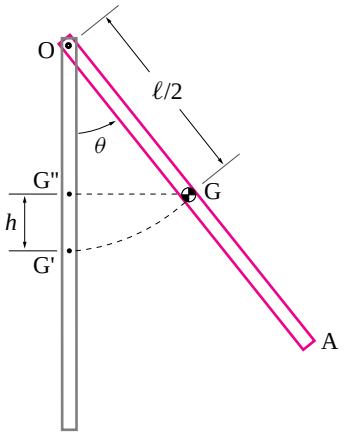


Figure 7.64: Work done by the force of gravity in moving from G′ to G $\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = -mg\hat{\boldsymbol{j}} \cdot h\hat{\boldsymbol{j}} = -mgh$.

(Filename:sfig5.4.2a)

Solution

(a) **Equation of motion using energy balance:** We use the power equation, $\dot{E_{\rm K}} = P$, to derive the equation of motion of the bar.

$$E_{\rm K} = \frac{1}{2} I_{zz/G}\, \omega^2 + \frac{1}{2} m v_G^2$$

where $\frac{1}{2} I_{zz/G}\, \omega^2$ = kinetic energy of the bar due to rotation about the z-axis passing through the mass center G,

and $\frac{1}{2} m v_G^2$ = kinetic energy of the bar due to translation of the mass center.

But $v_G = \omega r_G = \omega \frac{\ell}{2}$. Therefore,

$$E_{\rm K} = \frac{1}{2}\frac{m\ell^2}{12}\omega^2 + \frac{1}{2} m\omega^2 \frac{\ell^2}{4} = \frac{1}{6} m\ell^2\omega^2,$$

and

$$\dot{E_{\rm K}} = \frac{d}{dt}\left(\frac{1}{6} m\ell^2\omega^2\right) = \frac{1}{3} m\ell^2 \omega\, \dot{\omega} = \frac{1}{3} m\ell^2 \dot{\theta}\ddot{\theta}.$$

Calculation of power (P): There are only two forces acting on the bar, the reaction force, $\vec{\boldsymbol{R}} (= R_x\hat{\boldsymbol{i}} + R_y\hat{\boldsymbol{j}})$ and the force due to gravity, $-mg\hat{\boldsymbol{j}}$. Since the support point O does not move, no work is done by $\vec{\boldsymbol{R}}$. Therefore,

$$\begin{aligned} W &= \text{Work done by gravity force in moving from G}' \text{ to G.} \\ &= -mgh \end{aligned}$$

Note that the negative sign stands for the work done *against* gravity. Now,

$$h = OG' - OG'' = \frac{\ell}{2} - \frac{\ell}{2}\cos\theta = \frac{\ell}{2}(1 - \cos\theta).$$

Therefore,

$$\begin{aligned} W &= -mg\frac{\ell}{2}(1 - \cos\theta) \\ \text{and} \qquad P &= \dot{W} = \frac{dW}{dt} = -mg\frac{\ell}{2}\sin\theta\dot{\theta}. \end{aligned}$$

Equating $\dot{E_{\rm K}}$ and P we get

$$\begin{aligned} -\not{m}\, g\frac{\ell}{2}\sin\theta\not{\dot{\theta}} &= \frac{1}{3}\not{m}\, \ell^2 \not{\dot{\theta}}\ddot{\theta} \\ \text{or} \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. \end{aligned}$$

$$\boxed{\ddot{\theta} + \tfrac{3g}{2\ell}\sin\theta = 0}$$

This equation is, of course, the same as we obtained using balance of angular momentum in Sample 7.21.

(b) **Find ω at $\theta = 0$:** We are given that at $t = 0$, $\theta = \pi/2$ and $\dot{\theta} \equiv \omega = 0$ (released from rest). This position is (1) shown in Fig. 7.65. In position (2) $\theta = 0$, i.e., the rod is vertical. Since there are no dissipative forces, the total energy of the system remains constant. Therefore, taking datum for potential energy as shown in Fig. 7.65, we may write

$$\underbrace{E_{K1}}_{0} + V_1 = E_{K2} + \underbrace{V_2}_{0}$$

$$\begin{aligned} \text{or} \qquad mg\frac{\ell}{2} &= \frac{1}{2} I_{zz/G}\,\omega^2 + \frac{1}{2} m v_G^2 \\ &= \frac{1}{2}\frac{m\ell^2}{12}\omega^2 + \frac{1}{2} m \overbrace{\frac{\ell}{2}\omega}^{v_G} \\ &= \frac{1}{6} m\ell^2\omega^2 \\ \Rightarrow \qquad \omega &= \pm\sqrt{\frac{3g}{\ell}} \end{aligned}$$

$$\boxed{\omega = \pm\sqrt{\tfrac{3g}{\ell}}}$$

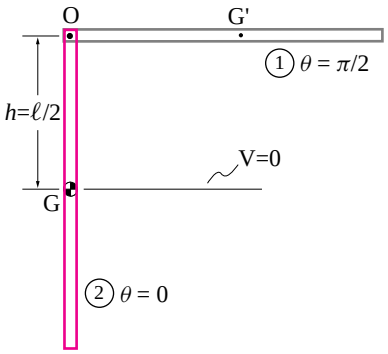


Figure 7.65: The total energy between positions (1) and (2) is constant.

(Filename:sfig5.4.2b)

(c) **Period of small oscillations:** The equation of motion is

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0.$$

For small θ, $\sin\theta \approx \theta$

$$\begin{aligned} \Rightarrow \qquad & \ddot{\theta} + \tfrac{3g}{2\ell}\theta = 0 \qquad (7.59) \\ \text{or} \qquad & \ddot{\theta} + \lambda^2\theta = 0 \\ \text{where} \qquad & \lambda^2 = \tfrac{3g}{2\ell}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{the circular frequency} &= \lambda = \sqrt{\frac{3g}{2\ell}}, \\ \text{and the time period T} &= \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{2\ell}{3g}}. \end{aligned}$$

$$\boxed{T = 2\pi\sqrt{\tfrac{2\ell}{3g}}}$$

[Say for $g = 9.81\,\text{m/s}^2$, $\ell = 1\,\text{m}$ we get $\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{2}{3}\frac{1}{9.81}}\,\text{s} = 0.4097\,\text{s}$]

SAMPLE 7.23 *The swinging stick: numerical solution of the equation of motion.* For the swinging stick considered in Samples 7.21 or 7.22, find the time that the rod takes to fall from $\theta = \pi/2$ to $\theta = 0$ if it is released from rest at $\theta = \pi/2$?

Solution $\pi/2$ is a big value of θ – big in that we cannot assume $\sin\theta \approx \theta$ (obviously $1 \neq 1.5708$). Therefore we may not use the linearized equation (7.59) to solve for t explicitly. We have to solve the full nonlinear equation (7.56) to find the required time. Unfortunately, we cannot get a *closed form* solution of this equation using mathematical skills you have at this level. Therefore, we resort to numerical integration of this equation. ①

① You may solve equation (7.56) by numerical integration using any computer software (*e.g.*, MathCad, MATHEMATICA, Maple, MATLAB, *etc*..) or your own program written in any language of your choice. To write your own program, however, you will have to know some method for numerical integration, *e.g.*, Euler's method, Runge-Kutta method, *etc*..

Here, we show how to do this integration and find the required time using the numerical solution. We assume that we have some numerical ODE solver, say `odesolver`, available to us that will give us the numerical solution given appropriate input.

The first step in numerical integration is to set up the given differential equation of second or higher order as a set of first order ordinary differential equations. To do so for Eqn. (7.56), we introduce ω as a new variable and write

$$\dot{\theta} = \omega \tag{7.60}$$

$$\dot{\omega} = -\frac{3g}{2\ell}\sin\theta \tag{7.61}$$

Thus, the second order ODE (7.56) has been rewritten as a set of two first order ODE's (7.60) and (7.61). We may write these first order equations in vector form by assuming $z = [\theta \quad \omega]^T$. That is,

$$z = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} \theta \\ \omega \end{Bmatrix} \qquad \Rightarrow \qquad \dot{z} = \begin{Bmatrix} \dot{\theta} \\ \dot{\omega} \end{Bmatrix} = \begin{Bmatrix} z_2 \\ -\frac{3g}{2\ell}\sin z_1 \end{Bmatrix}$$

To use any numerical integrator, we usually need to write a small program which will compute and return the value of **z** as output if t and z are supplied as input. Here is such a program called **barpend**, written as a pseudocode, for our equations. Anything in a line following a % sign is a comment.

```
  function zdot = barpend(t,z);
% ------------------------------------------------
% function to compute zdot for a bar pendulum
% where z = [theta; omega];
% ------------------------------------------------
g = 9.81; L = 1;       % these values can be changed
zdot = [z(2); -3*g/(2*L)*sin(z(1))];
```

Other than this function **barpend**, a numerical ODE solver will require the initial time $t0$, the final time tf, and initial conditions $z0$ as inputs. The following is a pseudocode, **runpend**, which contains all these input parameters and the commands to solve the ODEs and plot the results.

```
t0 = 0;  tf = 4;       % specify initial and final times
z0 = [pi/2; 0];        % initial conditions:
                       % [theta=pi/2; omega=0]

[t,z] = odesolver('barpend',t0,tf,z0);
                       % run odesolver. The output is [t,z]
```

```
plot(t,z)                % plots t vs. theta
                         % and t vs. omega together
xlabel('t'),ylabel('theta and omega')  % label axes
```

The results obtained from the numerical solution are shown in Fig. 7.66.

The problem of finding the time taken by the bar to fall from $\theta = \pi/2$ to $\theta = 0$ numerically is nontrivial. It is called a *boundary value problem*. We have only illustrated how to solve *initial value problems*. However, we can get fairly good estimate of the time just from the solution obtained. We first plot θ against time t to get the graph shown in Fig. 7.67. We find the values of t and the corresponding values of θ that bracket $\theta = 0$. Now, we can use linear interpolation to find the value of t at $\theta = 0$. Proceeding this way, we get $t = 0.4833$ (seconds), a little more than we get from the linear ODE in sample 7.22 of 0.40975. Additionally, we can get by interpolation that at $\theta = 0$

$$\omega = -5.4246\,\text{rad/s}.$$

How does this result compare with the analytical value of ω from sample 7.22 (which did not depend on the small angle approximates)? Well, we found that

$$\omega = -\sqrt{\frac{3g}{\ell}} = -\sqrt{\frac{3 \cdot 9.81\,\text{m/s}^2}{1\,\text{m}}} = -5.4249\,\text{s}^{-1}.$$

Thus, we get a fairly accurate value from numerical integration!

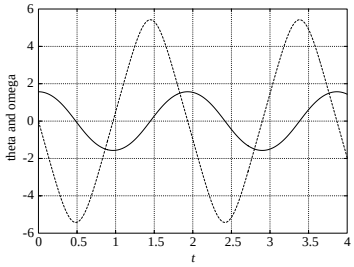


Figure 7.66: Numerical solution is shown by plotting θ and ω against time.

(Filename:sfig5.4.4a)

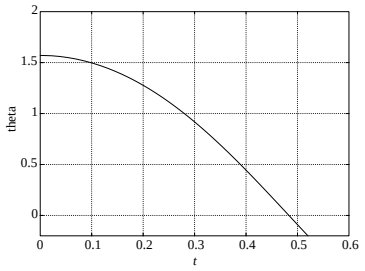


Figure 7.67: Graphic output of `plot(t, z(:,1))`, `xlabel('t')`, `ylabel('theta')`.

(Filename:sfig5.4.4b)

SAMPLE 7.24 *The swinging stick with a destabilizing torque.* Consider the swinging stick of Sample 7.21 once again.

(a) Find the equation of motion of the stick,if a torque $\vec{M} = M\hat{k}$ is applied at end O and a force $\vec{F} = F\hat{\imath}$ is applied at the other end A.

(b) Take $F = 0$ and $M = C\theta$. For $C = 0$ you get the equation of free oscillations obtained in Sample 7.21 or 7.22 For small C, does the period of the pendulum increase or decrease?

(c) What happens if C is big?

Solution

(a) A free body diagram of the bar is shown in Fig. 7.68. Once again, we can use $\sum \vec{M}_O = \dot{\vec{H}}_O$ to derive the equation of motion as in Sample 7.21. We calculated $\sum \vec{M}_O$ and $\dot{\vec{H}}_O$ in Sample 7.21. Calculation of $\dot{\vec{H}}_O$ remains the same in the present problem. We only need to recalculate $\sum \vec{M}_O$.

$$\begin{aligned}\sum \vec{M}_O &= M\hat{k} + \vec{r}_{G/O} \times mg(-\hat{\jmath}) + \vec{r}_{A/O} \times \vec{F} \\ &= M\hat{k} - \frac{\ell}{2} mg \sin\theta \hat{k} + F\ell \cos\theta \hat{k} \\ &= (M + F\ell\cos\theta - \frac{\ell}{2} mg \sin\theta)\hat{k}\end{aligned}$$

and

$$\dot{\vec{H}}_O = m\ddot{\theta}\frac{\ell^2}{3}\hat{k} \quad \text{(see Sample 7.21)}$$

Therefore, from $\sum \vec{M}_O = \dot{\vec{H}}_O$

$$\begin{aligned} M + F\ell\cos\theta - \frac{\ell}{2} mg\sin\theta &= m\ddot{\theta}\frac{\ell^2}{3} \\ \Rightarrow \quad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3F}{m\ell}\cos\theta - \frac{3M}{m\ell^2} &= 0. \end{aligned}$$

$$\boxed{\ddot{\theta} + \tfrac{3g}{2\ell}\sin\theta - \tfrac{3F}{m\ell}\cos\theta - \tfrac{3M}{m\ell^2} = 0}$$

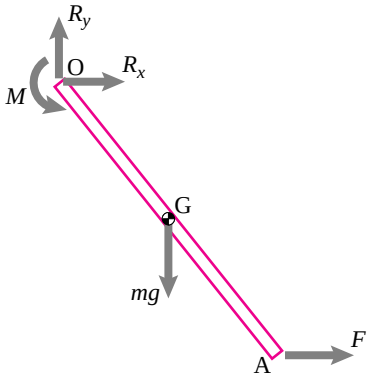


Figure 7.68: Free body diagram of the bar with applied torque $\vec{M}$ and force $\vec{F}$

(Filename:sfig5.4.5a)

(b) Now, setting $F = 0$ and $M = C\,\theta$ we get

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3C\theta}{m\ell^2} = 0 \tag{7.62}$$

Numerical Solution: We can numerically integrate (7.62) just as in the previous Sample to find $\theta(t)$. We will, however, need to modify function **barpend** to accommodate the new term due to M in the equation of motion. The new function, **barpendf** is as follows.

```
function zdot = barpendf(t,z);
% ----------------------------------------
% function to compute zdot for a bar pendulum
% with applied torque M = C * theta.
% where z = [theta; omega];
% ----------------------------------------

g = 9.81; L = 1; m = 1; C = 4;  % parameters
zdot = [ z(2) ;
         -(3*g/(2*L)) * sin(z(1)) + 3*C/(m*L^2) * z(1) ];
```

Using this function we run `odesolver` just as in the previous case to find the response of the pendulum. Fig. 7.69 shows different responses for various values of C. Note that for $C = 0$, it is the same case as unforced bar pendulum considered above. From Fig. 7.69 it is clear that the bar has periodic motion

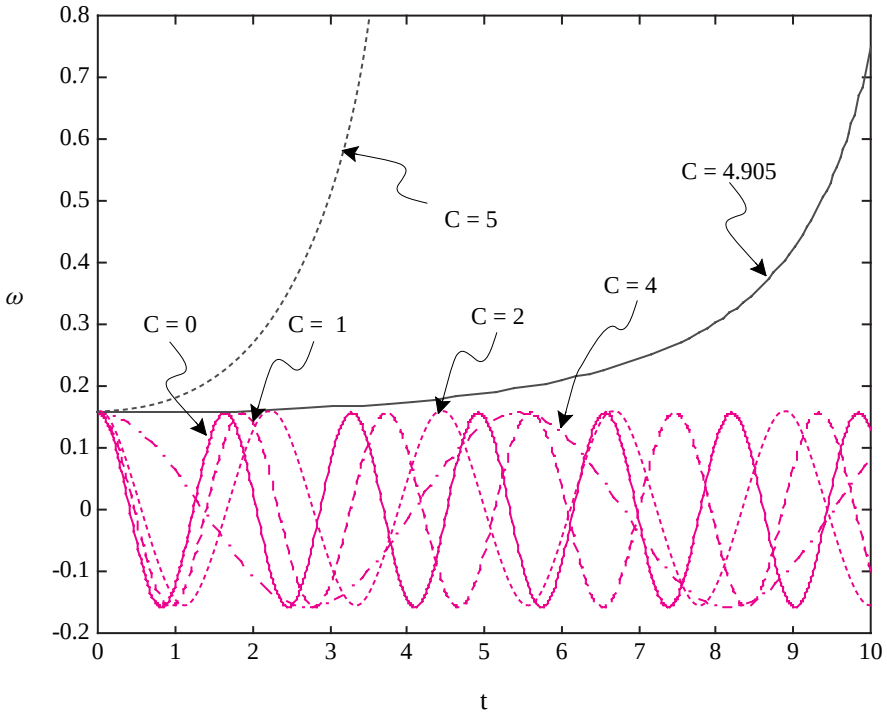


Figure 7.69: $\theta(t)$ with applied torque $M = C\,\theta$ for $C = 0,\ 1,\ 2,\ 4,\ 4.905,\ 5$. Note that for small C the motion is periodic but for large C ($C \geq 4.4$) the motion becomes aperiodic.

(Filename:sfig5.4.5b)

for small C, with the period of motion increasing with increasing values of C. It makes sense if you look at Eqn. (7.62) carefully. Gravity acts as a restoring force while the applied torque acts as a destabilizing force. Thus, with the resistance of the applied torque, the stick swings more sluggishly making its period of oscillation bigger.

(c) From Fig. 7.69, we see that at about $C \approx 4.9$ the stability of the system changes completely. $\theta(t)$ is not periodic anymore. It keeps on increasing at faster and faster rate, that is, the bar makes complete loops about point O with ever increasing speed. Does it make physical sense? Yes, it does. As the value of C is increased beyond a certain value (can you guess the value?), the applied torque overcomes any restoring torque due to gravity. Consequently, the bar is forced to rotate continuously in the direction of the applied force.

7.5 Polar moment of inertia: $I_{zz}^{\rm cm}$ and $I_{zz}^{\rm O}$

We know how to find the velocity and acceleration of every bit of mass on a 2-D rigid body as it spins about a fixed axis. So, as explained in the previous section, it is just a matter of doing integrals or sums to calculate the various motion quantities (momenta, energy) of interest. As the body moves and rotates the region of integration and the values of the integrands change. So, in principle, in order to analyze a rigid body one has to evaluate a different integral or sum at every different configuration. But there is a shortcut. A big sum (over all atoms, say), or a difficult integral is reduced to a simple multiplication a simple multiplication.

The moment of inertia matrix $[\boldsymbol{I}]$[①]is defined to simplify the expressions for the angular momentum, the rate of change of angular momentum, and the energy of a rigid body. For study of the analysis of flat objects in planar motion only one component of the matrix [**I**] is relevant, it is I_{zz}, called just I or J in elementary physics courses. Here are the results. A flat object spinning with $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ in the xy plane has a mass distribution which gives, by means of a calculation which we will discuss shortly, a moment of inertia I_{zz}^{cm} or just 'I' so that:

$$\vec{\boldsymbol{H}}_{\rm cm} = I\omega\hat{\boldsymbol{k}} \tag{7.63}$$

$$\dot{\vec{\boldsymbol{H}}}_{\rm cm} = I\dot{\omega}\hat{\boldsymbol{k}} \tag{7.64}$$

$$E_{{\rm K}/cm} = \frac{1}{2}\omega^2 I. \tag{7.65}$$

① In fact the moment of inertia for a given object depends on what reference point is used. Most commonly when people say 'the' moment of inertia they mean to use the center of mass as the reference point. For clarity this moment of inertia matrix is often notated as $[\boldsymbol{I}^{\rm cm}]$ in this book. If a different reference point, say point O is used, the matrix is notated as $[\boldsymbol{I}^{\rm O}]$.

The moments of inertia in 2-D : $[\boldsymbol{I}^{\rm cm}]$ and $[\boldsymbol{I}^{\rm O}]$.

We start by looking at the scalar I which is just the zz or 33 component of the matrix $[\boldsymbol{I}]$ that we will study later. The definition of I^{cm} is

$$I^{cm} \equiv = \int \underbrace{x^2+y^2}_{r^2}\, dm$$

$$= \int\int r^2 \overbrace{\underbrace{\left(\frac{m_{\rm tot}}{A}\right)}\, dA}^{dm} \qquad \text{for a uniform planar object}$$

The mass per unit area.

where x and y are the distances of the mass in the x and y direction measured from an origin, and r is the direct distance from that origin. If that origin is at the center of mass then we are calculating I^{cm}, if the origin is at a point labeled C or O then we are calculating I^C or I^O.

The term I_{zz} is sometimes called the *polar moment of inertia*, or *polar mass moment of inertia* to distinguish it from the I_{xx} and I_{yy} terms which have little utility in planar dynamics (but are all important when calculating the stiffness of beams!).

What, physically, is the moment of inertia? It is a measure of the extent to which mass is far from the given reference point. Every bit of mass contributes to I in proportion to the square of its distance from the reference point. Note from, say, eqn. (7.46) on page 390 that I is just the quantity we need to do mechanics problems.

Radius of gyration

Another measure of the extent to which mass is spread from the reference point, besides the moment of inertia, is the *radius of gyration*, r_{gyr}. The radius of gyration is sometimes called k but we save k for stiffness. The radius of gyration is defined as:

$$r_{gyr} \equiv \sqrt{I/m} \quad \Rightarrow \quad r_{gyr}^2 m = I.$$

That is, the radius of gyration of an object is the radius of an equivalent ring of mass that has the same I and the same mass as the given object.

Other reference points

For the most part it is I^{cm} which is of primary interest. Other reference points are useful

(a) if the rigid body is hinged at a fixed point O then a slight short cut in calculation of angular momentum and energy terms can be had; and

(b) if one wants to calculate the moment of inertia of a composite body about its center of mass it is useful to first find the moment of inertia of each of its parts about that point. But the center of mass of the composite is usually not the center of mass of any of the separate parts.

The box 8.2 on page 458 shows the calculation of I for a number of simple 2 dimensional objects.

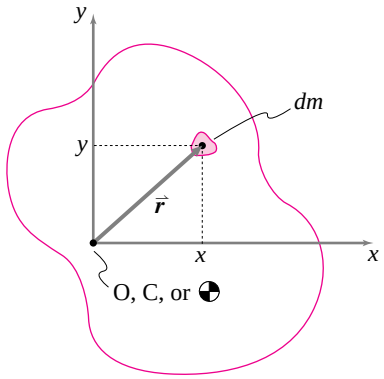


Figure 7.70: A general planar body.
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The parallel axis theorem for planar objects

The planar parallel axis theorem is the equation

$$I_{zz}^{C} = I_{zz}^{cm} + m_{\rm tot} \underbrace{r_{cm/C}^2}_{d^2}.$$

In this equation $d = r_{cm/C}$ is the distance from the center of mass to a line parallel to the z-axis which passes through point C. See box 7.4 on page 407 for a derivation of the parallel axis theorem for planar objects.

Note that $I_{zz}^{C} \geq I_{zz}^{cm}$, always.

One can calculate the moment of inertia of a composite body about its center of mass, in terms of the masses and moments of inertia of the separate parts. Say the position of the center of mass of m_i is (x_i, y_i) relative to a fixed origin, and the moment of inertia of that part about its center of mass is I_i. We can then find the moment of inertia of the composite I_{tot} about its center of mass (x_{cm}, y_{cm}) by the following sequence of calculations:

$$\begin{array}{lllll}
(1) & m_{\rm tot} & = & \sum m_i \\
(2) & x_{cm} & = & \left[\sum x_i m_i\right]/m_{\rm tot} \\
 & y_{cm} & = & \left[\sum y_i m_i\right]/m_{\rm tot} \\
(3) & d_i^2 & = & (x_i - x_{cm})^2 + (y_i - y_{cm})^2 \\
(4) & I_{tot} & = & \sum \left[I_i + m_i d_i^2\right].
\end{array}$$

Of course if you are mathematically inclined you can reduce this recipe to one grand formula with lots of summation signs. But you would end up doing the calculation in about this order in any case. As presented here this sequence of steps lends itself naturally to computer calculation with a spread sheet or any program that deals easily with arrays of numbers.

The tidy recipe just presented is actually more commonly used, with slight modification, in strength of materials than in dynamics. The need for finding area moments of inertia of strange beam cross sections arises more frequently than the need to find polar mass moment of inertia of a strange cutout shape.

The perpendicular axis theorem for planar rigid bodies

The perpendicular axis theorem for planar objects is the equation

$$I_{zz} = I_{xx} + I_{yy}$$

which is derived in box 7.4 on page 407. It gives the 'polar' inertia I_{zz} in terms of the inertias I_{xx} and I_{yy}. Unlike the parallel axis theorem, the perpendicular axis theorem does *not* have a three-dimensional counterpart. The theorem is of greatest utility when one wants to study the three-dimensional mechanics of a flat object and thus are in need of its full moment of inertia matrix.

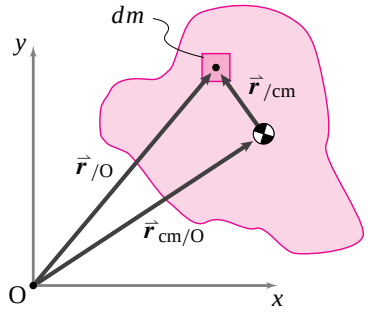
7.4 *THEORY*

The 2-D parallel axis theorem and the perpendicular axis theorem

Sometimes, one wants to know the moment of inertia relative to the center of mass and, sometimes, relative to some other point O, if the object is held at a hinge joint at O. There is a simple relation between these two moments of inertia known as *the parallel axis theorem*.

2-D parallel axis theorem

For the two-dimensional mechanics of two-dimensional objects, our only concern is I_{zz}^{o} and I_{zz}^{cm} and *not* the full moment of inertia matrix. In this case, $I_{zz}^{o} = \int r_{/o}^2\, dm$ and $I_{zz}^{cm} = \int r_{/cm}^2\, dm$. Now, let's prove the theorem in two dimensions referring to the figure.



$$
\begin{aligned}
I_{zz}^{O} &= \int r_{/O}^2\, dm \\
&= \int (x_{/O}^2 + y_{/O}^2) dm \\
&= \int [\underbrace{(x_{cm/O} + x_{/cm})}_{x_{/O}}^2 + \underbrace{(y_{cm/O} + y_{/cm})}_{y_{/O}}^2] dm \\
&= \int [(x_{cm/O}^2 + 2x_{cm/O}x_{/cm} + x_{/cm}^2) + \\
&\qquad (y_{cm/O}^2 + 2y_{cm/O}y_{/cm} + y_{/cm}^2)] dm \\
&= (x_{cm/O}^2 + y_{cm/O}^2) \underbrace{\int dm}_{m} + 2x_{cm/O} \underbrace{\int x_{/cm} dm}_{0} + \\
&\qquad 2y_{cm/O} \underbrace{\int y_{/cm} dm}_{0} + \int (x_{/cm}^2 + y_{/cm}^2) dm \\
&= r_{cm/O}^2 m + \underbrace{\int (x_{/cm}^2 + y_{/cm}^2) dm}_{I_{zz}^{cm}} \\
&= I_{zz}^{cm} + \underbrace{r_{cm/O}^2}_{d^2} m
\end{aligned}
$$

The cancellation $\int y_{/\text{cm}}\, dm = \int x_{/\text{cm}}\, dm = 0$ comes from the definition of center of mass.

Sometimes, people write the parallel axis theorem more simply as

$$I^0 = I^{cm} + md^2 \qquad \text{or} \qquad J_O = J_{cm} + md^2$$

using the symbol J to mean I_{zz}. One thing to note about the parallel axis theorem is that the moment of inertia about any point O is always *greater* than the moment of inertia about the center of mass. For a given object, the minimum moment of inertia is about the center of mass.

Why the name *parallel axis theorem*? We use the name because the two I's calculated are the moments of inertia about two parallel axes (both in the z direction) through the two points cm and O.

One way to think about the theorem is the following. The moment of inertia of an object about a point O not at the center of mass is the same as that of the object about the cm *plus* that of a point mass located at the center of mass. If the distance from O to the cm is larger than the outer radius of the object, then the d^2m term is larger than I_{zz}^{cm}. The distance of equality of the two terms is the radius of gyration, r_{gyr}.

Perpendicular axis theorem (applies to planar objects only)

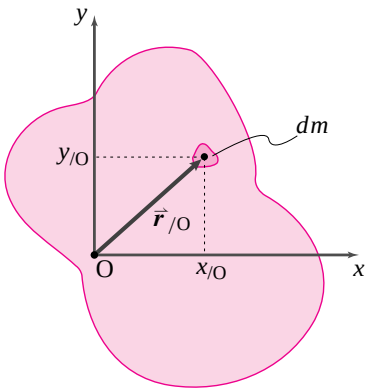
For planar objects,

$$
\begin{aligned}
I_{zz}^{O} &= \int |\vec{r}|^2\, dm \\
&= \int (x_{/O}^2 + y_{/O}^2) dm \\
&= \int x_{/O}^2 dm + \int y_{/O}^2 dm \\
&= I_{yy}^{O} + I_{xx}^{O}
\end{aligned}
$$

Similarly,

$$I_{zz}^{cm} = I_{xx}^{cm} + I_{yy}^{cm}.$$

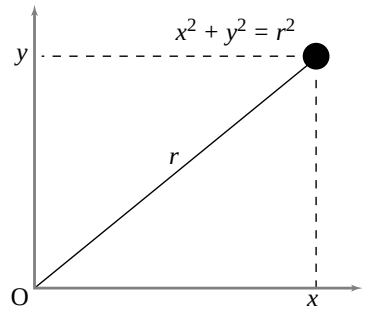
That is, the moment of inertia about the z-axis is the sum of the inertias about the two perpendicular axes x and y. Note that the objects must be planar ($z = 0$ everywhere) or the theorem would not be true. For example, $I_{xx}^{o} = \int (y_{/O}^2 + z_{/O}^2) dm \neq \int y_{/O}^2 dm$ for a three-dimensional object.



7.5 Some examples of 2-D Moment of Inertia

Here, we illustrate some simple moment of inertia calculations for two-dimensional objects. The needed formulas are summarized, in part, by the lower right corner components (that is, the elements in the third column and third row (3,3)) of the matrices in the table on the inside back cover.

One point mass



If we assume that all mass is concentrated at one or more points, then the integral

$$I^o_{zz} = \int r^2_{/o}\, dm$$

reduces to the sum

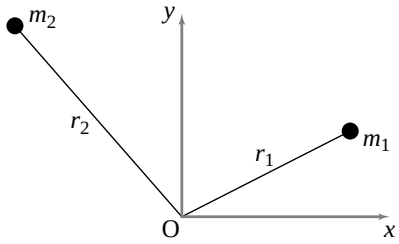
$$I^o_{zz} = \sum r^2_{i/o} m_i$$

which reduces to one term if there is only one mass,

$$I^o_{zz} = r^2 m = (x^2 + y^2)m.$$

So, if $x = 3\,\text{in}$, $y = 4\,\text{in}$, and $m = 0.1\,\text{lbm}$, then $I^o_{zz} = 2.5\,\text{lbm in}^2$. Note that, in this case, $I^{cm}_{zz} = 0$ since the radius from the center of mass to the center of mass is zero.

Two point masses

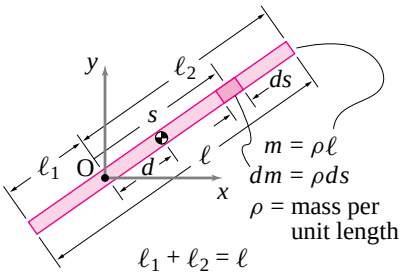


In this case, the sum that defines I^o_{zz} reduces to two terms, so

$$I^o_{zz} = \sum r^2_{i/o} m_i = m_1 r_1^2 + m_2 r_2^2.$$

Note that, if $r_1 = r_2 = r$, then $I^o_{zz} = m_{tot} r^2$.

A thin uniform rod



Consider a thin rod with uniform mass density, ρ, per unit length, and length ℓ. We calculate I^o_{zz} as

$$\begin{aligned} I^o_{zz} &= \int r^2 \overbrace{dm}^{\rho ds} \\ &= \int_{-\ell_1}^{\ell_2} s^2 \rho ds \quad (s = r) \\ &= \frac{1}{3}\rho s^3 \Big|_{-\ell_1}^{\ell_2} \quad (\text{since} \quad \rho \equiv const.) \\ &= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3). \end{aligned}$$

If either $\ell_1 = 0$ or $\ell_2 = 0$, then this expression reduces to $I^o_{zz} = \frac{1}{3}m\ell^2$. If $\ell_1 = \ell_2$, then O is at the center of mass and

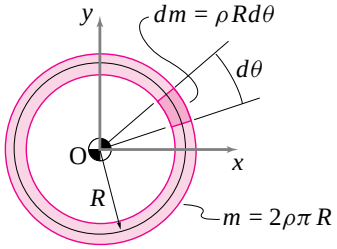
$$I^o_{zz} = I^{cm}_{zz} = \frac{1}{3}\rho\left(\left(\frac{\ell}{2}\right)^3 + \left(\frac{\ell}{2}\right)^3\right) = \frac{ml^2}{12}.$$

We can illustrate one last point. With a little bit of algebraic histrionics of the type that only hindsight can inspire, you can verify that the expression for I^0_{zz} can be arranged as follows:

$$\begin{aligned} I^0_{zz} &= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3) \\ &= \underbrace{\rho(\ell_1 + \ell_2)}_{m}\left(\underbrace{\frac{\ell_2 - \ell_1}{2}}_{d}\right)^2 + \underbrace{\rho\frac{(\ell_1 + \ell_2)^3}{12}}_{m\ell^2/12} \\ &= md^2 + m\frac{\ell^2}{12} \\ &= md^2 + I^{cm}_{zz} \end{aligned}$$

That is, the moment of inertia about point O is greater than that about the center of mass by an amount equal to the mass times the distance from the center of mass to point O squared. This derivation of the *parallel axis theorem* is for one special case, that of a uniform thin rod.

A uniform hoop



For a hoop of uniform mass density, ρ, per unit length, we might consider all of the points to have the same radius R. So,

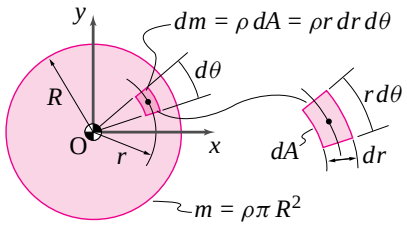
$$I_{zz}^{o} = \int r^2 dm = \int R^2 dm = R^2 \int dm = R^2 m.$$

Or, a little more tediously,

$$\begin{aligned} I_{zz}^{o} &= \int r^2 dm \\ &= \int_0^{2\pi} R^2 \rho R d\theta \\ &= \rho R^3 \int_0^{2\pi} d\theta \\ &= 2\pi\rho R^3 = \underbrace{(2\pi\rho R)}_{m} R^2 = mR^2. \end{aligned}$$

This I_{zz}^{o} is the same as for a single point mass m at a distance R from the origin O. It is also the same as for two point masses if they both are a distance R from the origin. For the hoop, however, O is at the center of mass so $I_{zz}^{o} = I_{zz}^{cm}$ which is not the case for a single point mass.

A uniform disk

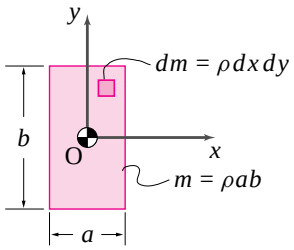


Assume the disk has uniform mass density, ρ, per unit area. For a uniform disk centered at the origin, the center of mass is at the origin so

$$\begin{aligned} I_{zz}^{o} = I_{zz}^{cm} &= \int r^2 dm \\ &= \int_0^R \int_0^{2\pi} r^2 \rho r d\theta dr \\ &= \int_0^R 2\pi\rho r^3 dr \\ &= 2\pi\rho \frac{r^4}{4}\Big|_0^R = \pi\rho\frac{R^4}{2} = (\pi\rho R^2)\frac{R^2}{2} \\ &= m\frac{R^2}{2}. \end{aligned}$$

For example, a 1 kg plate of 1 m radius has the same moment of inertia as a 1 kg hoop with a 70.7 cm radius.

Uniform rectangular plate



For the special case that the center of the plate is at point O, the center of mass of mass is also at O and $I_{zz}^{o} = I_{zz}^{cm}$.

$$\begin{aligned} I_{zz}^{o} = I_{zz}^{cm} &= \int r^2 dm \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2) \overbrace{\rho dx dy}^{dm} \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho\left(\frac{x^3}{3} + xy^2\right)\Big|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} dy \\ &= \rho\left(\frac{x^3 y}{3} + \frac{xy^3}{3}\right)\Big|_{x=-\frac{a}{2}}^{x=\frac{a}{2}}\Big|_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \\ &= \rho\left(\frac{a^3 b}{12} + \frac{ab^3}{12}\right) \\ &= \frac{m}{12}(a^2+b^2). \end{aligned}$$

Note that $\int r^2 dm = \int x^2 dm + \int y^2 dm$ for all planar objects (the *perpendicular axis theorem*). For a uniform rectangle, $\int y^2 dm = \rho \int y^2 dA$. But the integral $y^2 dA$ is just the term often used for I, the area moment of inertia, in strength of materials calculations for the stresses and stiffnesses of beams in bending. You may recall that $\int y^2 dA = \frac{ab^3}{12} = \frac{Ab^2}{12}$ for a rectangle. Similarly, $\int x^2 dA = \frac{Aa^2}{12}$. So, the polar moment of inertia $J = I_{zz}^{o} = m\frac{1}{12}(a^2+b^2)$ can be recalled by remembering the area moment of inertia of a rectangle combined with the perpendicular axis theorem.

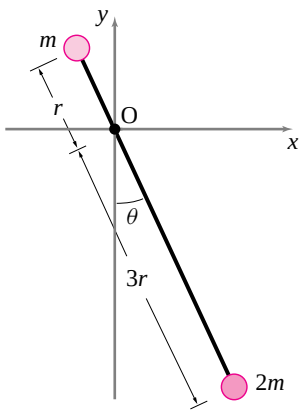


Figure 7.71: (Filename:sfig4.5.1)

SAMPLE 7.25 A pendulum is made up of two unequal point masses m and $2m$ connected by a massless rigid rod of length $4r$. The pendulum is pivoted at distance r along the rod from the small mass.

(a) Find the moment of inertia I_{zz}^O of the pendulum.

(b) If you had to put the total mass $3m$ at one end of the bar and still have the same I_{zz}^O as in (a), at what distance from point O should you put the mass? (This distance is known as the radius of gyration).

Solution Here we have two point masses. Therefore, the integral formula for I_{zz}^O ($I_{zz}^O = \int_m r_{/O}^2 dm$) gets replaced by a summation over the two masses:

$$\begin{aligned} I_{zz}^O &= \sum_{i=1}^{2} m_i r_{i/O}^2 \\ &= = m_1 r_{1/O}^2 + m_2 r_{2/O}^2 \end{aligned}$$

(a) For the pendulum, $m_1 = m,\ m_2 = 2m,\ r_{1/O} = r,\ r_{2/O} = 3r$.

$$\begin{aligned} I_{zz}^O &= mr^2 + 2m(3r)^2 \\ &= 19mr^2 \end{aligned}$$

$$\boxed{I_{zz}^O = 19mr^2}$$

(b) For the equivalent simple pendulum of mass $3m$, let the length of the massless rod (*i.e.*, the distance of the mass from O) be r_{gyr}.

$$(I_{zz}^O)_{\text{simple}} = (3m)\cdot r_{gyr}^2$$

Now we need $(I_{zz}^O)_{\text{simple}} = I_{zz}^O$ (from part (a))

$$\begin{aligned} \Rightarrow \quad 3\cancel{m} r_{gyr}^2 &= 19\cancel{m} r^2 \\ \Rightarrow \quad r_{gyr} &= \sqrt{\frac{19}{3}} r \\ &= 2.52r \end{aligned}$$

Thus the radius of gyration r_{gyr} of the given pendulum is $r_{gyr} = 2.52\,r$.

$$\boxed{r_{gyr} = 2.52\,r}$$

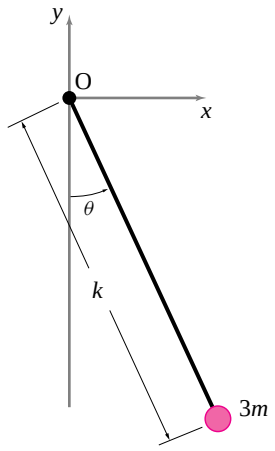


Figure 7.72: (Filename:sfig4.5.1a)

SAMPLE 7.26 A uniform rigid rod AB of mass $M = 2\,\text{kg}$ and length $3\ell = 1.5\,\text{m}$ swings about the z-axis passing through the pivot point O.

(a) Find the moment of inertia I_{zz}^{O} of the bar using the fundamental definition $I_{zz}^{O} = \int_m r_{/O}^2 dm$.

(b) Find I_{zz}^{O} using the parallel axis theorem given that $I_{zz}^{cm} = \frac{1}{12}m\ell^2$ where $m =$ total mass, and $\ell =$ total length of the rod. (You can find I_{zz}^{cm} for many commonly encountered objects in the table on the inside backcover of the text).

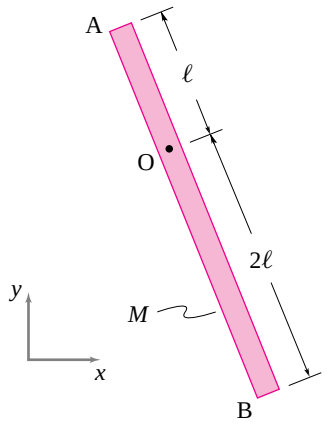


Figure 7.73: (Filename:sfig4.5.2)

Solution

(a) Since we need to carry out the integral, $I_{zz}^{O} = \int_m r_{/O}^2 dm$, to find I_{zz}^{O}, let us consider an infinitesimal length segment $d\ell'$ of the bar at distance ℓ' from the pivot point O. (see Figure 7.74). Let the mass of the infinitesimal segment be dm.

Now the mass of the segment may be written as

$$\begin{aligned} dm &= (\text{mass per unit length of the bar}) \cdot (\text{length of the segment}) \\ &= \frac{M}{3\ell}\,d\ell' \qquad \left(\text{Note: } \frac{\text{mass}}{\text{unit length}} = \frac{\text{total mass}}{\text{total length}}\right). \end{aligned}$$

We also note that the distance of the segment from point O, $r_{/O} = \ell$. Substituting the values found above for $r_{/O}$ and dm in the formula we get

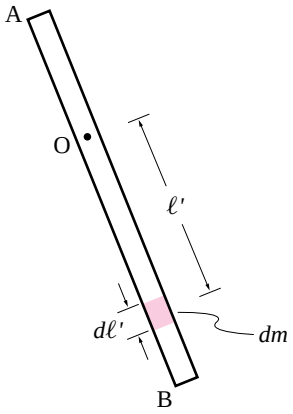


Figure 7.74: (Filename:sfig4.5.2a)

$$\begin{aligned} I_{zz}^{O} &= \int_{-\ell}^{2\ell} \underbrace{(\ell')^2}_{r_{/O}^2}\,\underbrace{\frac{M}{3\ell}d\ell'}_{dm} \\ &= \frac{M}{3\ell}\int_{-\ell}^{2\ell}(\ell')^2 d\ell' = \frac{M}{3\ell}\left[\frac{\ell'^3}{3}\right]_{-\ell}^{2\ell} \\ &= \frac{M}{3\ell}\left[\frac{8\ell^3}{3} - \left(-\frac{\ell^3}{3}\right)\right] = M\ell^2 \\ &= 2\,\text{kg}\cdot(0.5\,\text{m})^2 \\ &= 0.5\,\text{kg}\cdot\text{m}^2 \end{aligned}$$

$$\boxed{I_{zz}^{O} = 0.5\,\text{kg}\cdot\text{m}^2}$$

(b) The parallel axis theorem states that

$$I_{zz}^{O} = I_{zz}^{cm} + Mr_{O/cm}^2.$$

Since the rod is uniform, its center of mass is at its geometric center, *i.e.*, at distance $\frac{3\ell}{2}$ from either end. From the Fig 7.75 we can see that

$$r_{O/cm} = AG - AO = \frac{3\ell}{2} - \ell = \frac{\ell}{2}$$

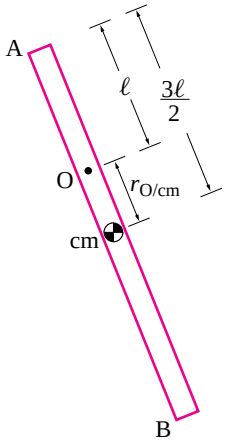


Figure 7.75: (Filename:sfig4.5.2b)

$$\begin{aligned} \text{Therefore, } I_{zz}^{O} &= \underbrace{\frac{1}{12}M(3\ell)^2}_{I_{zz}^{cm}} + M(\frac{\ell}{2})^2 \\ &= \frac{9}{12}M\ell^2 + M\frac{\ell^2}{4} = M\ell^2 \\ &= 0.5\,\text{kg}\cdot\text{m}^2 \quad \text{(same as in (a), of course)} \end{aligned}$$

$$\boxed{I_{zz}^{O} = 0.5\,\text{kg}\cdot\text{m}^2}$$

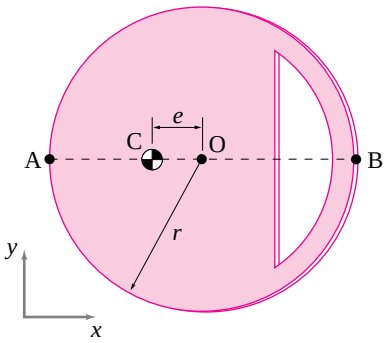


Figure 7.76: (Filename:sfig4.5.3)

SAMPLE 7.27 A uniform rigid wheel of radius $r = 1\,\text{ft}$ is made eccentric by cutting out a portion of the wheel. The center of mass of the eccentric wheel is at C, a distance $e = \frac{r}{3}$ from the geometric center O. The mass of the wheel (after deducting the cut-out) is 3.2 lbm. The moment of inertia of the wheel about point O, I_{zz}^{O}, is 1.8 lbm· ft^2. We are interested in the moment of inertia I_{zz} of the wheel about points A and B on the perimeter.

(a) Without any calculations, guess which point, A or B, gives a higher moment of inertia. Why?

(b) Calculate I_{zz}^{C}, I_{zz}^{A} and I_{zz}^{B} and compare with the guess in (a).

Solution

(a) The moment of inertia I_{zz}^{B} should be higher. Moment of inertia I_{zz} measures the geometric distribution of mass about the z-axis. But the distance of the mass from the axis counts more than the mass itself ($I_{zz}^{O} = \int_m r_{/O}^2 dm$). The distance $r_{/O}$ of the mass appears as a quadratic term in I_{zz}^{O}. The total mass is the same whether we take the moment of inertia about point A or about point B. However, the distribution of mass is not the same about the two points. Due to the cut-out being closer to point B there are more "dm's" at greater distances from point B than from point A. So, we guess that

$$\boxed{I_{zz}^{\text{B}} > I_{zz}^{\text{A}}}$$

(b) If we know the moment of inertia I_{zz}^{C} (about the center of mass) of the wheel, we can use the parallel axis theorem to find I_{zz}^{A} and I_{zz}^{B}. In the problem, we are given I_{zz}^{O}. But,

$$\begin{aligned}
I_{zz}^{\text{O}} &= I_{zz}^{\text{C}} + Mr_{O/C}^2 \quad \text{(parallel axis theorem)} \\
\Rightarrow \quad I_{zz}^{\text{C}} &= I_{zz}^{\text{O}} - Mr_{O/C}^2 \\
&= 1.8\,\text{lbm ft}^2 - 3.2\,\text{lbm} \underbrace{\left(\frac{1\,\text{ft}}{3}\right)^2}_{r_{O/C}=e=\frac{r}{3}} \\
&= 1.44\,\text{lbm}\cdot\text{ft}^2 \\
\text{Now,} \quad I_{zz}^{\text{A}} &= I_{zz}^{\text{C}} + Mr_{A/C}^2 = I_{zz}^{\text{C}} + M\left(\frac{2r}{3}\right)^2 \\
&= 1.44\,\text{lbm}\cdot\text{ft}^2 + 3.2\,\text{lbm}\left(\frac{2\,\text{ft}}{3}\right)^2 \\
&= 2.86\,\text{lbm}\cdot\text{ft}^2 \\
\text{and} \quad I_{zz}^{\text{B}} &= I_{zz}^{\text{C}} + Mr_{B/C}^2 = I_{zz}^{\text{C}} + M\left(r + \frac{r}{3}\right)^2 \\
&= 1.44\,\text{lbm}\cdot\text{ft}^2 + 3.2\,\text{lbm}\left(1\,\text{ft} + \frac{1\,\text{ft}}{3}\right)^2 \\
&= 7.13\,\text{lbm}\cdot\text{ft}^2
\end{aligned}$$

$$\boxed{I_{zz}^{\text{C}} = 1.44\,\text{lbm}\cdot\text{ft}^2,\ I_{zz}^{\text{A}} = 2.86\,\text{lbm}\cdot\text{ft}^2,\ I_{zz}^{\text{B}} = 7.13\,\text{lbm}\cdot\text{ft}^2}$$

Clearly, $I_{zz}^{\text{B}} > I_{zz}^{\text{A}}$, as guessed in (a).

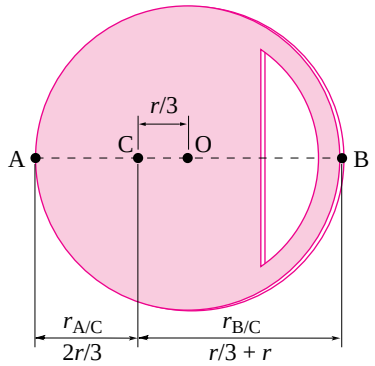


Figure 7.77: (Filename:sfig4.5.3a)

SAMPLE 7.28 *A sphere or a point?* A uniform solid sphere of mass m and radius r is attached to a massless rigid rod of length ℓ. The sphere swings in the xy plane. Find the error in calculating I_{zz}^{O} as a function of r/l if the sphere is treated as a point mass concentrated at the center of mass of the sphere.

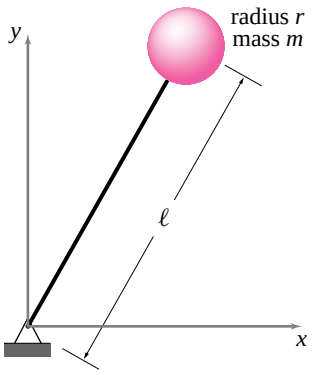


Figure 7.78: (Filename:sfig4.5.4)

Solution The exact moment of inertia of the sphere about point O can be calculated using parallel axis theorem:

$$\begin{aligned} I_{zz}^{O} &= I_{zz}^{cm} + ml^2 \\ &= \underbrace{\frac{2}{5}mr^2}_{\text{See table 4.10 of the text.}} + ml^2. \end{aligned}$$

If we treat the sphere as a point mass, he moment of inertia I_{zz}^{O} is

$$\tilde{I}_{zz}^{O} = ml^2.$$

Therefore, the relative error in I_{zz}^{O} is

$$\begin{aligned} \text{error} &= \frac{I_{zz}^{O} - \tilde{I}_{zz}^{O}}{I_{zz}^{O}} \\ &= \frac{\frac{2}{5}mr^2 + ml^2 - ml^2}{\frac{2}{5}mr^2 + ml^2} \\ &= \frac{\frac{2}{5}\frac{r^2}{l^2}}{\frac{2}{5}\frac{r^2}{l^2} + 1} \end{aligned}$$

From the above expression we see that for $r \ll l$ the error is very small. From the graph of error in Fig. 7.79 we see that even for $r = l/5$, the error in I_{zz}^{O} due to approximating the sphere as a point mass is less than 2%.

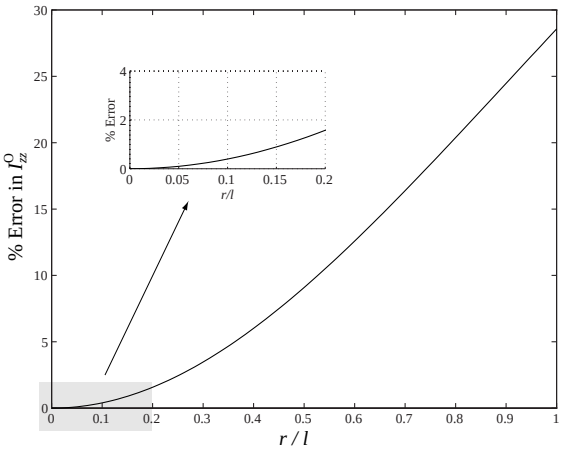


Figure 7.79: Relative error in I_{zz}^{O} of the sphere as a function of r/l. (Filename:sfig4.5.4a)

7.6 Using I_{zz}^{cm} and I_{zz}^{O} in 2-D circular motion dynamics

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from chapter 1.1. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. As you recall from chapter 3.6, the linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center of mass of the system of interest. One wishes for something so simple for the calculation of angular momentum.

It turns out that we are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies. The luck is not so great for 3-D rigid bodies but still there is some simplification. For general motion of non-rigid bodies there is no simplification to be had. The simplification is to use the moment of inertia for the bodies rather than evaluating the momenta and energy quantities as integrals and sums. Of course one may have to do a sum or integral to evaluate $I \equiv I_{zz}^{cm}$ or $[\boldsymbol{I}^{\text{cm}}]$ but once this calculation is done, one need not work with the integrals while worrying about the dynamics. At this point we will assume that you are comfortable calculating and looking-up moments of inertia. We proceed to use it for the purposes of studying mechanics. For constant rate rotation, we can calculate the velocity and acceleration of various points on a rigid body using $\vec{v} = \vec{\omega} \times \vec{r}$ and $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$. So we can calculate the various motion quantities of interest: linear momentum $\vec{L}$, rate of change of linear momentum $\dot{\vec{L}}$, angular momentum $\vec{H}$, rate of change of angular momentum $\dot{\vec{H}}$, and kinetic energy E_{K}.

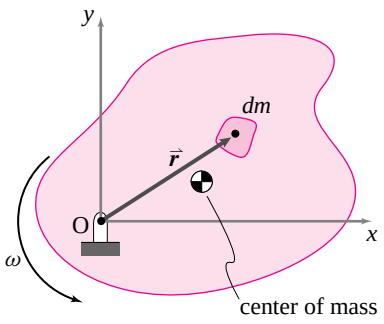


Figure 7.80: A two-dimensional body is rotating around the point O at constant rate ω. A differential bit of mass dm is shown. The center of mass is also shown.

(Filename:tfigure4.2Dinertia)

Consider a two-dimensional rigid body like that shown in figure 7.80. Now let us consider the various motion quantities in turn. First the linear momentum $\vec{L}$. The linear momentum of any system in any motion is $\vec{L} = \vec{v}_{cm} m_{tot}$. So, for a rigid body spinning at constant rate ω about point O (using $\vec{\omega} = \omega \hat{k}$):

$$\vec{L} = \vec{v}_{cm} m_{tot} = \vec{\omega} \times \vec{r}_{cm/o} m_{tot}.$$

Similarly, for any system, we can calculate the rate of change of linear momentum $\dot{\vec{L}}$ as $\dot{\vec{L}} = \vec{a}_{cm} m_{tot}$. So, for a rigid body spinning at constant rate,

$$\dot{\vec{L}} = \vec{a}_{cm} m_{tot} = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{cm/o}) m_{tot}.$$

That is, the linear momentum is correctly calculated for this special motion, as it is for all motions, by thinking of the body as a point mass at the center of mass.

Unlike the calculation of linear momentum, the angular momentum turns out to be something different than would be calculated by using a point mass at the center of mass. You can remember this important fact by looking at the case when the rotation is about the center of mass (point O coincides with the center of mass). In this case one can intuitively see that the angular momentum of a rigid body *is not* zero even

though the center of mass is not moving. Here's the calculation just to be sure:

$$\begin{aligned}
\vec{\boldsymbol{H}}_O &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{v}}\, dm && \text{(by definition of } \vec{\boldsymbol{H}}_O\text{)}\\
&= \int \vec{\boldsymbol{r}}_{/O} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/O})\, dm && \text{(using } \vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}\text{)}\\
&= \int (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}) \times \Big[(\omega\hat{\boldsymbol{k}}) \times (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}})\Big]\, dm && \text{(substituting } \vec{\boldsymbol{r}}_{/O} \text{ and } \vec{\boldsymbol{\omega}}\text{)}\\
&= \{\int (x_{/O}^2 + y_{/O}^2)\, dm\}\omega\hat{\boldsymbol{k}} && \text{(doing cross products)}\\
&= \{\int r_{/O}^2\, dm\}\omega\hat{\boldsymbol{k}}\\
&= \underbrace{I_{zz}^{O}}\,\omega\hat{\boldsymbol{k}}
\end{aligned}$$

I_{zz}^{O} is the 'polar' moment of inertia.

We have defined the 'polar' moment of inertia as $I_{zz}^{o} = \int r_{/o}^2\, dm$. In order to calculate I_{zz}^{o} for a specific body, assuming uniform mass distribution for example, one must convert the differential quantity of mass dm into a differential of geometric quantities. For a line or curve, $dm = \rho d\ell$; for a plate or surface, $dm = \rho dA$, and for a 3-D region, $dm = \rho dV$. $d\ell$, dA, and dV are differential line, area, and volume elements, respectively. In each case, ρ is the mass density per unit length, per unit area, or per unit volume, respectively. To avoid clutter, we do not define a different symbol for the density in each geometric case. The differential elements must be further defined depending on the coordinate systems chosen for the calculation; *e.g.*, for rectangular coordinates, $dA = dxdy$ or, for polar coordinates, $dA = rdrd\theta$.

Since $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{\omega}}$ always point in the $\hat{\boldsymbol{k}}$ direction for two dimensional problems people often just think of angular momentum as a scalar and write the equation above simply as '$H = I\omega$,' the form usually seen in elementary physics courses.

The derivation above has a feature that one might not notice at first sight. The quantity called I_{zz}^{O} does not depend on the rotation of the body. That is, the value of the integral does not change with time, so I_{zz}^{O} is a constant. So, perhaps unsurprisingly, a two-dimensional body spinning about the z-axis through O has constant angular momentum about O if it spins at a constant rate. ①

$$\dot{\vec{\boldsymbol{H}}}_O = \vec{\boldsymbol{0}}.$$

① Note that the angular momentum about some other point than O will not be constant unless the center of mass does not accelerate (*i.e.*, is at point O).

Now, of course we could find this result about constant rate motion of 2-D bodies somewhat more cumbersomely by plugging in the general formula for rate of change of angular momentum as follows:

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_O &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm\\
&= \int \vec{\boldsymbol{r}}_{/O} \times (\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/O}))\, dm\\
&= \int (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}) \times \Big[\omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}))\Big]\, dm\\
&= \vec{\boldsymbol{0}}.
\end{aligned} \tag{7.66}$$

Finally, we can calculate the kinetic energy by adding up $\frac{1}{2}m_i v_i^2$ for all the bits of mass on a 2-D body spinning about the z-axis:

$$E_K = \int \frac{1}{2}v^2\, dm = \int \frac{1}{2}(\omega r)^2\, dm = \frac{1}{2}\omega^2 \int r^2\, dm = \frac{1}{2}I_{zz}^{o}\omega^2 \quad . \tag{7.67}$$

If we accept the formulae presented for rigid bodies in the box at the end of chapter 7, we can find all of the motion quantities by setting $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{0}}$.

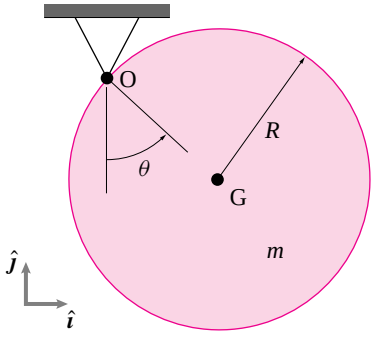


Figure 7.81: (Filename:tfigure5.3.pend.disk)

Example: **Pendulum disk**

For the disk shown in figure 7.81, we can calculate the rate of change of angular momentum about point O as

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} &= \vec{\boldsymbol{r}}_{G/O} \times m\vec{\boldsymbol{a}}_{cm} + I_{zz}^{cm}\alpha\hat{\boldsymbol{k}} \\ &= R^2 m\ddot{\theta}\hat{\boldsymbol{k}} + I_{zz}^{cm}\ddot{\theta}\hat{\boldsymbol{k}} \\ &= (I_{zz}^{cm} + R^2 m)\ddot{\theta}\hat{\boldsymbol{k}}.\end{aligned}$$

Alternatively, we could calculate directly

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} &= I_{zz}^{O}\alpha\hat{\boldsymbol{k}} \\ &= \underbrace{(I_{zz}^{cm} + R^2 m)}\,\ddot{\theta}\hat{\boldsymbol{k}}.\end{aligned}$$

by the parallel axis theroem

□

But you are cautioned against falling into the common misconception that the formula $M = I\alpha$ applies in three dimensions by just thinking of the scalars as vectors and matrices. That is, the formula

$$\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \underbrace{\dot{\vec{\boldsymbol{\omega}}}}_{\vec{\boldsymbol{\alpha}}} \tag{7.68}$$

is only correct when $\vec{\boldsymbol{\omega}}$ is zero or when $\vec{\boldsymbol{\omega}}$ is an eigenvector of $[I_{/O}]$. To repeat, the equation

$$\sum \text{Moments about O} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\alpha}} \tag{7.69}$$

is generally wrong, it only applies if there is some known reason to neglect $\vec{\boldsymbol{\omega}} \times \dot{\vec{\boldsymbol{H}}}_0$. For example, $\vec{\boldsymbol{\omega}} \times \dot{\vec{\boldsymbol{H}}}_0$ can be neglected when rotation is about a principal axis as for planar bodies rotating in the plane. The term $\vec{\boldsymbol{\omega}} \times \dot{\vec{\boldsymbol{H}}}_0$ can also be neglected at the start or stop of motion, that is when $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}}$.

The equation for linear momentum balance is the same as always, we just need to calculate the acceleration of the center of mass of the spinning body.

$$\dot{\vec{\boldsymbol{L}}} = m_{tot}\vec{\boldsymbol{a}}_{cm} = m_{tot}\left[\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{cm/O}) + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{cm/O}\right] \tag{7.70}$$

Finally, the kinetic energy for a planar rigid body rotating in the plane is:

$$E_{\mathrm{K}} = \frac{1}{2}\vec{\boldsymbol{\omega}} \cdot ([\boldsymbol{I}^{\mathrm{cm}}] \cdot \vec{\boldsymbol{\omega}}) + \frac{1}{2}m\,\underbrace{v_{\mathrm{cm}}^2}\,.$$

$\vec{\boldsymbol{v}}_{cm} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{cm/O}}$

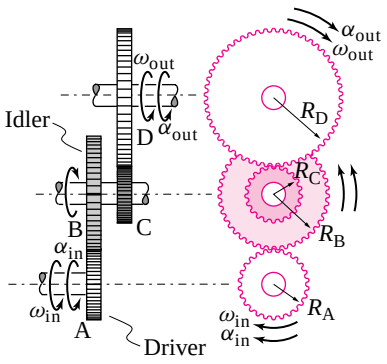


Figure 7.82: An accelerating compound gear train.

(Filename:sfig5.6.1)

SAMPLE 7.29 *An accelerating gear train.* In the gear train shown in Fig. 7.82, the torque at the input shaft is $M_{in} = 200\,\text{N·m}$ and the angular acceleration is $\alpha_{in} = 50\,\text{rad/s}^2$. The radii of the various gears are: $R_A = 5\,\text{cm}$, $R_B = 8\,\text{cm}$, $R_C = 4\,\text{cm}$, and $R_D = 10\,\text{cm}$ and the moments of inertia about the shaft axis passing through their respective centers are: $I_A = 0.1\,\text{kg m}^2$, $I_{BC} = 5I_A$, $I_D = 4I_A$. Find the output torque M_{out} of the gear train.

Solution Since the difference between the input power and the output power is used in accelerating the gears, we may write

$$P_{in} - P_{out} = \dot{E}_{\text{K}}$$

Let M_{out} be the output torque of the gear train. Then,

$$P_{in} - P_{out} = M_{in}\,\omega_{in} - M_{out}\,\omega_{out}. \tag{7.71}$$

Now,

$$\begin{aligned} \dot{E}_{\text{K}} &= \frac{d}{dt}(E_{\text{K}}) && (7.72)\\ &= \frac{d}{dt}\left(\frac{1}{2}I_A\,\omega_{in}^2 + \frac{1}{2}I_{BC}\,\omega_{BC}^2 + \frac{1}{2}I_D\,\omega_{out}^2\right)\\ &= I_A\,\omega_{in}\,\dot{\omega}_{in} + I_{BC}\,\omega_{BC}\,\dot{\omega}_{BC} + I_D\,\omega_{out}\,\dot{\omega}_{out}\\ &= I_A\,\omega_{in}\,\alpha_{in} + 5I_A\,\omega_{BC}\,\alpha_{BC} + 4I_A\,\omega_{out}\,\alpha_{out}. && (7.73) \end{aligned}$$

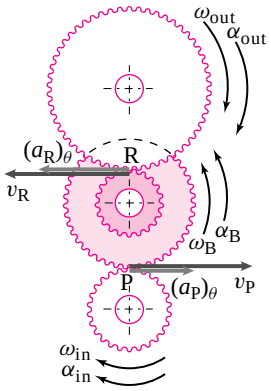


Figure 7.83: The velocity or acceleration of the point of contact between two meshing gears has to be the same irrespective of which meshing gear's geometry and motion is used to compute them.

(Filename:sfig5.6.1a)

The different ω's and the α's can be related by realizing that the linear speed or the tangential acceleration of the point of contact between any two meshing gears has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus, using the linear speed and tangential acceleration calculations for points P and R in Fig. 7.83, we find

$$\begin{aligned} v_P &= \omega_{in}\,R_A = \omega_B\,R_B\\ \Rightarrow\quad \omega_B &= \omega_{in}\cdot\frac{R_A}{R_B}\\ (a_P)_\theta &= \alpha_{in}\,R_A = \alpha_B\,R_B\\ \Rightarrow\quad \alpha_B &= \alpha_{in}\cdot\frac{R_A}{R_B}. \end{aligned}$$

Similarly,

$$\begin{aligned} v_R &= \omega_C\,R_C = \omega_{out}\,R_D\\ \Rightarrow\quad \omega_{out} &= \omega_C\cdot\frac{R_C}{R_D}\\ (a_R)_\theta &= \alpha_C\,R_C = \alpha_{out}\,R_D\\ \Rightarrow\quad \alpha_{out} &= \alpha_C\cdot\frac{R_C}{R_D}. \end{aligned}$$

But

$$\begin{aligned} \omega_C &= \omega_B = \omega_{BC}\\ \Rightarrow\quad \omega_{out} &= \omega_{in}\cdot\frac{R_A}{R_B}\cdot\frac{R_C}{R_D} \end{aligned}$$

and

$$\begin{aligned} \alpha_C &= \alpha_B = \alpha_{BC} \\ \Rightarrow \quad \alpha_{out} &= \alpha_{in} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D}. \end{aligned}$$

Substituting these expressions for ω_{out}, α_{out}, ω_{BC} and α_{BC} in equations (7.71) and (7.73), we get

$$\begin{aligned} P_{in} - P_{out} &= M_{in}\,\omega_{in} - M_{out}\omega_{in} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \\ &= \omega_{in}\left(M_{in} - M_{out} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D}\right). \\ \dot{E_{\mathrm{K}}} &= I_A\left[\omega_{in}\alpha_{in} + 5\omega_{in}\alpha_{in}(\frac{R_A}{R_B})^2 + 4\omega_{in}\alpha_{in}(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D})^2\right] \\ &= I_A\omega_{in}\left[\alpha_{in} + 5\alpha_{in}(\frac{R_A}{R_B})^2 + 4\alpha_{in}(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D})^2\right]. \end{aligned}$$

Now equating the two quantities, $P_{in} - P_{out}$ and $\dot{E_{\mathrm{K}}}$, and canceling ω_{in} from both sides, we obtain

$$\begin{aligned} M_{out}\frac{R_A}{R_B} \cdot \frac{R_C}{R_D} &= M_{in} - I_A\alpha_{in}\left[1 + 5(\frac{R_A}{R_B})^2 + 4(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D})^2\right] \\ M_{out}\frac{5}{8} \cdot \frac{4}{10} &= 200\,\mathrm{N{\cdot}m} - 5\,\mathrm{kg\ m^2 \cdot\ rad/s^2}\left[1 + 5(\frac{5}{8})^2 + 4(\frac{5}{8} \cdot \frac{4}{10})^2\right] \\ M_{out} &= 735.94\,\mathrm{N{\cdot}m} \\ &\approx 736\,\mathrm{N{\cdot}m}. \end{aligned}$$

$$\boxed{M_{out} = 736\,\mathrm{N{\cdot}m}}$$

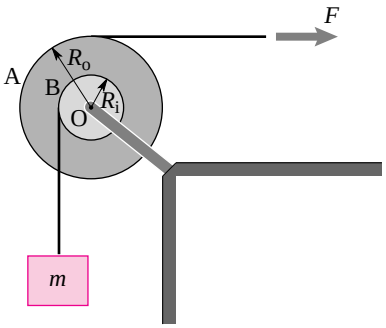


Figure 7.84: Two drums with strings wrapped around are used to pull up a mass m.

(Filename:sfig5.6.2)

SAMPLE 7.30 *Drums used as pulleys.* Two drums, A and B of radii $R_o = 200$ mm and $R_i = 100$ mm are welded together. The combined mass of the drums is $M = 20$ kg and the combined moment of inertia about the z-axis passing through their common center O is $I_{zz/O} = 1.6$ kg m^2. A string attached to and wrapped around drum B supports a mass $m = 2$ kg. The string wrapped around drum A is pulled with a force $F = 20$ N as shown in Fig. 7.84. Assume there is no slip between the strings and the drums. Find

(a) the angular acceleration of the drums,
(b) the tension in the string supporting mass m, and
(c) the acceleration of mass m.

Solution The free body diagram of the drums and the mass are shown in Fig. 7.85 separately where T is the tension in the string supporting mass m and O_x and O_y are the support reactions at O. Since the drums can only rotate about the z-axis, let

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} \quad \text{and} \quad \dot{\vec{\boldsymbol{\omega}}} = \dot{\omega}\hat{\boldsymbol{k}}.$$

Now, let us do angular momentum balance about the center of rotation O:

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$$

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= T R_i \hat{\boldsymbol{k}} - F R_o \hat{\boldsymbol{k}} \\ &= (T R_i - F R_o)\hat{\boldsymbol{k}}. \end{aligned}$$

Since the motion is restricted to the xy-plane (*i.e.*, 2-D motion), the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ may be computed as

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} &= I_{zz/cm}\,\dot{\omega}\hat{\boldsymbol{k}} + \vec{\boldsymbol{r}}_{cm/O} \times \vec{\boldsymbol{a}}_{cm}\ M_{total} \\ &= I_{zz/O}\,\dot{\omega}\hat{\boldsymbol{k}} + \underbrace{\vec{\boldsymbol{r}}_{O/O}}_{0} \times \underbrace{\vec{\boldsymbol{a}}_{cm}}_{0}\ M_{total} \\ &= I_{zz/O}\,\dot{\omega}\hat{\boldsymbol{k}}. \end{aligned}$$

Setting $\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ we get

$$T R_i - F R_o = I_{zz/O}\,\dot{\omega}. \tag{7.74}$$

Now, let us write linear momentum balance, $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, for mass m:

$$\underbrace{(T - mg)\hat{\boldsymbol{j}}}_{\sum \vec{\boldsymbol{F}}} = m\vec{\boldsymbol{a}}.$$

Do we know anything about acceleration $\vec{\boldsymbol{a}}$ of the mass? Yes, we know its direction ($\pm\hat{\boldsymbol{j}}$) and we also know that it has to be the same as the tangential acceleration $(\vec{\boldsymbol{a}}_D)_\theta$ of point D on drum B (why?). Thus,

$$\begin{aligned} \vec{\boldsymbol{a}} &= (\vec{\boldsymbol{a}}_D)_\theta \\ &= \dot{\omega}\hat{\boldsymbol{k}} \times (-R_i\hat{\boldsymbol{\imath}}) \\ &= -\dot{\omega} R_i \hat{\boldsymbol{\jmath}}. \end{aligned} \tag{7.75}$$

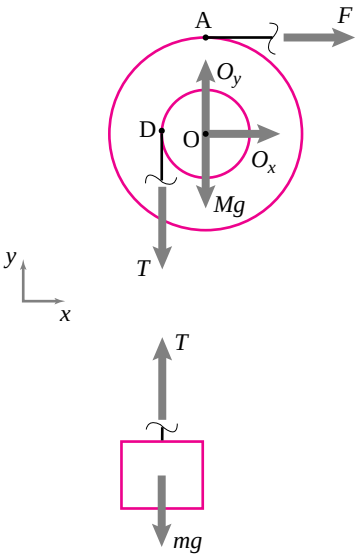


Figure 7.85: Free body diagram of the drums and the mass m. T is the tension in the string supporting mass m and O_x and O_y are the reactions of the support at O.

(Filename:sfig5.6.2a)

Therefore,

$$T - mg = -m\dot{\omega}R_i. \tag{7.76}$$

(a) **Calculation of $\dot{\omega}$:** We now have two equations, (7.74) and (7.76), and two unknowns, $\dot{\omega}$ and T. Subtracting R_i times Eqn.(7.76) from Eqn. (7.74) we get

$$\begin{aligned}
-FR_o + mgR_i &= (I_{zz/O} + mR_i^2)\dot{\omega} \\
\Rightarrow \qquad \dot{\omega} &= \frac{-FR_o + mgR_i}{(I_{zz/O} + mR_i^2)} \\
&= \frac{-20\,\text{N}\cdot 0.2\,\text{m} + 2\,\text{kg}\cdot 9.81\,\text{m/s}^2\cdot 0.1m}{1.6\,\text{kg m}^2 + 2\,\text{kg}\cdot(0.1\,\text{m})^2} \\
&= \frac{-2.038\,\text{kg m}^2/\text{s}^2}{1.62\,\text{kg m}^2} \\
&= -1.258\frac{1}{\text{s}^2}
\end{aligned}$$

$$\boxed{\vec{\dot{\boldsymbol{\omega}}} = -1.26\,\text{rad/s}^2\hat{\boldsymbol{k}}}$$

(b) **Calculation of tension T:** From equation (7.76):

$$\begin{aligned}
T &= mg - m\dot{\omega}R_i \\
&= 2\,\text{kg}\cdot 9.81\,\text{m/s}^2 - 2\,\text{kg}\cdot(-1.26\,\text{s}^{-2})\cdot 0.1\,\text{m} \\
&= 19.87\,\text{N}
\end{aligned}$$

$$\boxed{T = 19.87\,\text{N}}$$

(c) **Calculation of acceleration of the mass:** Since the acceleration of the mass is the same as the tangential acceleration of point D on the drum, we get (from eqn. (7.75))

$$\begin{aligned}
\vec{\boldsymbol{a}} &= (\vec{\boldsymbol{a}}_D)_\theta = -\dot{\omega}R_i\hat{\boldsymbol{j}} \\
&= -(-1.26\,\text{s}^{-2})\cdot 0.1\,\text{m} \\
&= 0.126\,\text{m/s}^2\hat{\boldsymbol{j}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = 0.13\,\text{m/s}^2\hat{\boldsymbol{j}}}$$

Comments: It is important to understand why the acceleration of the mass is the same as the tangential acceleration of point D on the drum. We have assumed (as is common practice) that the string is massless and inextensible. Therefore each point of the string supporting the mass must have the same linear displacement, velocity, and acceleration as the mass. Now think about the point on the string which is momentarily in contact with point D of the drum. Since there is no relative slip between the drum and the string, the two points must have the same vertical acceleration. This vertical acceleration for point D on the drum is the tangential acceleration $(\vec{\boldsymbol{a}}_D)_\theta$.

SAMPLE 7.31 *Energy Accounting:* Consider the pulley problem of Sample 7.30 again.

(a) What percentage of the input energy (work done by the applied force F) is used in raising the mass by 1 m?

(b) Where does the rest of the energy go? Provide an energy-balance sheet.

Solution

(a) Let W_i and W_h be the input energy and the energy used in raising the mass by 1 m, respectively. Then the percentage of energy used in raising the mass is

$$\% \text{ of input energy used} = \frac{W_h}{W_i} \times 100.$$

Thus we need to calculate W_i and W_h to find the answer. W_i is the work done by the force F on the system during the interval in which the mass moves up by 1 m. Let s be the displacement of the force F during this interval. Since the displacement is in the same direction as the force (we know it is from Sample 7.30), the input-energy is

$$W_i = F\, s.$$

So to find W_i we need to find s.

For the mass to move up by 1 m the inner drum B must rotate by an angle θ where

$$1\,\text{m} = \theta\, R_i \qquad \Rightarrow \qquad \theta = \frac{1\,\text{m}}{0.1\,\text{m}} = 10\,\text{rad}.$$

Since the two drums, A and B, are welded together, drum A must rotate by θ as well. Therefore the displacement of force F is

$$s = \theta\, R_o = 10\,\text{rad} \cdot 0.2\,\text{m} = 2\,\text{m},$$

and the energy input is

$$W_i = F\, s = 20\,\text{N} \cdot 2\,\text{m} = 40\text{J}.$$

Now, the work done in raising the mass by 1 m is

$$W_h = mgh = 2\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot 1\,\text{m} = 19.62\text{J}.$$

Therefore, the percentage of input-energy used in raising the mass

$$= \frac{19.62\,\text{N·m}}{40} \times 100 = 49.05\% \approx 49\%.$$

(b) The rest of the energy (= 51%) goes in accelerating the mass and the pulley. Let us find out how much energy goes into each of these activities. Since the initial state of the system from which we begin energy accounting is not prescribed (that is, we are not given the height of the mass from which it is to be raised 1 m, nor do we know the velocities of the mass or the pulley at that initial height), let us assume that at the initial state, the angular speed of the pulley is ω_o and the linear speed of the mass is v_o. At the end of raising the mass by 1 m from this state, let the angular speed of the pulley be ω_f and the linear speed of the mass be v_f. Then, the energy used in accelerating the pulley

is

$$\begin{aligned}
(\Delta E_K)_{\text{pulley}} &= \text{final kinetic energy} - \text{initial kinetic energy} \\
&= \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_o^2 \\
&= \frac{1}{2} I \underbrace{(\omega_f^2 - \omega_o^2)}
\end{aligned}$$

assuming constant acceleration, $\omega_f^2 = \omega_o^2 + 2\alpha\theta$, or $\omega_f^2 - \omega_o^2 = 2\alpha\theta$.

$$\begin{aligned}
&= I\alpha\theta \qquad \text{(from Sample 7.34, } \alpha = 1.258\,\text{rad/s}^2\text{.)} \\
&= 1.6\,\text{kg m}^2 \cdot 1.258\,\text{rad/s}^2 \cdot 10\,\text{rad} \\
&= 20.13\,\text{N·m} = 20.13\,\text{J.}
\end{aligned}$$

Similarly, the energy used in accelerating the mass is

$$\begin{aligned}
(\Delta E_K)_{\text{mass}} &= \text{final kinetic energy} - \text{initial kinetic energy} \\
&= \frac{1}{2} m v_f^2 - \frac{1}{2} m v_o^2 \\
&= \frac{1}{2} m \underbrace{(v_f^2 - v_o^2)}_{2ah} \\
&= mah \\
&= 2\,\text{kg} \cdot 0.126\,\text{m/s}^2 \cdot 1\,\text{m} \\
&= 0.25\,\text{J.}
\end{aligned}$$

We can calculate the percentage of input energy used in these activities to get a better idea of energy allocation. Here is the summary table:

Activities	**Energy Spent**	
	in Joule	as % of input energy
In raising the mass by 1 m	19.62	49.05%
In accelerating the mass	0.25	0.62 %
In accelerating the pulley	20.13	50.33 %
Total	40.00	100 %

So, what would you change in the set-up so that more of the input energy is used in raising the mass? Think about what aspects of the motion would change due to your proposed design.

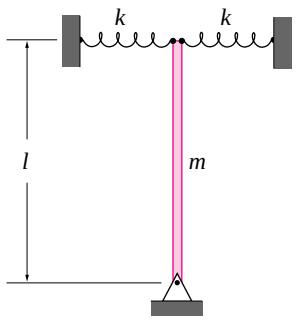


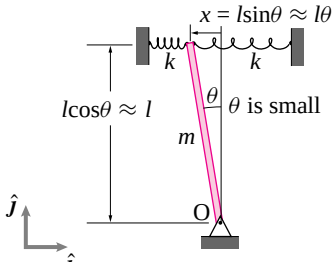
Figure 7.86: (Filename:sfig10.1.2)

SAMPLE 7.32 A uniform rigid bar of mass $m = 2\,\text{kg}$ and length $L = 1\,\text{m}$ is pinned at one end and connected to two springs, each with spring constant k, at the other end. The bar is tweaked slightly from its vertical position. It then oscillates about its original position. The bar is timed for 20 full oscillations which take 12.5 seconds. Ignore gravity.

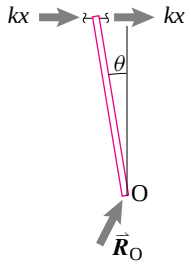
(a) Find the equation of motion of the rod.

(b) Find the spring constant k.

(c) What should be the spring constant of a torsional spring if the bar is attached to one at the bottom and has the same oscillating motion characteristics?



(a) Geometry



(b) Free body diagram

Figure 7.87: (Filename:sfig10.1.2a)

Solution

(a) Refer to the free body diagram in figure 7.87. Angular momentum balance for the rod about point O gives

$$\sum \vec{M}_{\text{O}} = \dot{\vec{H}}_{\text{O}}$$

$$\begin{aligned} \text{where} \quad \vec{M}_{\text{O}} &= -2k\overbrace{x}^{l\sin\theta}\cdot l\cos\theta\hat{\boldsymbol{k}} \\ &= -2kl^2\sin\theta\,\cos\theta\hat{\boldsymbol{k}}, \\ \text{and} \quad \dot{\vec{H}}_{\text{O}} &= I^O_{zz}\ddot{\theta}\hat{\boldsymbol{k}} = \underbrace{\frac{1}{3}ml^2}_{I^O_{zz}}. \end{aligned}$$

Thus

$$\frac{1}{3}ml^2\,\ddot{\theta} = -2kl^2\sin\theta\cos\theta.$$

However, for small θ, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$,

$$\Rightarrow \quad \ddot{\theta} + \frac{6k\cancel{l}^2}{m\cancel{l}^2}\theta = 0. \tag{7.77}$$

$$\boxed{\ddot{\theta} + \tfrac{6k}{m}\theta = 0}$$

(b) Comparing Eqn. (7.77) with the standard harmonic oscillator equation $\ddot{x} + \lambda^2 x = 0$, we get

$$\begin{aligned} \text{angular frequency} \quad \lambda &= \sqrt{\frac{6k}{m}}, \\ \text{and the time period} \quad T &= \frac{2\pi}{\lambda} \\ &= 2\pi\sqrt{\frac{m}{6k}}. \end{aligned}$$

From the measured time for 20 oscillations, the time period (time for one oscillation) is

$$T = \frac{12.5}{20}\,\text{s} = 0.625\,\text{s}$$

Now equating the measured T with the derived expression for T we get

$$\begin{aligned} 2\pi\sqrt{\frac{m}{6k}} &= 0.625\,\text{s} \\ \Rightarrow \qquad k &= 4\pi^2\cdot\frac{m}{6(0.625\,\text{s})^2} \\ &= \frac{4\pi^2\cdot 2\,\text{kg}}{6(0.625\,\text{s})^2} \\ &= 33.7\,\text{N/m}. \end{aligned}$$

$$\boxed{k = 33.7\,\text{N/m}}$$

(c) If the two linear springs are to be replaced by a torsional spring at the bottom, we can find the spring constant of the torsional spring by comparison. Let k_{tor} be the spring constant of the torsional spring. Then, as shown in the free body diagram (see figure 7.88), the restoring torque applied by the spring at an angular displacement θ is $k_{\text{tor}}\theta$. Now, writing the angular momentum balance about point O, we get

$$\begin{aligned} \sum \vec{M}_{\text{O}} &= \dot{\vec{H}}_{\text{O}} \\ -k_{\text{tor}}\theta\hat{\boldsymbol{k}} &= I_{zz}^{O}(\ddot{\theta}\hat{\boldsymbol{k}}) \\ \Rightarrow \qquad \ddot{\theta} + \frac{k_{\text{tor}}}{I_{zz}^{O}}\theta &= 0. \end{aligned}$$

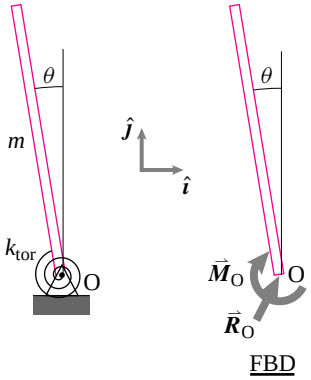


Figure 7.88: (Filename:sfig10.1.2b)

Comparing with the standard harmonic equation, we find the angular frequency

$$\lambda = \sqrt{\frac{k_{\text{tor}}}{I_{zz}^{O}}} = \sqrt{\frac{k_{\text{tor}}}{\frac{1}{3}ml^2}}$$

If this system has to have the same period of oscillation as the first system, the two angular frequencies must be equal, *i.e.*,

$$\begin{aligned} \sqrt{\frac{k_{\text{tor}}}{\frac{1}{3}ml^2}} &= \sqrt{\frac{6k}{m}} \\ \Rightarrow \qquad k_{\text{tor}} &= 6k\cdot\frac{1}{3}l^2 = 2kl^2 \\ &= 2\cdot(33.7\,\text{N/m})\cdot(1\,\text{m})^2 \\ &= 67.4\,\text{N·m} \end{aligned}$$

$$\boxed{k_{\text{tor}} = 67.4\,\text{N·m}}$$

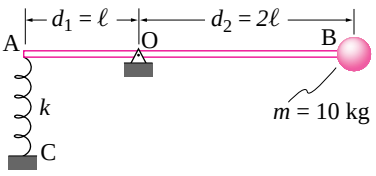


Figure 7.89: (Filename:sfig3.4.3)

SAMPLE 7.33 *Hey Mom, look, I can seesaw by myself.* A kid, modelled as a point mass with $m = 10\,\text{kg}$, is sitting at end B of a rigid rod AB of negligible mass. The rod is supported by a spring at end A and a pin at point O. The system is in static equilibrium when the rod is horizontal. Someone pushes the kid vertically downwards by a small distance y and lets go. Given that $AB = 3\,\text{m}$, $AC = 0.5\,\text{m}$, $k = 1\,\text{kN/m}$; find

(a) the unstretched (relaxed) length of the spring,
(b) the equation of motion (a differential equation relating the position of the mass to its acceleration) of the system, and
(c) the natural frequency of the system.

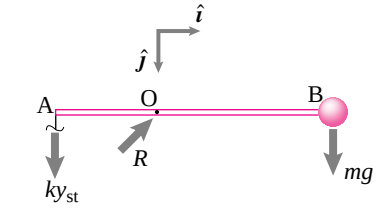
If the rod is pinned at the midpoint instead of at O, what is the natural frequency of the system? How does the new natural frequency compare with that of a mass m simply suspended by a spring with the same spring constant?

Solution

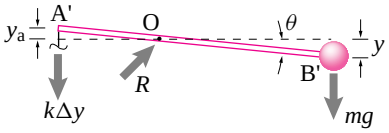
(a) **Static Equilibrium:** The FBD of the (rod + mass) system is shown in Fig. 7.90. Let the stretch in the spring in this position be y_{st} and the relaxed length of the spring be ℓ_0. The balance of angular momentum about point O gives:

$$\begin{aligned} \sum \vec{M}_{/O} = \dot{\vec{H}}_{/O} &= \vec{0} \quad \text{(no motion)} \\ \Rightarrow \quad (ky_{st})d_1 - (mg)d_2 &= 0 \\ \Rightarrow \quad y_{st} &= \frac{mg}{k} \cdot \frac{d_2}{d_1} \\ &= \frac{10\,\text{kg} \cdot 9.8\,\text{m/s}^2 \cdot 2\ell}{1000\,\text{N/m} \cdot \ell} = 0.196\,\text{m} \\ \text{Therefore,} \quad \ell_0 &= AC - y_{st} \\ &= 0.5\,\text{m} - 0.196\,\text{m} = 0.304\,\text{m}. \end{aligned}$$

$$\boxed{\ell_0 = 30.4\,\text{cm}}$$



(a) Static equilibrium



(b) Mass m is displaced downwards

Figure 7.90: Free body diagrams
(Filename:sfig3.4.3a)

(b) **Equation of motion:** As point B gets displaced downwards by a distance y, point A moves up by a proportionate distance y_a. From geometry, ①

$$\begin{aligned} y &\approx d_2\theta \quad \Rightarrow \quad \theta = \frac{y}{d_2} \\ y_a &\approx d_1\theta = \frac{d_1}{d_2}y \end{aligned}$$

Therefore, the total stretch in the spring, in this position,

$$\Delta y = y_a + y_{st} = \frac{d_1}{d_2}y + \frac{d_2}{d_1}\frac{mg}{k}$$

Now, Angular Momentum Balance about point O gives:

$$\begin{aligned} \sum \vec{M}_{/O} &= \dot{\vec{H}}_{/O} \\ \sum \vec{M}_{/O} &= \vec{r}_B \times mg\hat{j} + \vec{r}_A \times k\Delta y\hat{j} \\ &= (d_2 mg - d_1 k\Delta y)\hat{k} \qquad (7.78) \\ \dot{\vec{H}}_{/O} &= \vec{r}_B \times m\vec{a} = \vec{r}_B \times m\ddot{y}\hat{j} \qquad (7.79) \\ &= d_2 m\ddot{y}\hat{k} \qquad (7.80) \end{aligned}$$

① Here, we are considering a very small y so that we can ignore the arc the point mass B moves on and take its motion to be just vertical (i.e., $\sin\theta \approx \theta$ for small θ).

Equating (7.78) and (7.80) we get

$$\begin{aligned} & & d_2mg - d_1k\Delta y &= d_2m\ddot{y} \\ \text{or} & & d_2mg - d_1k\left(\frac{d_1}{d_2}y + \frac{d_2mg}{d_1k}\right) &= d_2m\ddot{y} \\ \text{or} & & d_2\cancel{mg} - k\frac{d_1^2}{d_2}y - d_2\cancel{mg} &= d_2m\ddot{y} \\ \text{or} & & \ddot{y} + \frac{k}{m}\frac{d_1^2}{d_2^2}y &= 0 \end{aligned}$$

$$\boxed{\ddot{y} + \tfrac{k}{m}\tfrac{d_1^2}{d_2^2}y = 0}$$

(c) **The natural frequency of the system:** We may also write the previous equation as

$$\ddot{y} + \lambda y = 0 \qquad \text{where} \qquad \lambda = \frac{k}{m}\frac{d_1^2}{d_2^2}. \tag{7.81}$$

Substituting $d_1 = \ell$ and $d_2 = 2\ell$ in the expression for λ we get the natural frequency of the system

$$\sqrt{\lambda} = \frac{1}{2}\sqrt{\frac{k}{m}} = \frac{1}{2}\sqrt{\frac{1000\,\text{N/m}}{10\,\text{kg}}} = 5\,\text{s}^{-1}$$

$$\boxed{\sqrt{\lambda} = 5\,\text{s}^{-1}}$$

(d) **Comparision with a simple spring mass system:** When $d_1 = d_2$, the equation

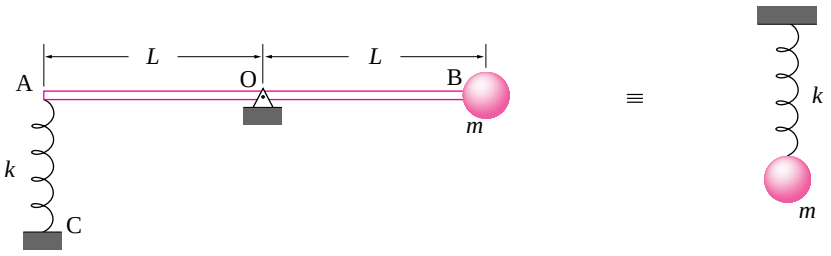


Figure 7.91: (Filename:sfig3.4.3b)

of motion (7.81) becomes

$$\ddot{y} + \frac{k}{m}y = 0$$

and the natural frequency of the system is simply

$$\sqrt{\lambda} = \sqrt{\frac{k}{m}}$$

which corresponds to the natural frequency of a simple spring mass system shown in Fig. 7.91.

In our system (with $d_1 = d_2$) any vertical displacement of the mass at B induces an equal amount of stretch or compression in the spring which is exactly the case in the simple spring-mass system. Therefore, the two systems are mechanically equivalent. Such equivalences are widely used in modeling complex physical systems with simpler mechanical models.

SAMPLE 7.34 *Energy method:* Consider the pulley problem of Sample 7.30 again. Use energy method to

(a) find the angular acceleration of the pulley, and
(b) the acceleration of the mass.

Solution In energy method we use speeds, not velocities. Therefore, we have to be careful in our thinking about the direction of motion. In the present problem, let us assume that the pulley rotates and accelerates clockwise. Consequently, the mass moves up against gravity.

(a) The energy equation we want to use is

$$P = \dot{E}_K.$$

The power P is given by $P = \sum \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{v}}_i$ where the sum is carried out over all external forces. For the mass and pulley system the external forces that do work are ① F and mg. Therefore,

① There are other external forces on the system: the reaction force of the support point O and the weight of the pulley—both forces acting at point O. But, since point O is stationary, these forces do no work.

$$\begin{aligned} P &= \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}_A + m\vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{v}}_m \\ &= F\hat{\boldsymbol{\imath}} \cdot v_A\hat{\boldsymbol{\imath}} + (-mg\hat{\boldsymbol{\jmath}}) \cdot \underbrace{v_D\hat{\boldsymbol{\jmath}}}_{\vec{\boldsymbol{v}}_m} \\ &= F v_A - mg v_D. \end{aligned}$$

The rate of change of kinetic energy is

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt}\Big(\underbrace{\frac{1}{2} m\, v_D^2}_{\text{K.E. of the mass}} + \underbrace{\frac{1}{2} I_{zz/O}\, \omega^2}_{\text{K.E. of the pulley}} \Big) \\ &= m\, v_D\, \dot{v}_D + I_{zz/O}\, \omega\, \dot{\omega}. \end{aligned}$$

Now equating the power and the rate of change of kinetic energy, we get

$$F\, v_A - mg\, v_D = m\, v_D\, \dot{v}_D + I_{zz/O}\, \omega\, \dot{\omega}$$

From kinematics, $v_A = \omega R_o$, $v_D = \omega R_i$ and $\dot{v}_D \equiv (a_D)_\theta = \dot{\omega} R_i$. Substituting these values in the above equation, we get

$$\begin{aligned} \omega(F R_o - mg R_i) &= \omega\, \dot{\omega}(m R_i^2 + I_{zz/O}) \\ \Rightarrow \quad \dot{\omega} &= \frac{F R_o - mg R_i}{(I_{zz/O} + m R_i^2)} \\ &= \frac{20\,\text{N} \cdot 0.2\,\text{m} - 2\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot 0.1m}{1.6\,\text{kg m}^2 + 2\,\text{kg} \cdot (0.1\,\text{m})^2} \\ &= 1.258 \frac{1}{\text{s}^2} \qquad \text{(same as the answer before.)} \end{aligned}$$

Since the sign of $\dot{\omega}$ is positive, our initial assumption of clockwise acceleration of the pulley is correct.

$$\boxed{\dot{\omega} = 1.26\,\text{rad/s}^2}$$

(b) From kinematics,

$$a_m = (a_D)_\theta = \dot{\omega} R_i = 0.126\,\text{m/s}^2.$$

$$\boxed{a_m = 0.13\,\text{m/s}^2}$$

SAMPLE 7.35 A flywheel of diameter 2 ft spins about the axis passing through its center and perpendicular to the plane of the wheel at 1000 rpm. The wheel weighs 20 lbf. Assuming the wheel to be a thin, uniform disk, find its kinetic energy.

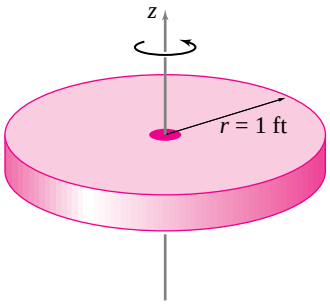


Figure 7.92: (Filename:sfig7.4.1)

Solution The kinetic energy of a 2-D rigid body spinning at speed ω about the z-axis passing through its mass center is

$$E_K = \frac{1}{2} I_{zz}^{cm} \omega^2$$

where I_{zz}^{cm} is the mass moment of inertia about the z-axis. For the flywheel,

$$\begin{aligned} I_{zz}^{cm} &= \frac{1}{2} m R^2 \quad \text{(from table IV at the back of the book)} \\ &= \frac{1}{2} \frac{W}{g} R^2 \quad \text{(where } W \text{ is the weight of the wheel)} \\ &= \frac{1}{2} \cdot \underbrace{\left(\frac{20\,\text{lbf}}{g}\right)}_{20\,\text{lbm}} \cdot (1\,\text{ft})^2 = 10\,\text{lbm}\cdot\text{ft}^2 \end{aligned}$$

The angular speed of the wheel is

$$\begin{aligned} \omega &= 1000\,\text{rpm} \\ &= 1000 \cdot \frac{2\pi}{60}\,\text{rad/s} \\ &= 104.72\,\text{rad/s}. \end{aligned}$$

Therefore the kinetic energy of the wheel is

$$\begin{aligned} E_K &= \frac{1}{2} \cdot (10\,\text{lbm}\cdot\text{ft}^2) \cdot (104.72\,\text{rad/s})^2 \\ &= 5.483 \times 10^4\,\text{lbm}\cdot\text{ft}^2/\text{s}^2 \\ &= \frac{5.483 \times 10^4}{32.2}\,\text{lbf}\cdot\text{ft} \\ &= 1.702 \times 10^3\,\text{ft}\cdot\text{lbf}. \end{aligned}$$

$$\boxed{1.702 \times 10^3\,\text{ft}\cdot\text{lbf}.}$$

8 Advanced topics in circular motion

In the last chapter we discussed the motions of particles and rigid bodies that rotate about a fixed axis from a planar point of view. In this chapter we again are going to think about fixed axis rotation, but now in three dimensions. The axis of rotation might be in any skew direction and the rotating bodies might be arbitrarily complicated three-dimensional shapes. As a cartoon, imagine a rigid body skewered with a rigid rod and then turned by a motor that speeds up and slows down. More practically think of the crankshaft in a car engine (Fig. 8.1). Other applications include accelerating or decelerating shafts of all kinds, gears, turbines, flywheels, pendula, and swinging doors.



Figure 8.1: A car crankshaft is a complex three-dimensional object which is well approximated for many purposes as rotating about a fixed axis. The relative timing of the reciprocating pistons is controlled by this complex shape. "Connecting rods" are pinned to the cylinders at one end and to the short offset cylinders on the crankshaft at the other.

(Filename:tfigure.crankshaft)

To understand this motion we need to take a little more care with the kinematics because it now involves three dimensions, although in some sense the basic ideas are unchanged from the previous two-dimensional chapter. The three dimensional mechanics naturally gets more involved.

As for all motions of all systems, the momentum balance equations apply to any system or any part of a system that has fixed axis rotation. So our mechanics results will be based on these familiar equations:

$$\text{Linear momentum balance:} \qquad \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}},$$

$$\text{Angular momentum balance:} \qquad \sum \vec{\boldsymbol{M}}_{i/O} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}.$$

and

$$\text{Power balance:} \qquad P = \dot{E}_{\mathrm{K}}.$$

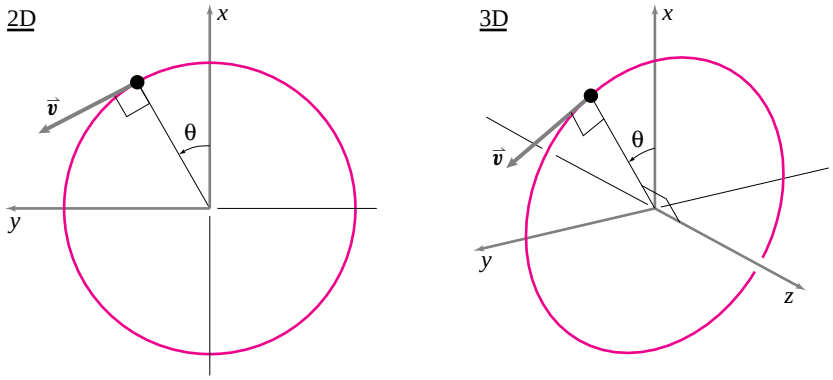
As always, we will evaluate the left hand sides of the momentum equations using the forces and moments in the free body diagram. We evaluate the right hand sides of

these equations using our knowledge of the velocities and accelerations of the various mass points.

The chapter starts with a discussion of kinematics. Then we consider the mechanics of systems with fixed axis rotation. The moment of inertia matrix is then introduced followed by a section where the moment of inertia is used as a shortcut in the evaluation of $\vec{\vec{H}}_O$. Finally we discuss dynamic balance, an important topic in machine design that is genuinely three-dimensional.

8.1 3-D description of circular motion

Let's first assume each particle is going in circles around the z axis, as in the previous chapter. The figure below shows this two dimensional situation first two dimensionally (left) and then as a two-dimensional motion in a three dimensional world (right).



The velocity and acceleration are described in the same way for three dimensional motion about a fixed axis as for two dimensional motion.

- the velocity is tangent to the circle it is going around and is proportional in magnitude to the radius of the circle and also to its angular speed. That is, the direction of the velocity is in the direction $\hat{\boldsymbol{e}}_\theta$ and has magnitude ωR, where ω is the angular rate of rotation and R is the radius of the circle that the particle is going around.
- the acceleration can be constructed as the sum of two vectors. One is pointed to the center of the circle and proportional in magnitude to both the square of the angular speed and to the radius. The other vector is tangent to the circle and equal in magnitude to the rate of increase of speed.

These two ideas are summarized by the following formulas:

$$\vec{\boldsymbol{v}} = \omega R \hat{\boldsymbol{e}}_\theta \qquad \text{and} \qquad \vec{\boldsymbol{a}} = -\underbrace{\omega^2 R}_{v^2/R}\,\hat{\boldsymbol{e}}_R + \underbrace{R\ddot{\theta}}_{\dot{v}}\,\hat{\boldsymbol{e}}_\theta \tag{8.1}$$

with

$$v = \omega R. \tag{8.2}$$

The axis of rotation might not be the z-axis of a convenient xyz coordinate system. So the xy plane of circles might not be the xy plane of the coordinate system you might want to use for some other reasons. Fortunately, we can write the formulas 8.1 in a way that rids us of these problems.

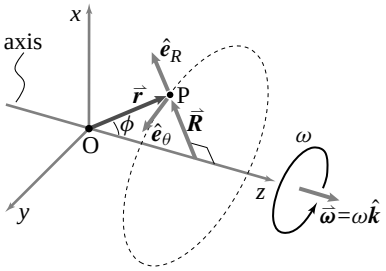


Figure 8.2: When a point P is going in circles about the z axis, we define the unit vector $\hat{\boldsymbol{e}}_R$ to be pointed from the axis to the point P. We define the unit vector $\hat{\boldsymbol{e}}_\theta$ to be tangent to the circle at P. Both of these vectors change in time as the point moves along its circular path.

(Filename:tfigure4.2)

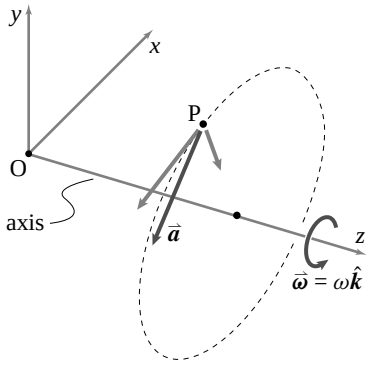


Figure 8.3: The acceleration is the sum of two components. One directed towards the center of the circle in the $-\hat{\boldsymbol{e}}_R$ direction, and one tangent to the circle in the $\hat{\boldsymbol{e}}_\theta$ direction.

(Filename:tfigure5.3)

Here are some formulas which are equivalent to the formulas 8.1 but which do not make use of the polar coordinate base vectors.

$$\vec{v} = \vec{\omega} \times \vec{r}, \tag{8.3}$$
$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}). \tag{8.4}$$

To check that equations 8.3 and 8.4 are really equivalent to 8.1? we need to verify that the vector $\vec{\omega} \times \vec{r}$ is equal to $\omega R \hat{e}_\theta$ and that the vector $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is equal to the vector $-\omega^2 R \hat{e}_R$.

First, define $\vec{r}$ as the position of the point of interest relative to any point on the axis of rotation. If this point happens to be the center of the circle for some particle then $\vec{r} = \vec{R}$. But, in general, $\vec{r} \neq \vec{R}$.

Now look at $\vec{v} = \vec{\omega} \times \vec{r}$. Looking at figure 8.2 and using the right hand rule, it is clear that the direction of the cross product $\vec{\omega} \times \vec{r}$ is in fact the $\hat{e}_\theta$ direction. What about the magnitude? The magnitude $|\vec{\omega} \times \vec{r}| = |\vec{\omega}||\vec{r}| \sin\phi$. But $|\vec{r}| \sin\phi = R$. So the magnitude of $\vec{\omega} \times \vec{r}$ is ωR. That is,

$$\vec{\omega} \times \vec{r} = (|\vec{\omega} \times \vec{r}|) \cdot (\text{unit vector in the direction of } \vec{\omega} \times \vec{r}) \tag{8.5}$$
$$= (\omega R) \cdot \hat{e}_\theta \tag{8.6}$$
$$= \vec{v} \tag{8.7}$$

Triple cross product

The formula for acceleration of a point on a rigid body includes the centripetal term $\vec{\omega} \times (\vec{\omega} \times \vec{r})$. This expression is a special case of the general vector expression

$$\vec{A} \times (\vec{B} \times \vec{C})$$

which is sometimes called the 'vector triple product' because its value is a vector (as opposed to the scalar value of the 'scalar triple product'). The primary useful identity with vector triple products is:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \tag{8.8}$$

This formula may be remembered by the semi-mnemonic device 'cab minus bac' since $\vec{A} \cdot \vec{C} = \vec{C} \cdot \vec{A}$ and $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. This formula is discussed in box 8.1 on page 437.

So now we can write

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} \times (\omega R) \cdot \hat{e}_\theta = -\omega^2 \underbrace{R\hat{e}_R}_{\vec{R}} = -\omega^2 \vec{R} = \vec{a}. \tag{8.9}$$

Note that equations 8.3 and 8.4 are *vector* equations. They do not make use of any coordinate system. So, for example, we can use them even if $\vec{\omega}$ is not in the z direction.

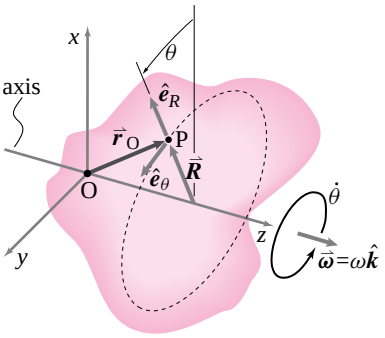


Figure 8.4: A rigid body spinning about the z axis. Every point on the body, like point P at $\vec{r}$, is going in circles. All of these circles have centers on the axis of rotation. All the points are going around at the same angular rate, $\dot{\theta} = \omega$.

(Filename:tfigure4.3D)

Angular velocity of a rigid body in 3D

If a rigid body is constrained to rotate about an axis then all points on the body have the same angular rate about that axis. Hence one says that the *body* has an angular velocity. So the measure of rotation rate of a three-dimensional rigid body is the body's angular velocity vector $\vec{\boldsymbol{\omega}}$. At any instant in time a given body has one and only one angular velocity $\vec{\boldsymbol{\omega}}$. Although we only discuss fixed axis rotation in this chapter, a given body has a unique angular velocity for general motion.

For rotations about the z-axis, $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. ω is the $\dot{\theta}$ shown in figure 8.4. Since all points of a rigid body have the same $\dot{\theta}$, even if they have different θ's, the definition is not ambiguous. We would like to make this idea precise enough to be useful for calculations. Why, one may ask, do we talk about rotation rate ω or $\vec{\boldsymbol{\omega}}$ instead of just using the derivative of an angle θ, $\dot{\theta}$? The answer is that for a rigid body one would have trouble deciding what angle θ to measure.

First recall the situation for a two dimensional rigid body. Consider all possible $\theta_1,\ \theta_2,\ \theta_3, \ldots$, the angles that all possible lines marked on a body could make with the positive x-axis, the positive y axis, or any other fixed line that does not rotate. As the body rotates all of these angles increment by the same amount. Therefore, each of these angles increases at the same rate. Because all these angular rates are the same, one need not define $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$, $\dot{\theta}_3 = \omega_3$, etc. for each of the lines. Every line attached to the body rotates at the same rate and we call this rate ω. So $\dot{\theta}_1 = \omega$, $\dot{\theta} = \omega$, $\dot{\theta}_3 = \omega$, etc. Rather than say the lengthy phrase 'the rate of rotation of every line attached to the rigid body is ω', we instead say 'the rigid body has angular velocity ω'. For use in vector equations, we define the angular velocity vector of a two-dimensional rigid body as $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and for a 3-D body rotating about an axis in the $\hat{\boldsymbol{\lambda}}$ direction as $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\lambda}}$.

What do we mean by these angles θ_i for crooked lines in a three-dimensional body? We simply look at shadows of lines drawn in or on the body of interest onto a plane perpendicular to the axis of rotation; *i.e.*, perpendicular to $\hat{\boldsymbol{\lambda}}$. See figure 8.5. The rate of change of their orientation ($\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3$) is ω, and $\vec{\boldsymbol{\omega}}$ is therefore $\omega\hat{\boldsymbol{\lambda}}$. This intuitive geometric definition of ω in terms of the rotation of shadows has run its course. It is not very convenient for developing formulas.

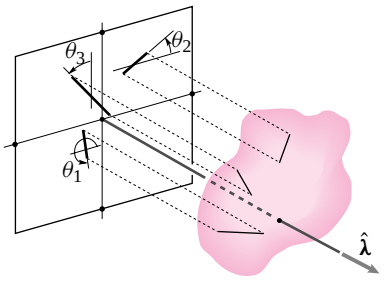


Figure 8.5: The shadows of lines marked in a 3-D rigid body are shown on a plane perpendicular to the axis of rotation. The shadows rotate on the plane at the rate $\dot{\theta}_1 = \dot{\theta}_2 = \omega$. The angular velocity vector is $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\lambda}}$.

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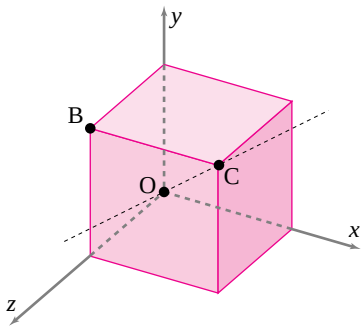


Figure 8.6: A spinning cube.

(Filename:tfigure4.cube)

Example: **What are the velocity and acceleration of one corner of a cube that is spinning about a diagonal?**

A one foot cube is spinning at 60 rpm about the diagonal OC. What are the velocity and acceleration of point B? First let's find the velocity using $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$:

$$\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \\
&= (60\,\text{rpm}\ \boldsymbol{\lambda}_{OC}) \times \vec{\boldsymbol{r}}_{OB} \\
&= \left(2\pi\ \text{s}^{-1}\frac{(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})}{\sqrt{3}}\right) \times (1\,\text{ft}(\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})) \\
&= (2\pi/\sqrt{3})(-\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})\,\text{ft/s}.
\end{aligned}$$

Now of course this equation could have been worked out with the first of equations **??** but it would have been quite tricky to find the vectors $\hat{\boldsymbol{e}}_\theta$, $\vec{\boldsymbol{R}}$, and $\hat{\boldsymbol{e}}_R$! To find the acceleration we just plug in the formula $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$ as follows:

$$\begin{aligned}
\vec{\boldsymbol{a}} &= \vec{\boldsymbol{\omega}} \times [\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}] \\
&= (60\,\text{rpm}\ \boldsymbol{\lambda}_{OC}) \times [(60\,\text{rpm}\ \boldsymbol{\lambda}_{OC}) \times \vec{\boldsymbol{r}}_{OB}]
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi\ \mathrm{s}^{-1}\frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}) \\
&\qquad \times \left[(\frac{2\pi\ \mathrm{s}^{-1}(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}) \times (1\,\mathrm{ft}(\hat{\jmath}+\hat{k}) \right] \\
&= (2\pi/\sqrt{3})^2(2\hat{\imath}-\hat{\jmath}-\hat{k})\,\mathrm{ft/s}^2.
\end{aligned}
$$

The last line of calculation is eased by the calculation of velocity above where the term in square brackets, the velocity, was already calculated. □

Relative motion of points on a rigid body

The relative velocity of two points A and B is defined to be

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A$$

So, the relative velocity of two points glued to one rigid body, as observed from a Newtonian frame, is given by

$$
\begin{aligned}
\vec{v}_{B/A} &\equiv \vec{v}_B - \vec{v}_A && (8.10)\\
&= \vec{\omega}\times\vec{r}_{B/O} - \vec{\omega}\times\vec{r}_{A/O} && (8.11)\\
&= \vec{\omega}\times(\vec{r}_{B/O} - \vec{r}_{A/O}) && (8.12)\\
&= \vec{\omega}\times\vec{r}_{B/A}, && (8.13)\\
& && (8.14)
\end{aligned}
$$

where point O is a point in the Newtonian frame on the fixed axis of rotation. Clearly, since points A and B are fixed in the body $\mathcal{B}$ their velocities and hence their relative velocity as observed in a reference frame fixed to $\mathcal{B}$ is $\vec{\mathbf{0}}$. But, point A has some absolute velocity that is different from the absolute velocity of point B, as viewed from point O in the fixed frame. The relative velocity of points A and B, the difference in absolute velocity of the two points, is due to the difference in their positions relative to point O. Similarly, the relative acceleration of two points glued to one rigid body spinning about a fixed axis is

$$\vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A = \vec{\omega}\times(\vec{\omega}\times\vec{r}_{\mathrm{B/A}}) + \dot{\vec{\omega}}\times\vec{r}_{\mathrm{B/A}}. \qquad (8.15)$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point O fixed on the axis. These kinematics results, 8.14 and 8.15, are useful for calculating angular momentum relative to the center of mass. They are also sometimes useful for the understanding of the motions of machines with moving connected parts.

To repeat, for two points on one rigid body we have that

$$\dot{\vec{r}}_{B/A} = \vec{\omega}\times\vec{r}_{B/A}. \qquad (8.16)$$

Equation (8.16) is the most fundamental equation for those desiring a deeper understanding of the three-dimensional rotation of rigid bodies. Unless one desires to pursue matrix representations of rotation, equation (8.16) is *the* defining equation for $\vec{\omega}$. There is always exactly one vector $\vec{\omega}$ so that equation (8.16) is true for any pair of points on a rigid body.

Equation (8.16) is not so simple a defining equation as one would hope for such an intuitive concept as spinning. But, besides the definition with shadows, its the simplest definition we have.

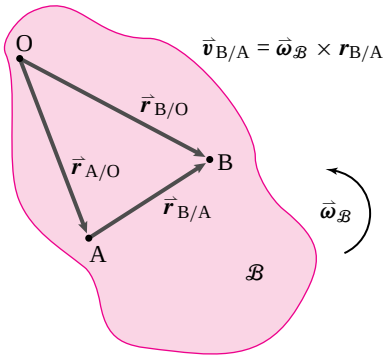


Figure 8.7: Two points on a rigid body

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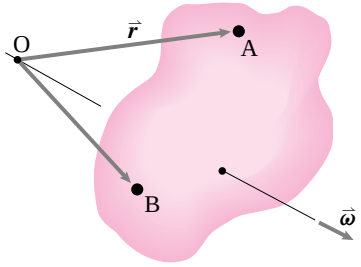


Figure 8.8: Two points, A and B on one body that has a fixed axis of rotation.

(Filename:tfigure.twoptsonabody.4.3)

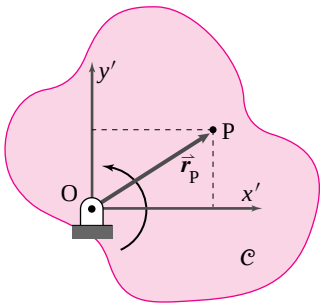


Figure 8.9: A rotating rigid body $\mathcal{C}$ with rotating frame $x'y'$ attached.

(Filename:tfigure4.intro.rot.frames)

Relative velocity and acceleration using rotating frames

If we glued a coordinate system $x'y'$ to a rotating rigid body $\mathcal{C}$, we would have what is called a rotating frame as shown in figure 8.9. The base vectors in this frame change in time the same way as did $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in section 7.1. That is

$$\frac{d}{dt}\hat{\boldsymbol{\imath}}' = \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}' \quad \text{and} \quad \frac{d}{dt}\hat{\boldsymbol{\jmath}}' = \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}'.$$

If we now write the relative position of B to A in terms of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, we have

$$\vec{\boldsymbol{r}}_{B/A} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'.$$

Since the coordinates x' and y' rotate with the body to which A and B are attached, they are constant with respect to that body,

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

So

$$\begin{aligned}
\frac{d}{dt}(\vec{\boldsymbol{r}}_{B/A}) &= \frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right) \\
&= \underbrace{\dot{x}'}_{0}\hat{\boldsymbol{\imath}}' + x'\frac{d}{dt}\hat{\boldsymbol{\imath}}' + \underbrace{\dot{y}'}_{0}\hat{\boldsymbol{\jmath}}' + y'\frac{d}{dt}\hat{\boldsymbol{\jmath}}' \\
&= x'(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}') \\
&= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \underbrace{(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')}_{\vec{\boldsymbol{r}}_{B/A}} \\
&= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}.
\end{aligned}$$

If we now try to calculate the rate of change of $\vec{\boldsymbol{v}}_{B/A}$, keeping $\vec{\boldsymbol{\omega}}$ constant for simplicity, we get

$$\begin{aligned}
\frac{d}{dt}(\vec{\boldsymbol{v}}_{B/A}) &= \frac{d}{dt}\left(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}\right) \\
&= \underbrace{\frac{d\vec{\boldsymbol{\omega}}_{\mathcal{C}}}{dt}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \frac{d\vec{\boldsymbol{r}}_{B/A}}{dt} \\
\vec{\boldsymbol{a}}_{B/A} &= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}).
\end{aligned}$$

$\vec{\boldsymbol{0}}$ for constant rate circular motion

We could have taken a short-cut in the calculation of acceleration $\vec{\boldsymbol{a}}$. Instead of using $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$, we could have used $\vec{\boldsymbol{a}} = -\omega^2\vec{\boldsymbol{R}}$ where $\vec{\boldsymbol{R}}$ is the radius of the circle each particle is traveling on. It is evident from the picture that the appropriate radius is $\vec{\boldsymbol{R}} = s\sin\phi\,\hat{\boldsymbol{\jmath}}$, so $\vec{\boldsymbol{a}} = -\omega^2 s\sin\phi\,\hat{\boldsymbol{\jmath}}$.

Mechanics

Now that we know the velocity and acceleration of every point in the system we are ready, in principle, to find $\vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{H}}_O$ in terms of the angular velocity vector $\vec{\boldsymbol{\omega}}$, its rate of change $\dot{\vec{\boldsymbol{\omega}}}$, and the position of all the mass in the system. This we do in the next section.

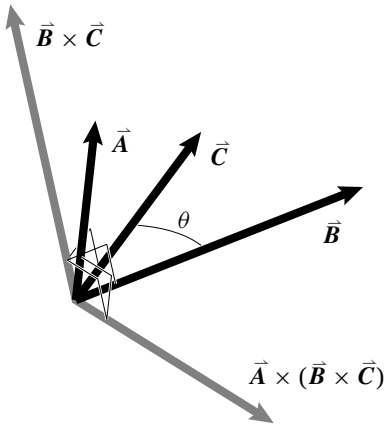
8.1 *THEORY*
The triple vector product $\vec{A} \times (\vec{B} \times \vec{C})$

The formula

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \qquad (8.17)$$

can be verified by writing each of the vectors in terms of its orthogonal components (*e.g.*, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$) and checking equality of the 27 terms on the two sides of the equations (only 12 are non-zero). If this 20 minute proof seems tedious it can be replaced by a more abstract geometric argument partly presented below that surely takes more than 20 minutes to grasp.

Geometry of the vector triple product



Because $\vec{B} \times \vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ it is perpendicular to the plane of $\vec{B}$ and $\vec{C}$, that is, it is 'normal' to the plane BC. $\vec{A} \times (\vec{B} \times \vec{C})$ is perpendicular to both $\vec{A}$ and $\vec{B} \times \vec{C}$, so it is perpendicular to the normal to the plane of BC. That is, it must be in the plane of $\vec{B}$ and $\vec{C}$. But any vector in the plane of $\vec{B}$ and $\vec{C}$ must be a combination of $\vec{B}$ and $\vec{C}$. Also, the vector triple product must be proportional in magnitude to each of $\vec{A}$, $\vec{B}$ and $\vec{C}$. Finally, the triple cross product of $\vec{A} \times (\vec{B} \times \vec{C})$ must be the negative of $\vec{A} \times (\vec{C} \times \vec{B})$ because $\vec{B} \times \vec{C} = -\vec{C} \times \vec{B}$. So the identity

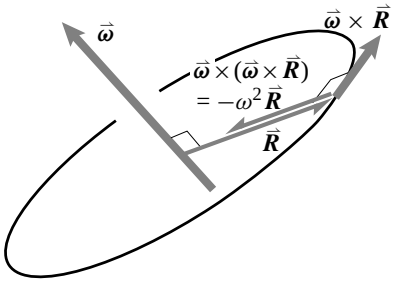
$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

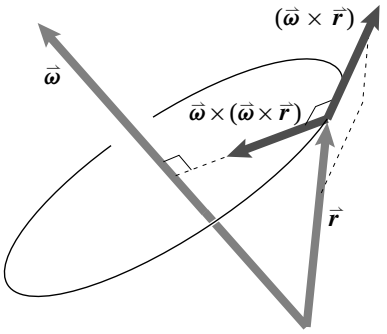
is almost natural: The expression above is almost the only expression that is a linear combination of $\vec{A}$ and $\vec{B}$ that is linear in both, also linear in $\vec{C}$ and switches sign if $\vec{B}$ and $\vec{C}$ are interchanged. These properties would be true if the whole expression were multiplied by any constant scalar. But a test of the equation with three unit vectors shows that such a multiplicative constant must be one. This reasoning constitutes an informal derivation of the identity 8.8.

Using the triple cross product in dynamics equations

We will use identity 8.8 for two purposes in the development of dynamics equations:

(a) In the 2D expression for acceleration, the centripetal acceleration is given by $\vec{\omega} \times (\vec{\omega} \times \vec{R})$ simplifies to $-\omega^2\vec{R}$ if $\vec{\omega} \perp \vec{R}$. This equation follows by setting $\vec{A} = \vec{\omega}$, $\vec{B} = \vec{\omega}$ and $\vec{C} = \vec{R}$ in equation 8.8 and using $\vec{R} \cdot \vec{\omega} = 0$ if $\vec{\omega} \perp \vec{R}$. In 3D $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ gives the vector shown in the lower figure below.





(b) The term $\vec{r} \times (\vec{\omega} \times \vec{r})$ will appear in the calculation of the angular momentum of a rigid body. By setting $\vec{A} = \vec{r}$, $\vec{B} = \vec{\omega}$ and $\vec{C} = \vec{r}$, in equation 8.8 and use $\vec{r} \cdot \vec{r} = r^2$ because $\vec{r} \parallel \vec{r}$ we get the useful result that $\vec{r} \times (\vec{\omega} \times \vec{r}) = r^2\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}$.

SAMPLE 8.1 For a particle in circular motion, we frequently use angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$ to describe its motion. You have probably learned in physics that the linear speed of the particle is $v = \omega r$, the tangential acceleration is $a_t = \alpha r$, and the centripetal or radial acceleration is $a_r = \omega^2 r$, where r is the radius of the circle. These formulae have scalar expressions. Their vector forms, as you will learn in Chapters 5 and 6, are $\vec{v} = \vec{\omega} \times \vec{r}$, $\vec{a}_t = \vec{\alpha} \times \vec{r}$, and $\vec{a}_r = \vec{\omega} \times (\vec{\omega} \times \vec{r})$. Using these definitions, find (i) $\vec{v}$, (ii) $\vec{a}_t$, and (iii) $\vec{a}_r$ and show the resulting vectors for $\vec{\omega} = 2\,\text{rad/s}\hat{k}$, $\vec{\alpha} = 4\,\text{rad/s}^2\hat{k}$ and $\vec{r}_G = 3\,\text{m}(\cos 30^\text{o}\hat{\imath} + \sin 30^\text{o}\hat{\jmath})$, where $\vec{r}_G$ is the position vector of the particle.

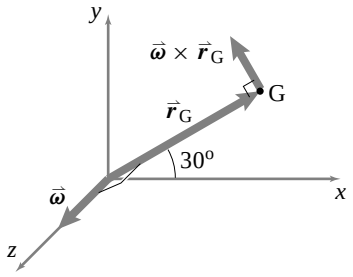


Figure 8.10: $\vec{v} = \vec{\omega} \times \vec{r}_G$.

(Filename:sfig1.2.10a)

Solution

$$\begin{aligned} \vec{\omega} &= \omega\hat{k} = 2\,\text{rad/s}\hat{k} \\ \vec{\alpha} &= \alpha\hat{k} = 4\,\text{rad/s}^2\hat{k} \\ \vec{r}_G &= r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath} = 3\,\text{m}(\cos 30^\text{o}\hat{\imath} + \sin 30^\text{o}\hat{\jmath}). \end{aligned}$$

(i) From the given formulae, the linear velocity

$$\begin{aligned} \vec{v} &= \vec{\omega} \times \vec{r}_G \\ &= \omega\hat{k} \times (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) = \omega r_{G_x}\hat{\jmath} + \omega r_{G_y}(-\hat{\imath}) \\ &= 6\,\text{m/s}(\cos 30^\text{o}\hat{\jmath} - \sin 30^\text{o}\hat{\imath}) \\ &= 3\,\text{m/s}(-\hat{\imath} + \sqrt{3}\hat{\jmath}). \end{aligned}$$

The velocity vector $\vec{v}$ is perpendicular to both $\vec{\omega}$ and $\vec{r}_G$. These vectors are shown in Fig. 8.10. You should use the right hand rule to confirm the direction of $\vec{v}$.

$$\boxed{\vec{v} = 3\,\text{m/s}(-\hat{\imath} + \sqrt{3}\hat{\jmath})}$$

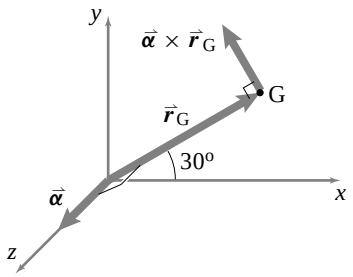


Figure 8.11: $\vec{a}_t = \vec{\alpha} \times \vec{r}_G$.

(Filename:sfig1.2.10b)

(ii) The tangential acceleration

$$\begin{aligned} \vec{a}_t &= \vec{\alpha} \times \vec{r}_G \\ &= \alpha\hat{k} \times (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) = \alpha r_{G_x}\hat{\jmath} - \alpha r_{G_y}\hat{\imath} \\ &= 8\,\text{m/s}^2(\cos 30^\text{o}\hat{\jmath} - \sin 30^\text{o}\hat{\imath}). \end{aligned}$$

Since $\vec{\omega}$ and $\vec{\alpha}$ are in the same direction, calculation of $\vec{a}_t$ is similar to that of $\vec{v}$ and $\vec{a}_t$ has to be in the same direction as $\vec{v}$. This vector is shown in Fig. 8.11. Once again, just as in the case of $\vec{v}$ we could easily check that $\vec{a}_t$ is perpendicular to both $\vec{\alpha}$ and $\vec{r}_G$.

$$\boxed{\vec{a}_t = 4\,\text{m/s}^2(3\,\text{m/s}(-\hat{\imath} + \sqrt{3}\hat{\jmath})}$$

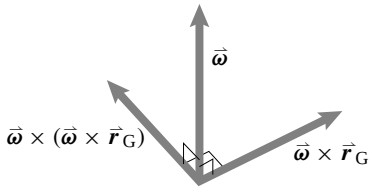


Figure 8.12: $\vec{a}_r = \vec{\omega} \times (\vec{\omega} \times \vec{r}_G)$

(Filename:sfig1.2.10c)

(iii) Finally, the radial acceleration

$$\begin{aligned} \vec{a}_r &= \vec{\omega} \times (\vec{\omega} \times \vec{r}_G) \\ &= \omega\hat{k} \times (\omega r_{G_x}\hat{\jmath} - \omega r_{G_y}\hat{\imath}) \\ &= \omega^2 r_{G_x}(-\hat{\imath}) - \omega^2 r_{G_y}\hat{\jmath} = -\omega^2\vec{r}_G \\ &= 12\,\text{m/s}^2(-\cos 30^\text{o}\hat{\imath} - \sin 30^\text{o}\hat{\jmath}). \end{aligned}$$

This cross product is illustrated in Fig. 8.12. Both from the illustration as well as the calculation you should be able to see that $\vec{a}_r$ is in the direction of $-\vec{r}_G$. In fact, you could show that $\vec{a}_r = -\omega^2\vec{r}_G$.

$$\boxed{\vec{a}_r = 6\,\text{m/s}^2(-\sqrt{3}\hat{\imath} - \hat{\jmath})}$$

SAMPLE 8.2 *Simple 3-D circular motion.* A system with two point masses A and B is mounted on a rod OC which makes an angle $\theta = 45^o$ with the horizontal. The entire assembly rotates about the y-axis with constant angular speed $\omega = 3\,\text{rad/s}$, maintaining the angle θ. Find the velocity of point A. What is the radius of the circular path that A describes? Assume that at the instant shown, AB is in the xy plane.

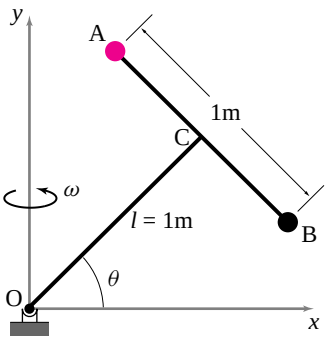


Figure 8.13: (Filename:sfig4.2.DH1)

Solution The angular velocity of the system is

$$\vec{\omega} = \omega\hat{j} = 3\,\text{rad/s}\hat{j}.$$

Let $\vec{r}$ be the position vector of point A. Then the velocity of point A is

$$\begin{aligned}
\vec{v} &= \vec{\omega} \times \vec{r} \\
&= \omega\hat{j} \times (\underbrace{l\cos\theta\hat{i} + l\sin\theta\hat{j}}_{\text{OC}} + \underbrace{d\cos\theta\hat{j} - d\sin\theta\hat{i}}_{\text{CA}}) \\
&= \omega\hat{j} \times [(l\cos\theta - d\sin\theta)\hat{i} + (l\sin\theta + d\cos\theta)\hat{j}] \\
&= -(\omega l\cos\theta - \omega d\sin\theta)\hat{k} \\
&= -[3\,\text{rad/s}(1\,\text{m}\cdot\cos 45^o - 0.5\,\text{m}\cdot\sin 45^o)] \\
&= -1.06\,\text{m/s}\hat{k}
\end{aligned}$$

$$\boxed{\vec{v} = -1.06\,\text{m/s}\hat{k}.}$$

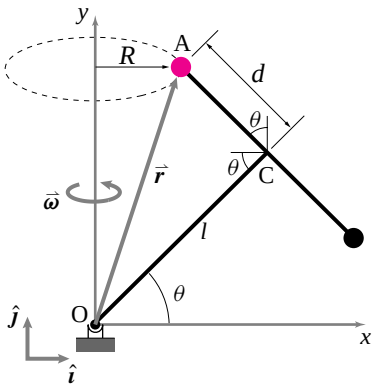


Figure 8.14: (Filename:sfig4.2.DH2)

We can find the radius of the circular path of A by geometry. However, we know that the velocity of A is also given by

$$\vec{v} = \omega R\hat{e}_\theta$$

where R is the radius of the circular path. At the instant of interest, $\hat{e}_\theta = -\hat{k}$ (see figure 8.15).

$$\text{Thus} \qquad \vec{v} = -\omega R\hat{k}.$$

Comparing with the answer obtained above, we get

$$\begin{aligned}
-1.06\,\text{m/s}\hat{k} &= -\omega R\hat{k} \\
\Rightarrow \qquad R &= \frac{1.06\,\text{m/s}}{3\,\text{rad/s}} \\
&= 0.35\,\text{m}.
\end{aligned}$$

$$\boxed{R = 0.35\,\text{m}}$$

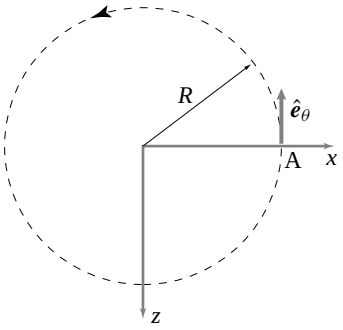


Figure 8.15: Circular trajectory of point A as seen by looking down along the y-axis. At the instant shown, $\hat{e}_\theta = -\hat{k}$.
(Filename:sfig4.2.DH3)

SAMPLE 8.3 *Kinematics in 3-D—some basic questions:* The following questions are about the velocity and acceleration formulae for the non-constant rate circular motion about a fixed axis:

$$\begin{aligned}\vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \\ \vec{\boldsymbol{a}} &= \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\end{aligned}$$

(a) In the formulae above, what is $\vec{\boldsymbol{r}}$? How is it different from $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R$ used in the formulae $\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta$?

(b) What is the difference between $\dot{\theta}$ and $\vec{\boldsymbol{\omega}}$, and $\ddot{\theta}$ and $\dot{\vec{\boldsymbol{\omega}}}$?

(c) Are the parentheses around the $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ term necessary in the acceleration formula?

(d) Under what condition(s) can a particle have only tangential acceleration?

Solution

(a) In the formulae for velocity and acceleration, $\vec{\boldsymbol{r}}$ refers to a vector from any point on the axis of rotation to the point of interest. Usually the origin of a coordinate system located on the axis of rotation is a convenient point to take as the base point for $\vec{\boldsymbol{r}}$. You can, however, choose any other point on the axis of rotation as the base point.
The vector $\vec{\boldsymbol{r}}$ is different from $\vec{\boldsymbol{R}}$ in that $\vec{\boldsymbol{R}}$ is the position vector of the point of interest with respect to the center of the circular path that the point traces during its motion. See Fig. 5.2 of the text.

(b) $\dot{\theta}$ and $\ddot{\theta}$ are the magnitudes of angular velocity and angular acceleration, respectively, in planar motion, *i.e.*, $\vec{\boldsymbol{\omega}} = \dot{\theta}\hat{\boldsymbol{k}}$ and $\dot{\vec{\boldsymbol{\omega}}} = \ddot{\theta}\hat{\boldsymbol{k}}$. We have introduced these notations to highlight the simple nature of planar circular motion. Of course, you are free to use $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\dot{\vec{\boldsymbol{\omega}}} = \alpha\hat{\boldsymbol{k}}$ if you wish.

(c) Yes, the parentheses around $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ in the acceleration formula are mandatory. The parentheses imply that this term has to be calculated before carrying out the cross product with $\vec{\boldsymbol{\omega}}$ in the formula. Since the term in the parentheses is the velocity, you may also write the acceleration formula as

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}.$$

Even if the formula is clear in your mind and you know which cross product to carry out first, it is a good idea to put the parentheses.

(d) First of all let us recognize the tangential and the normal (or radial) components of the acceleration:

$$\vec{\boldsymbol{a}} = \overbrace{\dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}}^{\text{tangential}} + \overbrace{\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})}^{\text{radial}}.$$

Clearly, for a particle to have only tangential acceleration, the second term must be zero. For the second term to be zero we must have either $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$ or $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}}$. But if $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$, then the tangential acceleration also becomes zero; the particle is on the axis of rotation and hence has no acceleration. Thus the condition that allows only tangential acceleration to survive is $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}}$. Now remember that $\dot{\vec{\boldsymbol{\omega}}}$ is *not* zero. Therefore, the condition we have found can be true only momentarily. This dissappearance of the radial acceleration happens at start-up motions and in direction-reversing motions. ①

① In all start-up motions, the velocity is zero but the acceleration is not zero at the start up ($t = 0$). In direction-reversing motions, such as that of the washing machine drum during the wash-cycle, just at the moment when the direction of motion reverses, velocity becomes zero but the acceleration is non-zero.

SAMPLE 8.4 *Velocity and acceleration in 3-D:* The rod shown in the figure rotates about the y-axis at angular speed 10 rad/s and accelerates at the rate of 2 rad/s^2. The dimensions of the rod are $L = h = 2$ m and $r = 1$ m. There is a small mass P glued to the rod at its free end. At the instant shown, the three segments of the rod are parallel to the three axes.

(a) Find the velocity of point P at the instant shown.
(b) Find the acceleration of point P at the instant shown.

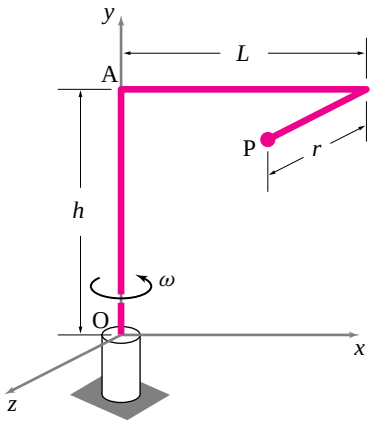


Figure 8.16: (Filename:sfig5.2.2)

Solution We are given:

$$\vec{\omega} = \omega\hat{\boldsymbol{\jmath}} = 10\,\text{rad/s}\,\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega}\hat{\boldsymbol{\jmath}} = 2\,\text{rad/s}^2\,\hat{\boldsymbol{\jmath}}.$$

(a) The velocity of point P is

$$\vec{\boldsymbol{v}} = \vec{\omega} \times \vec{\boldsymbol{r}}.$$

At the instant shown, the position vector of point P (the vector $\vec{\boldsymbol{r}}_{\text{P/O}}$) seems to be a good choice for $\vec{\boldsymbol{r}}$. ① Thus,

$$\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{r}}_{\text{P/O}} = L\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{k}}.$$

Therefore,

$$\begin{aligned}
\vec{\boldsymbol{v}} &= \omega\hat{\boldsymbol{\jmath}} \times (L\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{k}}) \\
&= \omega(-L\hat{\boldsymbol{k}} + r\hat{\boldsymbol{\imath}}) \\
&= 10\,\text{rad/s} \cdot (-2\,\text{m}\hat{\boldsymbol{k}} + 1\,\text{m}\hat{\boldsymbol{\imath}}) \\
&= (20\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{k}})\,\text{m/s}.
\end{aligned}$$

As a check, we look down the y-axis and draw a velocity vector at point P (tangent to the circular path at point P) without paying attention to the answer we got. From the top view in Fig. 8.17 we see that at least the signs of the components of $\vec{\boldsymbol{v}}$ seem to be correct.

$$\boxed{\vec{\boldsymbol{v}} = (20\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{k}})\,\text{m/s}}$$

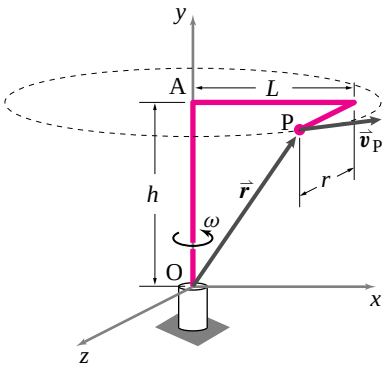
(b) The acceleration of point P is

$$\begin{aligned}
\vec{\boldsymbol{a}} &= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}) \\
&= \dot{\omega}\hat{\boldsymbol{\jmath}} \times (L\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{k}}) + \omega\hat{\boldsymbol{\jmath}} \times \underbrace{\omega(-L\hat{\boldsymbol{k}} + r\hat{\boldsymbol{\imath}})}_{\vec{\omega}\times\vec{\boldsymbol{r}}} \\
&= \dot{\omega}(-L\hat{\boldsymbol{k}} + r\hat{\boldsymbol{\imath}}) + \omega^2(-L\hat{\boldsymbol{\imath}} - r\hat{\boldsymbol{k}}) \\
&= 2\,\text{rad/s}^2(-2\,\text{m}\hat{\boldsymbol{k}} + 1\,\text{m}\hat{\boldsymbol{\imath}}) - 100(\,\text{rad/s})^2(2\,\text{m}\hat{\boldsymbol{\imath}} + 1\,\text{m}\hat{\boldsymbol{k}}) \\
&= -(98\hat{\boldsymbol{\imath}} + 104\hat{\boldsymbol{k}})\,\text{m/s}^2.
\end{aligned}$$

We can check the sign of the components of $\vec{\boldsymbol{a}}$ also. Note that the tangential acceleration, $\dot{\vec{\omega}} \times \vec{\boldsymbol{r}}$, is much smaller than the centripetal acceleration, $\vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}})$. Therefore, the total acceleration is almost in the same direction as the centripetal acceleration, that is, directed from point P to A. If you draw a vector from P to A, you should be able to see that it has negative components along both the x- and z-axes. Thus the answer we have got seems to be correct, at least in direction.

$$\boxed{\vec{\boldsymbol{a}} = -(98\hat{\boldsymbol{\imath}} + 104\hat{\boldsymbol{k}})\,\text{m/s}^2}$$

① An even better choice, perhaps, is the vector $\vec{\boldsymbol{r}}_{\text{P/A}}$. Remember, the only requirement on $\vec{\boldsymbol{r}}$ is that it must start at *some* point on the axis of rotation and must end at the point of interest.



Path of point P seen from the top:

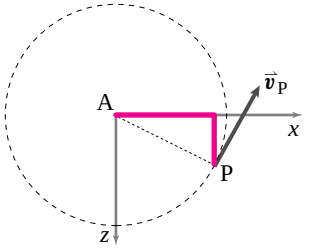


Figure 8.17: By drawing the velocity vector at point P (a vector tangent to the path) we see that $\vec{\boldsymbol{v}}_P$ must have a positive x-component and a negative z-component.
(Filename:sfig5.2.2a)

8.2 Dynamics of fixed-axis rotation

We now address mechanics questions concerning objects which are known *a priori* to spin about a fixed axis. We would like to calculate forces and moments if the motion is known. And we would like to determine the details of the motion, the angular acceleration in particular, if the applied forces and moments are known. Once the angular acceleration is known (as a function of some combination of time, angle and angular rate) the angular rate and angular position can be found by integration or solution of an ordinary differential equation.

The full content of the subject follows from the basic mechanics equations

$$\text{linear momentum balance,} \qquad \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}}$$

$$\text{angular momentum balance,} \qquad \sum \vec{\boldsymbol{M}}_{i/C} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}},$$

$$\text{and power balance:} \qquad P = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} + \dot{E}_{\mathrm{int}}.$$

The quantities $\dot{\vec{\boldsymbol{L}}}$ and $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$ are defined in terms of the position and acceleration of the system's mass (see the second page of the inside cover). To evaluate $\dot{\vec{\boldsymbol{L}}}$ and $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$ for fixed-axis rotation we can use the kinematics relations from the previous chapter which determine velocity and acceleration of points on a body spinning about a fixed axis in terms of the position $\vec{r}$ of the point of interest relative to any point on the axis.

$$\begin{aligned} \vec{v} &= \vec{\omega} \times \vec{r}, \\ \vec{a} &= \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}. \end{aligned}$$

For fixed-axis rotation $\vec{\omega} = \omega\hat{\boldsymbol{\lambda}}$ and $\dot{\vec{\omega}} = \dot{\omega}\hat{\boldsymbol{\lambda}}$ with $\hat{\boldsymbol{\lambda}}$ a constant unit vector along the axis of rotation.

To solve problems we draw a free body diagram, write the equations of linear and angular momentum balance, and evaluate the terms using the kinematics relations. In general this will lead to the evaluation of a sum or an integral. A short cut, the moment of inertia matrix, will be introduced in later sections.

Before proceeding to more difficult three-dimensional problems, let's first cover a simple 2D problem.

Example: **Spinning disk**

The round flat uniform disk in figure 8.18 is in the xy plane spinning at the constant rate $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ about its center. It has mass m_{tot} and radius R_0. What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

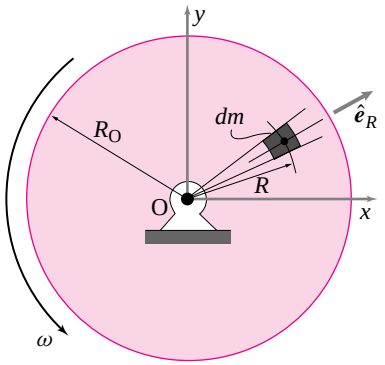
$$\sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{cm} = \vec{\boldsymbol{0}},$$

which we could also have calculated by evaluating the integral $\dot{\vec{\boldsymbol{L}}} \equiv \int \vec{\boldsymbol{a}}\, dm$ instead of using the general result that $\dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{cm}$. From

angular momentum balance we have:

$$\begin{aligned}
\sum \vec{M}_{i/O} &= \dot{\vec{H}}_{/O} \\
\Rightarrow \quad \vec{M} &= \int \vec{r}_{/O} \times \vec{a}\, dm \\
&= \int_0^{R_0} \int_0^{2\pi} (R\hat{e}_R) \times (-R\omega^2 \hat{e}_R) \underbrace{\frac{m_{\text{tot}}}{\pi R_O^2} \overbrace{R\, d\theta\, dR}^{dA}}_{dm} \\
&= \int \int \vec{0}\, d\theta\, dR \\
&= \vec{0}.
\end{aligned}$$

□



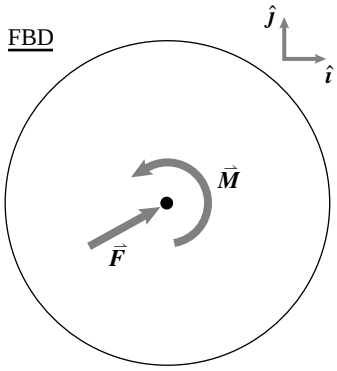


Figure 8.18: A uniform disk turned by a motor at a constant rate.

(Filename:tfigure4.3.motordisk)

Power balance is of limited use for constant rate circular motion. If all parts of a system move at constant angular rate at a constant radius then they all have constant speed. Thus the kinetic energy of the system is constant. So the power balance equation just says that the net power into the system is the amount dissipated inside (assuming no energy storage).

Example: **Spinning disk, again**

Consider the spinning disk from figure 8.18 and the previous example. The power balance equation III gives

$$P = \dot{E}_{\text{K}} + \underbrace{\dot{E}_{\text{P}}}_{0} + \underbrace{\dot{E}_{\text{int}}}_{0} \quad \Rightarrow \quad P = \int \vec{v} \cdot \vec{a}\, dm = \int 0\, dm = 0. \tag{8.18}$$

In this example there is no force or torque acting on the disk so the power P must turn out to be zero. In other constant rate problems the force and moment will *not* turn out to be zero, but the kinetic energy of the system will still be constant and so, assuming no energy storage or dissipation, we will still have $P = 0$. □

A stick sweeps out a cone

Now we consider a genuinely three-dimensional problem involving fixed-axis, rigid body rotation. Consider a long narrow stick swinging in circles so that it sweeps out a cone (Fig. 8.19). Each point on the stick is moving in circles around the z-axis at a constant rate ω. What is the relation between ω and the angle of the stick ϕ? The approach to this problem is, as usual, to draw a free body diagram, write momentum balance equations, evaluate the left and right hand sides, and then solve for quantities of interest. The hard part of this problem is evaluating the right hand side of the angular momentum balance equations.

To simplify calculation, we look at the pendulum at the instant it passes through the yz-plane, assuming the $x - y - z$ axes are fixed in space.

The free body diagram shown in figure 8.20 shows the gravity force at the center of mass, the reaction force at point O, and, consistent with the shown construction of the hinge, the moments at O perpendicular to the hinge.

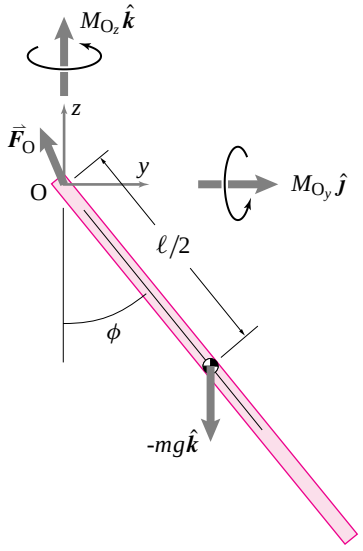


Figure 8.20: Free body diagram of the rod.

(Filename:tfigure4.spherical.pend.fbd)

Because we are interested in the relation between ϕ and ω and not the reaction force, at least for now, we look at angular momentum balance about point O.

$$\sum \vec{M}_O = \dot{\vec{H}}_O$$

First, we show and discuss the results of evaluating the equation of angular momentum balance. Then, we will show the details of calculating $\sum \vec{M}_O$ and the details of several methods for calculating $\dot{\vec{H}}_O$.

Now, let's look at the details of evaluating the terms in the angular momentum balance equation.

Evaluation of $\sum \vec{M}_O$

To find $\sum \vec{M}_O$, we had to find the moment of the gravity force. The most direct method is to use the definition

$$\begin{aligned}\sum \vec{M}_O &= \vec{r}_{/O} \times \vec{F} \\ &= \frac{\ell}{2}[\cos\phi(-\hat{k}) + \sin\phi\,\hat{j}] \times (-mg\hat{k}) \\ &= -\frac{mg\ell\sin\phi}{2}\hat{i}\end{aligned}$$

One could also 'slide' the gravity force to the level of O (a force displaced in its direction of action is mechanically equivalent). Then you can see from the figure that the force is perpendicular to its position relative to O. Moment is then force (mg) times distance ($\frac{\ell}{2}\sin\phi$) in the direction given by the right hand rule ($-\hat{i}$).

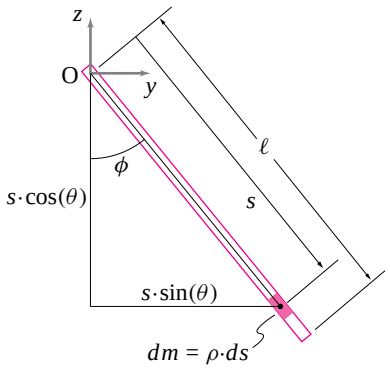


Figure 8.21: The rod is shown on the yz plane. We use this figure to locate the bit of mass dm corresponding to the bit of length of rod ds.

(Filename:tfigure4.sphere.ds)

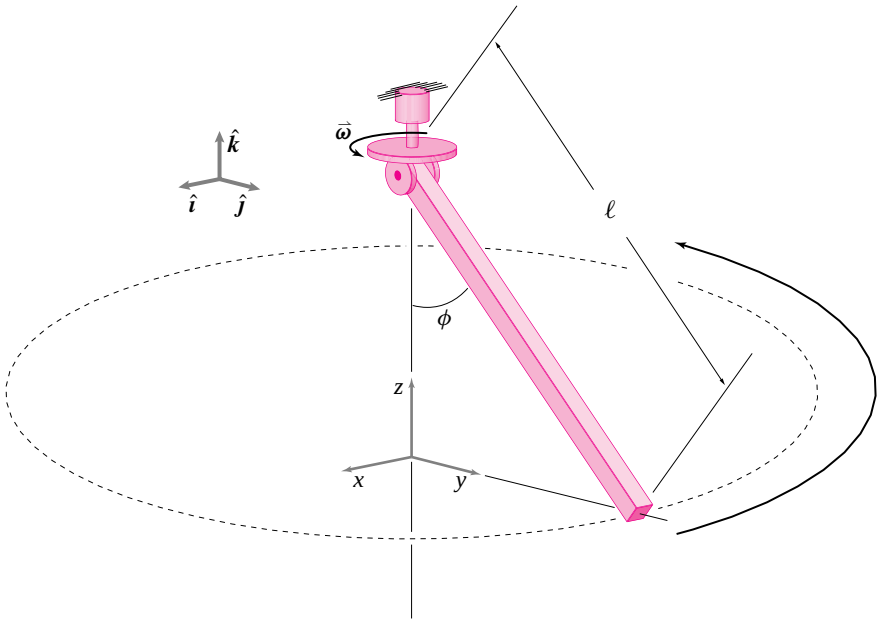


Figure 8.19: A spherical rigid body pendulum (uniform thin rod) going in circles at constant rate $\vec{\omega}$.

(Filename:tfigure4.spherical.pend)

Evaluation of $\dot{\vec{H}}_O$

We now evaluate $\dot{\vec{H}}_O$ by adding up the contribution to the sum from each bit of mass.

$\vec{r}_{/O} = -s\cos\phi\hat{k} + s\sin\phi\hat{j}$

$dm = \rho\, ds$, where
ρ = mass per unit length

$$\dot{\vec{H}}_O = \int \underbrace{\vec{r}_{/O}}\times\underbrace{\vec{a}}\;\underbrace{dm}$$

For constant rate circular motion,

$$\begin{aligned}\vec{a} &= \vec{\omega}\times(\vec{\omega}\times\vec{r}_{/O})\\ &= (\omega\hat{k})\times\Big[(\omega\hat{k})\times\\ &\qquad s(\sin\phi\hat{j}-\cos\phi\hat{k})\Big]\\ &= -\omega^2 s\sin\phi\hat{j}\end{aligned}$$

$$\begin{aligned}
&= \int_0^\ell \underbrace{(-s\cos\phi\hat{k}+s\sin\phi\hat{j})}_{\vec{r}_{/O}}\times\underbrace{(-\omega^2 s\sin\phi\hat{j})}_{\vec{a}}(\rho\,ds)\\
&= \int_0^\ell -s^2\cos\phi\sin\phi\;\omega^2\hat{\imath}\rho ds \qquad \text{(evaluating the cross product)}\\
&= -\cos\phi\sin\phi\;\omega^2\rho\hat{\imath}\int_0^\ell s^2\,ds \qquad (\phi,\ \rho,\text{ and }\omega\text{ do not vary with }s)\\
&= -\cos\phi\sin\phi\;\omega^2(\rho\frac{\ell^3}{3})\hat{\imath} \qquad \text{(evaluating the integral)}\\
\dot{\vec{H}}_O &= -\cos\phi\sin\phi\;\omega^2(\frac{m\ell^2}{3})\hat{\imath} \qquad \text{(because } m=\rho\ell).
\end{aligned}$$

We could have taken a short-cut in the calculation of acceleration $\vec{a}$. Instead of using $\vec{a} = \vec{\omega}\times(\vec{\omega}\times\vec{r})$, we could have used $\vec{a} = -\omega^2\vec{R}$ where $\vec{R}$ is the radius of the circle each particle is traveling on. It is evident from the picture that the appropriate radius is $\vec{R} = s\sin\phi\hat{j}$, so $\vec{a} = -\omega^2 s\sin\phi\hat{j}$.

We will show two more methods for calculating $\dot{\vec{H}}_O$ in section 8.4 on page 467 once you have studied the moment of inertia matrix in section 8.3.

The results for the conically swinging stick

We can now evaluate the terms in the angular momentum balance equation as

$$\overbrace{-mg\sin\phi\frac{\ell}{2}\hat{\imath}+M_{O_y}\hat{j}+M_{O_z}\hat{k}}^{\sum\vec{M}_O} = \overbrace{-\sin\phi\cos\phi\frac{m\ell^2}{3}\omega^2\hat{\imath}}^{\dot{\vec{H}}_{/O}}. \qquad (8.19)$$

We can get three scalar equations from eqn. 8.19 by dotting it with $\hat{j}$, $\hat{k}$, and $\hat{\imath}$ to get

$$\boxed{M_{O_y}=0} \quad \text{and} \quad \boxed{M_{O_z}=0}$$

and

$$\boxed{\omega^2 = \tfrac{3g}{2\ell\cos\phi}}.$$

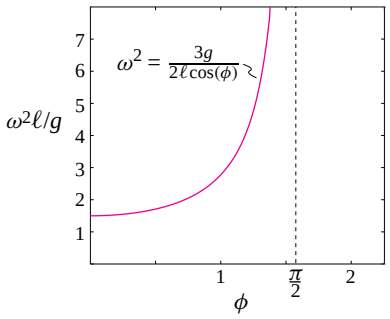


Figure 8.22: Plot of non-dimensional rotational speed $\frac{\omega^2\ell}{g}$ versus hang angle ϕ. For $\omega^2\ell/g < 3/2$ the only solution is $\phi = 0$ (hanging straight down). At or very close to $\omega^2\ell/g = 3/2$ a range of ϕ's is possible. As $\phi \to \pi/2$ and the rod becomes close to horizontal, the spin rate ω goes to infinity.

(Filename:tfigure4.spherical.vv)

①**Caution:** For more general three dimensional motion than rotation about a fixed axis the equation $M = I\alpha$ does not apply. Trying to vectorize by underlining various terms gives the wrong answer.

Note that $M_{O_y} = M_{O_z} = 0$. That is, for this special motion, the hinge joint at O could be replaced with a ball-and-socket joint.

Note that the solution for a point mass spherical pendulum is $\omega^2 = \frac{g}{\ell\cos\phi}$. That is, this stick would rotate at the same rate and angle as a point mass at the end of a rod of length $\frac{2\ell}{3}$. One could not easily anticipate this result. We point it out here to emphasize that the analysis of this rigid-body problem cannot be reduced *a priori* to any simple particle mechanics problem.

In figure 8.22, non-dimensional rotational speed $\frac{\omega^2\ell}{g}$ is plotted versus hang angle ϕ. As one might expect intuitively unless ω is high enough, ($\omega^2 > \frac{3g}{2\ell}$), the only solution is hanging straight down ($\phi = 0$). At the critical speed ($\omega^2 = \frac{3g}{2\ell}$), the curve is nearly flat, implying that a range of hang angles ϕ is possible all with nearly the same angular velocity. As is also intuitively plausible, the bar gets close to the horizontal (close to $\frac{\pi}{2}$), the spin rate goes to infinity.

The scalar equations governing rotation about an axis

For two dimensional motion of flat hinged objects we had the simple relation "$M = I\alpha$". This formula captures our simple intuitions about angular momentum balance. When you apply torque to a body its rate of rotation increases. It turns out that, for three-dimensional motion of a rigid body about a fixed axis, the same result applies if we interpret the terms correctly.①

If the axis of rotation goes through C and is in the direction $\hat{\boldsymbol{\lambda}}$ we can define $M = \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{M}}_{\mathrm{C}}$ as the moment about the axis of rotation. We can similarly look at the $\hat{\boldsymbol{\lambda}}$ component of $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$ (assume, for definiteness, that the system is continuous).

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} &= \hat{\boldsymbol{\lambda}} \cdot \int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}}\, dm \\
&= \hat{\boldsymbol{\lambda}} \cdot \int \vec{\boldsymbol{r}} \times \left[(\omega\hat{\boldsymbol{\lambda}}) \times \left((\omega\hat{\boldsymbol{\lambda}}) \times \vec{\boldsymbol{r}} \right) + (\dot{\omega}\hat{\boldsymbol{\lambda}}) \times \vec{\boldsymbol{r}} \right] dm. \\
&= \dot{\omega} \int R^2\, dm, \qquad (8.20)
\end{aligned}$$

where R is the distance of the mass points from the axis. The last line follows from the previous most simply by paying attention to directions and magnitudes when using the right-hand rule and the geometric definition of the cross product. We thus have derived the result that

$$M = I\alpha,$$

if by M we mean moment about the fixed axis and by I we mean $\int R^2\, dm$. Actually, the scalar we call I in the above equation is a manifestation of a more general matrix $[\boldsymbol{I}]$ that we will explore in the next section.

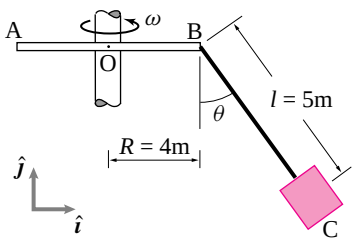
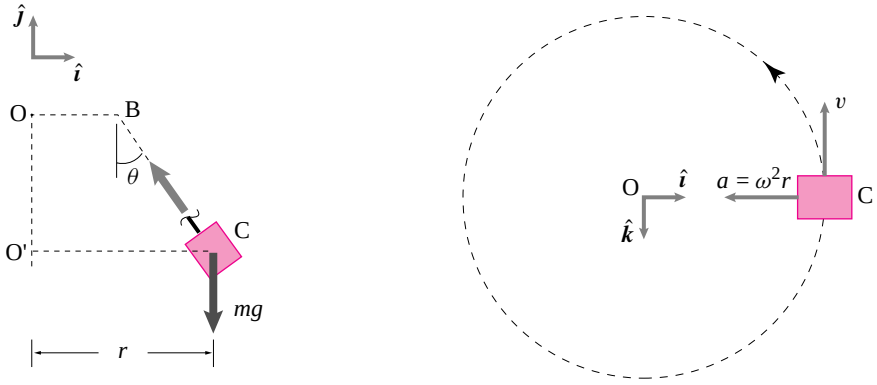


Figure 8.23: A carnival ride rotating at a constant speed

(Filename:sfig4.2.1)

SAMPLE 8.5 *Going on a carnival ride at a constant rate.* A carnival ride with roof AB and carriage BC is rotating about the vertical axis with constant angular velocity $\vec{\omega} = \omega\hat{j}$. If the carriage with its occupants has mass $m = 100\,\text{kg}$, find the tension in the inextensible and massless rod BC when $\theta = 30^o$. What is the required angular speed ω (in revolutions/minute) to maintain this angle?

Solution The free body diagram of the carriage is shown in Fig. 8.24(a). The



(a) FBD of C

(b) Geometry of motion looking down $\hat{j}$-axis

Figure 8.24: (Filename:sfig4.2.1a)

geometry of motion of the carriage is shown in Fig. 8.24(b). The carriage goes around a circle of radius $r = O'C$ with constant speed $v = \omega r$. The only acceleration that the carriage has is the centripetal acceleration and at the moment of interest $\vec{a} = -\omega^2 r\hat{i}$.

The linear momentum balance $(\sum \vec{F} = \dot{\vec{L}})$ for the carriage gives:

$$\begin{aligned} T\hat{\lambda}_{CB} - mg\hat{j} &= m\vec{a} \\ \text{or} \qquad T(-\sin\theta\hat{i} + \cos\theta\hat{j}) - mg\hat{j} &= -m\omega^2 r\hat{i} \end{aligned} \tag{8.21}$$

Scalar equations from eqn. (8.21) are:

$$\begin{aligned} [\text{eqn. (8.21)}]\cdot\hat{j} \quad \Rightarrow \quad T\cos\theta - mg &= 0 \\ \Rightarrow \quad T &= \frac{mg}{\cos\theta} \\ &= \frac{100\,\text{kg}\cdot 9.8\,\text{m/s}^2}{\sqrt{3}/2} \\ &= 1133\,\text{N}. \\ [\text{eqn. (8.21)}]\cdot\hat{i} \quad \Rightarrow \quad -T\sin\theta &= -m\omega^2 r \\ \Rightarrow \quad \omega^2 &= \frac{T\sin\theta}{mr} \\ &= \frac{T\sin\theta}{m(R + l\sin\theta)} \\ &= \frac{1133\,\text{N}\cdot\frac{1}{2}}{100\,\text{kg}(4\,\text{m} + 5\,\text{m}\cdot\frac{1}{2})} \\ &= 0.87\frac{1}{\text{s}^2} \end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad \omega &= 0.93\frac{\text{rad}}{\text{s}} \\
&= 0.93\frac{1}{\cancel{\text{s}}} \cdot \frac{1\text{rev}}{2\pi} \cdot \frac{60\cancel{\text{s}}}{1\,\text{min}} \\
&= 8.9\,\text{rpm}.
\end{aligned}$$

$$\boxed{T = 1133\,\text{N},\ \omega = 8.9\,\text{rpm}}$$

Alternatively,

we could also find the angular speed using angular momentum balance. The angular momentum balance about point B gives

$$\sum \vec{\boldsymbol{M}}_{/B} = \dot{\vec{\boldsymbol{H}}}_{/B}$$

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/B} &= \vec{\boldsymbol{r}}_{C/B} \times (-mg\hat{\boldsymbol{j}}) \\
&= -mgl\sin\theta\hat{\boldsymbol{k}} \\
\dot{\vec{\boldsymbol{H}}}_{/B} &= \vec{\boldsymbol{r}}_{C/B} \times (-m\omega^2 r\hat{\boldsymbol{\imath}}) \\
&= -m\omega^2 rl\cos\theta\hat{\boldsymbol{k}}
\end{aligned}$$

Equating the two quantities, we get

$$\begin{aligned}
\cancel{m}\omega^2 r\cancel{l}\cos\theta &= \cancel{m}g\cancel{l}\sin\theta \\
\Rightarrow \qquad \omega^2 &= \frac{g}{r}\tan\theta \\
&= \frac{g\tan\theta}{R + l\sin\theta} \\
&= \frac{9.8\,\text{m/s}^2 \cdot 0.577}{4\,\text{m} + 5\,\text{m}\frac{1}{2}} \\
&= 0.87\frac{1}{\text{s}^2} \\
\omega &= 0.93\,\text{s}^{-1} = 8.9\,\text{rpm}
\end{aligned}$$

which is the same value as we found using the linear momentum balance .

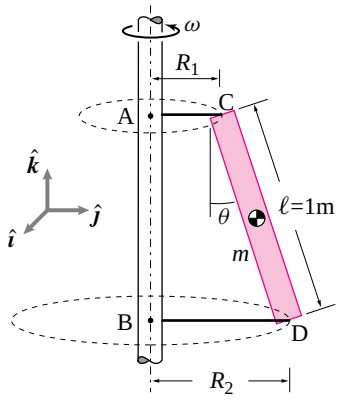


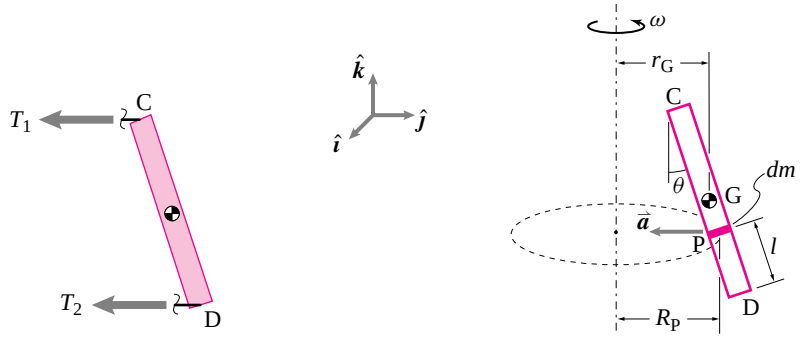
Figure 8.25: A bar held by two strings rotates in 3-D.

(Filename:sfig4.6.2)

SAMPLE 8.6 *A crooked bar rotating with a shaft in space.* A uniform rod CD of mass $m = 2\,\text{kg}$ and length $\ell = 1\,\text{m}$ is fastened to a shaft AB by means of two strings: AC of length $R_1 = 30\,\text{cm}$, and BD of length $R_2 = 50\,\text{cm}$. The shaft is rotating at a constant angular velocity $\vec{\boldsymbol{\omega}} = 5\,\text{rad/s}\hat{\boldsymbol{k}}$. There is no gravity. At the instant shown, find the tensions in the two strings.

Solution The free body diagram of the rod is shown in Fig. 8.26. The linear momentum balance ($\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$) for the rod gives:

$$T_1 + T_2 = m\omega^2 r_G. \tag{8.22}$$



(a) FBD (b) Geometry of motion

Figure 8.26: (Filename:sfig4.6.2a)

Usually, linear momentum balance gives us two scalar equations in 2-D and three scalar equations in 3-D. Unfortunately, in this case, it gives only one equation for two unknowns T_1 and T_2. Therefore, we need one more equation.

The angular momentum balance about point D gives:

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/D} &= \dot{\vec{\boldsymbol{H}}}_{/D}, \\
\text{where} \quad \sum \vec{\boldsymbol{M}}_{/D} &= \vec{\boldsymbol{r}}_{C/D} \times (-T_1\hat{\boldsymbol{j}}) \\
&= \ell(-\sin\theta\,\hat{\boldsymbol{j}} + \cos\theta\hat{\boldsymbol{k}}) \times (-T_1\hat{\boldsymbol{j}}) \\
&= \ell\, T_1 \cos\theta\hat{\boldsymbol{i}},
\end{aligned}$$

and

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/D} &= \int_m \vec{\boldsymbol{r}}_{P/D} \times (-\omega^2 R_P\,\hat{\boldsymbol{j}})dm \\
&= \int_0^\ell \overbrace{l(-\sin\theta\,\hat{\boldsymbol{j}} + \cos\theta\hat{\boldsymbol{k}})}^{\vec{\boldsymbol{r}}_{P/D}} \times (-\omega^2 \overbrace{(R_2 - l\sin\theta)}^{R_P}\,\hat{\boldsymbol{j}})\,\overbrace{\frac{m}{\ell}\,dl}^{dm} \\
&= \frac{m\omega^2}{\ell}\left(R_2\cos\theta\int_0^\ell l\,dl - \cos\theta\sin\theta\int_0^\ell l^2\,dl\right)\hat{\boldsymbol{i}}
\end{aligned}$$

$$= \frac{m\omega^2}{\ell}\left(R_2\cos\theta\frac{\ell^2}{2} - \cos\theta\sin\theta\frac{\ell^3}{3}\right)\hat{\boldsymbol{\imath}}$$
$$= m\omega^2\ell\cos\theta(\frac{1}{2}R_2 - \frac{1}{3}\ell\sin\theta)\hat{\boldsymbol{\imath}}.$$

Thus,

$$\not{\ell}\, T_1\cos\not{\theta} = m\omega^2\not{\ell}\cos\not{\theta}(\frac{1}{2}R_2 - \frac{1}{3}\ell\sin\theta)$$
$$\Rightarrow \quad T_1 = m\omega^2(\frac{1}{2}R_2 - \frac{1}{3}\ell\sin\theta).$$

Substituting in (8.22) we get

$$T_2 = m\omega^2(r_G - \frac{1}{2}R_2 + \frac{1}{3}\ell\sin\theta).$$

Plugging in the given numerical values and noting that $r_G = (R_1 + R_2)/2 = 40\,\text{cm}$ and $\ell\sin\theta = R_2 - R_1 = 20\,\text{cm}$, we get

$$T_1 = 2\,\text{kg}\cdot(5\frac{1}{s})^2\cdot(0.4\,\text{m} - \frac{1}{2}0.5\,\text{m} + \frac{1}{3}0.2\,\text{m})$$
$$= 9.17\frac{\text{kg}\cdot\text{m}}{\text{s}^2} = 9.17\,\text{N}$$
$$\text{and} \quad T_2 = 10.83\,\text{N}.$$

$$\boxed{T_1 = 9.1\,\text{N}, \quad T_2 = 10.9\,\text{N}}$$

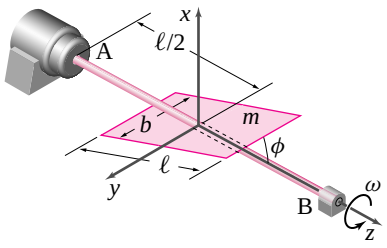
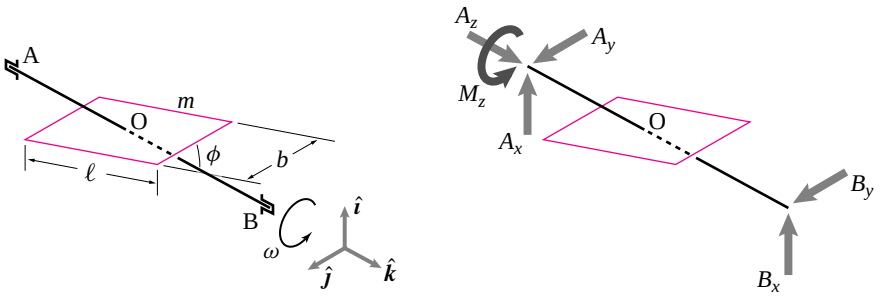


Figure 8.27: A rectangular plate, mounted rigidly at an angle ϕ on a shaft, wobbles as the shaft rotates at a constant speed.

(Filename:sfig4.6.6)

SAMPLE 8.7 *A crooked plate rotating with a shaft in space.* A rectangular plate of mass m, length ℓ, and width b is welded to a shaft AB in the center. The long edge of the plate is parallel to the shaft axis but is tipped by an angle ϕ with respect to the shaft axis. The shaft rotates with a constant angular speed ω. The end B of the shaft is free to move in the z-direction. Assume there is no gravity. Find the reactions at the supports.

Solution A simple line sketch and the Free Body Diagram of the system are shown in Fig. 8.28ab. The linear momentum balance equation for the shaft and the plate



(a) (b)

Figure 8.28: (Filename:sfig4.6.6a)

system is:

$$\sum \vec{F} = m_{total}\, \vec{a}_{cm}.$$

Since the center of mass is on the axis of rotation, $\vec{a}_{cm} = \vec{0}$. Therefore,

$$(A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + A_z\hat{k} = \vec{0}$$

$$\Rightarrow \qquad A_x + B_x = 0, \quad A_y + B_y = 0, \quad A_z = 0. \tag{8.23}$$

The angular momentum balance about the center of mass O is:

$$\sum \vec{M}_{\rm O} = \dot{\vec{H}}_{\rm O}$$

- **Calculation of $\sum \vec{M}_{\rm O}$:**

$$\begin{aligned} \sum \vec{M}_{\rm O} &= \vec{r}_{A/O} \times \vec{F}_A + \vec{r}_{B/O} \times \vec{F}_B + M_z\hat{k} \\ &= -\frac{\ell}{2}\hat{k} \times (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) + \frac{\ell}{2}\hat{k} \times (B_x\hat{\imath} + B_y\hat{\jmath}) + M_z\hat{k} \\ &= \frac{\ell}{2}(A_y - B_y)\hat{\imath} + \frac{\ell}{2}(B_x - A_x)\hat{\jmath} + M_z\hat{k} \end{aligned} \tag{8.24}$$

① $\dot{\vec{H}}$ could also be computed using the moment of inertia matrix of the body. See the next two text sections.

- **Calculation of $\dot{\vec{H}}_{\rm O}$:** $\dot{\vec{H}}_{\rm O}$ can be computed in various ways. ①Here, to compute $\dot{\vec{H}}_{\rm O}$, we use

$$\dot{\vec{H}}_{\rm O} = \int_M \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm,$$

the formula which we have used so far. To carry out this integration for the plate, we take, as usual, an infinitesimal mass dm of the body, calculate its angular momentum about O, and then integrate over the entire mass of the body:

$$\dot{\vec{H}}_{\rm O} = \int_M \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm$$

We need to write carefully each term in the integrand. Let us define an axis w (don't confuse this dummy variable w with ω) along the length of the plate (see Fig. 8.29(a)). We take an area element $dA = dw\,dy$ on the plate as our infinitesimal mass. Fig. 8.29(b) shows this element and its coordinates.

$$\begin{aligned} dm &= \rho\, dA = \frac{m}{\ell b} dw\, dy \quad (\rho = \text{mass per unit area}) \\ \vec{\boldsymbol{a}}_{dm} &= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{dm/O}) \\ \vec{\boldsymbol{r}}_{dm/O} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \end{aligned}$$

where

$$x = w\sin\phi, \quad y = y, \quad z = w\cos\phi. \tag{8.25}$$

Therefore,

$$\begin{aligned} \vec{\boldsymbol{a}}_{cm} &= \omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times (w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + w\cos\phi\hat{\boldsymbol{k}})) \\ &= \omega\hat{\boldsymbol{k}} \times (\omega\, w\sin\phi\hat{\boldsymbol{\jmath}} - \omega\, y\hat{\boldsymbol{\imath}}) = -\omega^2(w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}), \\ \vec{\boldsymbol{r}}_{dm/O} \times \vec{\boldsymbol{a}}_{dm} &= (w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + w\cos\phi\hat{\boldsymbol{k}}) \times [-\omega^2(w\sin\phi\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})] \\ &= \omega^2(-w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}} + wy\cos\phi\hat{\boldsymbol{\imath}}). \end{aligned}$$

Thus,

$$\begin{aligned} \vec{\boldsymbol{H}}_O &= \int_{-b/2}^{b/2}\int_{-\ell/2}^{\ell/2} \overbrace{\omega^2(-w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}} + wy\cos\phi\hat{\boldsymbol{\imath}})}^{\vec{\boldsymbol{r}}_{dm/O}\times\vec{\boldsymbol{a}}_{dm}} \overbrace{\frac{m}{\ell b} dw\, dy}^{dm} \\ &= \frac{m}{\ell b}\omega^2 \int_{-b/2}^{b/2} \left(\int_{-\ell/2}^{\ell/2} (-w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}} + wy\cos\phi\hat{\boldsymbol{\imath}}) dw \right) dy \\ &= \frac{m}{\ell b}\omega^2 \int_{-b/2}^{b/2} \left(-\sin\phi\cos\phi\, \frac{w^3}{3}\Big|_{-\ell/2}^{\ell/2} + y\cos\phi \underbrace{\frac{w^2}{2}\Big|_{-\ell/2}^{\ell/2}}_{0} \right) dy \\ &= -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}. \end{aligned} \tag{8.26}$$

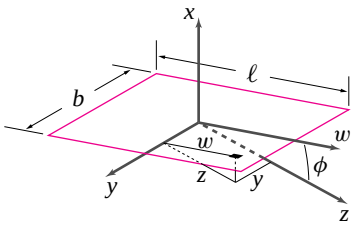
- **Now, back to angular momentum balance:** Now equating (8.24) and (8.26) and dotting both sides with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ we get

$$A_y - B_y = 0, \quad B_x - A_x = -\frac{m\omega^2\ell}{6}\sin\phi\cos\phi, \quad M_z = 0, \tag{8.27}$$

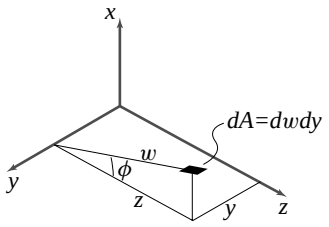
respectively. Solving (8.23) and (8.27) simultaneously we get

$$A_y = B_y = 0, \quad A_x = \frac{m\omega^2\ell}{12}, \quad \sin\phi\cos\phi B_x = -\frac{m\omega^2\ell}{12}\sin\phi\cos\phi.$$

$$\boxed{A_x = \tfrac{m\omega^2\ell}{12}\sin\phi\cos\phi,\ B_x = -\tfrac{m\omega^2\ell}{12}\sin\phi\cos\phi,\ A_y = B_y = A_z = M_z = 0}$$



(a)



(b)

Figure 8.29: Calculation of $\vec{\boldsymbol{H}}_O$: (a) mass element dm is shown on the plate, (b) the mass element as an area element and its geometry.

(Filename:sfig4.6.6b)

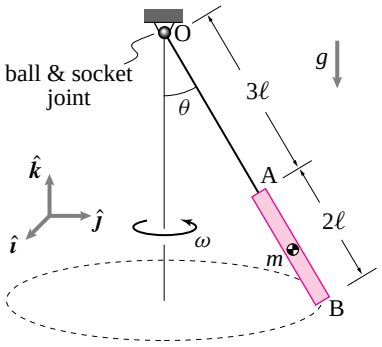
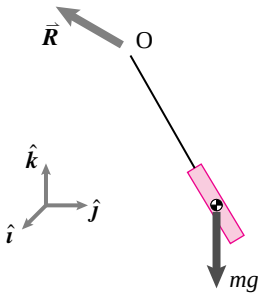
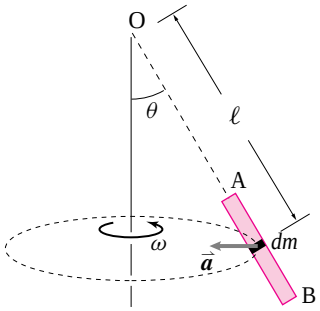


Figure 8.30: A short rod swings in 3-D.
(Filename:sfig4.6.1)



(a) FBD



(b) Calculation of $\dot{\vec{H}}$

Figure 8.31: (Filename:sfig4.6.1a)

SAMPLE 8.8 *A short rod as a 3-D pendulum.* A uniform rod AB of mass m and length 2ℓ is welded to a massless, inextensible thin rod OA at point A. Rod OA is attached to a ball and socket joint at point O. The rods are going around in a circle with constant speed maintaining a constant angle θ with the vertical axis. Assume that θ is small.

(a) How many revolutions does the system make in one second?

Solution

(a) The free body diagram of the system (rod OA + rod AB) is shown in Fig. 8.31. Let ω be the angular speed of the system. Then, the number of revolutions in one second is $n = \omega/(2\pi)$. Therefore, to find the answer we need to calculate ω.

The angular momentum balance about point O gives $\sum \vec{M}_O = \dot{\vec{H}}_O$. Now,

$$\begin{aligned}
\sum \vec{M}_O &= \vec{r}_{G/O} \times (-mg\hat{k}) \\
&= 4\ell(\sin\theta\,\hat{j} - \cos\theta\hat{k}) \times (-mg\hat{k}) \\
&= -4\ell\, mg\sin\theta\hat{i}.
\end{aligned}$$

and

$$\begin{aligned}
\dot{\vec{H}}_O &= \int_m \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm \\
&= \int_{3\ell}^{5\ell} \overbrace{\ell(\sin\theta\,\hat{j} - \cos\theta\hat{k})}^{\vec{r}_{dm/O}} \times \overbrace{(-\omega^2 l\sin\theta\,\hat{j})}^{\vec{a}_{dm}} \overbrace{\frac{m}{2\ell}\,d\ell}^{dm} \\
&= -\frac{m}{2\ell}\omega^2\sin\theta\cos\theta\hat{i}\int_{3\ell}^{5\ell}\ell^2\,d\ell \\
&= -\frac{49}{3}\ell^2 m\omega^2\sin\theta\cos\theta\hat{i}.
\end{aligned}$$

By equating the two quantities ($\sum \vec{M}_O = \dot{\vec{H}}_O$), we get

$$\begin{aligned}
-4\ell\, mg\sin\theta\hat{i} &= -\frac{49}{3}\ell^2 m\omega^2\sin\theta\cos\theta\hat{i} \\
\Rightarrow \quad \omega^2 &= \frac{12g}{49\ell\cos\theta}.
\end{aligned}$$

But for small θ, $\cos\theta \approx 1$. Therefore,

$$\omega = \sqrt{\frac{12g}{49\ell}} = \frac{2\sqrt{3}}{7}\sqrt{\frac{g}{\ell}}$$

and the number of revolutions per unit time is $n = \frac{2\sqrt{3}}{14\pi}\sqrt{\frac{g}{\ell}}$.

$$\boxed{n = \frac{2\sqrt{3}}{14\pi}\sqrt{\frac{g}{\ell}}}$$

(b) Note, the natural frequency of this rod swinging back and forth as a simple pendulum turns out to be the same as the angular speed ω of the rotating system above for small θ.

8.3 Moment of inertia matrices $[I^{\rm cm}]$ and $[I^{\rm O}]$

We now know how to find the velocity and acceleration of every bit of mass on a rigid body as it spins about a fixed axis. It is just a matter of doing integrals or sums to calculate the various motion quantities (momenta, energy) of interest. As the body moves and rotates the region of integration and the values of the integrands change. So, in principle, in order to analyze a rigid body one has to evaluate a different integral or sum at every different configuration. But there is a shortcut. A big sum (over all atoms, say), or a difficult integral is reduced to a simple multiplication using the moment of inertia. In three-dimensions this multiplication is a matrix multiplication.

[**I**], the moment of inertia matrix①, is defined for the purpose of simplifying the expressions for the angular momentum, the rate of change of angular momentum, and the energy of *a system which moves like a rigid body*. For study of the analysis of flat objects in planar motion only one component of the matrix [**I**] is relevant, it is I_{zz}, called just I or J in elementary physics courses. Here are the results. They are derived in the box 8.3 on page 462. To avoid intimidating you at the start we first review the results in 2-D . A flat object spinning with $\vec{\omega} = \omega\hat{k}$ in the xy plane has a mass distribution which gives, by means of a calculation which we will discuss shortly, a moment of inertia I_{zz}^{cm} or just 'I' so that:

$$\vec{H}_{\rm cm} = I\omega\hat{k} \tag{8.28}$$

$$\dot{\vec{H}}_{\rm cm} = \vec{0} \tag{8.29}$$

$$E_{{\rm K}/cm} = \frac{1}{2}\omega^2 I. \tag{8.30}$$

For a rigid body spinning in 3-D about a fixed axis with the angular velocity $\vec{\omega}$ we need matrix multiplication, where the determination of the needed matrix is the central topic of this section.

$$\vec{H}_{\rm cm} = [I^{cm}]\cdot\vec{\omega} \tag{8.31}$$

$$\dot{\vec{H}}_{\rm cm} = \vec{\omega}\times\underbrace{[I^{cm}]\cdot\vec{\omega}}_{\vec{H}_{\rm cm}} + [I^{cm}]\cdot\dot{\vec{\omega}} \tag{8.32}$$

$$E_{{\rm K}/cm} = \frac{1}{2}\vec{\omega}\cdot([I^{cm}]\cdot\vec{\omega}) = \frac{1}{2}\vec{\omega}\cdot\vec{H}_{\rm cm}. \tag{8.33}$$

In detail, for example,

$$\begin{bmatrix} H_{x/cm} \\ H_{y/cm} \\ H_{z/cm} \end{bmatrix} = \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix}_{xyz} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{8.34}$$

where $\vec{H}_{\rm cm} = H_{x/cm}\hat{\imath} + H_{y/cm}\hat{\jmath} + H_{z/cm}\hat{k}$. Note that the 2-D results are a special case of the 3-D results because, as you will soon see, for 2-D objects $I_{xz} = I_{yz} = 0$. We postpone the use of these equations till section 8.4.

① In fact the moment of inertia matrix for a given object depends on what reference point is used. Most commonly when people say 'the' moment of inertia they mean to use the center of mass as the reference point. For clarity this moment of inertia matrix is often written as $[I^{\rm cm}]$ in this book. If a different reference point, say point O is used, the matrix is notated as $[I^{\rm O}]$.

The moments of inertia in 2-D : $[I^{\rm cm}]$ and $[I^{\rm O}]$.

We start by looking at the scalar I which is just the zz or 33 component of the matrix $[I]$. The definition of I^{cm} is

$$I^{cm} \equiv = \int \underbrace{x^2 + y^2}_{r^2}\, dm$$

$$= \int\int r^2 \overbrace{\left(\frac{m_{\text{tot}}}{A}\right)}^{dm} dA \qquad \text{for a uniform planar object}$$

The mass per unit area.

where x and y are the distances of the mass in the x and y direction measured from an origin, and r is the direct distance from that origin. If that origin is at the center of mass then we are calculating I^{cm}, if the origin is at a point labeled C or O then we are calculating I^C or I^O.

The term I_{zz} is sometimes called the *polar moment of inertia*, or *polar mass moment of inertia* to distinguish it from the I_{xx} and I_{yy} terms which have little utility in planar dynamics (but are all important when calculating the stiffness of beams!).

What, physically, is the moment of inertia? It is a measure of the extent to which mass is far from the given reference point. Every bit of mass contributes to I in proportion to the square of its distance from the reference point.

Radius of gyration

Another measure of the extent to which mass is spread from the reference point, besides the moment of inertia, is the *radius of gyration*, r_{gyr}. The radius of gyration is sometimes called k but we save k for stiffness. The radius of gyration is defined as:

$$r_{gyr} \equiv \sqrt{I/m} \quad \Rightarrow \quad r_{gyr}^2 m = I.$$

That is, the radius of gyration of an object is the radius of an equivalent ring of mass that has the same I and the same mass as the given object.

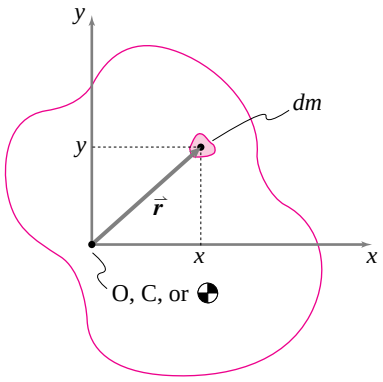


Figure 8.32: A general planar body.
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Other reference points

For the most part it is I^{cm} which is of primary interest. Other reference points are useful

(a) if the rigid body is hinged at a fixed point O then a slight short cut in calculation of angular momentum and energy terms can be had; and

(b) if one wants to calculate the moment of inertia of a composite body about its center of mass it is useful to first find the moment of inertia of each of its parts about that point. But the center of mass of the composite is usually not the center of mass of any of the separate parts.

The box 8.2 on page 458 shows the calculation of I for a number of simple 2 dimensional objects.

The parallel axis theorem for planar objects

The planar parallel axis theorem is the equation

$$I_{zz}^{C} = I_{zz}^{cm} + m_{\mathrm{tot}} \underbrace{r_{cm/C}^{2}}_{d^2}.$$

In this equation $d = r_{cm/C}$ is the distance from the center of mass to a line parallel to the z-axis which passes through point C. See box 7.4 on page 407 for a derivation of the parallel axis theorem for planar objects.

Note that $I_{zz}^{C} \geq I_{zz}^{cm}$, always.

One can calculate the moment of inertia of a composite body about its center of mass, in terms of the masses and moments of inertia of the separate parts. Say the position of the center of mass of m_i is (x_i, y_i) relative to a fixed origin, and the moment of inertia of that part about its center of mass is I_i. We can then find the moment of inertia of the composite I_{tot} about its center of mass (x_{cm}, y_{cm}) by the following sequence of calculations:

$$\begin{array}{lrcl}
(1) & m_{\mathrm{tot}} & = & \sum m_i \\
(2) & x_{cm} & = & \left[\sum x_i m_i\right] / m_{\mathrm{tot}} \\
 & y_{cm} & = & \left[\sum y_i m_i\right] / m_{\mathrm{tot}} \\
(3) & d_i^2 & = & (x_i - x_{cm})^2 + (y_i - y_{cm})^2 \\
(4) & I_{tot} & = & \sum \left[I_i + m_i d_i^2\right].
\end{array}$$

Of course if you are mathematically inclined you can reduce this recipe to one grand formula with lots of summation signs. But you would end up doing the calculation in about this order in any case. As presented here this sequence of steps lends itself naturally to computer calculation with a spread sheet or any program that deals easily with arrays of numbers.

The tidy recipe just presented is actually more commonly used, with slight modification, in strength of materials than in dynamics. The need for finding area moments of inertia of strange beam cross sections arises more frequently than the need to find polar mass moment of inertia of a strange cutout shape.

The perpendicular axis theorem for planar rigid bodies

The perpendicular axis theorem for planar objects is the equation

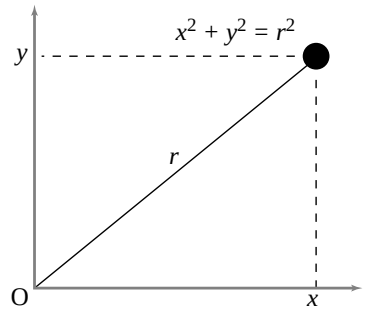
$$I_{zz} = I_{xx} + I_{yy}$$

which is derived in box 7.4 on page 407. It gives the 'polar' inertia I_{zz} in terms of the inertias I_{xx} and I_{yy}. Unlike the parallel axis theorem, the perpendicular axis theorem does *not* have a three-dimensional counterpart. The theorem is of greatest utility when one wants to study the three-dimensional mechanics of a flat object and thus are in need of its full moment of inertia matrix.

8.2 Some examples of 2-D Moment of Inertia

Here, we illustrate some simple moment of inertia calculations for two-dimensional objects. The needed formulas are summarized, in part, by the lower right corner components (that is, the elements in the third column and third row (3,3)) of the matrices in the table on the inside back cover.

One point mass



If we assume that all mass is concentrated at one or more points, then the integral

$$I^o_{zz} = \int r^2_{/o}\, dm$$

reduces to the sum

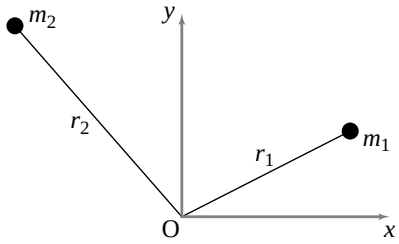
$$I^o_{zz} = \sum r^2_{i/o} m_i$$

which reduces to one term if there is only one mass,

$$I^o_{zz} = r^2 m = (x^2 + y^2)m.$$

So, if $x = 3\,\text{in}$, $y = 4\,\text{in}$, and $m = 0.1\,\text{lbm}$, then $I^o_{zz} = 2.5\,\text{lbm in}^2$. Note that, in this case, $I^{cm}_{zz} = 0$ since the radius from the center of mass to the center of mass is zero.

Two point masses

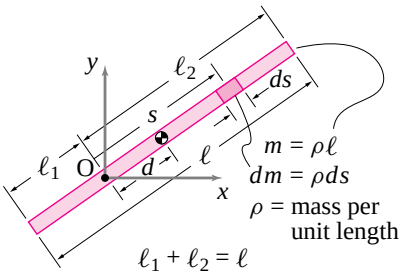


In this case, the sum that defines I^o_{zz} reduces to two terms, so

$$I^o_{zz} = \sum r^2_{i/o} m_i = m_1 r_1^2 + m_2 r_2^2.$$

Note that, if $r_1 = r_2 = r$, then $I^o_{zz} = m_{tot} r^2$.

A thin uniform rod



Consider a thin rod with uniform mass density, ρ, per unit length, and length ℓ. We calculate I^o_{zz} as

$$\begin{aligned} I^o_{zz} &= \int r^2 \overbrace{dm}^{\rho ds} \\ &= \int_{-\ell_1}^{\ell_2} s^2 \rho ds \quad (s = r) \\ &= \frac{1}{3}\rho s^3 \Big|_{-\ell_1}^{\ell_2} \quad (\text{since} \quad \rho \equiv const.) \\ &= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3). \end{aligned}$$

If either $\ell_1 = 0$ or $\ell_2 = 0$, then this expression reduces to $I^o_{zz} = \frac{1}{3}m\ell^2$. If $\ell_1 = \ell_2$, then O is at the center of mass and

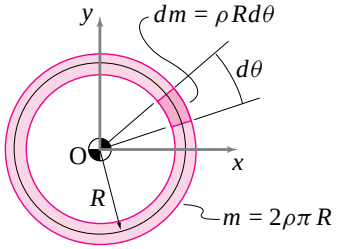
$$I^o_{zz} = I^{cm}_{zz} = \frac{1}{3}\rho\left(\left(\frac{\ell}{2}\right)^3 + \left(\frac{\ell}{2}\right)^3\right) = \frac{ml^2}{12}.$$

We can illustrate one last point. With a little bit of algebraic histrionics of the type that only hindsight can inspire, you can verify that the expression for I^0_{zz} can be arranged as follows:

$$\begin{aligned} I^0_{zz} &= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3) \\ &= \underbrace{\rho(\ell_1 + \ell_2)}_{m}\left(\underbrace{\frac{\ell_2 - \ell_1}{2}}_{d}\right)^2 + \underbrace{\rho\frac{(\ell_1 + \ell_2)^3}{12}}_{m\ell^2/12} \\ &= md^2 + m\frac{\ell^2}{12} \\ &= md^2 + I^{cm}_{zz} \end{aligned}$$

That is, the moment of inertia about point O is greater than that about the center of mass by an amount equal to the mass times the distance from the center of mass to point O squared. This derivation of the *parallel axis theorem* is for one special case, that of a uniform thin rod.

A uniform hoop



For a hoop of uniform mass density, ρ, per unit length, we might consider all of the points to have the same radius R. So,

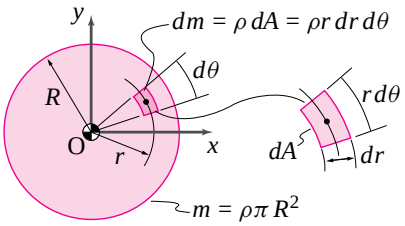
$$I^o_{zz} = \int r^2 dm = \int R^2 dm = R^2 \int dm = R^2 m.$$

Or, a little more tediously,

$$\begin{aligned} I^o_{zz} &= \int r^2 dm \\ &= \int_0^{2\pi} R^2 \rho R d\theta \\ &= \rho R^3 \int_0^{2\pi} d\theta \\ &= 2\pi\rho R^3 = \underbrace{(2\pi\rho R)}_{m} R^2 = mR^2. \end{aligned}$$

This I^o_{zz} is the same as for a single point mass m at a distance R from the origin O. It is also the same as for two point masses if they both are a distance R from the origin. For the hoop, however, O is at the center of mass so $I^o_{zz} = I^{cm}_{zz}$ which is not the case for a single point mass.

A uniform disk

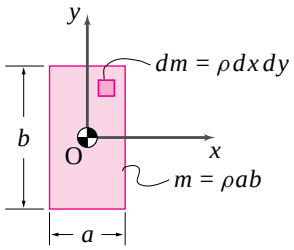


Assume the disk has uniform mass density, ρ, per unit area. For a uniform disk centered at the origin, the center of mass is at the origin so

$$\begin{aligned} I^o_{zz} = I^{cm}_{zz} &= \int r^2 dm \\ &= \int_0^R \int_0^{2\pi} r^2 \rho r d\theta dr \\ &= \int_0^R 2\pi\rho r^3 dr \\ &= 2\pi\rho \frac{r^4}{4}\Big|_0^R = \pi\rho\frac{R^4}{2} = (\pi\rho R^2)\frac{R^2}{2} \\ &= m\frac{R^2}{2}. \end{aligned}$$

For example, a 1 kg plate of 1 m radius has the same moment of inertia as a 1 kg hoop with a 70.7 cm radius.

Uniform rectangular plate



For the special case that the center of the plate is at point O, the center of mass of mass is also at O and $I^o_{zz} = I^{cm}_{zz}$.

$$\begin{aligned} I^o_{zz} = I^{cm}_{zz} &= \int r^2 dm \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2) \overbrace{\rho dx dy}^{dm} \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho\left(\frac{x^3}{3} + xy^2\right)\Big|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} dy \\ &= \rho\left(\frac{x^3 y}{3} + \frac{xy^3}{3}\right)\Big|_{x=-\frac{a}{2}}^{x=\frac{a}{2}}\Big|_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \\ &= \rho\left(\frac{a^3 b}{12} + \frac{ab^3}{12}\right) \\ &= \frac{m}{12}(a^2+b^2). \end{aligned}$$

Note that $\int r^2 dm = \int x^2 dm + \int y^2 dm$ for all planar objects (the *perpendicular axis theorem*). For a uniform rectangle, $\int y^2 dm = \rho \int y^2 dA$. But the integral $y^2 dA$ is just the term often used for I, the area moment of inertia, in strength of materials calculations for the stresses and stiffnesses of beams in bending. You may recall that $\int y^2 dA = \frac{ab^3}{12} = \frac{Ab^2}{12}$ for a rectangle. Similarly, $\int x^2 dA = \frac{Aa^2}{12}$. So, the polar moment of inertia $J = I^o_{zz} = m\frac{1}{12}(a^2+b^2)$ can be recalled by remembering the area moment of inertia of a rectangle combined with the perpendicular axis theorem.

The moment of inertias in 3-D: $[I^{\rm cm}]$ and $[I^{\rm O}]$

For the study of three-dimensional mechanics, including the simple case of constant rate rotation about a fixed axis, one often makes use of the moment of inertia matrix, defined below and motivated by the box 8.3 on page 462.

The distances x, y, z in the formulas below are the x, y, z components of the position of mass relative to a coordinate system which has either the center of mass (cm) or the point O as its origin. ①

$$[\boldsymbol{I}] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \tag{8.35}$$

$$= \begin{bmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)\,dm \end{bmatrix} \tag{8.36}$$

If all mass is on the xy plane then it is clear that $I_{xz} = I_{yz} = 0$ since $z = 0$ for the whole xy plane. If rotation is also about the z-axis then $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$.

Applying the formulas above we find that, if all of the mass is in the xy-plane and rotation is about the z-axis, the only relevant non-zero term in $[\boldsymbol{I}]$ is $I_{zz} = \int (x^2+y^2)\,dm$. And, I_{xx}, I_{yy}, and I_{xy} don't contribute to $\vec{\boldsymbol{H}}$, $\dot{\vec{\boldsymbol{H}}}$, or $E_{\rm K}$. In this manner you can check that the three dimensional equations, when applied to two-dimensional bodies, give the same results that we found directly for two-dimensional bodies.

① **Caution:** While we have $I_{xy} = -\int xy\,dm$, some old books define $I_{xy} = \int xy\,dm$. They then have minus signs in front of the off-diagonal terms in the moment of inertia matrix. They would say $I_{12} = -I_{xy}$. The numerical values in the matrix they write is the same as in the one we write. They just have a different sign convention in the definition of the components.

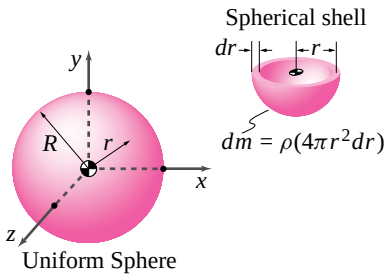


Figure 8.33: (Filename:tfigure4.3Dsphere)

Example: **Moment of inertia matrix for a uniform sphere**

A sphere is a special shape which is, naturally enough, spherically symmetric. Therefore,

$$I_{xx}^{cm} = I_{yy}^{cm} = I_{zz}^{cm}$$

and

$$I_{xy}^{cm} = I_{xz}^{cm} = I_{yz}^{cm} = 0.$$

So, all we need is I_{xx}^{cm} or I_{yy}^{cm} or I_{zz}^{cm}. Here is the trick:

$$\begin{aligned} I_{xx}^{cm} &= \frac{1}{3}(I_{xx}^{cm} + I_{yy}^{cm} + I_{zz}^{cm}) \\ &= \frac{1}{3}\left[\int (y^2+z^2)dm + \int (x^2+z^2)dm + \int (x^2+y^2)dm\right] \\ &= \frac{2}{3}\int (x^2+y^2+z^2)dm \\ &= \frac{2}{3}\int r^2\,dm \\ &= \frac{2}{3}\int_0^R r^2(4\rho\pi r^2\,dr) \\ &= \frac{8}{3}\rho\pi\int_0^R r^4\,dr \\ &= \frac{8}{15}\rho\pi R^5 \\ &= \frac{2}{5}mR^2 \qquad (m = \frac{4}{3}\rho\pi R^3). \end{aligned}$$

So,

$$[\boldsymbol{I}^{\rm cm}] = \frac{2}{5}mR^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

The parallel axis theorem for rigid bodies in three dimensions

The 3-D parallel axis theorem is stated below and in the table on the inside back cover. It is derived in box 8.4 on page 464. The parallel axis theorem for rigid bodies in three dimensions is the equation

$$[\boldsymbol{I}^{\mathrm{O}}] = [\boldsymbol{I}^{\mathrm{cm}}] + m \begin{bmatrix} y_{cm/o}^2 + z_{cm/o}^2 & -x_{cm/o}y_{cm/o} & -x_{cm/o}z_{cm/o} \\ -x_{cm/o}y_{cm/o} & x_{cm/o}^2 + z_{cm/o}^2 & -y_{cm/o}z_{cm/o} \\ -x_{cm/o}z_{cm/o} & -y_{cm/o}z_{cm/o} & x_{cm/o}^2 + y_{cm/o}^2 \end{bmatrix} \tag{8.37}$$

In this equation, $x_{cm/o}$, $y_{cm/o}$, and $z_{cm/o}$ are the x, y, and z coordinates, respectively, of the center of mass defined with respect to a coordinate system whose origin is located at some point O not at the center of mass *cm*. That is, if you know $[\boldsymbol{I}^{\mathrm{cm}}]$, you can find $[\boldsymbol{I}^{\mathrm{O}}]$ without doing any more integrals or sums. Like the 2-D parallel axis theorem. The primary utility of the 3-D parallel axis theorem is for the determination of [**I**] for an object that is a composite of simpler objects. Such are not beyond the scope of this book in principle. But in fact, given the finite time available for calculation, we do not leave much time for practice of this tedious but routine calculation.

Moment of inertia matrix and linear algebra

If you have taken a class in linear algebra you know that certain properties of matrices are important.

Eigenvectors of $[\boldsymbol{I}]$

First off, the moment of inertia matrix is always a symmetric matrix. This symmetry means that $[\boldsymbol{I}]$ always has a set of three mutually orthogonal eigenvectors.

Sometimes a pair of the eigenvalues are equal to each other implying that any vector in the plane of the corresponding eigenvectors is also an eigenvector.

The eigenvectors of the moment of inertia matrix of a given object have special meaning that is important for engineering. This will be discussed in section 8.5 on dynamic balance.

If the physical object has any natural symmetry directions these directions will usually manifest themselves in the dynamics of the body as being in the directions of the eigenvectors of object's moment of inertia matrix. For example, the dotted lines on figure 2.58 are all in directions of eigenvectors for the objects shown. But even if an object is wildly asymmetric in shape, its moment of inertia matrix is always symmetric and thus all objects have moment of inertia matrices with at least three different eigenvectors at least three of which are mutually orthogonal.

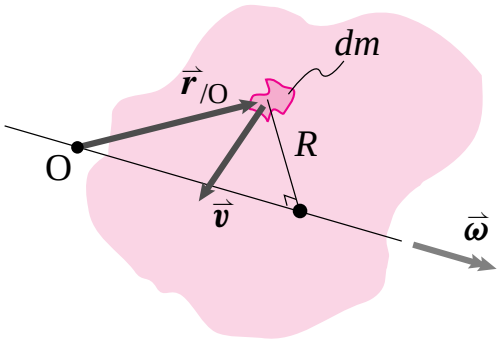
Properties of $[\boldsymbol{I}]$

For those with experience with linear algebra various properties of the moment of inertia matrix $[\boldsymbol{I}]$ are worth noting (although not worth proving here). Unless all mass is distributed on one straight line, the moment of inertia matrix is invertible (it is non-singular and has rank 3). Further, when invertible it is positive definite. In the special case that all the mass is on some straight line, the moment of inertia matrix is non-invertible and only positive semi-definite. The positive (semi) definiteness of the moment of inertia matrix is equivalent to the statement that the rotational kinetic energy of a body is always equal to or greater than zero. Finally, the eigenvalues of the moment of inertia matrix are all positive and have the property that no one can be greater than the sum of the other two.

8.3 Discovering the moment of inertia matrix

Derivation 1

Here we present a direct derivation of the moment of inertia matrix; that is, a derivation in which the moment of inertia matrix arises as a convenient short hand. Assume a rigid body is moving in such a way that point O is fixed (i.e., It is either on the line of a hinge or a ball-and-socket joint).



The most basic kinematic relation for a rigid body is that

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where $\vec{r}_{/O} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is the position of a point on the body relative to O and $\vec{\omega}$, the angular velocity of the body.

Now, we tediously calculate and arrange the terms in the angular momentum about point O,

$$\begin{aligned}
\vec{H}_O &= \int \vec{r}_{/O} \times \vec{v}\, dm \\
&= \int \vec{r}_{/O} \times (\vec{\omega} \times \vec{r}_{/O})\, dm \\
&= \int (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \times \\
&\quad \Big[(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}) \times (x\hat{\imath} + y\hat{\jmath} + z\hat{k})\Big]\, dm \\
&= \int (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \times \\
&\quad \Big[(\omega_y z - \omega_z y)\hat{\imath} + (\omega_z x - \omega_x z)\hat{\jmath} + (\omega_x y - \omega_y x)\hat{k}\Big]\, dm \\
&= \int \Big[\big(y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z)\big)\,\hat{\imath} \\
&\qquad + \big(z(\omega_y z - \omega_z y) - x(\omega_x y - \omega_y x)\big)\,\hat{\jmath} \\
&\qquad + \big(x(\omega_z x - \omega_x z) - y(\omega_y z - \omega_z y)\big)\,\hat{k}\Big]\, dm \\
&= \int \Big[\big((y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z\big)\,\hat{\imath} \\
&\qquad + \big(-yx\omega_x + (z^2 + x^2)\omega_y - yz\omega_z\big)\,\hat{\jmath} \\
&\qquad + \big(-zx\omega_x - xy\omega_y + (x^2 + y^2)\omega_z\big)\,\hat{k}\Big]\, dm
\end{aligned}$$

Since the integral is over the mass and $\vec{\omega}$ is constant over the body, we can pull $\vec{\omega}$ out of the integral so that we may write the equation in matrix form. Writing $\vec{H}_O$ as a column vector, we can rewrite the last equation as

$$\begin{bmatrix} H_{O_x} \\ H_{O_y} \\ H_{O_z} \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} \int (y^2 + z^2)\, dm & -\int xy\, dm & -\int xz\, dm \\ -\int xy\, dm & \int (x^2 + z^2)\, dm & -\int yz\, dm \\ -\int xz\, dm & -\int yz\, dm & \int (x^2 + y^2)\, dm \end{bmatrix}}_{[I^O]} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

Finally, defining $[I^O]$ by the matrix above, we can compactly write

$$\vec{H}_O = [I^O] \cdot \vec{\omega}$$

assuming O is a fixed point on the body where we represent $\vec{H}_O$ and $\vec{\omega}$ in terms of x, y, and z components.

Center-of-mass inertia matrix

For any system moving, distorting, and rotating any crazy way, we have the general result that

Contribution of the system to $\vec{H}_O$ if treated as a particle at the system center of mass

$$\vec{H}_O = \overbrace{\vec{r}_{cm/O} \times m_{tot}\vec{v}_{cm}} + \overbrace{\int (\vec{r}_{/cm} \times \vec{v}_{/cm})dm}^{\vec{H}_{cm}}$$

$$\vec{v}_{/cm} = (\vec{v} - \vec{v}_{cm})$$

as you can verify by substituting $\vec{v} = \vec{v}_{cm} + \vec{v}_{/cm}$ and $\vec{r} = \vec{r}_{cm} + \vec{r}_{/cm}$ into the general definition of $\vec{H}_O = \int \vec{r}_{/O} \times \vec{v}\, dm$. For a rigid body, we have

$$\vec{v}_{/cm} = \vec{\omega} \times \vec{r}_{/cm}.$$

So, by a derivation essentially identical to that for $\vec{H}_O$, we get

$$\vec{H}_{cm} = [I^{cm}] \cdot \vec{\omega}$$

with $[I^{cm}]$ being defined using x, y, and z, as the distances from the center of mass rather than from point O. So, for a rigid body in general motion, we can find the angular momentum by

$$\vec{H}_O = \vec{r}_{cm/O} \times \vec{v}_{cm} m_{tot} + [I^{cm}] \cdot \vec{\omega} \qquad (8.38)$$

Comment (aside)

In the special case that the body is rotating about point O, we also have

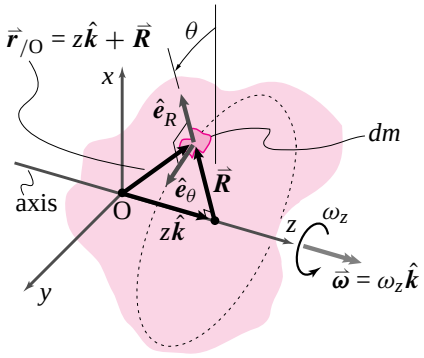
$$\vec{H}_O = [I^O] \cdot \vec{\omega}. \qquad (8.39)$$

You will see, if you look at the parallel axis theorem, that these two expressions 8.38 and 8.39 do in fact agree.

Derivation 2

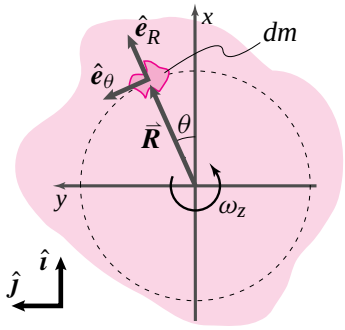
Here, we present a less direct but perhaps more intuitive derivation of the moment of inertia matrix. We start with the special case of a 3-D rigid body spinning in circles at constant rate about a fixed axis.

To see from where the moment of inertia matrix comes, we will first calculate the angular momentum about point O of a general 3-D rigid body spinning about the z-axis with constant rate $\dot\theta \equiv const. = \omega_z$ or $\vec{\boldsymbol{\omega}} = \omega_z\hat{\boldsymbol{k}}$. We will refer to this case as (1).



Starting with the definition of angular momentum, we get

$$\begin{aligned}\vec{\boldsymbol{H}}_O &= \int \vec{\boldsymbol{r}}_{/O}\times\vec{\boldsymbol{v}}\,dm\\ &= \int (z\hat{\boldsymbol{k}}+\vec{\boldsymbol{R}})\times(\dot\theta R\hat{\boldsymbol{e}}_\theta)dm.\end{aligned}$$



Looking down the z-axis in the figure, we see that

$$\begin{aligned}\vec{\boldsymbol{R}} &= R\hat{\boldsymbol{e}}_R = \overbrace{R\cos\theta}^{x}\hat{\boldsymbol{i}} + \overbrace{R\sin\theta}^{y}\hat{\boldsymbol{j}}, \quad \text{or}\\ \vec{\boldsymbol{R}} &= x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}.\end{aligned}$$

where $R = \sqrt{x^2+y^2}$. To compute the cross product in the integrand, we need

$$\begin{aligned}\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}_\theta &= \hat{\boldsymbol{k}}\times(-\sin\theta\hat{\boldsymbol{i}}+\cos\theta\hat{\boldsymbol{j}}) = -\hat{\boldsymbol{e}}_R \quad \text{and}\\ \vec{\boldsymbol{R}}\times\hat{\boldsymbol{e}}_\theta &= R\hat{\boldsymbol{e}}_R\times\hat{\boldsymbol{e}}_\theta = R\hat{\boldsymbol{k}}.\end{aligned}$$

Therefore, we now have

$$\vec{\boldsymbol{H}}_O = \int -z\dot\theta R\hat{\boldsymbol{e}}_R dm + \int \dot\theta R^2\hat{\boldsymbol{k}}\,dm$$

$$= \dot\theta\left[-\int z(\underbrace{R\cos\theta}_{x}\hat{\boldsymbol{i}} + \underbrace{R\sin\theta}_{y}\hat{\boldsymbol{j}})dm + \int (x^2+y^2)\hat{\boldsymbol{k}}\,dm\right]$$

$$= \dot\theta\left[\left(-\int zx\,dm\right)\hat{\boldsymbol{i}} + \left(-\int zy\,dm\right)\hat{\boldsymbol{j}} + \left(\int (x^2+y^2)dm\right)\hat{\boldsymbol{k}}\right].$$

To 'un-clutter' this expression, let's define the following:

$$\begin{aligned}I^O_{xz} &= -\int xz\,dm\\ I^O_{yz} &= -\int yz\,dm\\ I^O_{zz} &= -\int (x^2+y^2)\,dm\end{aligned}$$

So, now, we have for case (1)

$$(\vec{\boldsymbol{H}}_O)_1 = I^O_{xz}\omega_z\hat{\boldsymbol{i}} + I^O_{yz}\omega_z\hat{\boldsymbol{j}} + I^O_{zz}\omega_z\hat{\boldsymbol{k}}.$$

The substitutions we have defined form the elements of the third column of the inertia matrix, as we will see below in the general case. Let's now move on to general 3-D rigid body motion and infer the first and second columns of the inertia matrix.

In general, the angular velocity of a rigid body is given by

$$\vec{\boldsymbol{\omega}} = \omega_x\hat{\boldsymbol{i}} + \omega_y\hat{\boldsymbol{j}} + \omega_z\hat{\boldsymbol{k}}$$

So far, we have considered the special case above, $\vec{\boldsymbol{\omega}} = \omega_z\hat{\boldsymbol{k}}$. But, we could have looked at $\vec{\boldsymbol{\omega}} = \omega_x\hat{\boldsymbol{i}}$, case (2), and, similarly, would obtain instead the following angular momentum about point O

$$(\vec{\boldsymbol{H}}_O)_2 = I^O_{xx}\omega_x\hat{\boldsymbol{i}} + I^O_{yx}\omega_x\hat{\boldsymbol{j}} + I^O_{zx}\omega_x\hat{\boldsymbol{k}}.$$

Likewise, for $\vec{\boldsymbol{\omega}} = \omega_y\hat{\boldsymbol{j}}$, case (3), we would obtain

$$(\vec{\boldsymbol{H}}_O)_3 = I^O_{xy}\omega_y\hat{\boldsymbol{i}} + I^O_{yy}\omega_x\hat{\boldsymbol{j}} + I^O_{zy}\omega_x\hat{\boldsymbol{k}}.$$

Finally, for $\vec{\boldsymbol{\omega}} = \omega_x\hat{\boldsymbol{i}} + \omega_y\hat{\boldsymbol{j}} + \omega_z\hat{\boldsymbol{k}}$, we obtain

$$\begin{aligned}\vec{\boldsymbol{H}}_O &= (\vec{\boldsymbol{H}}_O)_1 + (\vec{\boldsymbol{H}}_O)_2 + (\vec{\boldsymbol{H}}_O)_3\\ &= I^O_{xz}\omega_z\hat{\boldsymbol{i}} + I^O_{yz}\omega_z\hat{\boldsymbol{j}} + I^O_{xz}\omega_z\hat{\boldsymbol{k}}\\ &\quad + I^O_{xx}\omega_x\hat{\boldsymbol{i}} + I^O_{yx}\omega_x\hat{\boldsymbol{j}} + I^O_{zx}\omega_x\hat{\boldsymbol{k}}\\ &\quad + I^O_{xy}\omega_y\hat{\boldsymbol{i}} + I^O_{yy}\omega_x\hat{\boldsymbol{j}} + I^O_{zy}\omega_x\hat{\boldsymbol{k}}.\end{aligned}$$

Collecting components, we get

$$\vec{\boldsymbol{H}}_O = H_{O_x}\hat{\boldsymbol{i}} + H_{O_y}\hat{\boldsymbol{j}} + H_{O_z}\hat{\boldsymbol{k}}$$

where

$$\begin{aligned}H_{O_x} &= I^O_{xx}\omega_x + I^O_{xy}\omega_y + I^O_{xz}\omega_z\\ H_{O_y} &= I^O_{yx}\omega_x + I^O_{yy}\omega_y + I^O_{yz}\omega_z\\ H_{O_z} &= I^O_{zx}\omega_x + I^O_{zy}\omega_y + I^O_{zz}\omega_z.\end{aligned}$$

We can combine the above results into a matrix representation. Representing $\vec{\boldsymbol{H}}_O$ as a column vector, we can re-write the above set of three equations as a product of a matrix and the angular velocity written as a column vector.

$$\begin{bmatrix}H_{O_x}\\ H_{O_y}\\ H_{O_z}\end{bmatrix} = \begin{bmatrix}I^O_{xx} & I^O_{xy} & I^O_{xz}\\ I^O_{yx} & I^O_{yy} & I^O_{yz}\\ I^O_{zx} & I^O_{zy} & I^O_{zz}\end{bmatrix}\cdot\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$

We define the coefficient matrix above to be the *moment of inertia matrix* about point O

$$[\boldsymbol{I}^{\rm O}] = \begin{bmatrix}I^O_{xx} & I^O_{xy} & I^O_{xz}\\ I^O_{yx} & I^O_{yy} & I^O_{yz}\\ I^O_{zx} & I^O_{zy} & I^O_{zz}\end{bmatrix}.$$

whose components are

$$\begin{array}{lll}I^O_{xx} = \int(y^2+z^2)dm & I^O_{xy} = -\int xy\,dm & I^O_{xz} = -\int xz\,dm\\ I^O_{yx} = -\int yx\,dm & I^O_{yy} = \int(x^2+z^2)dm & I^O_{yz} = -\int yz\,dm\\ I^O_{zx} = -\int zx\,dm & I^O_{zy} = -\int zy\,dm & I^O_{zz} = \int(x^2+y^2)dm.\end{array}$$

By inspection, one can see that $I^O_{xy} = I^O_{yx}$, $I^O_{xz} = I^O_{zx}$, and $I^O_{yz} = I^O_{zy}$. Thus, the inertia matrix is symmetric; i.e., $[\boldsymbol{I}^{\rm O}]^T = [\boldsymbol{I}^{\rm O}]$. So, there are always at most only six, not nine, independent components in the inertia matrix to compute.

8.4 *THEORY*
3-D parallel axis theorem

In three dimensions, the two matrices $[\boldsymbol{I}^{\rm O}]$ and $[\boldsymbol{I}^{\rm cm}]$ are related to each other in a way similar to the two-dimensional case. Since the inertia matrix has six independent entries in it, the derivation involves six integrals. Let's look at a typical term on the diagonal, say, I^O_{zz} and a typical off-diagonal term, say, I^O_{xy}. The calculation for the other terms is similar with a simple change of letters in the subscript notation.

First, $I^O_{zz} = \int (x^2_{/O} + y^2_{/O})dm = I^{cm}_{zz} + m(x^2_{cm/O} + y^2_{cm/O})$ by exactly the same reasoning used to derive the 2-D parallel axis theorem. We *cannot* do the last line in that derivation, however, since $r^2_{cm/O} = x^2_{cm/O} + y^2_{cm/O} + z^2_{cm/O} \neq x^2_{cm/O} + y^2_{cm/O}$, because now, for three-dimensional objects, $z_{cm/O} \neq 0$.

Now, let's look at an off-diagonal term.

$$\begin{aligned} I^O_{xy} &= -\int x_{/O} y_{/O} dm \\ &= -\int \overbrace{(x_{cm/O} + x_{/cm})}^{x_{/O}} \overbrace{(y_{cm/O} + y_{/cm})}^{y_{/O}} dm \\ &= -x_{cm/O} y_{cm/O} \underbrace{\int dm}_{m} - y_{cm/O} \underbrace{\int x_{/cm} dm}_{0} \\ &\quad - x_{cm/O} \underbrace{\int y_{/cm} dm}_{0} - \underbrace{\int x_{/cm} y_{/cm} dm}_{-I^{cm}_{xy}} \\ &= -m x_{cm/O} y_{cm/O} + I^{cm}_{xy} \end{aligned}$$

Similarly, we can calculate the other terms to get the whole 3-D parallel axis theorem.

$$[\boldsymbol{I}^{\rm O}] = [\boldsymbol{I}^{\rm cm}] + m \begin{bmatrix} y^2_{cm/o} + z^2_{cm/o} & -x_{cm/o} y_{cm/o} & -x_{cm/o} z_{cm/o} \\ -x_{cm/o} y_{cm/o} & x^2_{cm/o} + z^2_{cm/o} & -y_{cm/o} z_{cm/o} \\ -x_{cm/o} z_{cm/o} & -y_{cm/o} z_{cm/o} & x^2_{cm/o} + y^2_{cm/o} \end{bmatrix}.$$

Again, one can think of this result as follows. The moment of inertia matrix about point O is the same as that for parallel axes through the center of mass *plus* the moment of inertia matrix for a point mass at the center of mass.

Relation between 2-D and 3-D parallel axis theorems

The (3,3) (lower right corner) element in the matrix of the 3-D parallel axis theorem is the 2-D parallel axis theorem.

SAMPLE 8.9 For the dumbbell shown in Figure 8.34, take $m = 0.5\,\mathrm{kg}$ and $\ell = 0.4\,\mathrm{m}$. Given that at the instant shown $\theta = 30^{\mathrm{o}}$ and the dumbbell is in the yz-plane, find the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$, where O is the midpoint of the dumbbell.

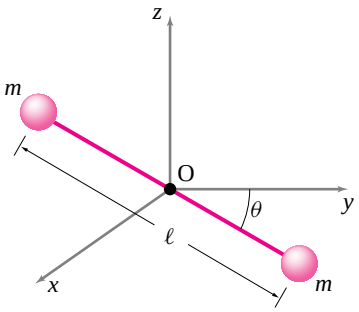


Figure 8.34: (Filename:sfig4.6.4)

Solution The dumbbell is made up of two point masses. Therefore we can calculate $[\boldsymbol{I}^{\mathrm{O}}]$ for each mass using the formula from the table on the inside back cover of the text and then adding the two matrices to get $[\boldsymbol{I}^{\mathrm{O}}]$ for the dumbbell.
Now, from Table 4.9 of the text,

$$[\boldsymbol{I}^{\mathrm{O}}] = m\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

For mass 1 (shown in Figure 8.35)

$$x = 0, \quad y = -\frac{\ell}{2}\cos\theta, \quad z = \frac{\ell}{2}\sin\theta.$$

Therefore,

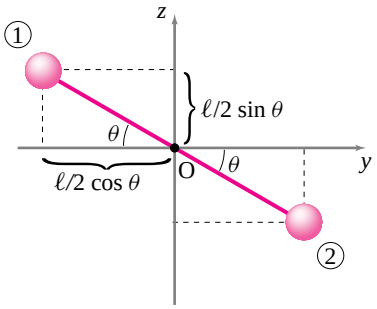


Figure 8.35: (Filename:sfig4.6.4a)

$$\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}]_{\mathrm{mass1}} &= m\begin{bmatrix} \frac{\ell^2}{4} & 0 & 0 \\ 0 & \frac{\ell^2}{4}\sin^2\theta & \frac{\ell^2}{4}\cos\theta\sin\theta \\ 0 & \frac{\ell^2}{4}\cos\theta\sin\theta & \frac{\ell^2}{4}\cos^2\theta \end{bmatrix} \\
&= 0.5\,\mathrm{kg}\begin{bmatrix} 0.04\,\mathrm{m}^2 & 0 & 0 \\ 0 & 0.04\,\mathrm{m}^2\cdot\frac{1}{4} & 0.04\,\mathrm{m}^2\cdot\frac{\sqrt{3}}{4} \\ 0 & 0.04\,\mathrm{m}^2\cdot\frac{\sqrt{3}}{4} & 0.04\,\mathrm{m}^2\cdot\frac{3}{4} \end{bmatrix} \\
&= 0.02\,\mathrm{kg}\cdot\mathrm{m}^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}
\end{aligned}$$

Similarly for mass 2,

$$x = 0, \quad y = \frac{\ell}{2}\cos\theta, \quad z = -\frac{\ell}{2}\sin\theta$$

$$\Rightarrow \quad [\boldsymbol{I}^{\mathrm{O}}]_{\mathrm{mass2}} = 0.02\,\mathrm{kg}\cdot\mathrm{m}^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}] &= [\boldsymbol{I}^{\mathrm{O}}]_{\mathrm{mass1}} + [\boldsymbol{I}^{\mathrm{O}}]_{\mathrm{mass2}} \\
&= 0.04\,\mathrm{kg}\cdot\mathrm{m}^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}.
\end{aligned}$$

◁

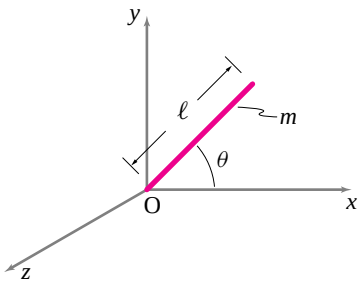


Figure 8.36: (Filename:sfig4.6.3)

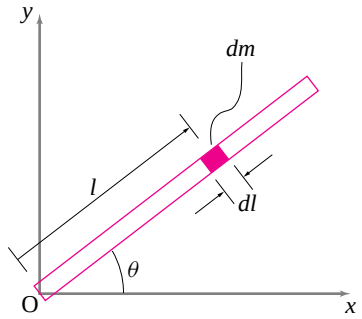


Figure 8.37: (Filename:sfig4.6.3a)

SAMPLE 8.10 A uniform rod of mass $m = 2\,\text{kg}$ and length $\ell = \frac{1}{2}\,\text{m}$ is pivoted at one of its ends. At the instant shown, the rod is in the xy-plane and makes an angle $\theta = 45^\text{o}$ with the x-axis. Find the moment of inertia matrix $[\boldsymbol{I}^\text{O}]$ for the rod.

Solution The moment of inertia matrix $[\boldsymbol{I}^\text{O}]$ for a continuous system is given by

$$[\boldsymbol{I}^\text{O}] = \int_{\text{over all mass}} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm.$$

Thus we need to carry out the integrals for the rod to find each component of the inertia matrix $[\boldsymbol{I}^\text{O}]$. Let us consider an infinitesimal length element dl of the rod at distance l from O (see Fig 8.37). The mass of this element is $dm = \underbrace{\frac{m}{\ell}}_{\text{mass/length}} \cdot dl$,

where m is the total mass.
The coordinates of this element are

$$x = l\cos\theta, \quad y = l\sin\theta, \quad z = 0 \quad \text{(since the rod is in the } xy\text{-plane)}.$$

Therefore,

$$\begin{aligned} I_{xx} &= \int_m (y^2 + \underbrace{z^2}_{0})dm = \int_0^\ell \underbrace{l^2\sin^2\theta}_{y^2} \cdot \underbrace{\frac{m}{\ell}dl}_{dm} \\ &= \frac{m}{\ell}\sin^2\theta \int_0^\ell l^2 dl = \frac{m}{\ell}\sin^2\theta \cdot \frac{\ell^3}{3} = \frac{m\ell^2}{3}\sin^2\theta. \end{aligned}$$

Similarly,

$$\begin{aligned} I_{yy} &= \int_m (x^2 + \underbrace{z^2}_{0})dm = \int_0^\ell l^2\cos^2\theta\frac{m}{\ell}dl = \frac{m\ell^2}{3}\cos^2\theta. \\ I_{zz} &= \int_m (x^2+y^2)dm = \int_0^\ell l^2\frac{m}{\ell}dl = \frac{m\ell^2}{3}. \\ I_{xy} &= -\int_m xy\,dm = -\int_0^\ell l^2\cos\theta\sin\theta\cdot\frac{m}{\ell}dl = -\frac{m\ell^2}{3}\cos\theta\sin\theta. \\ I_{xz} &= -\int_m x\underbrace{z}_{0}\,dm = 0. \quad I_{yz} = -\int_m y\underbrace{z}_{0}\,dm = 0. \end{aligned}$$

Thus,

$$[\boldsymbol{I}^\text{O}] = \frac{m\ell^2}{3}\begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting the values of m, ℓ and θ, we get

$$[\boldsymbol{I}^\text{O}] = 0.083\,\text{kg}\cdot\text{m}^2\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

◁

8.4 Mechanics using $[_I{}^{\rm cm}]$ and $[_I{}^{\rm O}]$

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from the inside cover. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. Linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center of mass of the system of interest. One would like something so simple for the calculation of angular momentum.

We are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies. The luck is not so great for 3-D rigid bodies but still there is some simplification①. The simplification is to use the moment of inertia for the bodies rather than evaluating the momenta and energy quantities as integrals and sums. Of course one may have to do a sum or integral to find $I \equiv I_{zz}^{cm}$ or $[\boldsymbol{I}^{\rm cm}]$ if a table is not handy. But once you know $[\boldsymbol{I}^{\rm cm}]$, say, you need not work with the integrals to evaluate angular momentum and its rate of change. Assuming that you are comfortable calculating and looking-up moments of inertia, we proceed to use it for the purposes of studying mechanics.

① For general motion of *non*-rigid bodies, a topic not covered in this book, the simplification for linear momentum still holds; linear momentum and its rate of change are given by the system mass and motion of the center of mass. But there is no general simplification for the sums in the evaluation of angular momentum and energy or their rates of change.

Lets now consider again the conically swinging rod of Fig. 8.19 on page 444. Method 1 for evaluating $\dot{\vec{\boldsymbol{H}}}_{\rm O}$ was evaluating the sums directly. If we accept the formulae presented for rigid bodies in Table I at the back of the book, we can find all of the motion quantities by setting $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{0}}$.

Method 2 of evaluating $\dot{\vec{\boldsymbol{H}}}_{\rm O}$: using the xyz moment of inertia matrix about point O

In section 8.1 we examined the conical swinging of a straight uniform rod. Now let's look at that example again using its moment of inertia matrix. For a rigid body in constant rate circular motion about an axis through O,

$$\dot{\vec{\boldsymbol{H}}}_{\rm O} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\rm O}$$

because the $\vec{\boldsymbol{H}}_{\rm O}$ vector rotates with the body. For a rigid body rotating about point O,

$$\vec{\boldsymbol{H}}_{\rm O} = [\boldsymbol{I}^{\rm O}] \cdot \vec{\boldsymbol{\omega}}.$$

We assumed at the outset that

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}.$$

So, the only trick is to find $[\boldsymbol{I}^{\rm O}]$ for a rod in the configuration shown. We recall, look up, or believe for now that

$$[\boldsymbol{I}^{\rm O}] = \begin{bmatrix} I_{xx}^O & I_{xy}^O & I_{xz}^O \\ I_{yx}^O & I_{yy}^O & I_{yz}^O \\ I_{zx}^O & I_{zy}^O & I_{zz}^O \end{bmatrix}.$$

$$I_{xy}^O = -\int \underbrace{x}_{0}\, y\, dm = 0 \qquad (x = 0, \text{ all mass in the } yz \text{ plane})$$

$$I_{xz}^O = -\int \underbrace{x}_{0}\, z\, dm = 0 \qquad (x = 0, \text{ all mass in the } yz \text{ plane})$$

$$
\begin{aligned}
I_{yy}^O &= \int (\underbrace{x^2}_{0} + z^2)dm \qquad (x = 0, \text{ all mass in the } yz \text{ plane}) \\
&= \int_0^\ell \underbrace{(-s \cdot \cos\phi)^2}_{z} \rho \, ds \\
&= \cos^2\phi \, \rho \frac{\ell^3}{3} = \cos^2\phi \, m \frac{\ell^2}{3} \\
I_{zz}^O &= \int (\underbrace{x^2}_{0} + y^2)dm \qquad (x = 0, \text{ all mass in the } yz \text{ plane}) \\
&= \int \underbrace{(s \cdot \sin\phi)^2}_{y} \rho \, ds \\
&= \sin^2\phi \, \rho \frac{\ell^3}{3} = \sin^2\phi \, m \frac{\ell^2}{3} \\
I_{xx}^O &= \int (y^2 + z^2)dm \qquad \text{(both integrals have been evaluated above)} \\
&= I_{yy}^O + I_{zz}^O \qquad \text{(perpendicular axis theorem)} \\
&= (\cos^2\phi + \sin^2\phi) m \frac{\ell^2}{3} \\
&= m \frac{\ell^2}{3} \\
I_{yz}^O &= -\int yz \, dm \\
&= -\int (s \cdot \sin\phi)(-s \cdot \cos\phi)^2 \, \rho \, ds \\
&= \sin\phi \cos\phi \cdot \rho \frac{\ell^3}{3} = \sin\phi \cos\phi \cdot m \frac{\ell^2}{3}.
\end{aligned}
$$

Putting these terms all together in the matrix, we get

$$
[\boldsymbol{I}^{\mathrm{O}}] = m \frac{\ell^2}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi & \cos\phi \sin\phi \\ 0 & \cos\phi \sin\phi & \sin^2\phi \end{bmatrix}.
$$

Now we can calculate $\vec{\boldsymbol{H}}_{\mathrm{O}}$ as

$$
\begin{aligned}
\vec{\boldsymbol{H}}_{\mathrm{O}} &= [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}} \\
&= m \frac{\ell^2}{3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi & \cos\phi \sin\phi \\ 0 & \cos\phi \sin\phi & \sin^2\phi \end{bmatrix}}_{[\boldsymbol{I}^{\mathrm{O}}]} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}}_{\vec{\boldsymbol{\omega}}} \\
&= m \frac{\ell^2}{3} \begin{bmatrix} 0 \\ \omega \cos\phi \sin\phi \\ \omega \sin^2\phi \end{bmatrix} \\
&= m\omega \frac{\ell^2}{3} \sin\phi (\cos\phi \, \hat{\boldsymbol{j}} + \sin\phi \hat{\boldsymbol{k}}).
\end{aligned}
$$

You may notice, by the way, that for this problem, $\vec{\boldsymbol{H}}_{\mathrm{O}}$ (in the direction of $\cos\phi \hat{\boldsymbol{j}} + \sin\phi \hat{\boldsymbol{k}}$) is perpendicular to the rod (in the direction of $\sin\phi \hat{\boldsymbol{j}} - \cos\phi \hat{\boldsymbol{k}}$). So $\vec{\boldsymbol{H}}_{\mathrm{O}}$ is *not* in the direction of $\vec{\boldsymbol{\omega}}$. ①

① Note: $\vec{\boldsymbol{H}}_{\mathrm{O}}$ is only parallel to $\vec{\boldsymbol{\omega}}$ if rotation is about one of the principal axes of the inertia matrix. For a symmetric body a principal axis corresponds with a symmetry direction for the body.

Now we calculate $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ as

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{O}} \\ &= (\omega\hat{\boldsymbol{k}}) \times \left[\omega m \frac{\ell^2}{3}(\cos\phi \sin\phi \hat{\boldsymbol{j}} + \sin^2\phi \hat{\boldsymbol{k}})\right] \\ &= -m\frac{\ell^2}{3}\cos\phi \sin\phi \omega^2 \hat{\boldsymbol{i}},\end{aligned}$$

the same result we got before.

Method 3 of calculating $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$: using $[\boldsymbol{I}^{\mathrm{O}}]$ and a coordinate system lined up with the rod

Here, as suggested above, we redo the problem using a rotated set of axes better aligned with the rod. This method makes calculation of the moment of inertia matrix quite a bit easier — we can even look it up in a table — but makes the determination of $\vec{\boldsymbol{\omega}}$ a little harder. We are stuck finding the components of $\vec{\boldsymbol{\omega}}$ in a rotated coordinate system. Referring to the table of moment of inertias on the inside of the back cover and taking care because different coordinates are used, the moment of inertia matrix about point O for a thin uniform rod, in terms of the rotated coordinates is

$$[\boldsymbol{I}^{\mathrm{O}}]_{x'y'z'} = m\frac{\ell^2}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

and the angular velocity in rotated coordinates is $\vec{\boldsymbol{\omega}} = -\omega\cos\phi\,\hat{\boldsymbol{j}}' + \omega\sin\phi\,\hat{\boldsymbol{k}}'$, which can be written in component form as

$$[\vec{\boldsymbol{\omega}}]_{x'y'z'} = \omega\begin{bmatrix}0\\ -\cos\phi\\ \sin\phi\end{bmatrix}.$$

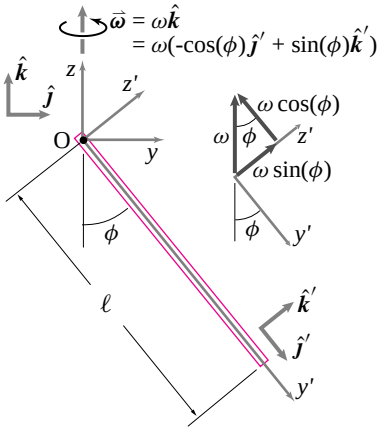


Figure 8.38: The spherical pendulum using $x'y'z'$ axes aligned with the rod.

(Filename:tfigure4.spherical.rotaxis)

We calculate the angular momentum about point O as

vector equation, independent of coordinates

$$\overbrace{\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}}}$$

$$\underbrace{\begin{bmatrix}H_{/O_{x'}}\\ H_{/O_{y'}}\\ H_{/O_{z'}}\end{bmatrix} = m\frac{\ell^2}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}_{x'y'z'}\cdot\omega\begin{bmatrix}0\\ -\cos\phi\\ \sin\phi\end{bmatrix}_{x'y'z'}}$$

all components in $x'y'z'$ coordinate system

$$\begin{bmatrix}H_{/O_{x'}}\\ H_{/O_{y'}}\\ H_{/O_{z'}}\end{bmatrix} = m\frac{\ell^2}{3}\omega\begin{bmatrix}0\\ 0\\ \sin\phi\end{bmatrix}_{x'y'z'}.$$

Finally, we calculate $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ as

$$\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{O}}$$

$$= [\omega(-\cos\phi\,\hat{\boldsymbol{j}}' + \sin\phi\hat{\boldsymbol{k}}'] \times (\omega m\frac{\ell^2}{3}\sin\phi\hat{\boldsymbol{k}}')$$
$$= -\sin\phi\cos\phi\; m\omega^2\frac{\ell^2}{3}\hat{\boldsymbol{\imath}}'.$$

This answer is again the same as what we got before because the x and x' axis are coincident so $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$.

The multitude of ways to calculate $\dot{\vec{\boldsymbol{H}}}_{\rm O}$

We just showed three ways to calculate $\dot{\vec{\boldsymbol{H}}}_{\rm O}$ for a conically swinging stick, but there are many more. You can get a sense of the possibilities by studying table I summarizing momenta and energy in the back of the book. Here are three basic choices.

$$\text{(1) } \dot{\vec{\boldsymbol{H}}}_{\rm O} = \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm, \qquad \text{or}$$
$$\text{(2) } \dot{\vec{\boldsymbol{H}}}_{\rm O} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\rm O}, \qquad \text{or}$$
$$\text{(3) } \dot{\vec{\boldsymbol{H}}}_{\rm O} = \vec{\boldsymbol{r}}_{cm/O} \times \vec{\boldsymbol{a}}_{cm} m_{tot} + \dot{\vec{\boldsymbol{H}}}_{\rm cm}.$$

Choice (3) is the safest choice to make if you are in doubt, since it is the only one of the three choices that does not depend on point O being a fixed point (which it was for this example). For option (2) above, we can calculate $\vec{\boldsymbol{H}}_{\rm O}$ various ways as

$$\text{(a) } \vec{\boldsymbol{H}}_{\rm O} = \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{v}}\, dm \qquad \text{or,}$$
$$\text{(b) } \vec{\boldsymbol{H}}_{\rm O} = [\boldsymbol{I}^{\rm O}] \cdot \vec{\boldsymbol{\omega}} \qquad \text{or,}$$
$$\text{(c) } \vec{\boldsymbol{H}}_{\rm O} = \vec{\boldsymbol{r}}_{cm/O} \times \vec{\boldsymbol{v}}_{cm} m_{tot} + \vec{\boldsymbol{H}}_{\rm cm}.$$

For option (3) above, we can calculate $\dot{\vec{\boldsymbol{H}}}_{\rm cm}$ as

$$\text{(d) } \dot{\vec{\boldsymbol{H}}}_{\rm cm} = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{a}}_{/cm} dm \qquad \text{or,}$$
$$\text{(e) } \dot{\vec{\boldsymbol{H}}}_{\rm cm} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\rm cm}.$$

For (c) and (e) above, we can calculate $\vec{\boldsymbol{H}}_{\rm cm}$ as

$$\text{(f) } \vec{\boldsymbol{H}}_{\rm cm} = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}_{/cm}\, dm \qquad \text{or,}$$
$$\text{(g) } \vec{\boldsymbol{H}}_{\rm cm} = [\boldsymbol{I}^{\rm cm}] \cdot \vec{\boldsymbol{\omega}}.$$

For either of options (b) or (g), we can calculate $[I]$ relative to the xyz axes (as we did for the second method on the previous pages) or relative to some rotated axes better aligned with the rod (as we did for the third method on the previous pages.)

As you can surmise from all the choices above, the list of options for the calculation of $\dot{\vec{\boldsymbol{H}}}_{\rm O}$ is too long and boring to show here.

The pros and cons of these methods depend on the problem at hand. If you want to avoid integration you are pretty much stuck using either $[\boldsymbol{I}^{\rm O}]$ or $[\boldsymbol{I}^{\rm cm}]$ with (e) and (g) above). Avoiding the integrals depends on your having a table of moments of inertia (like table IV in the back of this book).

Moment of inertia matrix

The moment of inertia matrix [**I**] has more uses. It can be used to calculate the rate of change of angular momentum, and the energy of *any system which moves like a rigid*

body. The formula for angular momentum is the same as for constant rate circular motion, but the formula for the rate of change of angular momentum has one new term. The formula for kinetic energy only contains velocities (not accelerations) and so is unchanged from the constant circular rate case. The sums and integrals which define the angular momentum $\vec{\boldsymbol{H}}_{\mathrm{O}}$, the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ and the kinetic energy E_{K} can now be evaluated. We skip the details and present the result of the calculation.

$$\vec{\boldsymbol{H}}_{\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}} \tag{8.40}$$

$$\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = \vec{\boldsymbol{\omega}} \times \underbrace{[\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}}}_{\vec{\boldsymbol{H}}_{\mathrm{O}}} + [\boldsymbol{I}^{\mathrm{O}}] \cdot \dot{\vec{\boldsymbol{\omega}}} \tag{8.41}$$

$$E_{\mathrm{K}} = \frac{1}{2} \vec{\boldsymbol{\omega}} \cdot ([\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}}) \tag{8.42}$$

Note that we are using the moment of inertia calculated relative to some point 0 that is on the axis of rotation. Point 0 is not necessarily the center of mass. You might jump ahead at this point and survey the table summarizing momenta and energy on the inside cover of the book.

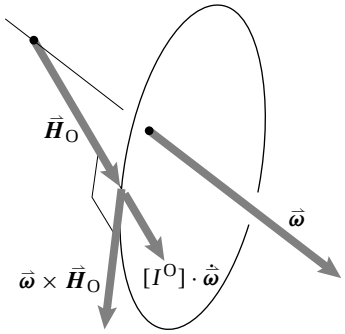


Figure 8.39: The two terms in the rate of change of angular momentum are shown.

(Filename:tfigure5.7)

Extended bodies in 3-D: the geometry of $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$

It is possible to understand the formula for the change of angular momentum geometrically. Here is one way of looking at it.

$$\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = \underbrace{\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{O}}}_{\text{Contribution from rotation of } \vec{\boldsymbol{H}}_{\mathrm{O}}} + \underbrace{[\boldsymbol{I}^{\mathrm{O}}] \cdot \dot{\vec{\boldsymbol{\omega}}}}_{\text{Due to the changing length of } \vec{\boldsymbol{H}}_{\mathrm{O}}}$$

The first term in the equation describes the rotation of the angular momentum vector. The second term describes its rate of change of length.

Since the body is spinning about a fixed axis, the orientation of the axis of rotation is not only fixed relative to the Newtonian frame, the room environment, but also relative to the body. At one instant of time we use a coordinate system to calculate $\vec{\boldsymbol{H}}_{\mathrm{O}}$. After the body has rotated a little, if we now used a coordinate system that rotated the same amount as the body, a coordinate system 'glued' to the body, we could calculate $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ again.

This new calculation of $\vec{\boldsymbol{H}}_{\mathrm{O}}$ will be almost identical to the calculation before the small rotation, however, because the moment of inertia matrix does not change in time relative to a coordinate system that moves with the body. Also, the coordinates of $\vec{\boldsymbol{\omega}}$ will be unchanged except for possibly a multiplication by a constant because the direction of $\vec{\boldsymbol{\omega}}$ doesn't change. Thus the only change of $\vec{\boldsymbol{H}}_{\mathrm{O}}$ as represented in this rotated coordinate system, is a possible change in its length due to a change in the spinning rate.

But this new coordinate system is a rotated coordinate system. To find the actual change in $\vec{\boldsymbol{H}}_{\mathrm{O}}$ we need to take this rotation into account. To picture this rotation consider the special case when the rotation rate is constant. Then the vector $\vec{\boldsymbol{H}}_{\mathrm{O}}$ is constant in the rotating coordinate system. That is, the vector $\vec{\boldsymbol{H}}_{\mathrm{O}}$ rotates with the body.

So the net change in $\vec{\boldsymbol{H}}_{\mathrm{O}}$ is a change due to rotation of the body added to a change due to the change in the rotation rate. For small angular changes, the direction of the first term is the same as the tangent to the circle that is traced by the tip of the angular momentum vector $\vec{\boldsymbol{H}}_{\mathrm{O}}$ as drawn on the body. To approximate the first term, consider the following reasoning. First, let's denote the first term by $(\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}})_{\mathrm{rot}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{O}}$,

where the subscript 'rot' indicates that this term is the contribution to $\dot{\vec{H}}_O$ from the rotation of $\vec{H}_O$. For small angular changes, $(\dot{\vec{H}}_O)_{rot} = (d\vec{H}_O/dt)_{rot} = \vec{\omega} \times \vec{H}_O \approx (\Delta\vec{H}_O)_{rot}/\Delta t$. Thus, the term due to rotation is approximately, for small Δt, $(\Delta\vec{H}_O)_{rot} = \Delta t(\vec{\omega} \times \vec{H}_0)$.

The contribution to $\dot{\vec{H}}_O$ due to change of length of $\vec{H}_O$ is $[\boldsymbol{I}^O] \cdot \dot{\vec{\omega}}$. Similarly, this term is approximately, for small Δt, $\Delta t([\boldsymbol{I}^O] \cdot \dot{\vec{\omega}})$.

Energy of things going in circles at variable rate

The energy only depends on the speeds of the parts of a system, not their accelerations. So, as with constant rate motion,

$$
\begin{aligned}
E_K &= \frac{1}{2}\int \underbrace{v^2}_{\vec{v}\cdot\vec{v}}\, dm \\
&= \frac{1}{2}\int (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})\, dm \\
&= \frac{1}{2}\omega^2 \int R^2\, dm
\end{aligned}
$$

R is the distance from the axis to the mass

$$
\begin{aligned}
&= \frac{1}{2}\vec{\omega} \cdot [\boldsymbol{I}^O] \cdot \vec{\omega} \\
&= \underbrace{\frac{1}{2}\omega^2 I_{zz}^o}
\end{aligned}
$$

For rotation about the z axis the 9 terms in the matrix formula reduce to this one simple term.

Rotation of a rigid body about a fixed axis: the general case

Consider a general rigid body of mass m and moment of inertia matrix with respect to the center of mass $[\boldsymbol{I}^{cm}]$ rotating about a fixed axis. Without loss of generality, let the axis of rotation be the $\hat{\boldsymbol{k}}$ axis.

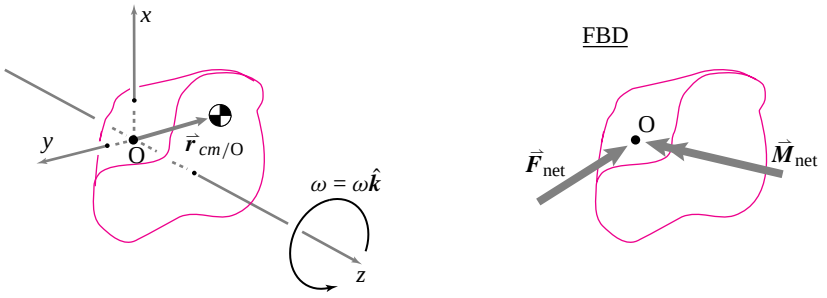


Figure 8.40: Free body diagram of a general rigid body rotating about the z-axis.

(Filename:tfigure5.gen.rigid.body)

Linear momentum balance

Referring to the free body diagram of the rigid body, linear momentum balance gives

$$\begin{aligned}\sum \vec{F} &= \dot{\vec{L}} \\ \vec{F}_{net} &= m\vec{a}_{cm} \\ &= m[\omega\hat{k} \times (\omega\hat{k} \times \vec{r}_{cm/O}) + \dot{\omega}\hat{k} \times \vec{r}_{cm/O}]. \end{aligned} \tag{8.43}$$

The first term on the right hand side of equation 8.43, the centripetal term, is directed from the center of mass (◓) through the axis of rotation; that is, it lies in the xy-plane (or has no $\hat{k}$ component). The second term on the right hand side of equation 8.43, the tangential term, is normal to the plane determined by the axis and the center of mass. It is tangent to the circle that the center of mass travels on. It is zero if the center of mass is on the axis.

Angular momentum balance

Angular momentum balance about point O gives

$$\begin{aligned}\sum \vec{M}_{O} &= \dot{\vec{H}}_{O} \\ \vec{M}_{net} &= \underbrace{\vec{r}_{cm/O} \times m\vec{a}_{cm}}_{\text{term (i)}} \\ &\quad + \underbrace{\omega\hat{k} \times \left\{[I^{cm}] \cdot \omega\hat{k}\right\}}_{\text{term (ii)}} \\ &\quad + \underbrace{[I^{cm}] \cdot (\dot{\omega}\hat{k})}_{\text{term (iii)}}. \end{aligned} \tag{8.44}$$

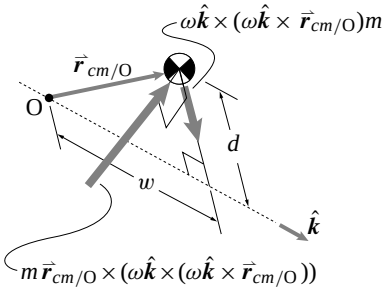


Figure 8.41: The first part of term (i)

(Filename:tfigure5.term1.a)

Let's look at each of the three terms on the right hand side in turn.

term (i)

The first term (i) on the right hand side of equation 8.44 is

$$\boxed{\vec{a}_{cm}=\omega\hat{k} \times (\omega\hat{k} \times \vec{r}_{cm/O}) + \dot{\omega}\hat{k} \times \vec{r}_{cm/O}}$$

$$\begin{aligned}\text{term (i)} &= \vec{r}_{cm/O} \times m\,\overbrace{\vec{a}_{cm}} \\ &= m[\vec{r}_{cm/O} \times (\omega\hat{k} \times (\omega\hat{k} \times \vec{r}_{cm/O})) \\ &\quad + \vec{r}_{cm/O} \times (\dot{\omega}\hat{k} \times \vec{r}_{cm/O})] \end{aligned}$$

Now, let's consider the two parts of term (i) in turn. The first part of term (i) is in the direction of $-\hat{k} \times \vec{r}_{cm/O}$. For example, if the center of mass is in the xz plane, this contribution to $\vec{M}_{net}$ is in the $-\hat{j}$ direction and could be accommodated by reaction forces on the axis in the $\hat{\imath}$ and $-\hat{\imath}$ directions. Now, the second part of term (i) can be decomposed into a part along the $\hat{k}$ direction and a part perpendicular to $\hat{k}$ (along d). The part along $\hat{k}$ is

$$\vec{r}_{cm/O} \times (\dot{\omega}\hat{k} \times \vec{r}_{cm/O}) \cdot \hat{k} = md^2\dot{\omega}\hat{k}.$$

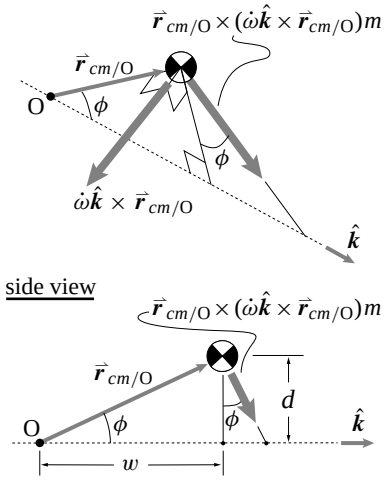


Figure 8.42: The second part of term (i)

(Filename:tfigure5.term1.b)

The part of term (i) perpendicular to $\hat{\boldsymbol{k}}$ has magnitude $m\dot{\omega}dw$.

term (ii)

Now, let's look at the second term in equation 8.44, term (ii), and expand it.

$$\begin{aligned}
\text{term (ii)} &= \omega\hat{\boldsymbol{k}} \times [\boldsymbol{I}^{\text{cm}}] \cdot \omega\hat{\boldsymbol{k}} \\
&= \omega\hat{\boldsymbol{k}} \times \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \\
&= \omega\hat{\boldsymbol{k}} \times \left[I_{xz}^{cm}\omega\hat{\boldsymbol{\imath}} + I_{yz}^{cm}\omega\hat{\boldsymbol{\jmath}} + I_{zz}^{cm}\omega\hat{\boldsymbol{k}} \right] \\
&= I_{xz}^{cm}\omega^2\hat{\boldsymbol{\jmath}} - I_{yz}^{cm}\omega^2\hat{\boldsymbol{\imath}}
\end{aligned}$$

So, the second term in equation 8.44, term (ii), has *no* $\hat{\boldsymbol{k}}$ component. The $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components are due to the off-diagonal terms in $[\boldsymbol{I}^{\text{cm}}]$. The off-diagonal terms are responsible for dynamic imbalance.

term (iii)

Now, the third term in equation 8.44, term (iii), is

$$\begin{aligned}
[\boldsymbol{I}^{\text{cm}}] \cdot \dot{\omega}\hat{\boldsymbol{k}} &= \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \dot{\omega} \end{bmatrix} \\
&= I_{xz}^{cm}\dot{\omega}\hat{\boldsymbol{\imath}} + I_{yz}^{cm}\dot{\omega}\hat{\boldsymbol{\jmath}} + I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}}.
\end{aligned}$$

Putting terms (i), (ii), and (iii) back together

Now, let's put the three terms back together into the equation of angular momentum balance about point O, equation 8.44. To help with the interpretation, assume the center of mass is in the xz plane. So, we start with

$$\vec{\boldsymbol{M}}_{net} = \dot{\vec{\boldsymbol{H}}}_{\text{O}}.$$

Breaking this vector equation into components, we get

$$\begin{aligned}
M_{xx} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{\imath}} &= \dot{\vec{\boldsymbol{H}}}_{\text{O}} \cdot \hat{\boldsymbol{\imath}} \\
M_{yy} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{\jmath}} &= \dot{\vec{\boldsymbol{H}}}_{\text{O}} \cdot \hat{\boldsymbol{\jmath}} \\
M_{zz} = \vec{\boldsymbol{M}}_{net} \cdot \hat{\boldsymbol{k}} &= \dot{\vec{\boldsymbol{H}}}_{\text{O}} \cdot \hat{\boldsymbol{k}}.
\end{aligned}$$

Now, using all the results so far, we find the torques about the x, y, and z axes. For the z-axis,

$$M_z = \underbrace{[m\overbrace{d^2}^{\text{d = distance of cm from axis of rotation}} + I_{zz}^{cm}]}_{\text{sometimes called '}[\boldsymbol{I}^{\text{O}}]\text{'}}\dot{\omega}. \tag{8.45}$$

The only torque about the z-axis, the axis of rotation, is due to the acceleration about that axis.

Next, for the x-axis,

$$M_x = -I_{yz}^{cm}\omega^2 + I_{xz}^{cm}\dot{\omega} + m\dot{\omega}dw. \tag{8.46}$$

There is a torque about the x-axis due to off-diagonal terms in $[\boldsymbol{I}^{\mathrm{cm}}]$. One term is the dynamic imbalance term I_{yz}^{cm} and the other is associated with angular acceleration. (Recall, the center of mass is on the xz plane.) There is also a term due to the center of mass tangential acceleration since the center of mass is on a plane that does not contain point O.

Finally, the torque about the y-axis is

$$M_y = I_{xz}^{cm}\omega^2 + I_{yz}^{cm}\dot{\omega} + (-m\omega^2 dw). \tag{8.47}$$

Here, we have nearly the same types of terms as in equation 8.46. In this case, though, the centripetal acceleration causes the center of mass motion to contribute to the torque.

So, if

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} \text{ and } \dot{\vec{\boldsymbol{\omega}}} = \dot{\omega}\hat{\boldsymbol{k}}$$

then

$$\begin{aligned}\vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} \quad = \quad & \left(-I_{yz}^{cm}\omega^2 + (I_{xz}^{cm} + mdw)\dot{\omega}\right)\hat{\boldsymbol{\imath}} \\ & +((I_{xz}^{cm} - mdw)\omega^2 + I_{yz}^{cm}\dot{\omega})\hat{\boldsymbol{\jmath}} \\ & +(md^2 + I_{zz}^{cm})\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

SAMPLE 8.11 *A scalar times a vector is a vector. A matrix times a vector is a vector. What is the difference?* Find the vectors $\vec{\boldsymbol{H}}_1 = I_G\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{H}}_2 = [\mathbf{I}_G]\vec{\boldsymbol{\omega}}$, if $\vec{\boldsymbol{\omega}} = 2\,\text{rad/s}\hat{\boldsymbol{\imath}} + 3\,\text{rad/s}\hat{\boldsymbol{\jmath}}$, $I_G = 10\,\text{kg m}^2$ and $[\mathbf{I}_G] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix}\text{kg m}^2$. Draw $\vec{\boldsymbol{\omega}}$, $\vec{\boldsymbol{H}}_1$ and $\vec{\boldsymbol{H}}_2$.

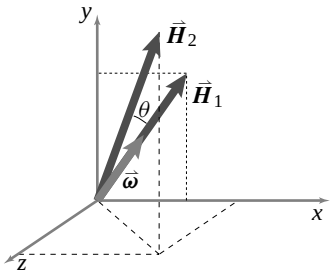


Figure 8.43: (Filename:sfig1.2.12)

Solution

$$\begin{aligned}
\vec{\boldsymbol{H}}_1 &= I_G\vec{\boldsymbol{\omega}} \\
&= 10\,\text{kg m}^2(2\,\text{rad/s}\hat{\boldsymbol{\imath}} + 3\,\text{rad/s}\hat{\boldsymbol{\jmath}}) \\
&= (20\hat{\boldsymbol{\imath}} + 30\hat{\boldsymbol{\jmath}})\,\text{kg m}^2/\,\text{s}. \\
\vec{\boldsymbol{H}}_2 &= [\mathbf{I}_G]\cdot\vec{\boldsymbol{\omega}} \\
&= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix}\text{kg m}^2 \cdot \begin{Bmatrix} 2 \\ 3 \\ 0 \end{Bmatrix}\text{rad/s} \\
&= (10\hat{\boldsymbol{\imath}} + 15\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}})\,\text{kg m}^2/\,\text{s}.
\end{aligned}$$

These two vectors, $\vec{\boldsymbol{H}}_1$ and $\vec{\boldsymbol{H}}_2$, are shown along with $\vec{\boldsymbol{\omega}}$ in Fig. 8.43. Note that $\vec{\boldsymbol{H}}_1$ has the same direction as $\vec{\boldsymbol{\omega}}$ but $\vec{\boldsymbol{H}}_2$ does not. $\vec{\boldsymbol{H}}_1$ and $\vec{\boldsymbol{\omega}}$ are both in the xy-plane but $\vec{\boldsymbol{H}}_2$ is not; it is in 3-D.

Comments: Multiplying a vector by a scalar does not change the direction of the vector but multiplying by a matrix does change the direction, in general. Find the angles between $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{H}}_1$ and between $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{H}}_2$ to convince yourself.

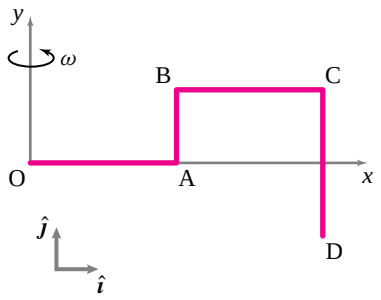


Figure 8.44: (Filename:sfig4.6.5)

SAMPLE 8.12 The composite rod OABCD shown in Figure 8.44 is made up of three identical rods OA, BC, CD of mass 0.5 kg and length 20 cm each, and the rod AB which is half of rod CD. The composite rod goes in circles about the y-axis at a constant rate $\omega = 5\,\text{rad/s}$. Find the angular momentum and the rate of change of angular momentum of the rod at the instant shown (*i.e.*, when the rod is in the xy-plane).

(a) Are all the components of $[\boldsymbol{I}^{\text{O}}]$ necessary to compute $\vec{\boldsymbol{H}}_{\text{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{\text{O}}$? Find $[\boldsymbol{I}^{\text{O}}]$ or the necessary components of $[\boldsymbol{I}^{\text{O}}]$.

(b) Find the angular momentum $\vec{\boldsymbol{H}}_{\text{O}}$.

(c) Find the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{\text{O}}$.

(d) If the rod were rotating about the z-axis instead, (*i.e.*, if the motion were in the xy-plane) which components of $[\boldsymbol{I}^{\text{O}}]$ would be required to find $\vec{\boldsymbol{H}}_{\text{O}}$? What would be the value of $\dot{\vec{\boldsymbol{H}}}_{\text{O}}$ in that case?

Solution Since the rod rotates about the y-axis, and O is a fixed point on this axis,

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{j}} \quad \text{and} \quad \vec{\boldsymbol{H}}_{\text{O}} = [\boldsymbol{I}^{\text{O}}]\cdot\vec{\boldsymbol{\omega}}.$$

(a) Since

$$\begin{aligned}\vec{\boldsymbol{H}}_{\text{O}} &= \begin{bmatrix} I^{\text{O}}_{xx} & I^{\text{O}}_{xy} & I^{\text{O}}_{xz} \\ I^{\text{O}}_{xy} & I^{\text{O}}_{yy} & I^{\text{O}}_{yz} \\ I^{\text{O}}_{xz} & I^{\text{O}}_{yz} & I^{\text{O}}_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \\ &= I^{\text{O}}_{xy}\omega\hat{\boldsymbol{i}} + I^{\text{O}}_{yy}\omega\hat{\boldsymbol{j}} + I^{\text{O}}_{yz}\omega\hat{\boldsymbol{k}} \\ &= (I^{\text{O}}_{xy}\hat{\boldsymbol{i}} + I^{\text{O}}_{yy}\hat{\boldsymbol{j}} + I^{\text{O}}_{yz}\hat{\boldsymbol{k}})\omega, \end{aligned} \tag{8.48}$$

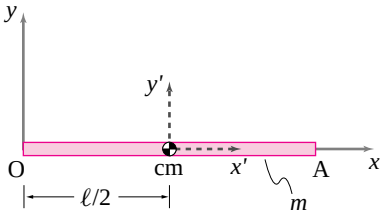


Figure 8.45: (Filename:sfig4.6.5a)

we only need to find three components of $[\boldsymbol{I}^{\text{O}}]$ to compute $\vec{\boldsymbol{H}}_{\text{O}}$, namely I^{O}_{xy}, I^{O}_{yy} and I^{O}_{yz}.

We can compute these components by considering each rod individually. For any rod, the components can be calculated using the values of the components about the center of mass of the rod (see table IV in the back of the book) and then using the parallel axis theorem.

$$\begin{aligned} \underline{\text{Rod OA}}: \quad I^{\text{O}}_{yy} &= I^{\text{cm}}_{y'y'} + m\frac{l^2}{4} = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 \\ I^{\text{O}}_{xy} &= \underbrace{I^{\text{cm}}_{x'y'}}_{0} + m(-x_{\text{cm/O}}\underbrace{y_{\text{cm/O}}}_{0}) = 0 \\ I^{\text{O}}_{yz} &= \underbrace{I^{\text{cm}}_{y'z'}}_{0} + m(-\underbrace{y_{\text{cm/O}}}_{0}\underbrace{z_{\text{cm/O}}}_{0}) = 0. \end{aligned}$$

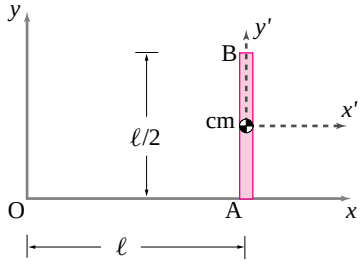


Figure 8.46: (Filename:sfig4.6.5b)

$$\begin{aligned} \underline{\text{Rod AB}}: \quad I^{\text{O}}_{yy} &= \underbrace{I^{\text{cm}}_{y'y'}}_{0} + \frac{m}{2}(x^2_{\text{cm/O}} + \underbrace{z^2_{\text{cm/O}}}_{0}) = \frac{ml^2}{2} \\ I^{\text{O}}_{xy} &= \underbrace{I^{\text{cm}}_{x'y'}}_{0} + \frac{m}{2}(-\underbrace{x_{\text{cm/O}}}_{l}\underbrace{y_{\text{cm/O}}}_{\frac{l}{4}}) = -\frac{ml^2}{8} \\ I^{\text{O}}_{yz} &= \underbrace{I^{\text{O}}_{y'z'}}_{0} + m(-y_{\text{cm/O}}\underbrace{z_{\text{cm/O}}}_{0}) = 0. \end{aligned}$$

$$\begin{aligned}
\underline{\text{Rod BC}}: \quad I^{\mathrm{O}}_{yy} &= I^{\mathrm{cm}}_{y'y'} + m(x^2_{\mathrm{cm/O}} + \underbrace{z^2_{\mathrm{cm/O}}}_{0}) = \frac{ml^2}{12} + m\frac{9l^2}{4} = \frac{7}{3}ml^2 \\
I^{\mathrm{O}}_{xy} &= \underbrace{I^{\mathrm{cm}}_{x'y'}}_{0} + m(-x_{\mathrm{cm/O}}y_{\mathrm{cm/O}}) = -m\frac{3l}{2}\cdot\frac{l}{2} = -\frac{3ml^2}{4} \\
I^{\mathrm{O}}_{yz} &= 0.
\end{aligned}$$

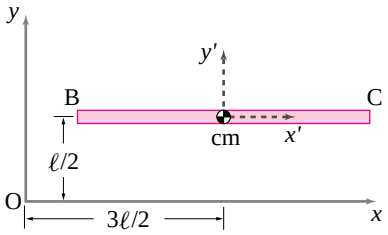


Figure 8.47: (Filename:sfig4.6.5c)

$$\begin{aligned}
\underline{\text{Rod CD}}: \quad I^{\mathrm{O}}_{yy} &= \underbrace{I^{\mathrm{cm}}_{y'y'}}_{0} + m(x^2_{\mathrm{cm/O}} + \underbrace{z^2_{\mathrm{cm/O}}}_{0}) = m(2l)^2 = 4ml^2 \\
I^{\mathrm{O}}_{xy} &= 0 \\
I^{\mathrm{O}}_{yz} &= 0.
\end{aligned}$$

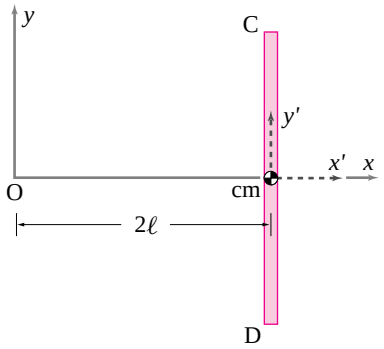


Figure 8.48: (Filename:sfig4.6.5d)

Thus for the entire rod,

$$\begin{aligned}
I^{\mathrm{O}}_{yy} &= \frac{1}{3}ml^2 + \frac{1}{2}ml^2 + \frac{7}{3}ml^2 + 4ml^2 = \frac{43}{6}ml^2 \\
I^{\mathrm{O}}_{xy} &= -\frac{1}{8}ml^2 - \frac{3}{4}ml^2 = -\frac{7}{8}ml^2 \\
I^{\mathrm{O}}_{yz} &= 0.
\end{aligned}$$

$$\boxed{I^{\mathrm{O}}_{yy} = \tfrac{43}{6}ml^2,\ I^{\mathrm{O}}_{xy} = -\tfrac{7}{8}ml^2,\ I^{\mathrm{O}}_{yz} = 0.}$$

(b)

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\mathrm{O}} &= (I^{\mathrm{O}}_{yy}\hat{\boldsymbol{\imath}} + I^{\mathrm{O}}_{xy}\hat{\boldsymbol{\jmath}} + I^{\mathrm{O}}_{yz}\hat{\boldsymbol{k}})\omega \quad \text{(from Eqn 8.48)} \\
&= ml^2\omega(\frac{43}{6}\hat{\boldsymbol{\jmath}} - \frac{7}{8}\hat{\boldsymbol{\imath}}) \\
&= 0.5\,\mathrm{kg}(0.2\,\mathrm{m})^2\cdot 5\,\mathrm{rad/s}(7.17\hat{\boldsymbol{\jmath}} - 0.87)\hat{\boldsymbol{\imath}}) \\
&= (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\mathrm{kg\cdot m^2/s}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}}_{\mathrm{O}} = (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\mathrm{kg\cdot m^2/s}.}$$

(c)

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{O}} = \omega\hat{\boldsymbol{\jmath}} \times \left([\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}}\right) \\
&= 5\,\mathrm{rad/s}\hat{\boldsymbol{\jmath}} \times (0.717\hat{\boldsymbol{\jmath}} - 0.087\hat{\boldsymbol{\imath}})\,\mathrm{kg\cdot m^2/s} \\
&= 0.435\,\mathrm{N\cdot m}\hat{\boldsymbol{k}}.
\end{aligned}$$

$$\boxed{\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = 0.435\,\mathrm{N\cdot m}\hat{\boldsymbol{k}}.}$$

(d) If the rod were rotating in the xy-plane, it would be planar circular motion. The only component of $[\boldsymbol{I}^{\mathrm{O}}]$ required for the caclculation of $\vec{\boldsymbol{H}}_{\mathrm{O}} = I^{\mathrm{O}}_{zz}\vec{\boldsymbol{\omega}}$ will be I^{O}_{zz}. Also,

$$\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{O}} = \vec{\boldsymbol{\omega}} \times I^{\mathrm{O}}_{zz}\vec{\boldsymbol{\omega}} = 0$$

since $\vec{\boldsymbol{\omega}}$ is parallel to $I^{\mathrm{O}}_{zz}\vec{\boldsymbol{\omega}}$. ◁

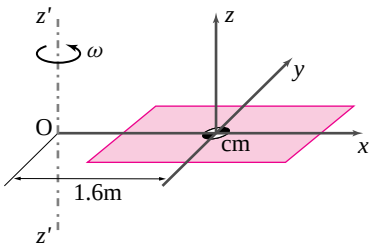


Figure 8.49: (Filename:sfig4.5.6)

SAMPLE 8.13 A 0.5 kg uniform rectangular solar panel rotates about an off-centered axis $z'z'$ with constant angular speed $\omega = 1.5\,\text{rad/s}$. Axis $z'z'$ is parallel to the transverse z-axis of the plate and is 1.6 m away from the center of mass of the plate. The in-plane moments of inertia I_{xx}^{cm} and I_{yy}^{cm} of the plate are given: $I_{xx}^{\text{cm}} = 4.0\,\text{kg}\cdot\text{m}^2$ and $I_{yy}^{\text{cm}} = 12.4\,\text{kg}\cdot\text{m}^2$. At the instant shown in Fig 8.49, calculate

(a) the linear momentum of the panel,
(b) the angular momentum of the panel, and
(c) the kinetic energy of the panel.

Solution Since the plate rotates about the $z'z'$-axis which is parallel to the z-axis,

$$\vec{\omega} = \omega\hat{k} = 1.5\,\text{rad/s}\hat{k}.$$

(a) **Linear momentum:**

$$\begin{aligned}\vec{L} &= m_{\text{tot}}\vec{v}_{\text{cm}} = m(\vec{\omega}\times\vec{r}_{\text{cm/O}})\\ &= 0.5\,\text{kg}\cdot(1.5\,\text{rad/s}\hat{k}\times 1.6\,\text{m}\hat{\imath})\\ &= 1.2\,\text{kg}\cdot\text{m/s}\hat{\jmath}\end{aligned}$$

$$\boxed{\vec{L} = 1.2\,\text{kg}\cdot\text{m/s}\hat{\jmath}}$$

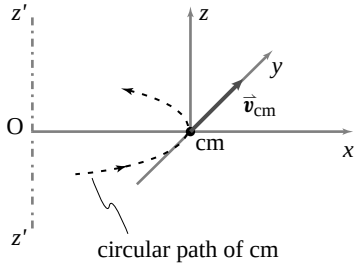


Figure 8.50: (Filename:sfig4.5.6a)

We can easily check the direction of $\vec{L}$ since $\vec{L} = m\vec{v}_{\text{cm}}$, it has to be in the same direction as $\vec{v}_{\text{cm}}$. The center of mass goes in circles about O, therefore, $\vec{v}_{\text{cm}}$ is tangential to the circular path, i.e. in the y-direction (see Fig 8.50).

(b) **Angular momentum:**

$$\vec{H} = I_{z'z'}^{\text{O}}\,\omega\hat{k}$$

Thus to find $\vec{H}$, we need to find $I_{z'z'}^{\text{O}}$. Since $z'z' \parallel zz$, we can use the parallel axis theorem to find $I_{z'z'}^{\text{O}}$ if we know I_{zz}^{cm}. We are given the in-plane moments of inertia I_{xx}^{cm} and I_{yy}^{cm}. Therefore, from the perpendicular axis theorem:

$$I_{zz}^{\text{cm}} = I_{xx}^{\text{cm}} + I_{yy}^{\text{cm}} = (4.0 + 12.4)\,\text{kg}\cdot\text{m}^2 = 16.4\,\text{kg}\cdot\text{m}^2.$$

Now using the parallel axis theorem,

$$\begin{aligned}I_{z'z'}^{\text{O}} &= I_{zz}^{\text{cm}} + Mr_{\text{cm/O}}^2\\ &= 16.4\,\text{kg}\cdot\text{m}^2 + 0.5\,\text{kg}(1.6\,\text{m})^2 = 17.68\,\text{kg}\cdot\text{m}^2.\\ \text{Therefore,}\quad \vec{H} &= 17.68\,\text{kg}\cdot\text{m}^2\cdot(1.5\,\text{rad/s}\hat{k})\\ &= 26.52\,\text{kg}\cdot\text{m}^2/\,\text{s}\hat{k}.\end{aligned}$$

$$\boxed{\vec{H} = 26.52\,\text{kg}\cdot\text{m}^2/\,\text{s}\hat{k}.}$$

(c) **Kinetic energy:**

$$\begin{aligned}E_{\text{K}} &= \frac{1}{2}I_{z'z'}^{\text{O}}\omega^2 = \frac{1}{2}(17.68\,\text{kg}\cdot\text{m}^2)(1.5\,\text{rad/s})^2\\ &= 19.89\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2} == 19.89\,\text{N}\cdot\text{m}\\ &= 19.89\ \text{J}\end{aligned}$$

$$\boxed{E_{\text{K}} = 19.89\ \text{J}}$$

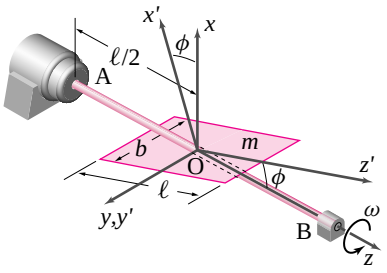


Figure 8.51: A rectangular plate of mass m rotates with shaft AB at a constant speed $\vec{\omega} = \omega\hat{k}$. A coordinate system $x'y'z'$ is aligned with the *principal axes* of the plate.

(Filename:sfig4.6.8)

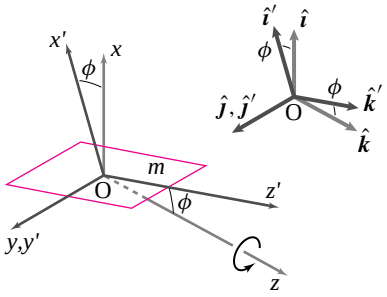


Figure 8.52: A line sketch of the rotating plate along with the two coordinate systems $x'y'z'$ and xyz. The basis vectors associated with the two systems are also shown.

(Filename:sfig4.6.8a)

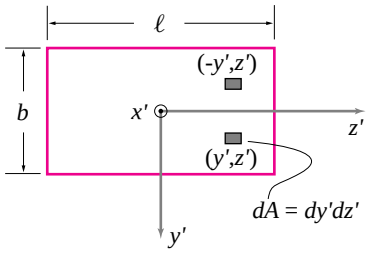


Figure 8.53: Symmetry about the z' axis implies that $I_{y'z'} = 0$. Same result holds if we consider the symmetry about the y' axis.

(Filename:sfig4.6.8b)

① You can use similar argument to find the off-diagonal terms in $[\boldsymbol{I}]$ whenever there is such symmetry with respect to the coordinate axes.

SAMPLE 8.14 *The rotating crooked plate again.* For the crooked plate considered in Sample 8.7 and shown again in Fig. 8.51,

(a) compute the moment of inertia matrix $[\boldsymbol{I}]$ in the $x'y'z'$ coordinate system,

(b) compute the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ using the moment of inertia $[\boldsymbol{I}]_{x'y'z'}$ computed above,

(c) express $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ as a vector in the xyz coordinate system.

Solution A line sketch of the plate and the two coordinate systems attached to it are shown in Fig. 8.52. The set of basis vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$ and $(\hat{\imath}', \hat{\jmath}', \hat{k}')$ associated with coordinate systems xyz and $x'y'z'$ respectively are shown separately for the sake of clarity. From the diagram of the two basis vector sets, we may write

$$\left.\begin{aligned} \hat{\imath}' &= \cos\phi\hat{\imath} - \sin\phi\hat{k} \\ \hat{\jmath}' &= \hat{\jmath} \\ \hat{k}' &= \sin\phi\hat{\imath} + \cos\phi\hat{k} \end{aligned}\right\}. \tag{8.49}$$

(a) **Calculation of $[\boldsymbol{I}]_{x'y'z'}$:**

$$[\boldsymbol{I}]_{x'y'z'} = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{x'y'} & I_{y'y'} & I_{y'z'} \\ I_{x'z'} & I_{y'z'} & I_{z'z'} \end{bmatrix}.$$

Let us first consider the off diagonal terms of $[\boldsymbol{I}]_{x'y'z'}$.
Since $x' = 0$ on the entire plate (the plate is in the $y'z'$ plane and the origin is on the plate.),

$$I_{x'y'} = \int_m \underbrace{x'}_{0}\, y'\, dm = 0, \quad I_{x'z'} = \int_m \underbrace{x'}_{0}\, z'\, dm = 0.$$

How about $I_{y'z'}$? Well, you can calculate it two ways: (a) carry out the integration $I_{y'z'} = \int_m y'z'\, dm$ over the entire plate mass and find that $I_{y'z'} = 0$, or (b) realize that for every mass element dm $(= \frac{m}{\ell b} dA)$ with coordinates $(+y'z')$, there exists another element dm at $(-y'z')$ such that the sum of their contributions to the integral is zero. Therefore, $I_{y'z'} = \int_m y'z'\, dm = 0$ ①Now the other terms:

$$\begin{aligned} I_{z'z'} &= \int_m (\underbrace{x'^2}_{0} + y'^2)\, dm = \int_{-\ell/2}^{\ell/2}\int_{-b/2}^{b/2} y'^2 \cdot \frac{m}{\ell b} dy'\, dz' \\ &= \frac{m}{\ell b}\int_{-\ell/2}^{\ell/2} \left(\frac{y'^3}{3}\Big|_{-b/2}^{b/2}\right) dz' = \frac{m}{\ell b}\cdot\frac{b^3}{12}\int_{-\ell/2}^{\ell/2} dz' \\ &= \frac{m}{\ell b}\cdot\frac{b^3}{12}\cdot\ell = \frac{mb^2}{12}, \end{aligned}$$

and similarly,

$$\begin{aligned} I_{y'y'} &= \int_m (\underbrace{x'^2}_{0} + z'^2)\, dm = \frac{m\ell^2}{12}, \\ I_{x'x'} &= \int_m (y'^2 + z'^2)\, dm = I_{y'y'} + I_{z'z'} = \frac{m}{12}(\ell^2 + b^2). \end{aligned}$$

Thus,

$$[\boldsymbol{I}]_{x'y'z'} = \frac{m}{12}\begin{bmatrix} \ell^2+b^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix}.$$

(b) **Calculation of $\dot{\vec{\boldsymbol{H}}}_{\text{O}}$:**

$$\dot{\vec{\boldsymbol{H}}}_{\text{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\text{O}} \quad \text{and} \quad \vec{\boldsymbol{H}}_{\text{O}} = [\boldsymbol{I}]_{x'y'z'}\{\vec{\boldsymbol{\omega}}\}_{x'y'z'}.$$

The subscripts $x'y'z'$ in $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ have been used to denote that both $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ are expressed in $x'y'z'$ coordinate system. ①We have calculated $[[\boldsymbol{I}]]_{x'y'z'}$ above. Now we need to find $\{\vec{\boldsymbol{\omega}}\}_{x'y'z'}$.

① We can express $[\boldsymbol{I}]$ and $\vec{\boldsymbol{\omega}}$ in any coordinate system of our choice but both of them must be in the same system for their product to be valid.

Let $\vec{\boldsymbol{\omega}} = \omega_{x'}\hat{\boldsymbol{i}}' + \omega_{y'}\hat{\boldsymbol{j}}' + \omega_{z'}\hat{\boldsymbol{k}}'$.

$$\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}} \quad \Rightarrow \quad \omega\hat{\boldsymbol{k}} = \omega_{x'}\hat{\boldsymbol{i}}' + \omega_{y'}\hat{\boldsymbol{j}}' + \omega_{z'}\hat{\boldsymbol{k}}' \tag{8.50}$$

Dotting both sides of Eqn. (8.50) with $\hat{\boldsymbol{i}}'$, $\hat{\boldsymbol{j}}'$, and $\hat{\boldsymbol{k}}'$ and using the relationships in (8.49) we get

$$\omega_{x'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{i}}'}^{-\sin\phi}) = -\omega\sin\phi,\ \omega_{y'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{j}}'}^{0}) = 0,\ \omega_{k'} = \omega(\overbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'}^{\cos\phi}) = \omega\cos\phi.$$

Thus,

$$\{\vec{\boldsymbol{\omega}}\}_{x'y'z'} = -\omega\sin\phi\hat{\boldsymbol{i}}' + \omega\cos\phi\hat{\boldsymbol{k}}'$$

So,

$$\begin{aligned} \vec{\boldsymbol{H}}_{\text{O}} &= \frac{m}{12}\begin{bmatrix} \ell^2+b^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & b^2 \end{bmatrix}\begin{Bmatrix} -\omega\sin\phi \\ 0 \\ \omega\cos\phi \end{Bmatrix} \\ &= \begin{Bmatrix} -\frac{m}{12}(\ell^2+b^2)\omega\sin\phi \\ 0 \\ \frac{m}{12}b^2\omega\cos\phi \end{Bmatrix} \\ \text{or} \qquad \vec{\boldsymbol{H}}_{\text{O}} &= \frac{m\omega}{12}[-(\ell^2+b^2)\sin\phi\hat{\boldsymbol{i}}' + b^2\cos\phi\hat{\boldsymbol{k}}']. \end{aligned}$$

Now we can easily compute $\dot{\vec{\boldsymbol{H}}}_{\text{O}}$ as follows:

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{\text{O}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\text{O}} \\ &= (-\omega\sin\phi\hat{\boldsymbol{i}}' + \omega\cos\phi\hat{\boldsymbol{k}}') \times \frac{m\omega}{12}[-(\ell^2+b^2)\sin\phi\hat{\boldsymbol{i}}' + b^2\cos\phi\hat{\boldsymbol{k}}'] \\ &= \frac{m\omega^2}{12}b^2\sin\phi\cos\phi\hat{\boldsymbol{j}}' - \frac{m\omega^2}{12}(\ell^2+b^2)\sin\phi\cos\phi\hat{\boldsymbol{j}}' \\ &= -\frac{m\omega^2}{12}\ell^2\sin\phi\cos\phi\hat{\boldsymbol{j}}'. \end{aligned}$$

(c) **Now, back to the xyz coordinate system:** Since $\hat{\boldsymbol{j}}' = \hat{\boldsymbol{j}}$ [Eqn (8.49)], we have

$$\dot{\vec{\boldsymbol{H}}}_{\text{O}} = -\frac{m\omega^2}{12}\ell^2\sin\phi\cos\phi\hat{\boldsymbol{j}}$$

which is the same result as obtained in Sample 8.7.

SAMPLE 8.15 *The calculation of $\dot{\vec{H}}$ using the moment of inertia matrix.* For the crooked plate considered in Sample 8.7, find the rate of change of angular momentum $\dot{\vec{H}}_{\mathrm{O}}$, using the moment of inertia $[\boldsymbol{I}^{\mathrm{O}}]$, calculated in the xyz coordinate system

Solution We calculate $\dot{\vec{H}}_{\mathrm{O}}$ using the following formula:

$$\dot{\vec{H}}_{\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \underbrace{\dot{\vec{\omega}}}_{\vec{\mathbf{0}}} + \vec{\omega} \times \overbrace{([\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\omega})}^{\vec{H}_{\mathrm{O}}}.$$

Note that point O is the center of mass of the plate. Let us write the expression for the angular momentum $\vec{H}_{\mathrm{O}}$ in matrix form.

$$\begin{Bmatrix} H_{x/O} \\ H_{y/O} \\ H_{z/O} \end{Bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix}.$$

It is clear from the components of $\vec{\omega}$ that we only need the last column of $[\boldsymbol{I}]$ matrix since other elements of $[\boldsymbol{I}]$ will multiply with zeros of $\vec{\omega}$ vector. Carrying out the multiplication we get

$$\begin{Bmatrix} H_{0x} \\ H_{0y} \\ H_{0z} \end{Bmatrix} = \begin{Bmatrix} I_{xz}\omega \\ I_{yz}\omega \\ I_{zz}\omega \end{Bmatrix}$$

or, in vector form

$$\vec{H}_{\mathrm{O}} = I_{xz}\omega\hat{\boldsymbol{\imath}} + I_{yz}\omega\hat{\boldsymbol{\jmath}} + I_{zz}\omega\hat{\boldsymbol{k}}.$$

Therefore,

$$\begin{aligned} \dot{\vec{H}}_{\mathrm{O}} &= \omega\hat{\boldsymbol{k}} \times \omega(I_{xz}\hat{\boldsymbol{\imath}} + I_{yz}\hat{\boldsymbol{\jmath}} + I_{zz}\hat{\boldsymbol{k}}) \\ &= -\omega^2 I_{yz}\hat{\boldsymbol{\imath}} + \omega^2 I_{xz}\hat{\boldsymbol{\jmath}}. \end{aligned}$$

But,

$$I_{xz} = -\int_m xz\,dm \quad \text{and} \quad I_{yz} = -\int_m yz\,dm$$

and $x = w\sin\phi$, $y = y$, $z = w\cos\phi$ (see Fig. 8.29), hence

$$\begin{aligned} \dot{\vec{H}}_{\mathrm{O}} &= -\omega^2 \int_m (-wy\cos\phi\hat{\boldsymbol{\imath}} + w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}})\,dm \\ &= \int_{-b/2}^{b/2}\int_{-\ell/2}^{\ell/2} \omega^2(-w^2\sin\phi\cos\phi\hat{\boldsymbol{\jmath}} + wy\cos\phi\hat{\boldsymbol{\imath}})\frac{m}{\ell b}dw\,dy \end{aligned}$$

which is the same inetgral as obtained in Sample 8.7 for $\dot{\vec{H}}_{\mathrm{O}}$. Therefore, the result is also the same:

$$\dot{\vec{H}}_{\mathrm{O}} = -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}.$$

$$\boxed{\dot{\vec{H}}_{\mathrm{O}} = -\frac{m\omega^2\ell^2}{12}\sin\phi\cos\phi\hat{\boldsymbol{\jmath}}}$$

SAMPLE 8.16 *Direct application of formula:* A rectangular plate is mounted on a massless shaft with the center of mass of the plate on the shaft axis. The shaft rotates about its axis with angular acceleration $\dot{\vec{\omega}} = 0.5\,\text{rpm/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$. At the instant of interest, the angular velocity is $\vec{\omega} = 100\,\text{rpm}(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}})$ and the components of the moment of inertia matrix of the plate are $I_{xx} = 2\,\text{kg·m}^2$, $I_{yy} = 4\,\text{kg·m}^2$, $I_{zz} = 6\,\text{kg·m}^2$ and $I_{xy} = I_{yz} = I_{xz} = 0$.

(a) Find the angular momentum of the plate about its mass-center and show that it is not in the same direction as the angular velocity.

(b) Find the net moment acting on the plate.

Solution We are given the angular velocity, the angular acceleration, and the moment of inertia matrix of the plate:

$$\begin{aligned}
\vec{\omega} &= 100\,\text{rpm}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) = 10.47\,\text{rad/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \\
\dot{\vec{\omega}} &= 5\,\text{rpm/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) = 0.52\,\text{rad/s}^2(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \\
[\boldsymbol{I}^{\text{cm}}] &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{kg·m}^2.
\end{aligned}$$

(a) **The angular momentum:** The angular momentum of the plate is

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\text{cm}} &= [\boldsymbol{I}^{\text{cm}}] \cdot \vec{\omega} \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{kg·m}^2 \cdot \begin{Bmatrix} 10.47 \\ 10.47 \\ 0 \end{Bmatrix} \text{rad/s} \\
&= (20.94\hat{\boldsymbol{\imath}} + 41.88\hat{\boldsymbol{\jmath}}) \underbrace{\text{kg·m}^2 \cdot \text{s}^{-1}}_{\text{N·m·s}} \\
&= (20.94\hat{\boldsymbol{\imath}} + 41.88\hat{\boldsymbol{\jmath}})\,\text{N·m} \cdot \text{s}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}}_{\text{cm}} = (20.94\hat{\boldsymbol{\imath}} + 41.88\hat{\boldsymbol{\jmath}})\,\text{N·m} \cdot \text{s}}$$

There are many ways of showing that $\vec{\boldsymbol{H}}_{\text{cm}}$ is not parallel to $\vec{\omega}$. We can simply draw the two vectors and show that they are not parallel We can, alternatively, take the cross product of the two vectors:

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\text{cm}} \times \vec{\omega} &= (20.94\hat{\boldsymbol{\imath}} + 41.88\hat{\boldsymbol{\jmath}})\,\text{N·m} \cdot \text{s} \times 10.47\,\text{rad/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \\
&= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 20.94 & 41.88 & 0 \\ 10.47 & 10.47 & 0 \end{vmatrix} \text{N·m} \\
&= \hat{\boldsymbol{k}}(219.24 - 438.48)\,\text{N·m} = -20.94\,\text{N·m}\hat{\boldsymbol{k}}
\end{aligned}$$

which is not zero, implying that the two vectors are not parallel.

(b) **The net moment:** The net moment on the plate can be found by applying angular momentum balance:

$$\begin{aligned}
\vec{\boldsymbol{M}}_{/\text{cm}} &= \dot{\vec{\boldsymbol{H}}}_{\text{cm}} = \vec{\omega} \times \underbrace{[\boldsymbol{I}^{\text{cm}}] \cdot \vec{\omega}}_{\vec{\boldsymbol{H}}_{\text{cm}}} + [\boldsymbol{I}^{\text{cm}}] \cdot \dot{\vec{\omega}} \\
&= \underbrace{20.94\,\text{N·m}\hat{\boldsymbol{k}}}_{\vec{\omega}\times\vec{\boldsymbol{H}}_{\text{cm}}} + \underbrace{1.04\,\text{N·m}\hat{\boldsymbol{\imath}} + 2.08\,\text{N·m}\hat{\boldsymbol{\jmath}}}_{[\boldsymbol{I}^{\text{cm}}]\cdot\dot{\vec{\omega}}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{M}}_{/\text{cm}} = (1.04\hat{\boldsymbol{\imath}} + 2.08\hat{\boldsymbol{\jmath}} + 20.94\hat{\boldsymbol{k}})\,\text{N·m}}$$

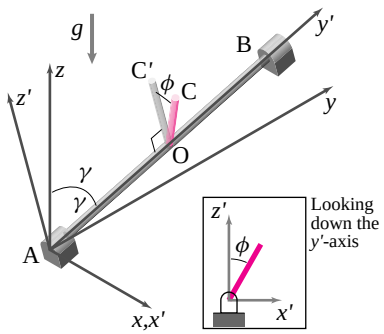


Figure 8.54: A rod, welded to a tipped shaft AB, swings around the shaft axis if it is tipped slightly from the vertical plane yz.

(Filename:sfig5.5.2)

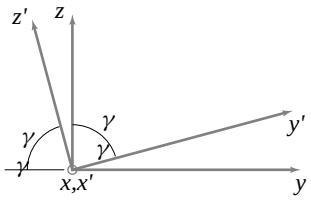


Figure 8.55: $y'z'$ axes are obtained by rotating yz axes counterclockwise about the x-axis by an angle $90^o - \gamma$.

(Filename:sfig5.5.2a)

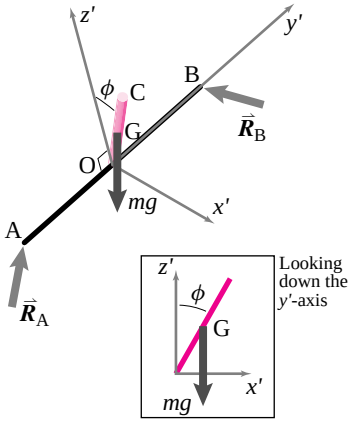


Figure 8.56: Free body diagram of the bar and shaft system. In the inset, only the weight of the bar, mg, is shown for clarity of geometry.

(Filename:sfig5.5.2b)

SAMPLE 8.17 *A rod swings around a tipped axis in 3-D.* A rigid shaft of negligible mass is connected to frictionless hinges at A and B. The shaft is tipped from the horizontal plane (xy-plane) such that the shaft axis makes an angle γ with the vertical axis. A uniform rod OC of mass m and length ℓ is welded to the shaft. At time $t = 0$, the rod is tipped by a small angle ϕ from its position OC' in the yz (or $y'z'$) plane. Find the equation of motion of the rod.

Solution Let $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ be the basis vectors associated with the $x'y'z'$ coordinate system. Since $y'z'$ axes can be obtained by rotating yz axes counterclockwise about the x-axis by an angle $90^o - \gamma$, we can relate the basis vectors of the two coordinate systems with the help of Fig. 8.55:

$$\hat{\imath}' = \hat{\imath}, \quad \hat{\jmath}' = \sin\gamma\,\hat{\jmath} + \cos\gamma\,\hat{k}, \quad \text{and} \quad \hat{k}' = -\cos\gamma\,\hat{\jmath} + \sin\gamma\,\hat{k}. \tag{8.51}$$

The free body diagram of the shaft with rod OC is shown in Fig. 8.56. We can write angular momentum balance for this system about any point on the axis of rotation AB. However, rather than writing angular momentum balance about a point, let us write angular momentum balance about axis AB. This 'trick' will eliminate reactions $\vec{R}_A$ and $\vec{R}_B$ from our equations. Angular Momentum Balance about axis AB is:

$$\hat{\lambda}_{AB} \cdot [\sum \vec{M}_O = \dot{\vec{H}}_O]$$

$$\text{or} \qquad \hat{\jmath}' \cdot \sum \vec{M}_O = \hat{\jmath}' \cdot \dot{\vec{H}}_O.$$

Calculation of $(\hat{\jmath}' \cdot \sum \vec{M}_O)$: Since $\vec{R}_A$ and $\vec{R}_B$ pass through axis AB, they do not produce any moment about this axis. Therefore,

$$\begin{aligned}
\hat{\jmath}' \cdot \sum \vec{M}_O &= \hat{\jmath}' \cdot [\vec{r}_{G/O} \times mg(-\hat{k})] \\
&= \hat{\jmath}' \cdot \left[\frac{L}{2}(\sin\phi\hat{\imath}' + \cos\phi\hat{k}') \times mg(-\hat{k})\right] \\
&= \overbrace{(\sin\gamma\,\hat{\jmath} + \cos\gamma\hat{k})}^{\hat{\jmath}'} \cdot [\frac{L}{2}mg\overbrace{(\sin\phi\,\hat{\jmath} + \cos\phi\,\cos\gamma\hat{\imath})}^{\text{using Eqn.(8.51)}}] \\
&= \frac{\ell}{2}mg\sin\phi\,\sin\gamma
\end{aligned}$$

Calculation of $\hat{\jmath}' \cdot \dot{\vec{H}}_O$: Since the rod rotates about axis AB, we may write

$$\begin{aligned}
\vec{\omega} &= \dot{\phi}\hat{\jmath}', \\
\dot{\vec{\omega}} &= \ddot{\phi}\hat{\jmath}'.
\end{aligned}$$

Now,

$$\dot{\vec{H}}_O = [I^O]\{\dot{\vec{\omega}}\} + \vec{\omega} \times [I^O]\{\vec{\omega}\}$$

where

$$[I^O]\{\dot{\vec{\omega}}\} = [I^O]_{x'y'z'}\,\{\dot{\vec{\omega}}\}_{x'y'z'}$$

$$
\begin{aligned}
&= \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{x'y'} & I_{y'y'} & I_{y'z'} \\ I_{x'z'} & I_{y'z'} & I_{z'z'} \end{bmatrix} \begin{Bmatrix} 0 \\ \ddot{\phi} \\ 0 \end{Bmatrix} \\
&= \begin{Bmatrix} I_{x'y'}\ddot{\phi} \\ I_{y'y'}\ddot{\phi} \\ I_{y'z'}\ddot{\phi} \end{Bmatrix},
\end{aligned}
$$

similarly,

$$
\begin{aligned}
[\boldsymbol{I}^{\mathrm{O}}]\{\vec{\boldsymbol{\omega}}\} &= \begin{Bmatrix} I_{x'y'}\dot{\phi} \\ I_{y'y'}\dot{\phi} \\ I_{y'z'}\dot{\phi} \end{Bmatrix}, \\
\Rightarrow \qquad \vec{\boldsymbol{\omega}} \times [\boldsymbol{I}^{\mathrm{O}}]\{\vec{\boldsymbol{\omega}}\} &= \dot{\phi}\hat{\boldsymbol{j}}' \times [I_{x'y'}\dot{\phi}\hat{\boldsymbol{i}}' + I_{y'y'}\dot{\phi}\hat{\boldsymbol{j}}' + I_{y'z'}\dot{\phi}\hat{\boldsymbol{k}}'] \\
&= -I_{x'y'}\dot{\phi}^2\hat{\boldsymbol{k}}' + I_{y'z'}\dot{\phi}^2\hat{\boldsymbol{i}}'.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{\boldsymbol{j}}' \cdot \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} &= \hat{\boldsymbol{j}}' \cdot [(I_{x'y'}\ddot{\phi}\hat{\boldsymbol{i}}' + I_{y'y'}\ddot{\phi}\hat{\boldsymbol{j}}' + I_{y'z'}\dot{\phi}\hat{\boldsymbol{k}}') + (-I_{x'y'}\dot{\phi}^2\hat{\boldsymbol{k}}' + I_{y'z'}\dot{\phi}^2\hat{\boldsymbol{i}}')] \\
&= I_{y'y'}\ddot{\phi}.
\end{aligned}
$$

Now, setting $\hat{\boldsymbol{j}}' \cdot \sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \hat{\boldsymbol{j}}' \cdot \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$, we get

$$
\begin{aligned}
\frac{L}{2} mg \sin\phi \, \sin\gamma &= I_{y'y'}\ddot{\phi} \\
\text{or} \qquad \ddot{\phi} &= \frac{mg\,(L/2)\,\sin\gamma}{I_{y'y'}} \sin\phi \\
\text{or} \qquad \ddot{\phi} &= C \sin\phi
\end{aligned}
$$

where

$$
C = \frac{mg\,(L/2)\,\sin\gamma}{I_{y'y'}}.
$$

For rod OC,

$$
I_{y'y'} = \int_m (x'^2 + z'^2)\, dm = \int_0^L l^2 \frac{m}{L}\, dl = \frac{1}{3} m L^2.
$$

Therefore, the equation of motion of the rod is

$$
\begin{aligned}
\ddot{\phi} - \frac{mg\,(L/2)\,\sin\gamma}{(1/3)mL^2} \sin\phi &= 0 \\
\ddot{\phi} - \frac{3g\,\sin\gamma}{2L} \sin\phi &= 0.
\end{aligned}
$$

$$
\boxed{\ddot{\phi} - \tfrac{3g\,\sin\gamma}{2L} \sin\phi = 0}
$$

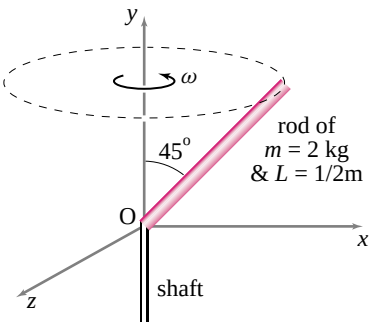


Figure 8.57: (Filename:sfig7.4.2)

SAMPLE 8.18 *Kinetic energy in 3-D rotation.* A thin rod of mass 2 kg and length $L = \frac{1}{2}$ m is welded to a massless shaft at an angle $\theta = 45^{\circ}$. The shaft rotates about its longitudinal axis (y-axis) at 100 rpm. The moment of inertia matrix of the rod about the weld point O is

$$[\boldsymbol{I}^{\mathrm{O}}] = 0.08\,\mathrm{kg}\cdot\mathrm{m}^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find the kinetic energy of the rod.

Solution The rod rotates about the fixed point O with angular velocity

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\jmath}} \quad \text{where} \quad \omega = 100\,\mathrm{rpm} = 10.47\,\mathrm{rad/s}.$$

The kinetic energy of a rigid body rotating about a fixed point O is given by

$$E_{\mathrm{K}} = \frac{1}{2}\vec{\boldsymbol{\omega}}\cdot\left[[\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}}\right]$$

For the given problem, let us write the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$ of the rod as ①

$$[\boldsymbol{I}^{\mathrm{O}}] = K_0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{where } K_0 = 0.08\,\mathrm{kg}\cdot\mathrm{m}^2.$$

$$\text{Now} \quad \vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\jmath}} = \{0 \quad \omega \quad 0\}^{\mathrm{T}}.$$

$$\text{Therefore,} \quad [\boldsymbol{I}^{\mathrm{O}}]\cdot\vec{\boldsymbol{\omega}} = K_0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} = K_0 \begin{Bmatrix} -\omega \\ \omega \\ 0 \end{Bmatrix}$$

$$= -K_0\omega\hat{\boldsymbol{\imath}} + K_0\omega\hat{\boldsymbol{\jmath}}.$$

① This step is simply to facilitate computation. We carry out the multiplication and at the end substitute the value of K_0.

Substituting the expression in the formula for E_{K} we get

$$\begin{aligned} E_{\mathrm{K}} &= \frac{1}{2}\omega\hat{\boldsymbol{\jmath}}\cdot(-K_0\omega\hat{\boldsymbol{\imath}} + K_0\omega\hat{\boldsymbol{\jmath}}) \\ &= \frac{1}{2}K_0\omega^2 \\ &= \frac{1}{2}\cdot 0.08\,\mathrm{kg}\cdot\mathrm{m}^2\cdot(10.47\,\mathrm{rad/s})^2 \\ &= 4.38\,\mathrm{N}\cdot\mathrm{m} = 4.38\mathrm{J} \end{aligned}$$

$$\boxed{E_{\mathrm{K}} = 4.38\mathrm{J}}$$

8.5 Dynamic balance

Sometimes when something spins it can shake the structure that holds it. A familiar example is an 'unbalanced' car tire. But also important are unbalanced rotors in all kinds of machines. Most often one seeks to eliminate or minimize the imbalance. For example the strange design of car crank-shafts is due in large part to an attempt to minimize its imbalance. But what does it mean, in the language of mechanics, to say a rotating body is 'unbalanced'? It means that forces and/or torques are required to hold the axis still while the object spins at constant rate. Going back to the basic momentum balance equations:

$$\text{Linear momentum balance:} \qquad \sum \vec{\boldsymbol{F}} = \dot{\vec{\boldsymbol{L}}}$$

and

$$\text{Angular momentum balance:} \qquad \sum \vec{\boldsymbol{M}}_{\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}},$$

we see that forces and torques are required if $\dot{\vec{\boldsymbol{L}}} \neq \vec{\mathbf{0}}$ or if $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} \neq \vec{\mathbf{0}}$.

So imbalance means $\dot{\vec{\boldsymbol{L}}}$ and/or $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$ is not zero. We break the concept of balance into the following two concepts:

(1) An object is said to be *statically balanced* with respect to a given axis of rotation if $\dot{\vec{\boldsymbol{L}}} = \vec{\mathbf{0}}$ when it spins at constant rate about that axis.
(2) An object is said to be *dynamically balanced* with respect to a given axis of rotation if both $\dot{\vec{\boldsymbol{L}}} = \vec{\mathbf{0}}$ and $\dot{\vec{\boldsymbol{H}}} = \vec{\mathbf{0}}$ for constant rate rotation about that axis.

The origin of the words 'static' balance and 'dynamic' balance is in how the imbalance can be measured. Static imbalance can be measured with a statics experiment, dynamic imbalance requires a dynamics experiment.

Static balance

If the center of mass of a rigid body is on the fixed axis of rotation then it will not accelerate. Thus, $\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot} = \vec{\mathbf{0}}$ and the object is statically balanced. An equivalent definition of static balance is that the *net* force on the spinning body is zero. Whether or not this net force is so can be tested with a statics experiment. Put the axis of rotation on good bearings and see if the mass hangs down in any preferred direction. In a tire shop, this kind of balancing is something called bubble balancing because of the bubble in the level measuring device.

Dynamic balance

The first condition for dynamic balance of an object with respect to spinning about an axis is that the object be statically balanced. The center of mass must lie on the axis of rotation. The second condition for dynamic balance, that $\dot{\vec{\boldsymbol{H}}} = \vec{\mathbf{0}}$, is a little more subtle. To make things more specific, let's calculate the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ with respect to a point O that is on the axis of rotation. So, for constant rate rotation about a fixed axis we have:

$$\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{\mathrm{O}} \tag{8.52}$$

because $\vec{\boldsymbol{H}}_{\mathrm{O}}$ spins with the body. $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ is evidently zero if the angular momentum, $\vec{\boldsymbol{H}}_{\mathrm{O}}$, is parallel to the angular velocity, $\vec{\boldsymbol{\omega}}$. However, $\vec{\boldsymbol{H}}_{\mathrm{O}} = [\boldsymbol{I}^{\mathrm{O}}] \cdot \vec{\boldsymbol{\omega}}$ which means $\vec{\boldsymbol{H}}_{\mathrm{O}}$ is parallel to $\vec{\boldsymbol{\omega}}$ only when $\vec{\boldsymbol{\omega}}$ is an eigenvector of $[\boldsymbol{I}^{\mathrm{O}}]$.

So an object which is spinning about a fixed axis is dynamically balanced if its center of mass is on the axis (static balance) and *the angular velocity vector $\vec{\boldsymbol{\omega}}$ is an eigenvector of the moment of inertia matrix.*

①**Caution:** A common misperception about angular momentum is that it is always parallel to angular velocity. Sometimes angular momentum *is* parallel to angular velocity. When? When $\vec{\boldsymbol{\omega}}$ is an *eigenvector* of the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$, which is always the case in 2-D . But in general, angular momentum is *not* parallel to angular velocity.

If we restrict ①our attention to cases where the axis of rotation is the z-axis then this condition is easy to recognize. It is when $I_{xz} = I_{yz} = 0$, that is, when the only non-zero element in the third column and in the third row of the $[\boldsymbol{I}^{\mathrm{O}}]$ matrix is I^{O}_{zz} in the lower right corner.

Since these terms cause wobbling if one of the coordinate axis is an axis of rotation, the terms that are off the main diagonal in the moment of inertia matrix are sometimes called the 'imbalance' terms. These terms lead to dynamic imbalance when the object is spun about the x, y, or z-axes. The off diagonal terms are also sometimes called the 'centrifugal' terms. This naming has an intuitive basis. One way to understand dynamic imbalance is to think of it being due to the unbalanced centrifugal pull of bits of mass that are spinning in circles. ②

②When you go to ACME garage and get your car tires balanced you can politely ask the mechanic: "Ma'am, could you make sure that one of the eigenvectors of my wheel's moment of inertia matrix is parallel to the car axle? Thanks." The mechanic, if she is a decent person, will make sure that it is the appropriate eigenvector that is parallel to the axle. Otherwise the wheel would point straight sideways with the axle piercing the rubber.

Often, but not always, you can tell by inspection if something is dynamically balanced for rotation about a certain axis. If, for every bit of mass that is not on the axis there is another equal bit of mass that is exactly opposite, with respect to the axis, then the object is balanced. Some objects that do not meet this symmetry condition are also dynamically balanced, however.

Example: **Some common objects**

All of the figures in figure 2.58 on page 65 are dynamically balanced about all the axes shown and about any axis perpendicular to any pair of these axes. □

Example: **A cube = a sphere**

Both a cube and a sphere have moment of inertia matrices proportional to the identity matrix

$$[\boldsymbol{I}^{\mathrm{cm}}]_{\mathrm{cube}} = \frac{m\ell^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\boldsymbol{I}^{\mathrm{cm}}]_{\mathrm{sphere}} = \frac{mD^2}{10}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, for any $\vec{\boldsymbol{\omega}}$, we get that $\vec{\boldsymbol{H}}_{\mathrm{cm}}$ is parallel to $\vec{\boldsymbol{\omega}}$ and, thus, both of these objects are dynamically balanced for rotation about any axis through their centers of mass.

Here is an example where the mathematics contradicts intuition; it does not seem like a cube should be balanced for rotation about a random skewed axis through its center of mass! □

SAMPLE 8.19 *Static and dynamic balance in 2-D motion.* A system with two equal masses at A and B rotates about point O at a constant rate $\omega = 10$ rpm. The system is shown in the figure. The rod connecting the two masses has negligible mass.

(a) Is the system statically balanced? If not, suggest a way to balance it.
(b) Is the system dynamically balanced? If not, suggest a way to balance it.

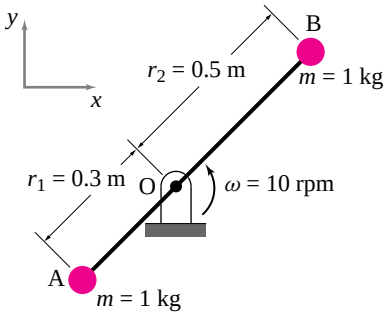


Figure 8.58: (Filename:sfig4.7.DH1)

Solution

(a) Since the two masses are equal, the center of mass of the system is at the geometric center of rod AB, i.e., at a point O′, halfway between A and B. Clearly, O′ is not on the axis of rotation which passes through O and is perpendicular to the plane of motion. Therefore, the system is *not* statically balanced.
To balance the system, we must move the center of mass to O or pivot the system at O′. Say we cannot move the pivot point. So, to move the center of mass O, we add mass m' to m at A. From the definition of center of mass:

$$\begin{aligned} (m+m')r_1 &= mr_2 \\ \Rightarrow \qquad m' &= \frac{m'(r_2-r_1)}{r_1} = \frac{1\,\text{kg}(0.5\,\text{m}-0.3\,\text{m})}{0.3\,\text{m}} \\ &= 0.67\,\text{kg}. \end{aligned}$$

Add 0.67 kg to mass at A.

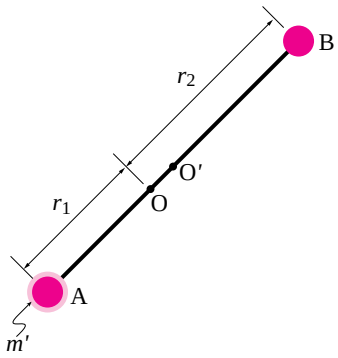


Figure 8.59: (Filename:sfig4.7.DH2)

(b) The system, as given, is *not* dynamically balanced since it is not statically balanced. Let us check if it is dynamically balanced after adding m' to A as suggested above.

$$\sum \vec{M}_O = \dot{\vec{H}}_O.$$

Let us calculate $\dot{\vec{H}}_O$ to see if it is zero:

$$\begin{aligned} \dot{\vec{H}}_O &= \vec{r}_1 \times m_1\vec{a}_1 + \vec{r}_2 \times m_2\vec{a}_2 \\ &= (-r_1\hat{e}_R) \times m_1(\omega^2 r_1\hat{e}_R) + (r_2\hat{e}_R) \times (-\omega^2 r_2\hat{e}_R)m_2 \\ &= -m_1\omega^2 r_1^2\underbrace{(\hat{e}_R \times \hat{e}_R)}_{\vec{0}} - m_2\omega^2 r_2^2\underbrace{(\hat{e}_R \times \hat{e}_R)}_{\vec{0}} \\ &= \vec{0}. \end{aligned}$$

The net torque on the system is zero, therefore it is dynamically balanced. Note that $\dot{\vec{H}}_O = \vec{0}$ irrespective of the values of m_1, m_2, r_1, and r_2. Thus $\dot{\vec{H}}_O = \vec{0}$ even for the system as given. But the system, as given, is not dynamically balanced because *static balance is a necessary condition for dynamic balance*.

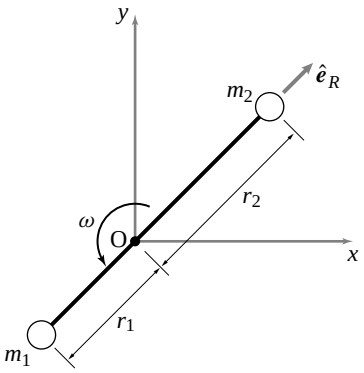


Figure 8.60: (Filename:sfig4.7.DH3)

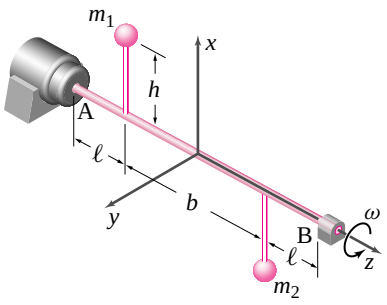


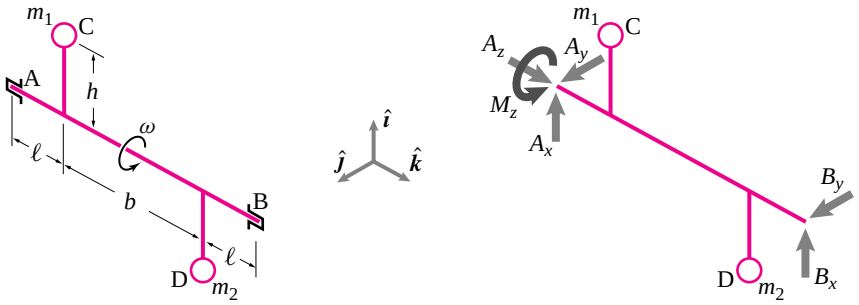
Figure 8.61: Imbalance of a rotor due to point masses spinning at a constant rate.
(Filename:sfig4.7.1)

SAMPLE 8.20 *Imbalance of a rotor due to rotating masses.* Two masses m_1 and m_2 are attached to a massless shaft AB by massless rigid rods of length h each. The two masses are separated by distance b along the shaft and are in the same plane but on the opposite sides of the shaft. The shaft is rotating with a constant angular speed ω. The shaft is free to move along the z-axis at point B. Ignore gravity.

(a) Is the system *statically balanced*?
(b) What is the torque required to keep the motion going about the z-axis (i.e., M_z) ?
(c) What are the reactions at the support points of the shaft?

Solution

(a) A simple line sketch and the free body diagram of the system are shown in Fig. 8.62. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the system gives:



(a) (b)

Figure 8.62: (a) A simple line diagram of the system. (b) Free Body Diagram of the system.
(Filename:sfig4.7.1a)

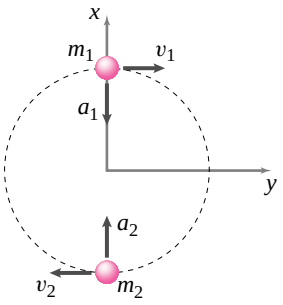


Figure 8.63: Accelerations of m_1 and m_2.
(Filename:sfig4.7.1b)

$$
\begin{aligned}
(A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + A_z\hat{k} &= m_1\vec{a}_1 + m_2\vec{a}_2 \\
&= m_1\omega^2 h(-\hat{\imath}) + m_2\omega^2 h\hat{\imath} \\
\Rightarrow \quad A_x + B_x &= \omega^2 h(m_2 - m_1) \qquad (8.53) \\
A_y + B_y &= 0 \qquad (8.54) \\
A_z &= 0
\end{aligned}
$$

Clearly, $\sum \vec{F} \neq \vec{0}$ which means the system is not statically balanced.

(b) M_z can be easily calculated by writing angular momentum balance about axis AB. In fact, we do not even need to write the equation in this case; since all the forces pass through this axis and the accelerations of m_1 and m_2 also pass through it,

$$\sum M_{/AB}(\text{due to reaction forces}) = 0 \quad \text{and} \quad \dot{H}_{/AB} = 0.$$

But

$$
\begin{aligned}
\sum M_{/AB} &= \dot{H}_{/AB} \\
\text{or} \quad M_z &+ \sum M_{/AB}(\text{due to reaction forces}) = \dot{H}_{/AB} \\
\Rightarrow \quad M_z &= 0.
\end{aligned}
$$

$$\boxed{M_z = 0}$$

(c) For the four unknown reactions at A and B (we have already found A_z and M_z) we have two scalar equations so far (from Linear Momentum Balance). Angular Momentum Balance about point A gives:

$$\sum \vec{\boldsymbol{M}}_{/A} = \dot{\vec{\boldsymbol{H}}}_{/A}$$

Now,

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/A} &= \vec{\boldsymbol{r}}_{B/A} \times \sum \vec{\boldsymbol{F}}_B \\
&= (2\ell + b)\hat{\boldsymbol{k}} \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}}) \\
&= B_x(2\ell + b)\hat{\boldsymbol{\jmath}} - B_y(2\ell + b)\hat{\boldsymbol{\imath}} \\
\dot{\vec{\boldsymbol{H}}}_{/A} &= \vec{\boldsymbol{r}}_{C/A} \times m_1 \vec{\boldsymbol{a}}_1 + \vec{\boldsymbol{r}}_{D/A} \times m_2 \vec{\boldsymbol{a}}_2 \\
&= (\ell\hat{\boldsymbol{k}} + h\hat{\boldsymbol{\imath}}) \times m_1(-\omega^2 h\hat{\boldsymbol{\imath}}) + ((\ell + b)\hat{\boldsymbol{k}} - h\hat{\boldsymbol{\imath}}) \times m_2(\omega^2 h\hat{\boldsymbol{\imath}}) \\
&= \omega^2 h(m_2(\ell + b) - m_1\ell)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

Equating $\sum \vec{\boldsymbol{M}}_{/A}$ and $\dot{\vec{\boldsymbol{H}}}_{/A}$ we get

$$B_x(2\ell + b)\hat{\boldsymbol{\jmath}} - B_y(2\ell + b)\hat{\boldsymbol{\imath}} = \omega^2 h(m_2(\ell + b) - m_1\ell)\hat{\boldsymbol{\jmath}}$$

Dotting both sides of the equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, we get

$$\begin{aligned}
B_y &= 0 \\
\text{and} \qquad B_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1\ell].
\end{aligned}$$

Substituting B_x and B_y in (8.53) and (8.54) we find

$$\begin{aligned}
A_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2\ell - m_1(\ell + b)] \\
A_y &= 0
\end{aligned}$$

$$\boxed{A_x = \tfrac{\omega^2 h}{(2\ell+b)}[m_2\ell - m_1(\ell + b)]}$$

$$\boxed{B_x = \tfrac{\omega^2 h}{(2\ell+b)}[m_2(\ell + b) - m_1\ell]}$$

$$\boxed{A_y = B_y = A_z = M_z = 0}$$

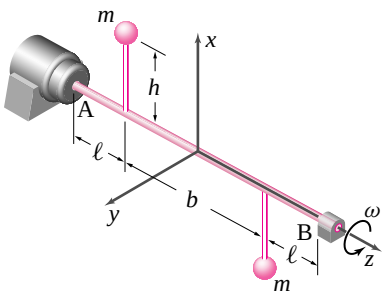


Figure 8.64: Imbalance of a rotor due to point masses spinning at a constant rate.

(Filename:sfig4.7.2)

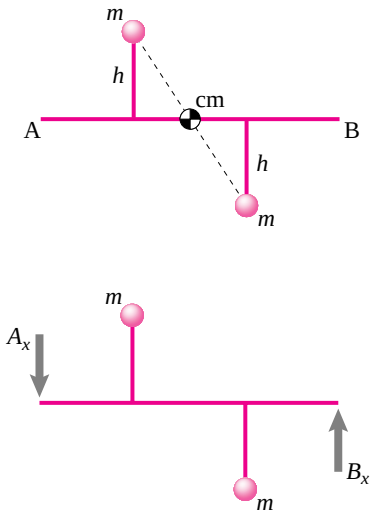


Figure 8.65: Reaction forces at A and B are required to keep the motion going.

(Filename:sfig4.7.2a)

① Balancing a rotor by adding two masses in two selected transverse (to the shaft axis) planes is quite a general method. Even if the rotor has more than two spinning masses the net angular momentum due to all masses can be reduced to the angular momentum produced by an equivalent system with just two masses such as the one under consideration.

SAMPLE 8.21 *Balancing a rotor with spinning masses.* Consider the same system as in the previous sample problem (Sample 8.20). Assume that $m_1 = m_2 = m$.

(a) Set $m_1 = m_2$ in the reactions calculated. Is the system statically balanced now? Explain.

(b) Is the system *dynamically balanced*? Explain.

(c) Balance the system dynamically by (a) adjusting the geometry, (b) adding two masses to the system.

Solution

(a) Recall from the solution of Sample 8.20 that

$$\begin{aligned} A_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2\ell - m_1(\ell + b)] \\ B_x &= \frac{\omega^2 h}{(2\ell + b)}[m_2(\ell + b) - m_1\ell] \\ A_y &= B_y = A_z = M_z = 0 \end{aligned}$$

Setting $m_1 = m_2 = m$ in the above expressions for the reactions, we get

$$A_x = \frac{\omega^2 h}{(2\ell + b)}(-mb) \quad \text{and} \quad B_x = \frac{\omega^2 h}{(2\ell + b)}(mb).$$

Thus, $A_x + B_x = 0$. Therefore, $\sum \vec{F} = \vec{0}$ which means the system is statically balanced.

Static Balance: When the two masses are equal, the center of mass of the system is on the axis of rotation. Therefore, the static reactions are zero irrespective of the orientations of the two masses.

The requirement for static balance is $\sum \vec{F} = \vec{0}$ in any static orientation of the system. As long as the center of mass of the system lies on the axis of rotation, the system will be in static balance.

Dynamic Balance: When the masses are rotating at constant speed, reaction forces at A and B are required to keep the motion going. Clearly, $\sum \vec{M} \neq \vec{0}$ now. Therefore, the system is not in dynamic balance.

At the instant shown the reaction forces are A_x and B_x as shown in Fig. 8.65, but they change directions as the position of the masses changes. For example, if the masses are in the y–z plane the reaction forces will be A_y and $-B_y$. The magnitudes of these forces will remain the same. Thus, we have rotating reaction forces at A and B. These rotary forces cause wear in the bearings and induce vibrations in the frame of the machine and the supporting structure. Therefore, these forces are undesirable.

If we can somehow make these *dynamic reactions* zero, the bearings, the machine frame, the supporting structure, the machine operator and the company will all be very happy. So, how do we do it? Read on.

(b) There are many ways in which the rotor can be dynamically balanced:

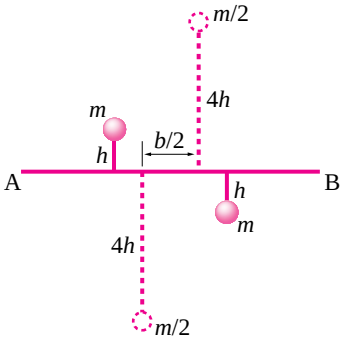
- **Trivial solution:** Remove both the masses. Of course, there is nothing left to produce any nonzero $\dot{\vec{H}}$. Thus, $\sum \vec{M}_{/\text{any point}} = \vec{0}$. Also, $\sum \vec{F} = \vec{0}$.
- **Adjust geometry:** Set $b = 0$. The center of mass is still on the axis of rotation $\Rightarrow$ $\sum \vec{F} = \vec{0}$. In addition, $\dot{\vec{H}}_{/A} = \vec{0}$ $\Rightarrow$ $\sum \vec{M}_{/A} = \vec{0}$.
- **Add two masses:** If b is required to be nonzero, we can balance the rotor by adding two masses in any two selected transverse (to the shaft-axis) planes to make $\dot{\vec{H}}_{/A} = \vec{0}$. ①For example:

– We can take two equal masses m and m, and place them in the opposite directions of the masses already on the shaft as shown in Fig. 8.66(a). It should be clear that the net angular momentum about any point on the shaft will be zero now.
– We can add two masses m' and m' different from the masses attached to the shaft and place them a distance c apart on the shaft. Let the length of the connecting rods of the new masses be h'. For the net angular momentum to be zero we need

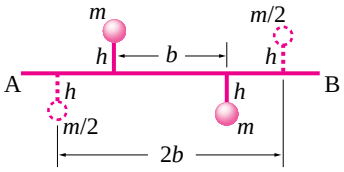
$$\begin{aligned} \overbrace{\dot{\vec{H}}_{\text{due to } m'\text{s}}}^{m'\omega^2 h' c} &= \overbrace{\dot{\vec{H}}_{\text{due to } m\text{s}}}^{m\omega^2 h b} \\ \Rightarrow \quad m'\,h'\,c &= m\,h\,b \end{aligned}$$

Thus, we have the freedom ① of selecting any combination of m', h' and c to give the required product. Here are two examples:

(i) Let $m' = m/2$, $c = b/2$, then $h' = \frac{mhb}{m'c} = 4h$.
(ii) Let $m' = m/2$, $c = 2b$, then $h' = \frac{mhb}{m'c} = h$.



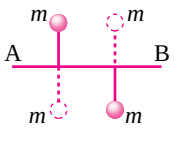
(a)



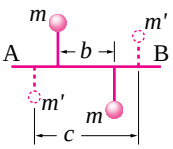
(b)

Figure 8.67: Dynamic balancing of the rotor by adding two masses. (a) $m' = m/2$, $c = b/2$ and $h' = 4h$. (b) $m' = m/2$, $c = 2b$ and $h' = h$.

(Filename:sfig4.7.2c)



(a)



(b)

Figure 8.66: Dynamic balancing of the rotor by adding two masses. (a) The two added masses (shown by dotted circles) are the same as the original masses on the shaft. (b) The two added masses are different from the original masses ($m' \neq m$).

(Filename:sfig4.7.2b)

① In practice, this freedom is usually restricted by geometric and space constraints.

9 General planar motion of a rigid body

Many parts of practical machines and structures move in ways that can be idealized as straight-line or parallel motion (Chapter 6) or circular motion (Chapters 7 and 8). But often an engineer must analyze parts with more general motions a plane in unsteady flight or a connecting rod in a car engine. Of course, the same basic laws of mechanics still apply. The chapter starts with the kinematics of a planar rigid body in two dimensions and then progresses to the mechanics and analysis of motions. Almost throughout, we will use two modeling approximations:

- The objects are planar, or symmetric with respect to a plane; and
- They have planar motions in that plane.

A *planar object* is one where the whole body is flat and all its matter is confined to one plane, say the xy plane. This is surely a palatable approximation for a flat cut piece of sheet metal. For more substantial real objects, like a full car, the approximation seems at a glance to be terrible. But, it turns out, so long as the motion is planar the squishing of the car into the xy plane has no effect on the dynamics.

A *planar motion* is one where the velocities of all points are in the same constant plane, say a fixed xy plane, at all times. Note that the positions of the points do not have to be in same plane for a planar motion.

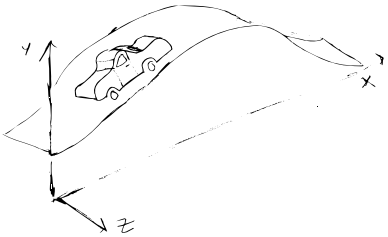


Figure 9.1: Planar motion of a 3D car.

(Filename:tfigure.2D3Dcar)

Example: **A car going over a hill**

A car driving on a road is going over a bump. Assume the road is straight in map view, say in the x direction. So long as the bump crosses the road (so the road surface is made of line segments in the z direction). Although

the car is clearly not planar, the car motion is likely well modeled as planar, with the velocities of all points in the car in the xy plane. □

Example: **Skewered sphere**

A sphere skewered and rotating about a fixed axes in the $\hat{\boldsymbol{k}}$ direction has a planar motion. The points on the object do not live in a common plane. But all of the velocities are orthogonal to $\hat{\boldsymbol{k}}$ and thus in the xy plane. This problem does fit in with the methods of this chapter. The symmetry of the sphere with respect to the xy plane makes it so that the three-dimensional mass distribution does not invalidate the two-dimensional analysis. □

Example: **Skewered plate**

Imagine a flat rectangular plate with normal $\hat{\boldsymbol{n}}$. Imagine it has a fixed axis of rotation in the direction $\hat{\boldsymbol{\lambda}}$ that makes a 45^{o} to $\hat{\boldsymbol{n}}$. This is a planar object (a plane normal to $\hat{\boldsymbol{n}}$) in planar motion (all velocities are in the plane normal to $\hat{\boldsymbol{\lambda}}$. But the plane of motion is not the plane of the mass distribution. So this example does not fit into the discussion of this chapter.
[A two-dimensional analysis of the plate in this example we would project all the plates mass into the plane normal to the $\hat{\boldsymbol{\lambda}}$ direction. Effects of static imbalance would be correctly predicted, but effects of dynamic imbalance are essentially three-dimensional, and would be lost in this projection.] □

Of course no real object is exactly planar and no real motion is exactly a planar motion. But some objects are relatively flat or thin and, more importantly, many motions are well approximated as planar motions. Thus many, if not most, engineering analysis are based on the planar motion approximation and thus use, as a convenience of thought but not actually a further approximation, a planar mass distribution as well. For bodies that are approximately symmetric about the xy plane of motion (such as a car, if the asymmetrically placed driver's mass is neglected), there is no loss in doing a two-dimensional rather than full three dimensional analysis.

But even if one desires in the end to do a three-dimensional analysis, the easiest way to gain the needed skills is to master the analysis of two-dimensional models using flat bodies in planar motions.

9.1 Kinematics of planar rigid-body motion

The approximation that an object moves as if was were rigid is common in mechanics. And useful too, that's why we have a whole book on the topic. We now continue our study of rigid body mechanics by considering the kinematics of planar motion. Namely, the motions of points. We study how points on a rigid body move for two, somewhat independent, reasons. First, one needs to know about the motion of mass points in order to apply the momentum-balance equations. Second, formulas involving the motion of points on a rigid body are useful to understand mechanisms, machines where rigid bodies are attached in various ways to each other.

The velocities of points on a rigid body are not arbitrary. That the various points maintain constant distance with respect to each other restricts the relative velocities

and accelerations of the points. In this section you will learn three important concepts about this restriction: the concept of angular velocity, the vector formula for relative velocity of points on a rigid body, and the vector formula for relative acceleration of points on a rigid body. All of these are familiar topics, just now in a broader context.

Displacement and rotation

When a planar body (read, say, machine part) $\mathcal{B}$ moves from one point in the plane it has a displacement and a rotation. For definiteness, we start in some reference position. We mark a reference point on the body at a point that, in the reference configuration, coincides with a fixed reference point, say 0. We mark a (directed) line on the body that, in the reference configuration, coincides with a fixed reference line, say the positive x axis. When a plane flies from New York to Bombay its position is measured relative to a point in the Gulf of Guinea 1000 miles west of Gabon,① even though the plane never goes there (nor does anyone want it to). Similarly, a body never needs to be at its reference position or orientation as it moves.

We could measure rotation by measuring the rotations of any line that connected two points fixed to the body. For each line we keep track of the angle the line makes with a line fixed in space, say the positive x or y axis. And, as for the case of pure rotation, its simplest just to stick to the convention that counter-clockwise rotations are positive (Fig. 9.3). The angles $\theta_1, \theta_2 \ldots$, all change with time and are all different from each other. But all the angles change at the the same amount, just like in section 7.3. We lose nothing by picking any line we like for reference and measuring the body rotation by the rotation of that line.

So

> the net motion of a rigid planar body is described by the vector displacement of a reference point from a reference position $\vec{r}_{0'/0} = \vec{r}_{00'}$ and a rotation of the body from the reference position, θ.

① That's O^o longitude and O^o latitude.

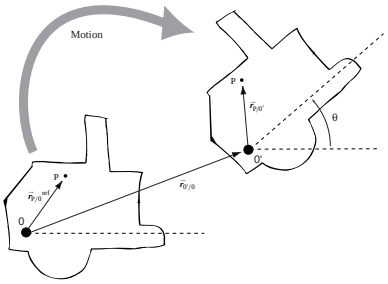


Figure 9.2: The displacement and rotation of a planar body relative to a reference configuration.

(Filename:tfigure.dispandrot)

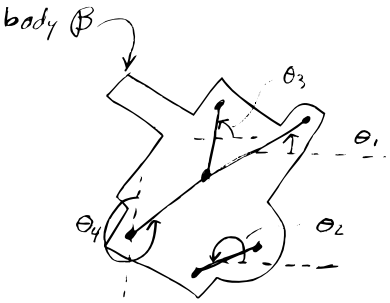


Figure 9.3: Rotation of body $\mathcal{B}$ is measured by the rotation of real or imagined lines marked on the body. The lines make different angles: $\theta_1 \neq \theta_2$, $\theta_2 \neq \theta_3$ etc, but $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \ldots$. Angular velocity is defined as $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ with $\omega \equiv \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \ldots$.

(Filename:tfigure.2Drotation)

The position of a point on a moving rigid body.

Given that a point P on the body is at a known position $\vec{r}_{\mathrm{P}/0}{}^{\mathrm{ref}}$ in the reference configuration, where is it after the body has been displaced by $\vec{r}_{0'/0}$ and rotated an angle θ? An easy way treat this is to write the position $\vec{r}_{\mathrm{P}}$ of P as

$$\vec{r}_{\mathrm{P}/0} = \vec{r}_{0'/0} + \vec{r}_{\mathrm{P}/0'}.$$

One way to think of this is as the body first displacing without rotation and then rotating without displacing. Don't mind that this is not the route by which the body moved to its new configuration, the end state is the same. The displacement of the body is given by $\vec{r}_{0'/0}$ so we already have half our answer. The other term $\vec{r}_{\mathrm{P}/0'}$ we find by rotating $\vec{r}_{\mathrm{P}/0}{}^{\mathrm{ref}}$ as we did in Sect. 7.3. Thus, we can describe the coordinates of a point as,

$$\left[\vec{r}_{\mathrm{P}/0}\right]_{xy} = \underbrace{\left[\vec{r}_{0'/0}\right]_{xy}}_{\text{displacement}} + \underbrace{\left[R(\theta)\right]\left[\vec{r}_{\mathrm{P}/0'}\right]_{x'y'}}_{\text{rotation}}. \tag{9.1}$$

where $\left[R(\theta)\right] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. As the motion progresses the displacement changes with time as does the rotation angle θ.

Angular velocity

Because all lines body $\mathcal{B}$ rotate at the same rate (at a given instant) $\mathcal{B}$'s rotation rate is the single number we call $\omega_{\mathcal{B}}$ ('omega b'). In order to make various formulas work out we define a vector angular velocity with magnitude $\omega_{\mathcal{B}}$ which is perpendicular to the xy plane:

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \underbrace{\omega_{\mathcal{B}}}_{\dot{\theta}} \hat{\boldsymbol{k}}$$

where $\dot{\theta}$ is the rate of change of the angle of *any* line marked on body $\mathcal{B}$.

So long as you are careful to define angular velocity by the rotation of line segments and not by the motion of individual particles, the concept of angular velocity in general motion is defined exactly as for a body rotating about a fixed axis. A legitimate way to think about planar motion of a rigid body is that any given point is moving in circles about any other given point (relative to that point). When a rigid body moves it always has an angular velocity (possible zero). If we call the body $\mathcal{B}$ (script B), we then call the body's angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$. The angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is a vector that describes the rate of rotation of the body.

Relative velocity of two points on a rigid body

For any two points A and B glued to a rigid body, $\mathcal{B}$ the relative velocity of the points ('the velocity of B relative to A') is given by the cross product of the angular velocity of the body with the relative position of the two points:

$$\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}. \tag{9.2}$$

This formula says that the relative velocity of two points on a rigid body is the same as would be predicted for one of the points if the other were stationary.

Absolute velocity of a point on a rigid body

If one knows the velocity of one point on a rigid body and one also knows the angular velocity of the body, then one can find the velocity of any other point. How?

$$\begin{aligned} \vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + (\vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A) \\ &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\ &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \underbrace{\vec{\boldsymbol{r}}_{B/A}}_{\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A} \end{aligned}$$

That is, the absolute velocity of the point B is the absolute velocity of the point A plus the velocity of the point B relative to the point A. Because B and A are on the same rigid body, their relative velocity is given by formula 9.2 above. The equation is valid no matter what values are known and which are not known. It can be used in various ways depending on what is known about the motion of the body being studied and what is known about the motion of the chosen points A and B.

An alternative approach is to differentiate the coordinate expression eqn. (9.1) as follows.

$$\begin{aligned}
\left[\vec{\boldsymbol{v}}_{\mathrm{P}/0}\right]_{xy} &= \frac{d}{dt}\left[\vec{\boldsymbol{r}}_{\mathrm{P}/0}\right]_{xy} && (9.3)\\
&= \frac{d}{dt}\begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \frac{d}{dt}\left\{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}\right\} && (9.4)\\
&= \begin{bmatrix} \dot{x}_{0'/0} \\ \dot{y}_{0'/0} \end{bmatrix} + \begin{bmatrix} -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta \\ -\dot{\theta}\cos\theta & -\dot{\theta}\sin\theta \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} && (9.5)\\
&= \begin{bmatrix} \dot{x}_{0'/0} \\ \dot{y}_{0'/0} \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} && (9.6)\\
&= \begin{bmatrix} \dot{x}_{0'/0} \\ \dot{y}_{0'/0} \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\left[\vec{\boldsymbol{r}}_{\mathrm{P}/0'}\right]_{xy}. && (9.7)
\end{aligned}$$

Thus, matrix product $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\left[\vec{\boldsymbol{r}}_{\mathrm{P}/0'}\right]_{xy}$ is equivalent to the vector product $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{P}/0'}$ and the matrix $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$ is sometimes called the angular velocity matrix. It is an example of a so-called *skew symmetric* matrix because it is the negative of its own transpose.

Angular acceleration

We define the angular acceleration $\vec{\boldsymbol{\alpha}}$ of a rigid body as the rate of change of angular velocity, $\vec{\boldsymbol{\alpha}} = \dot{\vec{\boldsymbol{\omega}}}$. The angular acceleration of a body $\mathcal{B}$ is called $\vec{\boldsymbol{\alpha}}_{\mathcal{B}}$. For two-dimensional bodies moving in the plane both the angular velocity and the angular acceleration are always perpendicular to the plane. That is $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}} = \alpha\hat{\boldsymbol{k}} = \dot{\omega}\hat{\boldsymbol{k}}$. In this case the angular acceleration is only due to the speeding up or slowing down of the rotation rate; i.e., $\alpha = \dot{\omega} = \ddot{\theta}$.

Relative acceleration of two points on a rigid body

For any two points A and B glued to a rigid body $\mathcal{B}$, the acceleration of B relative to A is

$$\begin{aligned}
\vec{\boldsymbol{a}}_{B/A} &= \frac{d}{dt}\vec{\boldsymbol{v}}_{\mathrm{B/A}}\\
&= \frac{d}{dt}\left\{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}\right\}\\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{v}}_{\mathrm{B/A}}),\\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}),\\
&= \alpha_{\mathcal{B}}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A} + -\omega_{\mathcal{B}}^2\vec{\boldsymbol{r}}_{\mathrm{B/A}}), && (9.8)
\end{aligned}$$

If point A has no acceleration, this formula is the same as that for the acceleration of a point going in circles from chapter 7. On a rigid body in 2D all two points on rigid body can do relative to each other is to go in circles.

Absolute acceleration of a point on a rigid body

If one knows the acceleration of one point on a rigid body *and* the angular velocity and acceleration of the body, then one can find the acceleration of any other point. How?

$$\begin{aligned} \vec{a}_B &= \vec{a}_A + (\vec{a}_B - \vec{a}_A) = \vec{a}_A + \vec{a}_{B/A} \\ &= \vec{a}_A + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A}) + \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A} \\ &= \vec{a}_A - \omega_{\mathcal{B}}^2 \vec{r}_{B/A} + \alpha_{\mathcal{B}} \hat{k} \times \vec{r}_{B/A} \end{aligned} \tag{9.9}$$

Equation 9.9 is often called the three term acceleration formula. The acceleration of a point B on a rigid body is the sum of three terms. The first, $\vec{a}_A$, is the acceleration of some point A on the body. The second term, $\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A})$, is the centripetal acceleration of B going in circles relative to A. It is directed from B towards A. The third term, $\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A}$, is due to the change in the magnitude of the angular velocity and is in the direction normal to the line from A to B.

Summary of the kinematics of one rigid body in general 2D motion

The key idea in this section are that if you know the position, velocity, and acceleration of one point on a rigid body, and you know the rotation angle, angular rate and angular acceleration then you can find the position, velocity and acceleration of any point on the body. Thus, when we study the mechanics of a rigid body, we can use the position of one reference point and the rotation of the body as simple kinematic measures of the entire motion of the body.

SAMPLE 9.1 *Velocity of a point on a rigid body in planar motion.* A plate ABC, of an equilateral triangle geometry, is in motion in the x-y plane. At the instant shown in the figure, point B has velocity $\vec{v}_B = 0.3\,\text{m/s}\hat{\imath} + 0.6\,\text{m/s}\hat{\jmath}$ and the plate has angular velocity $\vec{\omega} = 2\,\text{rad/s}\hat{k}$. Find the velocity of point A.

Solution We are given $\vec{v}_B$ and ω, and we need to find $\vec{v}_A$, the velocity of point A on the same rigid body. We know that,

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$

Thus, to find $\vec{v}_A$, we need to find $\vec{r}_{A/B}$. Now,

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2\,\text{m}\hat{\imath}) = -0.2\,\text{m}\hat{\imath}$$

Thus,

$$\begin{aligned}
\vec{v}_A &= \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B} \\
&= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s} + 2\,\text{rad/s}\hat{k} \times (-0.2\hat{\imath})\,\text{m} \\
&= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s} - 0.4\hat{\jmath}\,\text{m/s} \\
&= (0.3\hat{\imath} + 0.2\hat{\jmath})\,\text{m/s}.
\end{aligned}$$

$$\boxed{\vec{v}_A = (0.3\hat{\imath} + 0.2\hat{\jmath})\,\text{m/s}}$$

SAMPLE 9.2 *The instantaneous center of rotation.* A point C on a rigid body, rotating with angular velocity $\omega = 5\,\text{rad/s}\hat{k}$ has velocity $\vec{v}_C = 2\,\text{m/s}\hat{\imath} - 5\,\text{m/s}\hat{\jmath}$. Find a point on the body, assuming it exists ①, that has zero velocity.

① The point with zero velocity is called the instantaneous center of rotation. Sometimes this point may lie outside the body.

Solution Let the point of zero velocity be O, with position vector $\vec{r}_{O/C}$ with respect to point C. Since $\vec{v}_O = \vec{v}_C + \vec{\omega} \times \vec{r}_{O/C}$, for $\vec{v}_O$ to be zero, $\vec{\omega} \times \vec{r}_{O/C}$ must be parallel to and in the opposite direction of $\vec{v}_C$. Since $\vec{\omega}$ is out of plane, $\vec{r}_{O/C}$ must be normal to $\vec{v}_C$ for the cross product to be parallel to $\vec{v}_C$. Now, let $\vec{v}_C = v_C\hat{\lambda}$. Then, $\vec{r}_{O/C} = r\hat{n}$ where $\hat{n} \perp \hat{\lambda}$ and $r = |\vec{r}_{O/C}|$. Thus,

$$v_C\hat{\lambda} + \omega\hat{k} \times r\hat{n} = \vec{v}_O = \vec{0} \tag{9.10}$$

Dotting eqn. (9.10) with $\hat{\lambda}$, we get

$$v_C = \omega r \quad \Rightarrow \quad r = \frac{v_C}{\omega} = \frac{\sqrt{29}\,\text{m/s}}{5\,\text{rad/s}} = 1.08\,\text{m}.$$

Since $\hat{\lambda} = \vec{v}_C/|\vec{v}_C| = 0.37\hat{\imath} - 0.93\hat{\jmath}$, $\hat{n} = 0.93\hat{\imath} + 0.37\hat{\jmath}$. Thus

$$\vec{r}_{O/C} = r\hat{n} = 1.08\,\text{m}(0.93\hat{\imath} + 0.37\hat{\jmath}) = 1\,\text{m}\hat{\imath} + 0.4\,\text{m}\hat{\jmath}.$$

$$\boxed{\vec{r}_{O/C} = 1\,\text{m}\hat{\imath} + 0.4\,\text{m}\hat{\jmath}}$$

Note: It is also possible to find $\vec{r}_{O/C}$ purely by vector algebra. Assuming $\vec{r}_{O/C} = (x\hat{\imath} + y\hat{\jmath})\,\text{m}$ and plugging into $\vec{v}_O = \vec{v}_C + \vec{\omega} \times \vec{r}_{O/C}$ along with the given values, we get $\vec{0} = (2 - 5y)\,\text{m/s}\hat{\imath} + (-5 + 5x)\,\text{m/s}\hat{\jmath}$. Dotting this equation with $\hat{\imath}$ and $\hat{\jmath}$, we get $2 - 5y = 0$ and $-5 + 5x = 0$, which give $x = 1$ and $y = 0.4$. Thus, $\vec{r}_{O/C} = 1\,\text{m}\hat{\imath} + 0.4\,\text{m}\hat{\jmath}$ as obtained above.

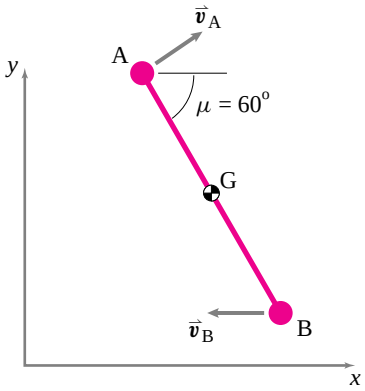
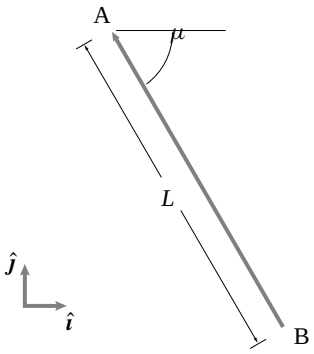
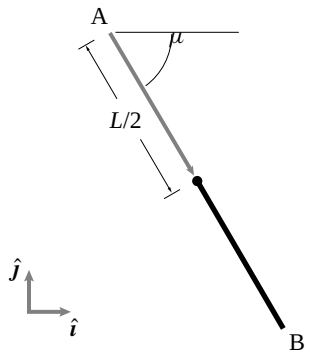


Figure 9.4: (Filename:sfig7.1.1)



$\vec{r}_{A/B} = L(-\cos\theta\hat{i} + \sin\theta\hat{j})$

Figure 9.5: (Filename:sfig7.1.1a)



$\vec{r}_{G/A} = L/2(\cos\theta\hat{i} - \sin\theta\hat{j})$

Figure 9.6: (Filename:sfig7.1.1b)

SAMPLE 9.3 A cheerleader throws her baton up in the air in the vertical xy-plane. At an instant when the baton axis is at $\theta = 60^o$ from the horizontal, the velocity of end A of the baton is $\vec{v}_A = 2\,\text{m/s}\hat{i} + \sqrt{3}\,\text{m/s}\hat{j}$. At the same instant, end B of the baton has velocity in the negative x-direction (but $|\vec{v}_B|$ is not known). If the length of the baton is $L = \frac{1}{2}$ m and the center of mass is in the middle of the baton, find the velocity of the center of mass.

Solution

$$\begin{aligned} \text{We are given:} \quad \vec{v}_A &= (2\hat{i} + \sqrt{3}\hat{j})\,\text{m/s} \\ \text{and} \quad \vec{v}_B &= -v_B\hat{i} \end{aligned}$$

where $v_B = |\vec{v}_B|$ is unknown. We need to find $\vec{v}_G$. Using the relative velocity formula for two points on a rigid body, we can write:

$$\vec{v}_G = \vec{v}_A + \vec{\omega} \times \vec{r}_{G/A} \tag{9.11}$$

Here, $\vec{v}_A$ and $\vec{r}_{G/A}$ are known. Thus, to find $\vec{v}_G$, we need to find $\vec{\omega}$, the angular velocity of the baton. Since the motion is in the vertical xy-plane, let $\vec{\omega} = \omega\hat{k}$. Then,

$$\begin{aligned} \vec{v}_B &= \vec{v}_A + \vec{\omega} \times \vec{r}_{A/B} = \vec{v}_A + \omega\hat{k} \times \underbrace{L(-\cos\theta\hat{i} + \sin\theta\hat{j})}_{\vec{r}_{A/B}} \\ \text{or} \quad -v_B\hat{i} &= (2\hat{i} + \sqrt{3}\hat{j})\,\text{m/s} - \omega L(\cos\theta\hat{j} + \sin\theta\hat{i}) \\ &= (2\hat{i} + \sqrt{3}\hat{j})\,\text{m/s} - \omega\cdot\frac{1}{2}\,\text{m}\cdot(\frac{1}{2}\hat{j} + \frac{\sqrt{3}}{2}\hat{i}) \end{aligned}$$

Dotting both sides of this equation with $\hat{j}$ we get:

$$\begin{aligned} 0 &= \sqrt{3}\,\text{m/s} - \frac{\omega}{2}\,\text{m}\cdot\frac{1}{2} \\ \Rightarrow \quad \omega &= \sqrt{3}\frac{\text{m}}{\text{s}}\cdot\frac{4}{1\,\text{m}} = 4\sqrt{3}\,\text{rad/s}. \end{aligned}$$

Now substituting the appropriate values in Eqn 9.11 we get:

$$\begin{aligned} \vec{v}_G &= \vec{v}_A + \omega\hat{k} \times \underbrace{\frac{L}{2}(\cos\theta\hat{i} - \sin\theta\hat{j})}_{\vec{r}_{G/A}} \\ &= \vec{v}_A + \frac{\omega L}{2}(\cos\theta\hat{j} + \sin\theta\hat{i}) \\ &= (2\hat{i} + \sqrt{3}\hat{j})\,\text{m/s} + \sqrt{3}\,\text{m/s}\cdot(\frac{1}{2}\hat{j} + \frac{\sqrt{3}}{2}\hat{i}) \\ &= (2 + \frac{3}{2})\,\text{m/s}\hat{i} + (\sqrt{3} + \frac{\sqrt{3}}{2})\,\text{m/s}\hat{j} \\ &= 3.5\,\text{m/s}\hat{i} + 2.6\,\text{m/s}\hat{j} \end{aligned}$$

$$\boxed{\vec{v}_G = (3.5\hat{i} + 2.6\hat{j})\,\text{m/s}}$$

SAMPLE 9.4 *Falling ladder:* The ends of a ladder of length $L = 3\,\text{m}$ slip along the frictionless wall and floor shown in Figure 9.7. At the instant shown, when $\theta = 60^o$, the angular speed $\dot\theta = 1.15\,\text{rad/s}$ and the angular acceleration $\ddot\theta = 2.5\,\text{rad/s}^2$. Find the absolute velocity and acceleration of end B of the ladder.

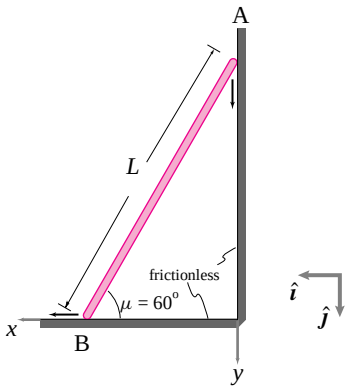


Figure 9.7: (Filename:sfig7.2.2)

Solution Since the ladder is falling, it is rotating clockwise. From the given information:

$$\begin{aligned}\vec{\omega} &= \dot\theta\hat{k} = -1.15\,\text{rad/s}\hat{k}\\ \dot{\vec{\omega}} &= \ddot\theta\hat{k} = -2.5\,\text{rad/s}^2\hat{k}.\end{aligned}$$

We need to find $\vec{v}_B$, the absolute velocity of end B, and $\vec{a}_B$, the absolute acceleration of end B.

Since the end A slides along the wall and end the B slides along the floor, we know the directions of $\vec{v}_A$, $\vec{v}_B$, $\vec{a}_A$ and $\vec{a}_B$.

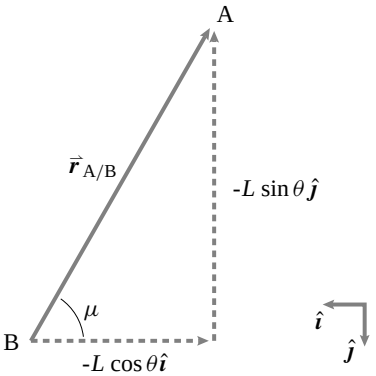
Let $\vec{v}_A = v_A\hat{j}$, $\vec{a}_A = a_A\hat{j}$, $\vec{v}_B = v_B\hat{i}$ and $\vec{a}_B = a_B\hat{i}$ where the scalar quantities v_A, a_A, v_B and a_B are unknown.

$$\begin{aligned}\text{Now,}\quad \vec{v}_A &= \vec{v}_B + \vec{v}_{A/B} = \vec{v}_B + \vec{\omega}\times\vec{r}_{A/B}\\ \text{or}\quad v_A\hat{j} &= v_B\hat{i} + \dot\theta\hat{k}\times\underbrace{L(-\cos\theta\hat{i} - \sin\theta\hat{j})}_{\vec{r}_{A/B}}\\ &= (v_B + \dot\theta L\sin\theta)\hat{i} - \dot\theta L\cos\theta\hat{j}.\end{aligned}$$

Dotting both sides of the equation with $\hat{i}$, we get:

$$\begin{aligned}v_A\underbrace{\hat{j}\cdot\hat{i}}_{0} &= (v_B + \dot\theta L\sin\theta)\underbrace{\hat{i}\cdot\hat{i}}_{1} + \dot\theta L\cos\theta\underbrace{\hat{j}\cdot\hat{i}}_{0}\\ \Rightarrow\quad 0 &= v_B + \dot\theta L\sin\theta\\ \Rightarrow\quad v_B &= -\dot\theta L\sin\theta = -(-1.15\,\text{rad/s})\cdot 3\,\text{m}\cdot\frac{\sqrt{3}}{2}\\ &= 2.99\,\text{m/s}.\end{aligned}$$

$$\boxed{\vec{v}_B = 2.9\,\text{m/s}\hat{i}}$$



$\vec{r}_{A/B}$ = -L cos θ î - L sin θ ĵ

Figure 9.8: (Filename:sfig7.2.2b)

Similarly,

$$\begin{aligned}\vec{a}_A &= \vec{a}_B + \dot{\vec{\omega}}\times\vec{r}_{A/B} + \overbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r}_{A/B})}^{-\omega^2\vec{r}_{A/B}}\\ a_A\hat{j} &= a_B\hat{i} + \ddot\theta\hat{k}\times L(-\cos\theta\hat{i} - \sin\theta\hat{j}) - \dot\theta^2 L(-\cos\theta\hat{i} - \sin\theta\hat{j})\\ &= (a_B + \ddot\theta L\sin\theta + \dot\theta^2 L\cos\theta)\hat{i} + (-\ddot\theta L\cos\theta + \dot\theta^2 L\sin\theta)\hat{j}.\end{aligned}$$

Dotting both sides of this equation with $\hat{i}$ (as we did for velocity) we get:

$$\begin{aligned}0 &= a_B + \ddot\theta L\sin\theta + \dot\theta^2 L\cos\theta\\ \Rightarrow\quad a_B &= -\ddot\theta L\sin\theta - \dot\theta^2 L\cos\theta\\ &= -(-2.5\,\text{rad/s}^2\cdot 3\,\text{m}\cdot\frac{\sqrt{3}}{2}) - (-1.15\,\text{rad/s})^2\cdot 3\,\text{m}\cdot\frac{1}{2}\\ &= 4.51\,\text{m/s}^2.\end{aligned}$$

$$\boxed{\vec{a}_B = 4.51\,\text{m/s}^2}$$

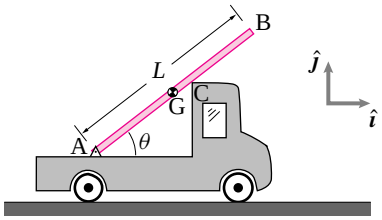


Figure 9.9: (Filename:sfig7.2.2a)

SAMPLE 9.5 *A board in the back of an accelerating truck.* A 10 ft long board AB rests in the back of a flat-bed truck as shown in Fig. 9.9. End A of the board is hinged to the bed of the truck. The truck is going on a level road at 55 mph. In preparation for overtaking a vehicle in the front the trucker accelerates at a constant rate 3 mph/s. At the instant when the speed of the truck is 60 mph, the magnitude of the relative velocity and relative acceleration of end B with respect to the bed of the truck are 10 ft/s and 12 ft/s^2, respectively. There is wind and at this instant, the board has lost contact with point C. If the angle θ between the board and the bed is 45^o at the instant mentioned, find

(a) the angular velocity and angular acceleration of the board,
(b) the acceleration of the center of mass of the board.

Solution At the instant of interest

$$\begin{aligned}
\vec{\boldsymbol{v}}_A &= \text{velocity of the truck} = 60\,\text{mph}\,\hat{\boldsymbol{\imath}} \\
&= 88\,\text{ft/s}\,\hat{\boldsymbol{\imath}} \\
\vec{\boldsymbol{a}}_A &= \text{acceleration of the truck} = 3\,\text{mph}/s \\
&= 4.4\,\text{ft/s}^2\,\hat{\boldsymbol{\imath}} \\
|\vec{\boldsymbol{v}}_{B/A}| &= v_{B/A} = \text{magnitude of relative velocity of B} \\
&= 10\,\text{ft/s} \\
|\vec{\boldsymbol{a}}_{B/A}| &= a_{B/A} = \text{magnitude of relative acceleration of B} \\
&= 12\,\text{ft/s}^2.
\end{aligned}$$

Let $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ be the angular velocity and $\dot{\vec{\boldsymbol{\omega}}} = \dot{\omega}\hat{\boldsymbol{k}}$ be the angular acceleration of the board.

(a) The relative velocity of end B of the board with respect to end A is

$$\begin{aligned}
\vec{\boldsymbol{v}}_{B/A} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \\
&= \omega\hat{\boldsymbol{k}} \times L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\
&= \omega L(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) \\
\Rightarrow \quad |\vec{\boldsymbol{v}}_{B/A}| &= \omega L \\
\Rightarrow \quad \omega &= \frac{|\vec{\boldsymbol{v}}_{B/A}|}{L} = \frac{v_{B/A}}{L} \\
&= \frac{10\,\text{ft/s}}{10\,\text{ft}} = 1\,\text{rad/s}.
\end{aligned}$$

Note that we have taken the positive value for ω because the board is rotating counterclockwise at the instant of interest (it is given that the board has lost contact with point C).
Similarly, we can compute the angular acceleration:

$$\begin{aligned}
\vec{\boldsymbol{a}}_{B/A} &= \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{B/A} - \omega^2\vec{\boldsymbol{r}}_{B/A} \\
&= \dot{\omega}\hat{\boldsymbol{k}} \times L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) - \omega^2 L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\
&= \dot{\omega}L(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) - \omega^2 L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\
\Rightarrow \quad |\vec{\boldsymbol{a}}_{B/A}| &= \sqrt{(\dot{\omega}L)^2 + (\omega^2 L)^2} = a_{B/A}\ \text{(given)} \\
\Rightarrow \quad a_{B/A}^2 &= (\dot{\omega}L)^2 + (\omega^2 L)^2 \\
\Rightarrow \quad \dot{\omega} &= \sqrt{\frac{a_{B/A}^2}{L^2} - \omega^4}
\end{aligned}$$

$$= \sqrt{\left(\frac{12\,\text{ft/s}^2}{10\,\text{ft}}\right)^2 - (1\,\text{rad/s})^4}$$
$$= \pm 0.663\,\text{rad/s}^2.$$

Once again, we select the positive value for $\dot{\omega}$ since we assume that the board accelerates counterclockwise.

$$\boxed{\vec{\boldsymbol{\omega}} = 1\,\text{rad/s}\hat{\boldsymbol{k}},\ \dot{\vec{\boldsymbol{\omega}}} = 0.663\,\text{rad/s}^2\hat{\boldsymbol{k}}}$$

(b) Now, we can easily calculate the acceleration of the center of mass as follows.

$$\begin{aligned}
\vec{\boldsymbol{a}}_G &= \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{G/A} \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{G/A} - \omega^2\vec{\boldsymbol{r}}_{G/A} \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\omega}\hat{\boldsymbol{k}} \times \frac{L}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) - \omega^2\frac{L}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\omega}\frac{L}{2}(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) - \omega^2\frac{L}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\
&= 4.4\,\text{ft/s}^2\hat{\boldsymbol{\imath}} + 0.663\,\text{rad/s}^2 \cdot \frac{10\,\text{ft}}{2} \cdot \left(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}}\right) \\
&\quad -(1\,\text{rad/s})^2 \cdot \frac{10\,\text{ft}}{2} \cdot \left(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right) \\
&= -1.48\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 1.19\,\text{ft/s}^2\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_G = -(1.48\hat{\boldsymbol{\imath}} + 1.19\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2}$$

Comments: This problem is admittedly artificial. We, however, solve this problem to show kinematic calculations.

9.2 Unconstrained dynamics of 2-D rigid-body planar motion

We will now use the kinematics of a rigid body (Sect. 9.1) to allow us to "evaluate" the motion quantities, namely linear momentum, angular momentum, kinetic energy, and their rates of change. By "evaluate" we mean that we would like to express the motion quantities in terms of *simple kinematic measures*① rather than as sums over Avogadro's number of particles (There are on the order of 10^{23} atoms in a typical engineering part.). Even neglecting atoms and viewing matter as continuous we could be stuck with integrals over complex domains. The simple measures of motion will be the displacement, velocity and acceleration of one reference point $0'$ on the body ($\vec{r}_{0'}$, $\vec{v}_{0'}$ and $\vec{a}_{0'}$) and the rotation, angular velocity, and angular acceleration of the body (θ, $\vec{\omega}$ and $\vec{\alpha}$).

① **Advanced aside:** What we call "simple kinematic measures" are examples of "generalized coordinates" in more advanced treatments. The idea sounds intimidating, but is simply this. If something can only move so many ways, you should only keep track of the motion with so many variables.

We will treat all bodies as if they are squished into the plane and moving in the plane. But the analysis is perfectly sensible for bodies that are symmetric with respect to the plane containing the velocities (see Box 9.2 on page 509).

Momenta balance for a rigid body

As always, once you have defined the system and the forces acting on it by drawing a free body diagram, the basic momentum balance equations are applicable (and exact for engineering purposes). Namely,

$$\text{Linear momentum balance:} \qquad \sum \vec{F}_i = \dot{\vec{L}} \qquad \text{and}$$
$$\text{Angular momentum balance:} \qquad \sum \vec{M}_{i/0} = \dot{\vec{H}}_{\mathrm{O}}.$$

Where the same point 0, any point, is used on both sides of the angular momentum balance equation.

We also have the power balance equations, which, for a system with no internal energy or dissipation is

$$\text{Power balance:} \qquad P = \dot{E}_{\mathrm{K}}.$$

The left hand sides of the momentum balance equations are evaluated the same way, whether the system is composed of one body or many, whether the bodies are deformable or not, and whether the points move in straight lines, circles, hither and thither, or not at all. It is the right hand sides of the momentum equations that involve the motion. Similarly, in the energy balance equations the applied power P only depends on the position of the forces and the motions of the material points they touch. But the kinetic energy E_{K} and its rate of change depend on the motion of the whole system. You already know how to evaluate the momenta and energy, and their rates of change, for a variety of special cases, namely

- Systems composed of particles where all the accelerations are known (Chapter 5);
- Systems where all points have the same acceleration. That is, the system moves like a rigid body that does not rotate (Chapter 6); and
- Systems where all points move in circles about the same fixed axis. That is, the system moves like a rigid body that is rotating about a fixed skewer (Chapters 7 and 8).

Now we go on to consider the general 2-D motions of a planar rigid body. Its now a little harder to evaluate $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{\mathrm{O}}$, $\dot{\vec{H}}_{\mathrm{O}}$, E_{K} and $\dot{E}_{\mathrm{K}}$. But not much.

Summary of the motion quantities

As for most of the special cases described in the chapters, table I in the back of the book describes the motion quantities for the planar motions we consider in this chapter. Most relevant is row (7).

The basic idea is that the momenta for general motion, which involves translation and rotation, is the sum of the momenta (both linear and angular, and their rates of change too) from those two effects. Namely, the linear momentum is described, as for all systems, by the motion of the center of mass

$$\vec{\boldsymbol{L}} = m_{\text{tot}}\vec{\boldsymbol{v}}_{\text{cm}} \quad \text{and} \quad \dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}},$$

and the angular momentum has two contributions, one from the motion of the center of mass and one from rotation about the center of mass,

$$\vec{\boldsymbol{H}}_{\text{O}} = \underbrace{\vec{\boldsymbol{r}}_{\text{cm/O}} \times (m_{\text{tot}}\vec{\boldsymbol{v}}_{\text{cm}})}_{\text{Angular momentum due to motion of the center of mass}} + \underbrace{I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}}_{\text{Angular momentum due to motion relative to the center of mass}} \quad \text{and} \quad \dot{\vec{\boldsymbol{H}}}_{\text{O}} = \vec{\boldsymbol{r}}_{\text{cm/O}} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}) + I_{zz}^{\text{cm}}\dot{\vec{\boldsymbol{\omega}}}.$$

The kinetic energy and its rate of change are given by

$$E_{\text{K}} = \underbrace{\frac{1}{2}m_{\text{tot}}v_{cm}^2}_{\text{kinetic energy from center of mass motion}} + \underbrace{\frac{1}{2}I_{zz}^{\text{cm}}\omega^2}_{\text{kinetic energy relative to the center of mass}} \quad \text{and} \quad \dot{E}_{\text{K}} = m_{\text{tot}}v\dot{v} + I_{zz}^{\text{cm}}\omega\dot{\omega}$$

9.1 *THEORY*
The 2D mechanics of 3D objects

Precisely speaking, the situation is this. For a three dimensional object with planar motion in the xy plane then the two dimensional analysis of this chapter is the exact projection of a proper three dimensional analysis. All forces and displacements are projected onto the xy plane and all moment terms are projected in the z direction.

9.2 *THEORY*
The work of a moving force and of a couple

The work of a force acting on a body from state one to state two is

$$W_{12} = \int_{t_1}^{t_2} P dt.$$

But sometimes we like to think not of the time integral of the power, but of the path integral of the moving force. So we rearrange this integral as follows.

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} P dt \\ &= \int_{t_1}^{t_2} \vec{\boldsymbol{F}} \cdot \underbrace{\vec{\boldsymbol{v}} dt}_{d\vec{\boldsymbol{r}}} \\ &= \int_{\vec{\boldsymbol{r}}_1}^{vr_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \qquad (9.12) \end{aligned}$$

(9.13)

The validity of equation 9.13 depends on the force acting on the same material point of the moving body as it moves from position 1 to position 2; i.e., the force moves with the body. If the material point of force application changes with time, equation 9.13 is senseless and should be replaced with the following more generally applicable equation:

$$W_{12} \equiv \int_{t_1}^{t_2} P\ dt = \int_{t_1}^{t_2} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} dt = \int P dt \qquad (9.14)$$

where $\vec{\boldsymbol{v}}$ is the velocity of the material point at the instantaneous location of the applied force.

The distinction between 9.13 and 9.14 is subtle. As an example think of standing still and dragging your hand on a passing train. Your hand slows down the train. It does (negative) work on the train. Naively you might think that the work of your hand on the train is zero because your hand is not moving. But if you read the paragraph above carefully you will see that the power of the hand force on the train is the force on the train dotted with the velocity of the train (*not* with the velocity of your hand). Thus, your hand does negative work on the train. In this case equation 9.14 applies to the train and 9.13 does not. Simultaneously, the train does *no* work on your hand since your hand is not moving as both 9.13 and 9.14 correctly give.

Work of an applied torque

By thinking of an applied torque as really a distribution of forces, the work of an applied torque is the sum of the contributions of the applied forces. If a collection of forces equivalent to a torque is applied to one rigid body the power of these forces turns out to be $\vec{\boldsymbol{M}} \cdot \vec{\boldsymbol{\omega}}$. At a given angular velocity a bigger torque applies more. And a given torque applies more power to a faster spinning object.

Here's a quick derivation for a collection of forces $\vec{\boldsymbol{F}}_\mathrm{i}$ that add to zero acting at points with positions $\vec{\boldsymbol{r}}_\mathrm{i}$ relative to a reference point on the body o'.

$$\begin{aligned} P &= \sum \vec{\boldsymbol{F}}_\mathrm{i} \cdot \vec{\boldsymbol{v}}_\mathrm{i} && (9.15) \\ &= \sum \vec{\boldsymbol{F}}_\mathrm{i} \cdot \left(\vec{\boldsymbol{v}}_{\mathrm{o}'} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{i/o}'} \right) && (9.16) \\ &= \vec{\boldsymbol{v}}_{\mathrm{o}'} \cdot \underbrace{\sum \vec{\boldsymbol{F}}_\mathrm{i}}_{\vec{\boldsymbol{0}}} + \sum \vec{\boldsymbol{F}}_\mathrm{i} \cdot \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{i/o}'} && (9.17) \\ &= \sum \vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{r}}_{\mathrm{i/o}'} \times \vec{\boldsymbol{F}}_\mathrm{i} && (9.18) \\ &= \vec{\boldsymbol{\omega}} \cdot \sum \vec{\boldsymbol{r}}_{\mathrm{i/o}'} \times \vec{\boldsymbol{F}}_\mathrm{i} && (9.19) \\ &= \vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{M}}_{\mathrm{o}'} && (9.20) \end{aligned}$$

(9.21)

Work of a general force distribution

A general force distribution has, by reasoning close to that above, a power of:

$$P = \vec{\boldsymbol{F}}_\mathrm{tot} \cdot \vec{\boldsymbol{v}}_{\mathrm{o}'} + \vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{M}}_{\mathrm{o}'}. \qquad (9.22)$$

(9.23)

For a given force system applied to a given body in a given motion any point o' can be used. The terms in the formula above will depend on o', but the sum does not.

SAMPLE 9.6 *A plate tumbling in space.* A rectangular plate of mass $m = 0.5\,\text{kg}$, $I_{zz}^{\text{cm}} = 2.08 \times 10^{-3}\,\text{kg}\cdot\text{m}^2$, and dimensions $a = 200\,\text{mm}$ and $b = 100\,\text{mm}$ is pushed by a force $\vec{F} = 0.5\,\text{N}\hat{\imath}$, acting at $d = 30\,\text{mm}$ away from the mass-center, as shown in the figure. Assume that the force remains constant in magnitude and direction but remains attached to the material point P of the plate. There is no gravity.

(a) Find the initial acceleration of the mass-center.
(b) Find the initial angular acceleration of the plate.
(c) Write the equations of motion of the plate (for both linear and angular motion).

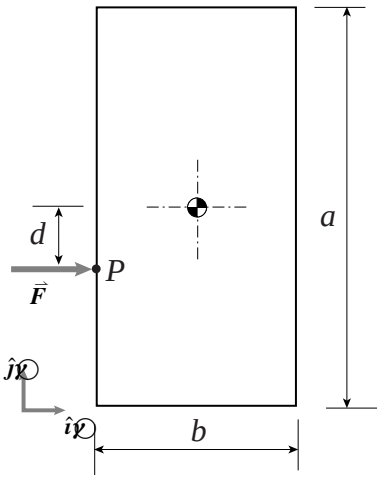


Figure 9.10: (Filename:sfig9.tumblingplate1)

Solution The only force acting on the plate is the applied force $\vec{F}$. Thus, Fig. 9.10 is also the free body diagram of the plate at the start of motion.

(a) From the linear momentum balance we get,

$$\begin{aligned} \sum \vec{F} &= m\vec{a}_{\text{cm}} \\ \Rightarrow \quad \vec{a}_{\text{cm}} &= \frac{\sum \vec{F}}{m} \\ &= \frac{0.5\,\text{N}\hat{\imath}}{0.5\,\text{kg}} = 1\,\text{m/s}^2\hat{\imath} \end{aligned}$$

which is the initial acceleration of the mass-center.

$$\boxed{\vec{a}_{\text{cm}} = 1\,\text{m/s}^2\hat{\imath}}$$

(b) From the angular momentum balance about the mass-center, we get

$$\begin{aligned} \vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}} \\ Fd\hat{k} &= I_{zz}^{\text{cm}}\dot{\vec{\omega}} \\ \Rightarrow \quad \dot{\vec{\omega}} &= \frac{Fd}{I_{zz}^{\text{cm}}}\hat{k} \\ &= \frac{0.5\,\text{N}\cdot 0.03\,\text{m}}{2.08\,\text{kg}\cdot\text{m}^2} \\ &= 7.2\,\text{rad/s}^2\hat{k} \end{aligned}$$

which is the initial angular acceleration of the plate.

$$\boxed{\dot{\vec{\omega}} = 7.2\,\text{rad/s}^2\hat{k}}$$

(c) To find the equations of motion, we can use the linear momentum balance and the angular momentum balance as we have done above. So, why aren't the equations obtained above for the linear acceleration, $\vec{a}_{\text{cm}} = F/m\hat{\imath}$, and the angular acceleration, $\dot{\vec{\omega}} = Fd/I_{zz}^{\text{cm}}\hat{k}$, qualified to be called equations of motion? Because, they are not valid for a general configuration of the plate during its motion. The expressions for the accelerations are valid only in the initial configuration (and hence those are initial accelerations).

Let us first draw a free body diagram of the plate in a general configuration during its motion (see Fig. 9.11). Assume the center of mass to be displaced by $x\hat{\imath}$ and $y\hat{\jmath}$, and the longitudinal axis of the plate to be rotated by $\theta\hat{k}$ with respect to the vertical (inertial y-axis). The applied force remains horizontal and attached to the material point P, as stated in the problem. The linear momentum balance gives

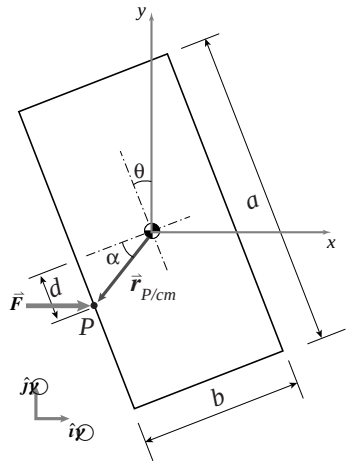


Figure 9.11: (Filename:sfig9.tumblingplate1.a)

$$\begin{aligned} \sum \vec{F} &= m\vec{a}_{\text{cm}} \\ \Rightarrow \quad \vec{a}_{\text{cm}} &= \frac{\sum \vec{F}}{m} \end{aligned}$$

$$\begin{aligned} \text{or} \qquad \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} &= \frac{F}{m}\hat{\boldsymbol{\imath}} \\ \Rightarrow \quad \ddot{x} &= \frac{F}{m} \\ \ddot{y} &= 0 \end{aligned}$$

Since F/m is constant, the equations of motion of the center of mass indicate that the acceleration is constant and that the mass-center moves in the x-direction.

Similarly, we now use angular momentum balance to determine the rotation (angular motion) of the plate. The angular momentum balance about the mass-center give

$$\begin{aligned} \vec{\boldsymbol{M}}_{\text{cm}} &= \dot{\vec{\boldsymbol{H}}}_{\text{cm}} \\ \vec{\boldsymbol{r}}_{\text{P/cm}} \times \vec{\boldsymbol{F}} &= I_{zz}^{\text{cm}}\ddot{\theta}\hat{\boldsymbol{k}} \end{aligned}$$

Now,

$$\begin{aligned} \vec{\boldsymbol{r}}_{\text{P/cm}} &= -r[\cos(\theta+\alpha)\hat{\boldsymbol{\imath}} + \sin(\theta+\alpha)\hat{\boldsymbol{\jmath}}] \\ \vec{\boldsymbol{F}} &= F\hat{\boldsymbol{\imath}} \\ \Rightarrow \quad \vec{\boldsymbol{r}}_{\text{P/cm}} \times \vec{\boldsymbol{F}} &= Fr\sin(\theta+\alpha)\hat{\boldsymbol{k}} \\ &= F(\underbrace{r\sin\alpha}_{d}\cos\theta + \underbrace{r\cos\alpha}_{b}\sin\theta)\hat{\boldsymbol{k}} \\ &= F(d\cos\theta + b\sin\theta)\hat{\boldsymbol{k}} \end{aligned}$$

Thus,

$$\ddot{\theta} = \frac{Fd}{I_{zz}^{\text{cm}}}\left(\cos\theta + \frac{b}{d}\sin\theta\right).$$

Thus, we have got the equations of motion for both linear and the angular motion.

$$\boxed{\ddot{x} = \tfrac{F}{m}, \quad \ddot{y} = 0, \quad \ddot{\theta} = \tfrac{Fd}{I_{zz}^{\text{cm}}}\left(\cos\theta + \tfrac{b}{d}\sin\theta\right)}$$

9.3 Special topics in planar kinematics

Pure rolling in 2-D

In this section, we would like to add to the vocabulary of special motions by considering *pure rolling*. Most commonly, one discusses pure rolling of round objects on flat ground, like wheels and balls, but we will also mention more advanced topics.

2-D rolling of a round wheel on level ground

The simplest case, the no-slip rolling of a round wheel, is an instructive starting point. First, we define the geometric and kinematic variables as shown in Fig. 9.12. For convenience, we pick a point D which was at $x_D = 0$ at the start of rolling, when $x_C = 0$. The key to the kinematics is that:

> *The arc length traversed on the wheel is the distance traveled by the wheel center.*

That is,

$$\begin{aligned} x_C &= s_D \\ &= R\phi \\ \Rightarrow v_C = \dot{x}_C &= R\dot{\phi} \\ \Rightarrow a_C = \dot{v}_C = \ddot{x}_C &= R\ddot{\phi} \end{aligned}$$

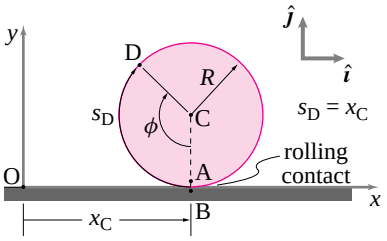


Figure 9.12: Pure rolling of a round wheel on a level support.

(Filename:tfigure7.2D.pure.rolling)

So the rolling condition amounts to the following set of restrictions on the position of C, $\vec{r}_C$, and the rotations of the wheel ϕ:

$$\vec{r}_C = R\phi\hat{\imath} + R\hat{\jmath}, \quad \vec{v}_C = R\dot{\phi}\hat{\imath}, \quad \vec{a}_C = R\ddot{\phi}\hat{\imath}, \quad \vec{\omega} = -\dot{\phi}\hat{k}, \text{ and } \quad \vec{\alpha} = \dot{\vec{\omega}} = -\ddot{\phi}\hat{k}.$$

If we want to track the motion of a particular point, say D, we could do so by using the following parametric formula:

$$\begin{aligned} \vec{r}_D &= \vec{r}_C + \vec{r}_{D/C} \\ &= R(\phi\hat{\imath} + \hat{\jmath}) + R(-\sin\phi\hat{\imath} - \cos\phi\hat{\jmath}) \\ &= R\left[(\phi - \sin\phi)\hat{\imath} + (1 - \cos\phi)\hat{\jmath})\right] \\ \Rightarrow \vec{v}_D &= R\left[(\dot{\phi}(1 - \cos\phi)\hat{\imath} + \dot{\phi}\sin\phi\hat{\jmath})\right] \qquad (9.24) \\ \Rightarrow \vec{a}_D &= R\dot{\phi}^2(\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}). \end{aligned}$$

assuming $\dot{\phi}$= constant

Note that if $\phi = 0$ or 2π or 4π, etc., then the point D is on the ground and eqn. (9.24) correctly gives that

$$\vec{v}_D = R\left[\dot{\phi}(\underbrace{1 - \overbrace{\cos(2n\pi)}^{1}}_{0})\right]\hat{\imath} + \dot{\phi}\underbrace{\sin(2n\pi)}_{0}\hat{\jmath} = \vec{0}.$$

Instantaneous Kinematics

Instead of tracking the wheel from its start, we could analyze the kinematics at the instant of interest. Here, we make the observation that the wheel rolls without slip. Therefore, the point on the wheel touching the ground has no velocity relative to the ground.

Velocity of point on the wheel touching the ground

Velocity of ground = $\vec{\mathbf{0}}$

$$\underbrace{\vec{v}_A}_{\vec{v}_A} = \underbrace{\vec{v}_B}_{\vec{\mathbf{0}}} \tag{9.25}$$

Now, we know how to calculate the velocity of points on a rigid body. So,

$$\vec{v}_A = \vec{v}_C + \vec{v}_{A/C},$$

where, since A and C are on the same rigid body (Fig. 9.12), we have from eqn. (8.14) that

$$\vec{v}_{A/C} = \vec{\omega} \times \vec{r}_{A/C}.$$

Putting this equation together with eqn. (9.25), we get

$$\begin{aligned} \vec{v}_A &= \vec{v}_B \\ \Rightarrow \underbrace{\vec{v}_C}_{v_C\hat{\imath}} + \underbrace{\vec{\omega}}_{\omega\hat{k}} \times \underbrace{\vec{r}_{A/C}}_{-R\hat{\jmath}} &= \vec{\mathbf{0}} \\ \Rightarrow v_C\hat{\imath} + \omega R\hat{\imath} &= \vec{\mathbf{0}} \\ \Rightarrow \boxed{v_C = -\omega R}. & \end{aligned}$$

We use $\vec{v}_C = v\hat{\imath}$ since the center of the wheel goes neither up nor down. Note that if you measure the angle by ϕ, like we did before, then $\vec{\omega} = -\dot{\phi}\hat{k}$ so that positive rotation rate is in the counter-clockwise direction. Thus, $v_C = -\omega R = -(-\dot{\phi})R = \dot{\phi}R$.

Since there is always *some* point of the wheel touching the ground, we know that $v_C = -\omega R$ for all time. Therefore,

$$\vec{a}_C = \dot{v}_C\hat{\imath} = -\dot{\omega}R\hat{\imath}.$$

Rolling of round objects on round surfaces

For round objects rolling on or in another round object, the analysis is similar to that for rolling on a flat surface. A common application is the so-called epicyclic, hypocyclic, or planetary gears (See Box 9.3 on planetary gears on page 516). Referring to Fig. 9.13, we can calculate the velocity of C with respect to a fixed frame two ways and compare:

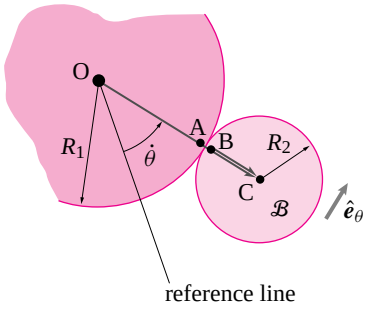


Figure 9.13: (Filename:tfigure7.rolling.on.another)

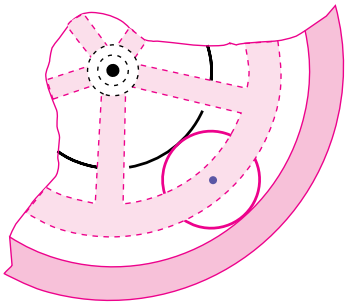
$$\begin{aligned} \vec{v}_C &= \vec{v}_B + \vec{v}_{C/B} \\ \vec{v}_C &= \overbrace{\underbrace{\vec{v}_A}_{\vec{\mathbf{0}}} + \underbrace{\vec{v}_{B/A}}_{\vec{\mathbf{0}}}}^{\vec{v}_B} + \vec{v}_{C/B}. \\ \dot{\theta}(R_1+R_2)\hat{e}_\theta &= \omega_{\mathcal{B}} R_2 \hat{e}_\theta \\ \Rightarrow \omega_{\mathcal{B}} &= \frac{\dot{\theta}(R_1+R_2)}{R_2} = \dot{\theta}(1+\frac{R_1}{R_2}). \end{aligned}$$

Example: **Two quarters.**

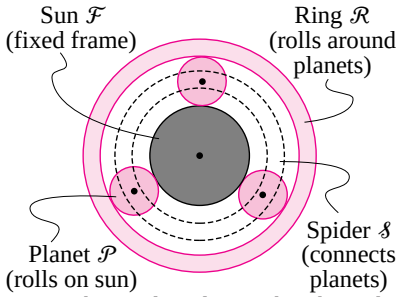
The formula above can be tested in the case of $R_1 = R_2$ by using two quarters or two dimes on a table. Roll one quarter, call it $\mathcal{B}$, around around another quarter pressed fast to the table. You will see that as the rolling quarter $\mathcal{B}$ travels around the stationary quarter one time, it makes two full revolutions. That is, the orientation of $\mathcal{B}$ changes twice as fast as the angle of the line from the center of the stationary quarter to its center. Or, in the language of the calculation above, $\omega_{\mathcal{B}} = 2\dot{\theta}$. □

9.3 The Sturmey-Archer hub

In 1903, the year the Wright Brothers first flew powered airplanes, the Sturmey-Archer company patented the internal-hub transmission. This marvel of engineering was sold on the best bikes until finicky but fast racing bicycles using derailleurs started to push them out of the market in the 1960's. The internal-hub Sturmey-Archer gear utilizes a system called *planetary gears*, gears which roll around other gears. See the figure below. In order to understand



this gear system, we need to understand its kinematics—the motion of its parts. Referring to figure above, the central 'sun' gear $\mathcal{F}$ is stationary, at least we treat it as stationary in this discussion since it is fixed to the bike frame, so it is fixed in body $\mathcal{F}$. The 'planet' gears roll around the sun gear. Let's call one of these planets $\mathcal{P}$. The spider $\mathcal{S}$ connects the centers of the rolling planets. Finally, the ring gear $\mathcal{R}$ rotates around the sun.



The gear transmission steps up the angular velocity when the spider $\mathcal{S}$ is driven and ring $\mathcal{R}$, which moves faster, is connected to the wheel. The transmission steps down the angular velocity when the ring gear is driven and the slower spider is connected to the wheel. The third 'speed' in the three-speed gear transmission is direct drive (the wheel is driven directly).

What are the 'gear ratios' in the planetary gear system? The 'trick' is to recognize that for rolling contact that the contacting points have the same velocity, $\vec{v}_A = \vec{v}_B$ and $\vec{v}_D = \vec{v}_E$. Let's define some terms.

$$\vec{\omega}_{\mathcal{S}} = \omega_{\mathcal{S}}\hat{k} \quad \text{angular velocity of the spider}$$
$$\vec{\omega}_{\mathcal{P}} = \omega_{\mathcal{P}}\hat{k} \quad \text{angular velocity of the planet}$$
$$\vec{\omega}_{\mathcal{R}} = \omega_{\mathcal{R}}\hat{k} \quad \text{angular velocity of the ring}$$

Now, we can find the relation of these angular velocities as follows.

Look at the velocity of point C in two ways. First,

A point on the spider	A point on the planetary gear

$$\begin{aligned} \vec{v}_C &= \vec{v}_C \\ \Rightarrow \vec{\omega}_{\mathcal{S}} \times \vec{r}_C &= \underbrace{\vec{v}_B}_{\vec{0}} + \vec{\omega}_{\mathcal{P}} \times \vec{r}_{C/B} \\ \Rightarrow \omega_{\mathcal{S}} r_C &= \omega_{\mathcal{P}} R_P \\ \Rightarrow \omega_{\mathcal{P}} &= \frac{r_C}{R_P}\omega_{\mathcal{S}} \end{aligned} \tag{9.26}$$

Next, let's look at point D and E:

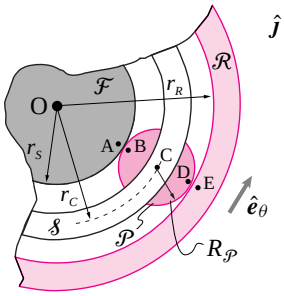
$$\begin{aligned} \vec{v}_D &= \vec{v}_E \\ \vec{v}_A + \vec{v}_{D/A} &= \vec{\omega}_{\mathcal{R}} \times \vec{r}_R \\ \vec{0} + \vec{\omega}_{\mathcal{P}} \times \vec{r}_{D/A} &= \omega_{\mathcal{R}}\hat{k} \times \vec{r}_R \\ \omega_{\mathcal{P}}(2R_P)\hat{e}_\theta &= \omega_{\mathcal{R}} r_R \hat{e}_\theta \\ \Rightarrow \omega_{\mathcal{R}} &= \frac{2R_P}{r_R}\underbrace{\omega_{\mathcal{P}}}_{\omega_{\mathcal{P}} = \frac{r_C}{R_P}\omega_{\mathcal{S}}} \end{aligned}$$

$r_C = r_S + R_P$

$$\begin{aligned} \omega_{\mathcal{R}} &= \frac{2R_P}{r_R}\frac{\overbrace{r_C}^{r_C = r_S + R_P}}{R_P}\omega_{\mathcal{S}} \\ &= \frac{2(r_S + R_P)}{\underbrace{r_R}_{r_R = r_S + 2R_P}}\omega_{\mathcal{S}} \end{aligned}$$

$$\Rightarrow \frac{\omega_{\mathcal{R}}}{\omega_{\mathcal{S}}} = 2\frac{1 + \frac{R_P}{r_S}}{1 + \frac{2R_P}{r_S}} = \text{angular velocity step-up.}$$

Typically, the gears



Rolling contact:
$\vec{v}_A = \vec{v}_B$
$\vec{v}_D = \vec{v}_E$
$\vec{v}_C = \vec{v}_C$

have radius ratio of $\frac{R_P}{r_S} = \frac{3}{2}$ which gives a gear ratio of $\frac{5}{4}$. Thus, the ratio of the highest gear to the lowest gear on a Sturmey-Archer hub is $\frac{5}{4}/\frac{4}{5} = \frac{25}{16} = 1.5625$. You might compare this ratio to that of a modern mountain bike, with eighteen or twenty-one gears, where the ratio of the highest gear to the lowest is about 4:1.

SAMPLE 9.7 A cylinder of diameter 500 mm rolls down an inclined plane with uniform acceleration (of the center of mass) $a = 0.1\,\text{m/s}^2$. At an instant t_0, the mass-center has speed $v_0 = 0.5\,\text{m/s}$.

(a) Find the angular speed ω and the angular acceleration $\dot{\omega}$ at t_0.
(b) How many revolutions does the cylineder make in the next 2 seconds?
(c) What is the distance travelled by the center of mass in those 2 seconds?

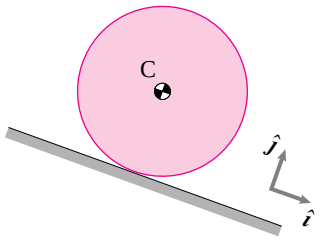


Figure 9.14: (Filename:sfig9.rolling.may00)

Solution This problem is about simple kinematic calculations. We are given the velocity, $\dot{x}$, and the acceleration, $xddot$, of the center of mass. We are supposed to find angular velocity ω, angular acceleration $\dot{\omega}$, angular displacement θ in 2 seconds, and the corresponding linear distance x along the incline. The radius of the cylinder $R = \text{diameter}/2 = 0.25\,\text{m}$.

(a) From the kinematics of pure rolling,

$$\begin{aligned} \omega &= \frac{\dot{x}}{R} = \frac{0.5\,\text{m/s}}{0.25\,\text{m}} = 2\,\text{rad/s}, \\ \dot{\omega} &= \frac{\ddot{x}}{R} = \frac{0.1\,\text{m/s}^2}{0.25\,\text{m}} = 0.4\,\text{rad/s}^2. \end{aligned}$$

$$\boxed{\omega = 2\,\text{rad/s}, \quad \dot{\omega} = 0.4\,\text{rad/s}^2}$$

(b) We can find the number of revolutions the cylinder makes in 2 seconds by solving for the angular dispcament θ in this time period. Since,

$$\ddot{\theta} \equiv \dot{\omega} = \text{constant},$$

we integrate this equation twice and substitute the initial conditions, $\dot{\theta}(t = 0) = \omega = 2\,\text{rad/s}$ and $\theta(t = 0) = 0$, to get

$$\begin{aligned} \theta(t) &= \omega t + \frac{1}{2}\dot{\omega}t^2 \\ \Rightarrow \quad \theta(t = 2\,\text{s}) &= (2\,\text{rad/s}) \cdot (2\,\text{s}) + \frac{1}{2}(0.4\,\text{rad/s}) \cdot (4\,\text{s}^2) \\ &= 4.8\,\text{rad} = \frac{4.8}{2\pi}\,\text{rev} = 0.76\,\text{rev}. \end{aligned}$$

$$\boxed{\theta = 0.76\,\text{rev}}$$

(c) Now that we know the angular displacement θ, the distance travelled by the mass-center is the arc-length corresponding to θ, *i.e.*,

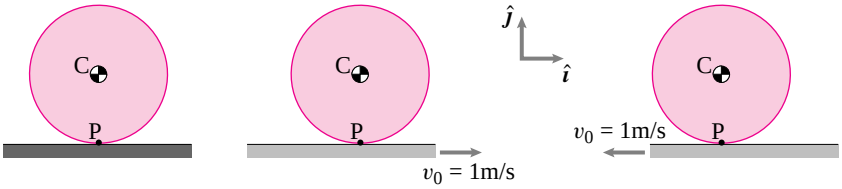
$$x = R\theta = (0.25\,\text{m}) \cdot (4.8) = 1.2\,\text{m}.$$

$$\boxed{x = 1.2\,\text{m}}$$

Note that we could have found the distance travelled by the mass-center by integrating the equation $\ddot{x} = 0.1\,\text{m/s}^2$ twice

SAMPLE 9.8 *Condition of pure rolling.* A cylinder of radius $R = 20\,\text{cm}$ rolls on a flat surface with absolute angular speed $\omega = 12\,\text{rad/s}$ under the conditions shown in the figure (In cases (ii) and (iii), you may think of the 'flat surface' as a conveyor belt). In each case,

(a) Write the condition of pure rolling.
(b) Find the velocity of the center C of the cylinder.



(i) Fixed base (ii) Base moves to the right (iii) Base moves to the left

Figure 9.15: (Filename:sfig7.rolling1)

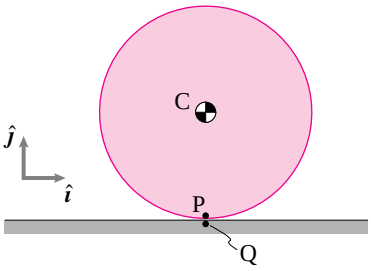


Figure 9.16: The cylinder rolls on the flat surface. Instantaneously, point P on the cylinder is in contact with point Q on the flat surface. For pure rolling, points P and Q must have the same velocity.

(Filename:sfig7.rolling1a)

Solution At any instant during rolling, the cylinder makes a point-contact with the flat surface. Let the point of instantaneous contact on the cylinder be P, and let the corresponding point on the flat surface be Q. The condition of pure rolling, in each case, is $\vec{v}_P = \vec{v}_Q$, that is, there is no relative motion between the two contacting points (a relative motion will imply slip). Now, we analyze each case.

Case(i) In this case, the bottom surface is fixed. Therefore,

(a) The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = \vec{0}$.
(b) Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = \vec{0} + (-\omega\hat{k}) \times R\hat{\jmath} \\ &= \omega R\hat{\imath} = (12\,\text{rad/s}) \cdot (0.2\,\text{m})\hat{\imath} = 2.4\,\text{m/s}\hat{\imath}.\end{aligned}$$

Case(ii) In this case, the bottom surface moves with velocity $\vec{v} = 1\,\text{m/s}\hat{\imath}$. Therefore, $\vec{v}_Q = 1\,\text{m/s}\hat{\imath}$. Thus,

(a) The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = 1\,\text{m/s}\hat{\imath}$.
(b) Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = v_0\hat{\imath} + \omega R\hat{\imath} \\ &= 1\,\text{m/s}\hat{\imath} + 2.4\,\text{m/s}\hat{\imath} = 3.4\,\text{m/s}\hat{\imath}.\end{aligned}$$

Case(iii) In this case, the bottom surface moves with velocity $\vec{v} = -1\,\text{m/s}\hat{\imath}$. Therefore, $\vec{v}_Q = -1\,\text{m/s}\hat{\imath}$. Thus,

(a) The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = -1\,\text{m/s}\hat{\imath}$.
(b) Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = -v_0\hat{\imath} + \omega R\hat{\imath} \\ &= -1\,\text{m/s}\hat{\imath} + 2.4\,\text{m/s}\hat{\imath} = 1.4\,\text{m/s}\hat{\imath}.\end{aligned}$$

(a): (i) $\vec{v}_P = \vec{0}$, (ii) $\vec{v}_P = 1\,\text{m/s}\hat{\imath}$, (iii) $\vec{v}_P = -1\,\text{m/s}\hat{\imath}$,

(b): (i) $\vec{v}_C = 2.4\,\text{m/s}\hat{\imath}$, (ii) $\vec{v}_C = 3.4\,\text{m/s}\hat{\imath}$, (iii) $\vec{v}_C = 1.4\,\text{m/s}\hat{\imath}$

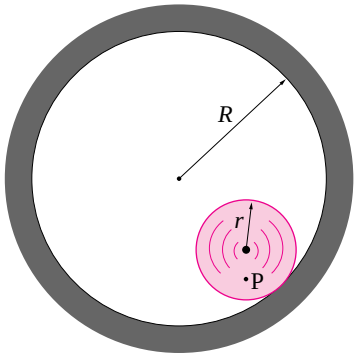


Figure 9.17: A uniform disk of radius r rolls without slipping inside a fixed cylinder.

(Filename:sfig6.5.3)

SAMPLE 9.9 *Motion of a point on a disk rolling inside a cylinder.* A uniform disk of radius r rolls without slipping with constant angular speed ω inside a fixed cylinder of radius R. A point P is marked on the disk at a distance ℓ ($\ell < r$) from the center of the disk. at a general time t during rolling, find

(a) the position of point P,
(b) the velocity of point P, and
(c) the acceleration of point P

Solution Let the disk be vertically below the center of the cylinder at $t = 0$ s such that point P is vertically above the center of the disk (Fig. 9.18). At this instant, Q is the point of contact between the disk and the cylinder. Let the disk roll for time t such that at instant t the line joining the two centers (line OC) makes an angle ϕ with its vertical position at $t = 0$ s. Since the disk has rolled for time t at a constant angular speed ω, point P has rotated counter-clockwise by an angle $\theta = \omega t$ from its original vertical position P'.

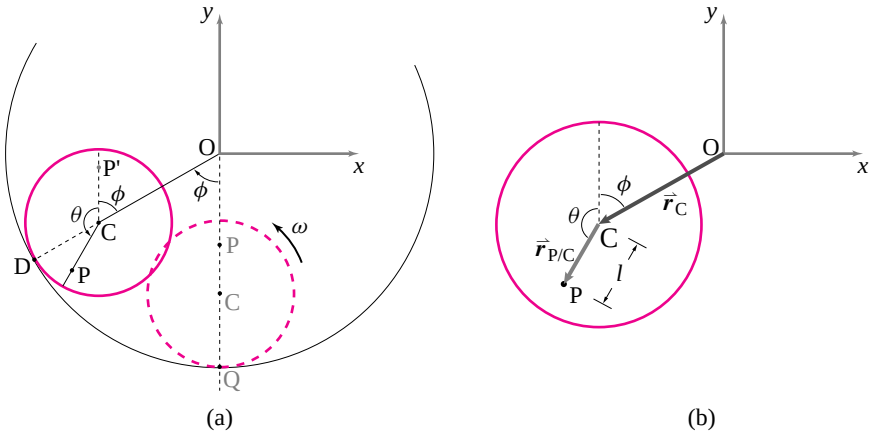


Figure 9.18: Geometry of motion: keeping track of point P while the disk rolls for time t, rotating by angle $\theta = \omega t$ inside the cylinder.

(Filename:sfig6.5.3a)

(a) **Position of point P:** From Fig. 9.18(b) we can write

$$\vec{r}_P = \vec{r}_C + \vec{r}_{P/C} = (R - r)\hat{\boldsymbol{\lambda}}_{OC} + \ell\hat{\boldsymbol{\lambda}}_{CP}$$

where

$$\begin{aligned} \hat{\boldsymbol{\lambda}}_{OC} &= \text{a unit vector along } OC = -\sin\phi\hat{\boldsymbol{\imath}} - \cos\phi\hat{\boldsymbol{\jmath}}, \\ \hat{\boldsymbol{\lambda}}_{CP} &= \text{a unit vector along } CP = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}. \end{aligned}$$

Thus,

$$\vec{r}_P = [-(R - r)\sin\phi - \ell\sin\theta]\hat{\boldsymbol{\imath}} + [-(R - r)\cos\phi + \ell\cos\theta]\hat{\boldsymbol{\jmath}}.$$

We have thus obtained an expression for the position vector of point P as a function of ϕ and θ. Since we also want to find velocity and acceleration of point P, it will be nice to express $\vec{r}_P$ as a function of t. As noted above, $\theta = \omega t$; but how do we find ϕ as a function of t? Note that the center of the disk C is going around point O in circles with angular velocity $-\dot{\phi}\hat{\boldsymbol{k}}$. The disk,

however, is rotating with angular velocity $\vec{\omega} = \omega\hat{k}$ about the instantaneous center of rotation, point D. Therefore, we can calculate the velocity of point C in two ways:

$$\begin{aligned} & & \vec{v}_C &= \vec{v}_C \\ \text{or} & & \vec{\omega} \times \vec{r}_{C/D} &= -\dot{\phi}\hat{k} \times \vec{r}_{C/O} \\ \text{or} & & \omega\hat{k} \times r(-\hat{\lambda}_{OC}) &= -\dot{\phi}\hat{k} \times (R-r)\hat{\lambda}_{OC} \\ \text{or} & & -\omega r(\hat{k} \times \hat{\lambda}_{OC}) &= -\dot{\phi}(R-r)(\hat{k} \times \hat{\lambda}_{OC}) \\ & \Rightarrow & \frac{r}{R-r}\omega &= \dot{\phi}. \end{aligned}$$

Integrating the last expression with respect to time, we obtain

$$\phi = \frac{r}{R-r}\omega t.$$

Let

$$q = \frac{r}{R-r},$$

then, the position vector of point P may now be written as

$$\vec{r}_P = [-(R-r)\sin(q\omega t) - \ell\sin(\omega t)]\hat{\imath} + [-(R-r)\cos(q\omega t) + \ell\cos(\omega t)]\hat{\jmath}. \tag{9.27}$$

(b) **Velocity of point P:** Differentiating Eqn. (9.27) once with respect to time we get

$$\vec{v}_P = -\omega[(R-r)q\cos(q\omega t) + \ell\cos(\omega t)]\hat{\imath} + \omega[(R-r)q\sin(q\omega t) - \ell\sin(\omega t)]\hat{\jmath}.$$

Substituting $(R-r)q = r$ in $\vec{v}_P$ we get

$$\vec{v}_P = -\omega r[\{\cos(q\omega t) + \frac{\ell}{r}\cos(\omega t)\}\hat{\imath} - \{\sin(q\omega t) - \frac{\ell}{r}\sin(\omega t)\}\hat{\jmath}]. \tag{9.28}$$

(c) **Acceleration of point P:** Differentiating Eqn. (9.28) once with respect to time we get

$$\vec{a}_P = -\omega^2 r[-\{q\sin(q\omega t) + \frac{\ell}{r}\sin(\omega t)\}\hat{\imath} - \{q\cos(q\omega t) - \frac{\ell}{r}\cos(\omega t)\}\hat{\jmath}]. \tag{9.29}$$

$$\boxed{\begin{aligned} \vec{r}_P &= [-(R-r)\sin(q\omega t) - \ell\sin(\omega t)]\hat{\imath} + [-(R-r)\cos(q\omega t) + \ell\cos(\omega t)]\hat{\jmath} \\ \vec{v}_P &= -\omega r[\{\cos(q\omega t) + \tfrac{\ell}{r}\cos(\omega t)\}\hat{\imath} - \{\sin(q\omega t) - \tfrac{\ell}{r}\sin(\omega t)\}\hat{\jmath}] \\ \vec{a}_P &= -\omega^2 r[-\{q\sin(q\omega t) + \tfrac{\ell}{r}\sin(\omega t)\}\hat{\imath} - \{q\cos(q\omega t) - \tfrac{\ell}{r}\cos(\omega t)\}\hat{\jmath}] \end{aligned}}$$

SAMPLE 9.10 *The rolling disk: instantaneous kinematics.* For the rolling disk in Sample 9.9, let $R = 4$ ft, $r = 1$ ft and point P be on the rim of the disk. Assume that at $t = 0$, the center of the disk is vertically below the center of the cylinder and point P is on the vertical line joining the two centers. If the disk is rolling at a constant speed $\omega = \pi$ rad/s, find

(a) the position of point P and center C at $t = 1$ s, 3 s, and 5.25 s,
(b) the velocity of point P and center C at those instants, and
(c) the acceleration of point P and center C at the same instants as above.

Draw the position of the disk at the three instants and show the velocities and accelerations found above.

Solution The general expressions for position, velocity, and acceleration of point P obtained in Sample 9.9 can be used to find the position, velocity, and acceleration of any point on the disk by substituting an appropriate value of ℓ in equations (9.27), (9.28), and (9.29). Since $R = 4r$,

$$q = \frac{r}{R-r} = \frac{1}{3}.$$

Now, point P is on the rim of the disk and point C is the center of the disk. Therefore,

$$\begin{aligned} \text{for point } P: \quad \ell &= r, \\ \text{for point } C: \quad \ell &= 0. \end{aligned}$$

Substituting these values for ℓ, and $q = 1/3$ in equations (9.27), (9.28), and (9.29) we get the following.

(a) **Position:**

$$\begin{aligned} \vec{\boldsymbol{r}}_C &= -3r\left[\sin\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\imath}} + \cos\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\jmath}}\right], \\ \vec{\boldsymbol{r}}_P &= \vec{\boldsymbol{r}}_C + r\left[-sin\,(\omega t)\,\hat{\boldsymbol{\imath}} + \cos(\omega t)\,\hat{\boldsymbol{\jmath}}\right]. \end{aligned}$$

(b) **Velocity:**

$$\begin{aligned} \vec{\boldsymbol{v}}_C &= -\omega r\left[\cos\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\imath}} - \sin\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\jmath}}\right], \\ \vec{\boldsymbol{v}}_P &= -\omega r\left[\left\{\cos\left(\frac{\omega t}{3}\right) + \cos(\omega t)\right\}\hat{\boldsymbol{\imath}} - \left\{\sin\left(\frac{\omega t}{3}\right) - \sin(\omega t)\right\}\hat{\boldsymbol{\jmath}}\right]. \end{aligned}$$

(c) **Acceleration:**

$$\begin{aligned} \vec{\boldsymbol{a}}_C &= \frac{\omega^2 r}{3}\left[\sin\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\imath}} + \cos\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\jmath}}\right], \\ \vec{\boldsymbol{a}}_P &= \omega^2 r\left[\left\{\frac{1}{3}\sin\left(\frac{\omega t}{3}\right) + \sin(\omega t)\right\}\hat{\boldsymbol{\imath}} + \left\{\frac{1}{3}\cos\left(\frac{\omega t}{3}\right) - \cos(\omega t)\right\}\hat{\boldsymbol{\jmath}}\right]. \end{aligned}$$

We can now use these expressions to find the position, velocity, and acceleration of the two points at the instants of interest by substituting $r = 1$ ft, $\omega = \pi$ rad/s, and appropriate values of t. These values are shown in Table 9.1.

The velocity and acceleration of the two points are shown in Figures 9.19(a) and (b) respectively.

It is worthwhile to check the directions of velocities and the accelerations by thinking about the velocity and acceleration of point P as a vector sum of the velocity (same for acceleration) of the center of the disk and the velocity (same for acceleration) of point P with respect to the center of the disk. Since the motions involved are circular motions at constant rate, a visual inspection of the velocities and the accelerations is not very difficult. Try it.

t	1 s	3 s	5.25 s
$\vec{r}_C$ (ft)	$3(-\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath})$	$3\hat{\jmath}$	$3(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{r}_P$ (ft)	$\vec{r}_C - \hat{\jmath}$	$\vec{r}_C - \hat{\jmath}$	$4(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_C$ (ft/s)	$\pi(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$	$\pi\hat{\imath}$	$\pi(-\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_P$ (ft/s)	$\pi(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$	$2\pi\hat{\imath}$	$\vec{0}$
$\vec{a}_C$ (ft/s^2)	$\frac{\pi^2}{3}(\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath})$	$-\frac{\pi^2}{3}\hat{\jmath}$	$\frac{\pi^2}{3}(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_P$ (ft/s^2)	$11.86(.24\hat{\imath} + .97\hat{\jmath})$	$\frac{2\pi^2}{3}\hat{\jmath}$	$13.16(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$

Table 9.1: Position, velocity, and acceleration of point P and point C

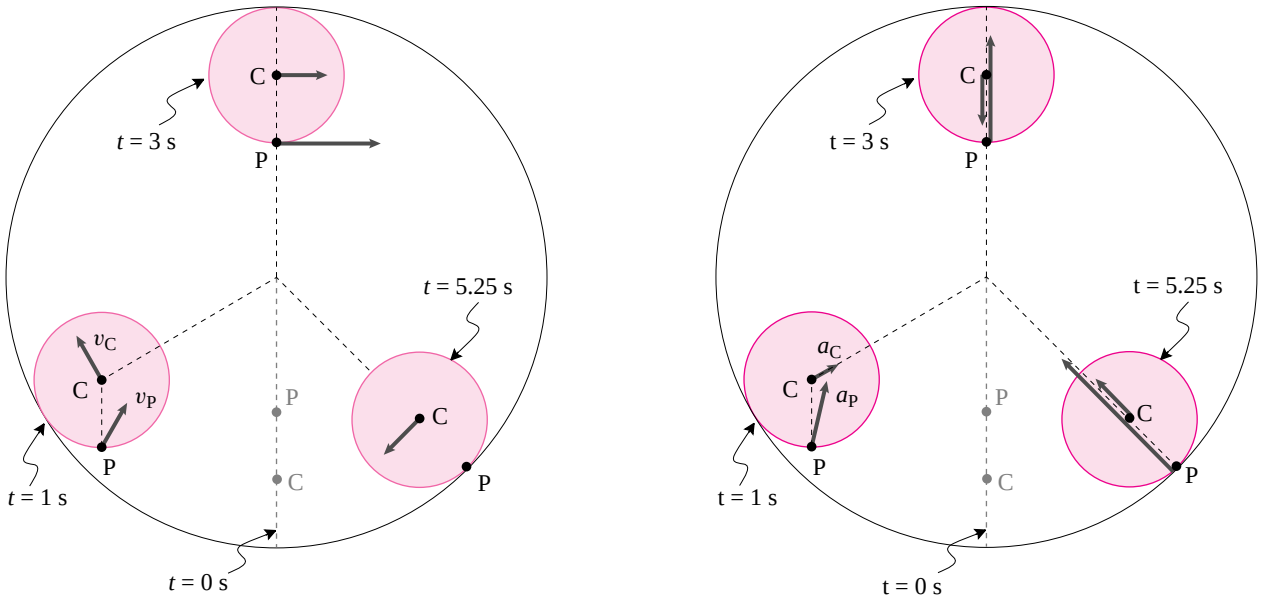


Figure 9.19: (a) Velocity and (b) Acceleration of points P and C at $t = 1\,\text{s}$, $3\,\text{s}$, and $5.25\,\text{s}$. (Filename:sfig6.5.4a)

SAMPLE 9.11 *The rolling disk: path of a point on the disk.* For the rolling disk in Sample 9.9, take $\omega = \pi$ rad/s. Draw the path of a point on the rim of the disk for one complete revolution of the center of the disk around the cylinder for the following conditions:

(a) $R = 8r$,
(b) $R = 4r$, and
(c) $R = 2r$.

Solution In Sample 9.9, we obtained a general expression for the position of a point on the disk as a function of time. By computing the position of the point for various values of time t up to the time required to go around the cylinder for one complete cycle, we can draw the path of the point. For the various given conditions, the variable that changes in Eqn. (9.27) is q. We can write a computer program to generate the path of any point on the disk for a given set of R and r. Here is a function written in MATLAB to generate the required path according to Eqn. (9.27).

MATLAB function to plot the path of a point on the disk:

```
program rollingdisk
%-------------------------------------------------------------
% This program will plots the path of any point on a disk of radius
% 'r' rolling with speed 'w' inside a cylinder of radius 'R'.
% The point of interest is distance 'l' away from the center of
% the disk. The coordinates x and y of the specified point P are
% calculated according to the relation mentioned above.
%-------------------------------------------------------------

phi = pi/50*[1,2,3,...,100]   % make a vector phi from 0 to 2*pi
x = R*cos(phi)                % create points on the outer cylinder
y = R*sin(phi)
plot y vs x                   % plot the outer cylinder
hold this plot                % hold to overlay plots of paths

q = r/(R-r)                   % calculate q.
T = 2*pi/(q*w)                % calculate time T for going aound-
                              % the cylinder once at speed 'w'.
t = T/100*[1,2,3, ..., 100]   % make a time vector t from 0 to T-
                              % taking 101 points.


rcx = -(R-r)*(sin(q*w*t))     % find the x coordinates of pt. C.
rcy = -(R-r)*(cos(q*w*t))     % find the y coordinates of pt. C.
rpx = rcx-l*sin(w*t)          % find the x coordinates of pt. P.
rpy = rcy + l*cos(q*t)        % find the y coordinates of pt. P.

plot rpy vs rpx               % plot the path of P and the path
plot rcy vs rcx               % of C. For path of C
```

Once coded, we can use this program to plot the paths of both the center and the point P on the rim of the disk for the three given situations. Note that for any point on the rim of the disk $l = r$ (see Fig 9.18).

(a) Let $R = 4$ units. Then $r = 0.5$ for $R = 8r$. To plot the required path, we run our program `rollingdisk` with desired input,

```
R = 4
r = 0.5
w = pi
l = 0.5
```
execute `rollingdisk`

The plot generated is shown in Fig.9.20 with a few graphic elements added for illustrative purposes.

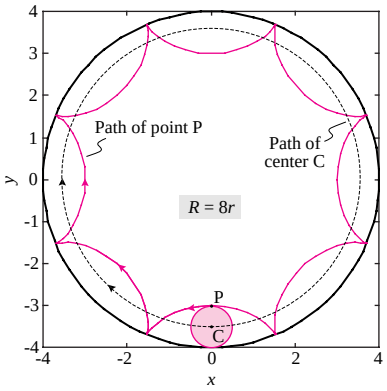


Figure 9.20: Path of point P and the center C of the disk for $R = 8r$.

(Filename:sfig6.5.5a)

(b) Similarly, for $R = 4r$ we type:

```
R = 4
r = 1
w = pi
l = 1
```
execute `rollingdisk`

to plot the desired paths. The plot generated in this case is shown in Fig.9.21

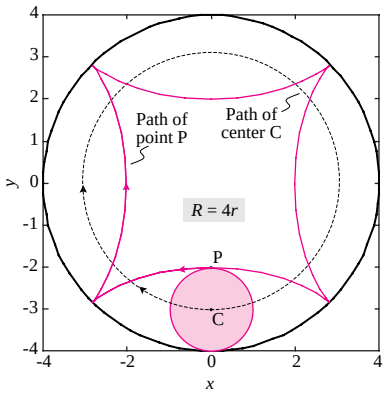


Figure 9.21: Path of point P and the center C of the disk for $R = 4r$.

(Filename:sfig6.5.5b)

(c) The last one is the most interesting case. The plot obtained in this case by typing:

```
R = 4
r = 2
w = pi
l = 2
```
execute `rollingdisk`

is shown in Fig.9.22. Point P just travels on a straight line! In fact, every point on the rim of the disk goes back and forth on a straight line. Most people find this motion odd at first sight. You can roughly verify the result by cutting a whole twice the diameter of a coin (say a US quarter or dime) in a piece of cardboard and rolling the coin around inside while watching a marked point on the perimeter.

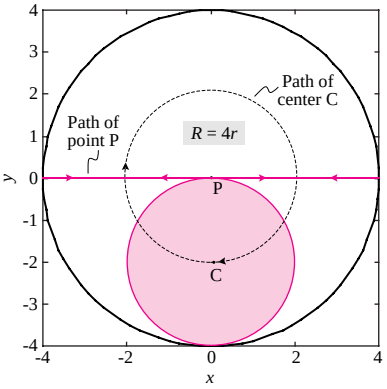


Figure 9.22: Path of point P and the center C of the disk for $R = 2r$.

(Filename:sfig6.5.5c)

A curiosity. We just discovered something simple about the path of a point on the edge of a circle rolling in another circle that is twice as big. The edge point moves in a straight line. In contrast one might think about the motion of the center G of a *straight* line segment that *slides* against two *straight* walls as in sample 9.20. A problem couldn't be more different. Naturally the path of point G is a circle (as you can check physically by looking at the middle of a ruler as you hold it as you sliding against a wall-floor corner).

9.4 Mechanics of contacting bodies: rolling and sliding

A typical machine part has forces that come from contact with other parts. In fact, with the major exception of gravity, most of the forces that act on bodies of engineering interest come from contact. Many of the forces you have drawn in free body diagrams have been contact forces: The force of the ground on an ideal wheel, of an axle on a bearing, etc.

We'd now like to consider some mechanics problems that involve sliding or rolling contact. Once you understand the kinematics from the previous section, there is nothing new in the mechanics. As always, the mechanics is linear momentum balance, angular momentum balance and energy balance. Because we are considering single rigid bodies in 2D the expressions for the motion quantities are especially simple (as you can look up in Table I at the back of the book): $\dot{\vec{L}} = m_{\text{tot}}\vec{a}_{\text{cm}}$, $\dot{\vec{H}}_{\text{C}} = \vec{r}_{\text{cm/C}} \times (m_{\text{tot}}\vec{a}_{\text{cm}}) + I\dot{\omega}\hat{k}$ (where $I = I_{zz}^{\text{cm}}$), and $E_{\text{K}} = m_{\text{tot}}v_{cm}^2/2 + I\omega^2/2$.

The key to success, as usual, is the drawing of appropriate free body diagrams (see Chapter 3 pages 88-91 and Chapter 6 pages 328-9). The two cases one needs to consider as possible are rolling, where the contact point has no relative velocity and the tangential reaction force is unknown but less than μN, and sliding where the relative velocity could be anything and the tangential reaction force is usually assumed to have a magnitude of μN but oppose the relative motion.

For friction forces in rolling refer to chapter 2 on free body diagrams. Note that in pure rolling contact, the contact force does no work because the material point of contact has no velocity. However, when there is sliding mechanical energy is dissipated unless there is no friction.

Work-energy relations and impulse-momentum relations are useful to solve some problems.

As for various problems occurring earlier in the text, it is often a savings of calculation to use angular momentum balance relative to a contact point.

SAMPLE 9.12 *A rolling wheel with mass.* Consider the wheel with mass m shown in figure **??**. The free body diagram of the wheel is shown here again. Write the equation of motion of the wheel.

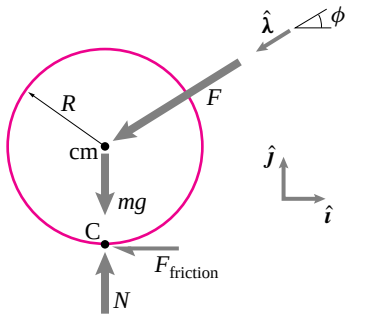


Figure 9.23: FBD of a wheel with mass m. Force F is applied by the axle.

(Filename:tfigure2.wheel.mass.lhs)

Solution We can write the equation of motion of the wheel in terms of either the center of mass position x or the angular displacement of the wheel θ. Since in pure rolling, these two variables share a simple relationship ($x = R\theta$), we can easily get the equation of motion in terms of x if we have the equation in terms of θ and vice versa. Since all the forces are shown in the free body diagram , we can easily write the angular momentum balance for the wheel. We choose the point of contact C as our reference point for the angular momentum balance (because the gravity force, $-mg\hat{\boldsymbol{j}}$, the friction force $-F_{friction}\hat{\boldsymbol{\imath}}$, and the normal reaction of the ground $N\hat{\boldsymbol{j}}$, all pass through the contact point C and therefore, produce no moment about this point). We have

$$\sum \vec{\boldsymbol{M}}_{\text{C}} = \dot{\vec{\boldsymbol{H}}}_{\text{C}}$$

where

$$\begin{aligned}\sum \vec{\boldsymbol{M}}_{\text{C}} &= \overbrace{\vec{\boldsymbol{r}}_{cm/C}}^{R\hat{\boldsymbol{j}}} \times (F\hat{\boldsymbol{\lambda}}) \\ &= R\hat{\boldsymbol{j}} \times F(-\cos\phi\hat{\boldsymbol{\imath}} - \sin\phi\hat{\boldsymbol{j}}) \\ &= FR\cos\phi\hat{\boldsymbol{k}}\end{aligned}$$

and

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{\text{C}} &= \vec{\boldsymbol{r}}_{\text{cm/C}} \times m\vec{\boldsymbol{a}}_{\text{cm}} + I_{zz}^{\text{cm}}\dot{\vec{\boldsymbol{\omega}}} \\ &= R\hat{\boldsymbol{j}} \times m\underbrace{\ddot{x}}_{\dot{\omega}R}\hat{\boldsymbol{\imath}} - I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\ &= -m\dot{\omega}R^2\hat{\boldsymbol{k}} - I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\ &= -(I_{zz}^{\text{cm}} + mR^2)\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

Thus,

$$\begin{aligned}FR\cos\phi\hat{\boldsymbol{k}} &= -(I_{zz}^{\text{cm}} + mR^2)\dot{\omega}\hat{\boldsymbol{k}} \\ \Rightarrow \quad \dot{\omega} \equiv \ddot{\theta} &= \frac{FR\cos\phi}{I_{zz}^{\text{cm}} + mR^2}\end{aligned}$$

which is the equation of motion we are looking for. Note that we can easily substitute $\ddot{\theta} = \ddot{x}/R$ in the equation of motion above to get the equation of motion in terms of the center of mass displacement x as

$$\ddot{x} = \frac{FR^2\cos\phi}{I_{zz}^{\text{cm}} + mR^2}.$$

$$\boxed{\ddot{\theta} = \frac{FR\cos\phi}{I_{zz}^{\text{cm}} + mR^2}}$$

Comments: We could have, of course, used linear momentum balance also to derive the equation of motion. Note, however, that the linear momentum balance will essentially give two scalar equations in the x and y directions involving all forces shown in the free body diagram . The angular momentum balance , on the other hand, gets rid of some of them. Depending on which forces are known, we may or may not need to use all the three scalar equations. In the final equation of motion, we must have only one unknown.

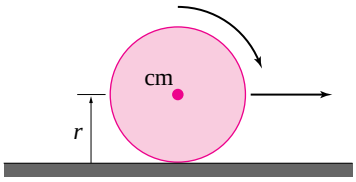


Figure 9.24: (Filename:sfig7.4.1a)

SAMPLE 9.13 *Energy and power of a rolling wheel.* A wheel of diameter 2 ft and mass 20 lbm rolls without slipping on a horizontal surface. The kinetic energy of the wheel is 1700 ft·lbf. Assume the wheel to be a thin, uniform disk.

(a) Find the rate of rotation of the wheel.
(b) Find the average power required to bring the wheel to a complete stop in 5 s.

Solution

(a) Let ω be the rate of rotation of the wheel. Since the wheel rotates without slip, its center of mass moves with speed $v_{\text{cm}} = \omega r$. The wheel has both translational and rotational kinetic energy. The total kinetic energy is

$$\begin{aligned} E_{\text{K}} &= \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}I^{\text{cm}}\omega^2 \\ &= \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}I^{\text{cm}}\omega^2 \\ &= \frac{1}{2}(mr^2 + \underbrace{I^{\text{cm}}}_{\frac{1}{2}mr^2})\omega^2 \\ &= \frac{3}{4}mr^2\omega^2 \\ \Rightarrow \quad \omega^2 &= \frac{4E_{\text{K}}}{3mr^2} \\ &= \frac{4 \times 1700\,\text{ft}\cdot\text{lbf}}{3 \times 20\,\text{lbm}\cdot 1\,\text{ft}^2} \\ &= \frac{4 \times 1700 \times 32.2\,\text{lbm}\cdot\text{ft}/\text{s}^2}{3 \times 20\,\text{lbm}\cdot\text{ft}} \\ &= 3649.33\frac{1}{\text{s}^2} \\ \Rightarrow \quad \omega &= 60.4\,\text{rad/s}. \end{aligned}$$

$$\boxed{\omega = 60.4\,\text{rad/s}}$$

Note: This rotational speed, by the way, is extremely high. At this speed the center of mass moves at 60.4 ft/s!

(b) Power is the rate of work done on a body or the rate of change of kinetic energy. Here we are given the initial kinetic energy, the final kinetic energy (zero) and the time to achieve the final state. Therefore, the avarage power is,

$$\begin{aligned} P &= \frac{E_{\text{K1}} - E_{\text{K2}}}{\Delta t} \\ &= \frac{1700\,\text{ft}\cdot\text{lbf} - 0}{5\,\text{s}} = 340\,\text{ft}\cdot\text{lbf/s} \\ &= 340\,\text{ft}\cdot\text{lbf/s} \cdot \frac{1\,\text{hp}}{550\,\text{ft}\cdot\text{lbf/s}} \\ &= 0.62\,\text{hp} \end{aligned}$$

$$\boxed{P = 0.62\,\text{hp}}$$

SAMPLE 9.14 *Equation of motion of a rolling wheel from energy balance.* Consider the wheel with mass m from figure **??**. The free body diagram of the wheel is shown here again. Derive the equation of motion of the wheel using energy balance.

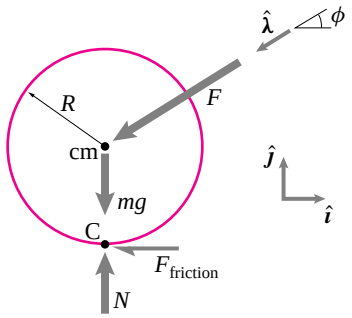


Figure 9.25: FBD of a rolling wheel.
(Filename:tfigure2.wheel.mass.lhs.en)

Solution From energy balance, we have

$$P = \dot{E}_{\mathrm{K}}$$

where

$$\begin{aligned} P &= \sum \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{v}}_i \\ &= F_{friction}\hat{\boldsymbol{\imath}} \cdot \overbrace{\vec{\boldsymbol{0}}}^{\vec{\boldsymbol{v}}_C} + N\hat{\boldsymbol{\jmath}} \cdot \overbrace{\vec{\boldsymbol{0}}}^{\vec{\boldsymbol{v}}_C} - mg\hat{\boldsymbol{\jmath}} \cdot \overbrace{v\hat{\boldsymbol{\imath}}}^{\vec{\boldsymbol{v}}_{cm}} + F\hat{\boldsymbol{\lambda}} \cdot \overbrace{v\hat{\boldsymbol{\imath}}}^{\vec{\boldsymbol{v}}_{cm}} \\ &= -mgv(\underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}}_{0}) + Fv(\underbrace{\hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{\imath}}}_{-\cos\phi}) \\ &= -Fv\cos\phi \end{aligned}$$

and

$$\begin{aligned} \dot{E}_{\mathrm{K}} &= \frac{d}{dt}(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{zz}^{\mathrm{cm}} \overbrace{\dot{\omega}^2}^{(\dot{x}/R)^2}) \\ &= \frac{1}{2}\frac{d}{dt}[(m + I_{zz}^{\mathrm{cm}}/R^2)\dot{x}^2] \\ &= (m + I_{zz}^{\mathrm{cm}}/R^2)\dot{x}\ddot{x}. \end{aligned}$$

Thus,

$$\begin{aligned} -Fv\cos\phi &= (m + I_{zz}^{\mathrm{cm}}/R^2)\dot{x}\ddot{x} \\ \text{or} \quad -F\cancel{\dot{x}}\cos\phi &= (m + I_{zz}^{\mathrm{cm}}/R^2)\cancel{\dot{x}}\ddot{x} \\ \Rightarrow \quad \ddot{x} &= -\frac{F\cos\phi}{m + I_{zz}^{\mathrm{cm}}/R^2}. \end{aligned}$$

We can also write the equation of motion in terms of θ by replacing $\ddot{x}$ with $\ddot{\theta}R$ giving,

$$\ddot{\theta} = \frac{FR\cos\phi}{m + I_{zz}^{\mathrm{cm}}/R^2}.$$

$$\boxed{\ddot{x} = -\frac{F\cos\phi}{m + I_{zz}^{\mathrm{cm}}/R^2}}$$

Comments: In the equations above (for calculating P), we have set $\vec{\boldsymbol{v}}_C = \vec{\boldsymbol{0}}$ because in pure rolling, the instantaneous velocity of the contact point is zero. Note that the force due to gravity is normal to the direction of the velocity of the center of mass. So, the only power supplied to the wheel is due to the force $F\hat{\boldsymbol{\lambda}}$ acting at the center of mass.

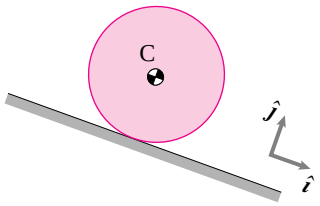


Figure 9.26: (Filename:sfig9.rolling.incline1)

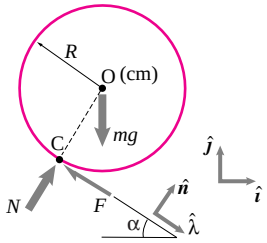


Figure 9.27: (Filename:sfig9.rolling.incline1a)

SAMPLE 9.15 *Equation of motion of a rolling disk on an incline.* A unifrom circular disk of mass $m = 1\,\text{kg}$ and radius $R = 0.4\,\text{m}$ rolls down an inclined shown in the figure. Write the equation of motion of the disk assuming pure rolling, and find the distance travelled by the center of mass in 2 s.

Solution The free body diagram of the disk is shown in Fig. 9.27. In addition to the base unit vectors $\hat{\imath}$ and $\hat{\jmath}$, let us use unit vectors $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ along the plane and perpendicular to the plane, respectively, to express various vectors. We can write the equation of motion using linear momentum balance or angular momentum balance. However, note that if we use linear momentum balance we have two unknown forces in the equation. On the otherhand, if we use angular momentum balance about the contact point C, these forces do not show up in the equation. So, let us use angular momentum balance about point C:

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

where

$$\begin{aligned}\sum \vec{M}_C &= \vec{r}_{O/C} \times m\vec{g} = R\hat{\boldsymbol{n}} \times (-mg\hat{\boldsymbol{\jmath}}) \\ &= -Rmg\sin\alpha\hat{\boldsymbol{k}}\end{aligned}$$

and

$$\begin{aligned}\dot{\vec{H}}_C &= -I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}} + \overbrace{\vec{r}_{O/C}}^{R\hat{\boldsymbol{n}}} \times m \overbrace{\vec{a}_{cm}}^{R\dot{\omega}\hat{\boldsymbol{\lambda}}} \\ &= -I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}} + mR^2\dot{\omega}(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\lambda}}) \\ &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

Thus,

$$\begin{aligned}-Rmg\sin\alpha\hat{\boldsymbol{k}} &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{\boldsymbol{k}} \\ \Rightarrow \quad \dot{\omega} &= \frac{g\sin\alpha}{R[1 + I_{zz}^{cm}/(mR^2)]}\end{aligned}$$

$$\boxed{\dot{\omega} = \frac{g\sin\alpha}{R[1+I_{zz}^{cm}/(mR^2)]}}$$

Note that in the above equation of motion, the right hand side is constant. So, we can solve the equation for ω and θ by simply integrating this equation and substituting the initial conditions $\omega(t=0) = 0$ and $\theta(t=0) = 0$. Let us write the equation of motion as $\dot{\omega} = \beta$ where $\beta = g\sin\alpha/R(1 + I_{zz}^{cm}/mR^2)$. Then,

$$\begin{aligned}\omega &\equiv \dot{\theta} = \beta t + C_1 \\ \theta &= \frac{1}{2}\beta t^2 + C_1 t + C_2.\end{aligned}$$

Substituting the given initial condtions $\dot{\theta}(0) = 0$ and $\theta(0) = 0$, we get $C_1 = 0$ and $C_2 = 0$, which implies that $\theta = \frac{1}{2}\beta t^2$. Now, in pure rolling, $x = R\theta$. Therefore,

$$\begin{aligned}x(t) &= R\theta(t) = \frac{1}{2}\beta t^2 = \cancel{R}\cdot\frac{1}{2}\frac{g\sin\alpha}{\cancel{R}(1 + I_{zz}^{cm}/mR^2)}t^2 \\ &= \frac{1}{2}\frac{g\sin\alpha}{1 + \frac{\frac{1}{2}mR^2}{mR^2}}t^2 = \frac{1}{3}(g\sin\alpha)t^2 \\ x(2\,\text{s}) &= \frac{1}{3}\cdot 9.8\,\text{m/s}^2\cdot\sin(30^o)\cdot(2\,\text{s})^2 = 6.53\,\text{m}\end{aligned}$$

$$\boxed{x(2\,\text{s}) = 6.53\,\text{m}}$$

SAMPLE 9.16 *Using Work and energy in pure rolling.* Consider the disk of Sample 9.15 rolling down the incline again. Suppose the disk starts rolling from rest. Find the speed of the center of mass when the disk is 2 m down the inclined plane.

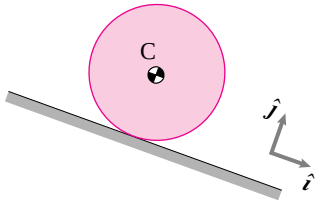


Figure 9.28: (Filename:sfig9.rolling.incline2)

Solution We are given that the disk rolls down, starting with zero initial velocity. We are to find the speed of the center of mass after it has travelled 2 malong the incline. We can, of course, solve this problem using equation of motion, by first solving for the time t the disk takes to travel the given distance and then evaluating the expression for speed $\omega(t)$ or $x(t)$ at that t. However, it is usually easier to use work energy principle whenever positions are specified at two instants, speed is specified at one of those instants, and speed is to be found at the other instant. This is because we can, presumably, compute the work done on the system in travelling the specified distance and relate it to the change in kinetic energy of the system between the two instants. In the problem given here, let ω_1 and ω_2 be the initial and final (after rolling down by $d = 2\,\text{m}$) angular speeds of the disk, respectively. We know that in rolling, the kinetic energy is given by

$$E_\text{K} = \frac{1}{2}m\overbrace{v_\text{cm}^2}^{(\omega R)^2} + \frac{1}{2}I_{zz}^\text{cm}\omega^2 = \frac{1}{2}(mR^2 + I_{zz}^\text{cm})\omega^2.$$

Therefore,

$$\Delta E_\text{K} = E_{\text{K}2} - E_{\text{K}1} = \frac{1}{2}(mR^2 + I_{zz}^\text{cm})(\omega_2^2 - \omega_1^2) \tag{9.30}$$

Now, let us calculate the work done by all the forces acting on the disk during the displacement of he mass-center by d along the plane. Note that in ideal rolling, the contact forces do no work. Therefore, the work done on the disk is only due to the gravitational force:

$$W = (-mg\hat{\boldsymbol{j}})\cdot(d\hat{\boldsymbol{\lambda}}) = -mgd(\overbrace{\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{\lambda}}}^{-\sin\alpha}) = mgd\sin\alpha \tag{9.31}$$

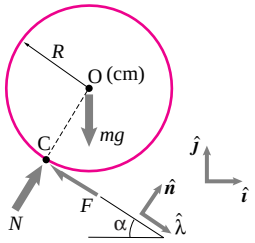


Figure 9.29: (Filename:sfig9.rolling.incline2a)

From work-energy principle (integral form of power balance, $P = \dot{E}_\text{K}$), we know that $W = \Delta E_\text{K}$. Therefore, from eqn. (9.30) and eqn. (9.31), we get

$$\begin{aligned} mgd\sin\alpha &= \frac{1}{2}(mR^2 + I_{zz}^\text{cm})(\omega_2^2 - \omega_1^2) \\ \Rightarrow\quad \omega_2^2 &= \omega_1^2 + \frac{2mgd\sin\alpha}{mR^2 + I_{zz}^\text{cm}} = \omega_1^2 + \frac{2gd\sin\alpha}{R^2\left(1 + \frac{I_{zz}^\text{cm}}{mR^2}\right)} \\ &= \omega_1^2 + \frac{4gd\sin\alpha}{3R^2} \end{aligned}$$

Substituting the values of g, d, α, R, etc., and setting $\omega_1 = 0$, we get

$$\begin{aligned} \omega_2^2 &= \frac{4\cdot(9.8\,\text{m/s}^2)\cdot(2\,\text{m})\cdot(\sin(30^\text{o})}{3\cdot(0.4\,\text{m})^2} = 81.67/s^2 \\ \Rightarrow\quad \omega_2 &= 9.04\,\text{rad/s}. \end{aligned}$$

The corresponding speed of the center of mass is

$$v_\text{cm} = \omega_2 R = 9.04\,\text{rad/s}\cdot 0.4\,\text{m} = 3.61\,\text{m/s}.$$

$$\boxed{v_\text{cm} = 3.61\,\text{m/s}}$$

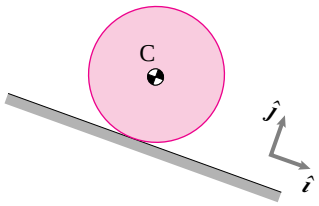


Figure 9.30: (Filename:sfig9.rolling.incline3)

SAMPLE 9.17 *Impulse and momentum calculations in pure rolling.* Consider the disk of Sample 9.15 rolling down the incline again. Find an expression for the rolling speed (ω) of the disk after a finite time Δt, given the initial rolling speed ω_1.

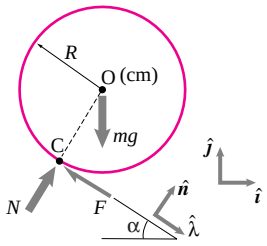


Figure 9.31: (Filename:sfig9.rolling.incline3a)

Solution Once again, this problem can be solved by integrating the equation of motion (as done in Sample 9.15). However, we will solve this problem here using impulse-momentum relationship. Note that we need the speed of the disk ω_2, after a finite time Δt, given the initial speed ω_1. Since the forces acting on the disk do not change during this time (assuming pure rolling), it is easy to calculate impulse and then relate it to the change in the momenta of the disk between the two instants. Now, from the linear impulse momentum relatioship, $\sum \vec{F} \cdot \Delta t = \vec{L}_2 - \vec{L}_1$, we have

$$(-F\hat{\boldsymbol{\lambda}} + N\hat{\boldsymbol{\imath}} - mg\hat{\boldsymbol{\jmath}})\Delta t = m(v_2 - v_1)\hat{\boldsymbol{\lambda}} \tag{9.32}$$

Dotting eqn. (9.32) with $\hat{\boldsymbol{\lambda}}$ gives

$$\begin{aligned} (-F - mg(\underbrace{\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\lambda}}}_{-\sin\alpha}))\Delta t &= m(v_2 - v_1) \\ (-F + mg\sin\alpha)\Delta t &= mR(\omega_2 - \omega_1) \end{aligned} \tag{9.33}$$

Similarly, the angular impulse-momentum relationship about the mass-center, $\vec{M}_O \Delta t = (\vec{H}_O)_2 - (\vec{H}_O)_1$, gives

$$\begin{aligned} (-FR\hat{\boldsymbol{k}})\Delta t &= -I_{zz}^{cm}(\omega_2 - \omega_1)\hat{\boldsymbol{k}} \\ \Rightarrow \quad FR\Delta t &= I_{zz}^{cm}(\omega_2 - \omega_1) \end{aligned} \tag{9.34}$$

Note that the other forces (N and mg) do not produce any moment about the mass-center as they pass through this point. We can now eliminate the unknown force F from eqn. (9.33) and eqn. (9.34) by multiplying eqn. (9.33) with R and adding to eqn. (9.34):

$$\begin{aligned} mgR\sin\alpha\Delta t &= (I_{zz}^{cm} + mR^2)(\omega_2 - \omega_1) \\ \text{or} \qquad g\sin\alpha\Delta t &= R\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)(\omega_2 - \omega_1) \\ \Rightarrow \quad \omega_2 &= \omega_1 + \frac{g\sin\alpha}{R\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)}\Delta t \end{aligned}$$

$$\boxed{\omega_2 = \omega_1 + \frac{g\sin\alpha}{R\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)}\Delta t}$$

SAMPLE 9.18 *Work and energy calculations in sliding.* A block of mass $m = 2.5$ kg slides down a frictionless incline from a 5 m height. The block encounters a frictional bed AB of length 1 m on the ground. If the speed of the block is 9 m/s at point B, find the coefficient of friction between the block and the frictional surface AB.

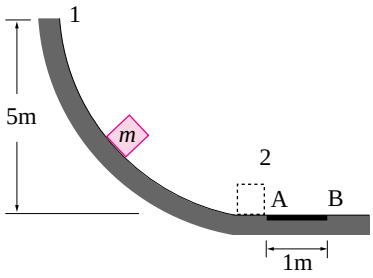


Figure 9.32: (Filename:sfig2.9.1a)

Solution We can divide the problem in two parts: We first find the speed of the block as it reaches point A using conservation of energy for its motion on the inclined surface, and then use work-energy principle to find the speed at B. Let the ground level be the datum for P.E. and let v be the speed at A. For the motion on the incline;

$$\begin{aligned} E_{K1} + E_{P1} &= E_{K2} + E_{P2} \\ 0 + mgh &= \frac{1}{2}mv^2 + 0 \\ \Rightarrow \quad v &= \sqrt{2gh} \\ &= \sqrt{2 \cdot 9.81\,\text{m/s}^2 \cdot 5\,\text{m}} \\ &= 9.90\,\text{m/s}. \end{aligned}$$

Now, as the block slides on the surface AB, a force of friction $= \mu\,\text{N} = \mu mg$ (since $\text{N} = mg$, from linear momentum balance in the vertical direction) acts in the opposite direction of motion (see Fig. 9.33. Work done by this force on the block is,

$$\begin{aligned} W &= \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{r}} \\ &= -\mu mg\hat{\boldsymbol{\imath}} \cdot L\hat{\boldsymbol{\imath}} \\ &= -\mu mgL \end{aligned}$$

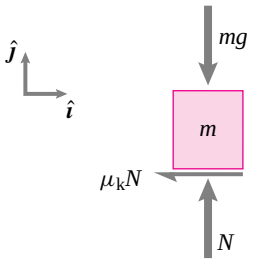


Figure 9.33: Free body diagram of the block when the block is on the rough surface.

(Filename:sfig2.9.1b)

From work energy relationship we have,

$$\begin{aligned} W &= \Delta E_K = E_{K2} - E_{K1} \\ \Rightarrow \quad E_{K2} &= E_{K1} + W \\ \frac{1}{2}mv_B^2 &= \frac{1}{2}mv^2 - \mu mgL \\ -\mu mgL &= \frac{1}{2}m(v_B^2 - v^2) \\ \Rightarrow \quad \mu &= \frac{1}{2gL}(v_B^2 - v^2) \\ &= \frac{(9.90\,\text{m/s})^2 - (9\,\text{m/s})^2}{2 \cdot 9.81\,\text{m/s}^2 \cdot 1\,\text{m}} \\ &= 0.87 \end{aligned}$$

$$\boxed{\mu = 0.87}$$

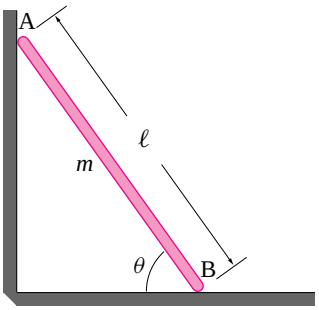


Figure 9.34: A ladder, modeled as a uniform rod of mass m and length ℓ, falls from a rest position at $\theta = \theta_o$ ($< \pi/2$) such that its ends slide along frictionless vertical and horizontal surfaces.

(Filename:sfig7.3.1)

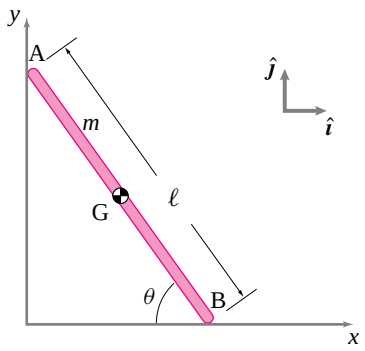


Figure 9.35: (Filename:sfig7.3.1a)

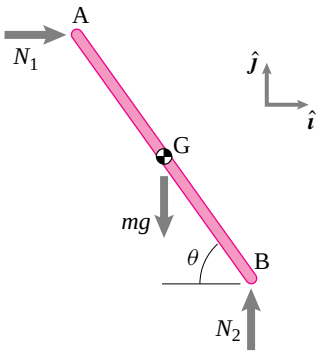


Figure 9.36: The free body diagram of the ladder.

(Filename:sfig7.3.1b)

SAMPLE 9.19 *Falling ladder.* A ladder AB, modeled as a uniform rigid rod of mass m and length ℓ, rests against frictionless horizontal and vertical surfaces. The ladder is released from rest at $\theta = \theta_o$ ($\theta_o < \pi/2$). Assume the motion to be planar (in the vertical plane).

(a) As the ladder falls, what is the path of the center of mass of the ladder?
(b) Find the equation of motion (e.g., a differential equation in terms of θ and its time derivatives) for the ladder.
(c) How does the angular speed ω ($= \dot{\theta}$) depend on θ?

Solution Since the ladder is modeled by a uniform rod AB, its center of mass is at G, half way between the two ends. As the ladder slides down, the end A moves down along the vertical wall and the end B moves out along the floor. Note that it is a single degree of freedom system as angle θ (a single variable) is sufficient to determine the position of every point on the ladder at any instant of time.

(a) **Path of the center of mass:** Let the origin of our x-y coordinate system be the intersection of the two surfaces on which the ends of the ladder slide (see Fig. 9.35). The position vector of the center of mass G may be written as

$$\begin{aligned} \vec{r}_G &= \vec{r}_B + \vec{r}_{G/B} \\ &= \ell\cos\theta\hat{\imath} + \frac{\ell}{2}(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\ &= \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}). \end{aligned} \tag{9.35}$$

Thus the coordinates of the center of mass are

$$x_G = \frac{\ell}{2}\cos\theta \quad \text{and} \quad y_G = \frac{\ell}{2}\sin\theta,$$

from which we get

$$x_G^2 + y_G^2 = \frac{\ell^2}{4}$$

which is the equation of a circle of radius $\frac{\ell}{2}$. Therefore, the center of mass of the ladder follows a circular path of radius $\frac{\ell}{2}$ centered at the origin. Of course, the center of mass traverses only that part of the circle which lies between its initial position at $\theta = \theta_o$ and the final position at $\theta = 0$.

(b) **Equation of motion:** The free body diagram of the ladder is shown in Fig. 9.36. Since there is no friction, the only forces acting at the end points A and B are the normal reactions from the contacting surfaces. Now, writing the the linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the ladder we get

$$N_1\hat{\imath} + (N_2 - mg)\hat{\jmath} = m\vec{a}_G = m\ddot{\vec{r}}_G.$$

Differentiating eqn. (9.35) twice we get $\ddot{\vec{r}}_G$ as

$$\ddot{\vec{r}}_G = \frac{\ell}{2}[(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta)\hat{\imath} + (\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\hat{\jmath}].$$

Substituting this expression in the linear momentum balance equation above and dotting both sides of the equation by $\hat{\imath}$ and then by $\hat{\jmath}$ we get

$$\begin{aligned} N_1 &= -\frac{1}{2}m\ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \\ N_2 &= \frac{1}{2}m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + mg. \end{aligned}$$

Next, we write the angular momentum balance for the ladder about its center of mass, $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$, where

$$\begin{aligned}\sum \vec{M}_{/G} &= \left(-N_1\frac{\ell}{2}\sin\theta + N_2\frac{\ell}{2}\cos\theta\right)\hat{k} \\ &= \frac{1}{2}m\ell(\ddot{\theta}\,\sin\theta + \dot{\theta}^2\cos\theta)\frac{\ell}{2}\sin\theta\hat{k} \\ &\quad + \left[\frac{1}{2}m\ell(\ddot{\theta}\,\cos\theta - \dot{\theta}^2\sin\theta) + mg\right]\frac{\ell}{2}\cos\theta\hat{k} \\ &= \left(\frac{1}{4}m\ell^2\ddot{\theta} + \frac{1}{2}mg\ell\cos\theta\right)\hat{k}\end{aligned}$$

and

$$\dot{\vec{H}}_{/G} = I_{zz/G}\dot{\vec{\omega}} = \frac{1}{12}m\ell^2\ddot{\theta}(-\hat{k}),$$

where $\dot{\vec{\omega}} = \ddot{\theta}(-\hat{k})$ because θ is measured positive in the clockwise direction $(-\hat{k})$. Now, equating the two quantities $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$ and dotting both sides with $\hat{k}$ we get

$$\begin{aligned}\frac{1}{4}\not{m}\ell^2\ddot{\theta} + \frac{1}{2}\not{m}g\ell\cos\theta &= -\frac{1}{12}\not{m}\ell^2\ddot{\theta} \\ \text{or} \qquad (\frac{1}{12} + \frac{1}{4})\ell^2\ddot{\theta} &= -\frac{1}{2}g\ell\cos\theta \\ \text{or} \qquad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta \qquad (9.36)\end{aligned}$$

which is the required equation of motion. Unfortunately, it is a nonlinear equation which does not have a nice closed form solution for $\theta(t)$.

(c) **Angular Speed of the ladder:** To solve for the angular speed ω $(= \dot{\theta})$ as a function of θ we need to express eqn. (9.36) in terms of ω, θ, and derivatives of ω with respect to θ. Now,

$$\ddot{\theta} = \dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}.$$

Substituting in eqn. (9.36) and integrating both sides from the initial rest position to an arbitrary position θ we get

$$\begin{aligned}\int_0^\omega \omega\,d\omega &= -\int_{\theta_0}^{\theta}\frac{3g}{2\ell}\cos\theta\,d\theta \\ \Rightarrow \qquad \frac{1}{2}\omega^2 &= -\frac{3g}{2\ell}(\sin\theta - \sin\theta_0) \\ \Rightarrow \qquad \omega &= \pm\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}.\end{aligned}$$

Since end B is sliding to the right, θ is decreasing; hence it is the negative sign in front of the square root which gives the correct answer, *i.e.*,

$$\vec{\omega} = \dot{\theta}(-\hat{k}) = -\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}\,\hat{k}.$$

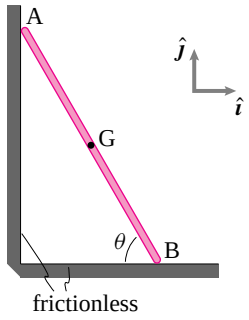


Figure 9.37: (Filename:sfig7.4.3)

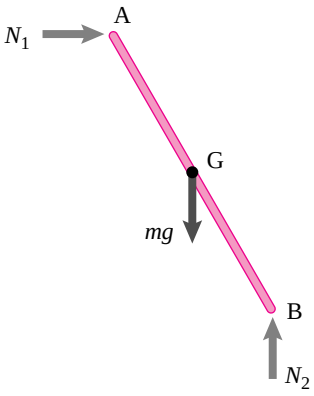
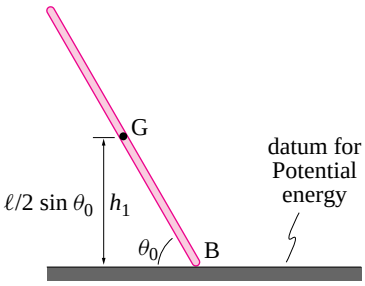
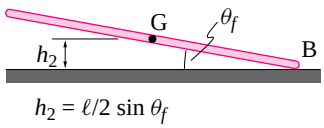


Figure 9.38: (Filename:sfig7.4.3a)





$h_2 = \ell/2 \sin\theta_f$

Figure 9.39: (Filename:sfig7.4.3b)

SAMPLE 9.20 *The falling ladder again.* Consider the falling ladder of Sample 9.4 again. The mass of the ladder is m and the length is ℓ. The ladder is released from rest at $\theta = 80^o$.

(a) At the instant when $\theta = 45^o$, find the speed of the center of mass of the ladder using energy.

(b) Derive the equation of motion of the ladder using work-energy balance.

Solution

(a) Since there is no friction, there is no loss of energy between the two states: $\theta_0 = 80^o$ and $\theta_f = 45^o$. The only external forces on the ladder are N_1, N_2, and mg as shown in the free body diagram. Since the displacements of points A and B are perpendicular to the normal reactions of the walls, N_1 and N_2, respectively, no work is done by these forces on the ladder. The only force that does work is the force due to gravity. But this force is conservative. Therefore, the conservation of energy holds between any two states of the ladder during its fall.

Let E_1 and E_2 be the total energy of the ladder at θ_0 and θ_f, respectively. Then

$$E_1 = E_2 \quad \text{(conservation of energy)}$$

$$\begin{aligned} \text{Now} \quad E_1 &= \underbrace{E_{K_1}}_{\text{K.E.}} + \underbrace{E_{P_1}}_{\text{P.E.}} \\ &= 0 + mgh_1 \\ &= mg\frac{\ell}{2}\sin\theta_0 \\ \text{and} \quad E_2 &= E_{K_2} + E_{P_2} = \underbrace{\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^G\omega^2}_{E_{K_2}} + mgh_2 \end{aligned}$$

Equating E_1 and E_2 we get

$$\begin{aligned} \cancel{m}g\frac{\ell}{\cancel{2}}(\sin\theta_0 - \sin\theta_f) &= \frac{1}{\cancel{2}}(\cancel{m}v_G^2 + \underbrace{\frac{1}{12}\cancel{m}\ell^2}_{I_{zz}^G}\omega^2) \\ \text{or} \quad g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\ell^2\omega^2 \end{aligned} \tag{9.37}$$

Clearly, we cannot find v_G from this equation alone because the equation contains another unknown, ω. So we need to find another equation which relates v_G and ω. To find this equation we turn to kinematics. Note that

$$\begin{aligned} \vec{r}_G &= \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\ \Rightarrow \quad \vec{v}_G &= \dot{\vec{r}}_G = \frac{\ell}{2}(-\sin\theta\cdot\dot{\theta}\hat{\imath} + \cos\theta\cdot\dot{\theta}\hat{\jmath}) \\ \Rightarrow \quad v_G &= |\vec{v}_G| = \sqrt{\frac{\ell^2}{4}(\cos^2\theta + \sin^2\theta)\,\dot{\theta}^2} \\ &= \frac{\ell}{2}\dot{\theta} = \frac{\ell}{2}\omega \\ \Rightarrow \quad \omega &= \frac{2v_G}{\ell} \end{aligned}$$

Substituting the expression for ω in Eqn 9.37 we get

$$\begin{aligned} g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\cancel{\ell^2}\cdot\frac{4v_G^2}{\cancel{\ell^2}} \\ &= \frac{4}{3}v_G^2 \\ \Rightarrow \quad v_G &= \sqrt{\frac{3g\ell}{4}(\sin\theta_0 - \sin\theta_f)} \\ &= 0.46\sqrt{g\ell} \end{aligned}$$

$$\boxed{v_G = 0.46\sqrt{g\ell}}$$

(b) Equation of motion: Since the ladder is a single degree of freedom system, we can use the power equation to derive the equation of motion:

$$P = \dot{E}_K$$

For the ladder, the only force that does work is mg. This force acts on the center of mass G. Therefore,

$$\begin{aligned} P &= \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}} = -mg\hat{\boldsymbol{j}}\cdot\vec{\boldsymbol{v}}_G \\ &= -mg\hat{\boldsymbol{j}}\cdot\left[\frac{\ell}{2}(-\sin\theta\hat{\boldsymbol{i}} + \cos\theta\hat{\boldsymbol{j}})\,\dot{\theta}\right] \\ &= -mg\frac{\ell}{2}\dot{\theta}\cos\theta \end{aligned}$$

The rate of change of kinetic energy

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt}(\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^{\mathrm{G}}\omega^2) \\ &= \frac{d}{dt}(\frac{1}{2}m\frac{\ell^2\omega^2}{4} + \frac{1}{2}\frac{m\ell^2}{12}\omega^2) \\ &= \frac{m\ell^2}{4}\omega\dot{\omega} + \frac{m\ell^2}{12}\omega\dot{\omega} \\ &= \frac{m\ell^2}{3}\omega\dot{\omega} \equiv \frac{m\ell^2}{3}\dot{\theta}\ddot{\theta} \quad (\text{since } \omega = \dot{\theta} \text{ and } \dot{\omega} = \ddot{\theta}) \end{aligned}$$

Now equating P and $\dot{E}_K$ we get

$$\begin{aligned} \frac{\cancel{m}\ell^2}{3}\cancel{\dot{\theta}}\ddot{\theta} &= -\cancel{m}g\frac{\ell}{2}\cancel{\dot{\theta}}\cos\theta \\ \Rightarrow \quad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta \end{aligned}$$

which is the same expression as obtained in Sample 9.19 (b).

$$\boxed{\ddot{\theta} = -\tfrac{3g}{2\ell}\cos\theta}$$

Note: To do this problem we have assumed that the upper end of the ladder stays in contact with the wall as it slides down. One might wonder if this is a consistent assumption. Does this assumption correspond to the non-physical assumption that the wall is capable of pulling on the ladder? Or in other words, if a real ladder was sliding against a slippery wall and floor would it lose contact? The answer is yes. One way of finding when contact would be lost is to calculate the normal reaction N_1 and finding out at what value of θ it passes through zero. It turns out that N_1 is zero at about $\theta = 41$o.

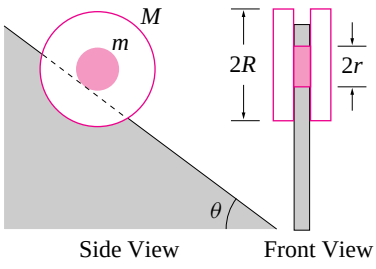


Figure 9.40: A composite wheel made of three uniform disks rolls down an inclined wedge without slipping.

(Filename:sfig7.3.3)

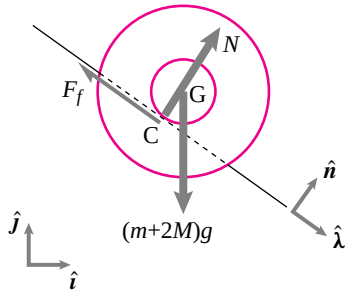


Figure 9.41: Free body diagram of the wheel.

(Filename:sfig7.3.3a)

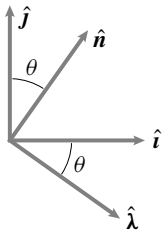


Figure 9.42: Geometry of unit vectors. This diagram can be used to find various dot and cross products between any two unit vectors. For example, $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{j}} = \sin\theta\hat{\boldsymbol{k}}$.

(Filename:sfig7.3.3b)

SAMPLE 9.21 *Rolling on an inclined plane.* A wheel is made up of three uniform disks— the center disk of mass $m = 1\,\text{kg}$, radius $r = 10\,\text{cm}$ and two identical outer disks of mass $M = 2\,\text{kg}$ each and radius R. The wheel rolls down an inclined wedge without slipping. The angle of inclination of the wedge with horizontal is $\theta = 30^o$. The radius of the bigger disks is to be selected such that the linear acceleration of the wheel center does not exceed 0.2g. Find the radius R of the bigger disks.

Solution Since a bound is prescribed on the linear acceleration of the wheel and the radius of the bigger disks is to be selected to satisfy this bound, we need to find an expression for the acceleration of the wheel (hopefully) in terms of the radius R.

The free body diagram of the wheel is shown in Fig. 9.41. In addition to the weight $(m + 2M)g$ of the wheel and the normal reaction N of the wedge surface there is an unknown force of friction F_f acting on the wheel at point C. This friction force is necessary for the condition of rolling motion. You must realize, however, that $F_f \neq \mu N$ because there is neither slipping nor a condition of impending slipping. Thus the magnitude of F_f is not known yet.

Let the acceleration of the center of mass of the wheel be

$$\vec{\boldsymbol{a}}_G = a_G\hat{\boldsymbol{\lambda}}$$

and the angular acceleration of the wheel be

$$\dot{\vec{\boldsymbol{\omega}}} = -\dot{\omega}\hat{\boldsymbol{k}}.$$

We assumed $\dot{\vec{\boldsymbol{\omega}}}$ to be in the *negative* $\hat{\boldsymbol{k}}$ direction. But, if this assumption is wrong, we will get a negative value for $\dot{\omega}$.

Now we write the equation of linear momentum balance for the wheel:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m_{\text{total}}\,\vec{\boldsymbol{a}}_{\text{cm}} \\ -(m+2M)g\hat{\boldsymbol{j}} + N\hat{\boldsymbol{n}} - F_f\hat{\boldsymbol{\lambda}} &= (m+2M)a_G\hat{\boldsymbol{\lambda}} \end{aligned}$$

This 2-D vector equation gives (at the most) two independent scalar equations. But we have three unknowns: N, F_f, and a_G. Thus we do not have enough equations to solve for the unknowns including the quantity of interest a_G. So, we now write the equation of angular momentum balance for the wheel about the point of contact C (using $\vec{\boldsymbol{r}}_{G/C} = r\hat{\boldsymbol{n}}$):

$$\sum \vec{\boldsymbol{M}}_{\text{C}} = \dot{\vec{\boldsymbol{H}}}_{\text{C}}$$

where

$$\begin{aligned} \vec{\boldsymbol{M}}_{\text{C}} &= \vec{\boldsymbol{r}}_{G/C} \times (m+2M)g(-\hat{\boldsymbol{j}}) \\ &= r\hat{\boldsymbol{n}} \times (m+2M)g(-\hat{\boldsymbol{j}}) \\ &= -(m+2M)gr\sin\theta\hat{\boldsymbol{k}} \quad \text{(see Fig. 9.42)} \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{\text{C}} &= I^G_{zz}\,\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{r}}_{G/C} \times m_{\text{total}}\,\vec{\boldsymbol{a}}_G \\ &= I^G_{zz}\,(-\dot{\omega}\hat{\boldsymbol{k}}) - m_{\text{total}}\,\dot{\omega}r^2\hat{\boldsymbol{k}} \\ &= (I^G_{zz} + m_{\text{total}}r^2)(-\dot{\omega}\hat{\boldsymbol{k}}) \\ &= [(\frac{1}{2}mr^2 + 2\cdot\frac{1}{2}MR^2) + \overbrace{(m+2M)}^{m_{\text{total}}}r^2](-\dot{\omega}\hat{\boldsymbol{k}}) \\ &= -[\frac{3}{2}mr^2 + M(R^2+2r^2)]\dot{\omega}\hat{\boldsymbol{k}}. \end{aligned}$$

Thus,

$$\begin{aligned} -(m+2M)gr\sin\theta\hat{\boldsymbol{k}} &= -[\frac{3}{2}mr^2 + M(R^2+2r^2)]\dot{\omega}\hat{\boldsymbol{k}} \\ \Rightarrow \quad \dot{\omega} &= \frac{(m+2M)gr\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}. \end{aligned} \tag{9.38}$$

Now we need to relate $\dot{\omega}$ to a_G. From the kinematics of rolling,

$$a_G = \dot{\omega}r.$$

Therefore, from Eqn. (9.38) we get

$$a_G = \frac{(m+2M)gr^2\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}.$$

Now we can solve for R in terms of a_G:

$$\begin{aligned} \frac{3}{2}mr^2 + M(R^2+2r^2) &= \frac{(m+2M)gr^2\sin\theta}{a_G} \\ \Rightarrow \quad M(R^2+2r^2) &= \frac{(m+2M)g}{a_G}r^2\sin\theta - \frac{3}{2}mr^2 \\ \Rightarrow \quad R^2 &= \frac{(m+2M)g}{Ma_G}r^2\sin\theta - \frac{3m}{2M}r^2 - 2r^2. \end{aligned}$$

Since we require $a_G \leq 0.2g$ we get

$$\begin{aligned} R^2 &\geq \left(\frac{(m+2M)g}{M\cdot 0.2g}\sin\theta - \frac{3m}{2M} - 2\right)r^2 \\ &\geq \left(\frac{5\,\text{kg}}{0.4\,\text{kg}}\cdot\frac{1}{2} - \frac{3\,\text{kg}}{4\,\text{kg}} - 2\right)(0.1\,\text{m})^2 \\ &\geq 0.035\,\text{m}^2 \\ \Rightarrow \quad R &\geq 0.187\,\text{m}. \end{aligned}$$

Thus the outer disks of radius 20 cm will do the job.

$$\boxed{R \geq 18.7\,\text{cm}}$$

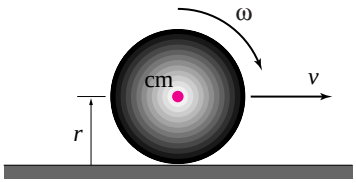


Figure 9.43: (Filename:sfig9.rollandslide.ball1)

SAMPLE 9.22 *Which one starts rolling first — a marble or a bowling ball?* A marble and a bowling ball, made of the same material, are launched on a horizontal platform with the same initial velocity, say v_0. The initial velocity is large enough so that both start out sliding. Towards the end of their motion, both have pure rolling motion. If the radius of the bowling ball is 16 times that of the marble, find the instant, for each ball, when the sliding motion changes to rolling motion.

Solution Let us consider one ball, say the bowling ball, first. Let the radius of the ball be r and mass m. The ball starts with center of mass velocity $\vec{v}_o = v_0\hat{\imath}$. The ball starts out sliding. During the sliding motion, the force of friction acting on the ball must equal μN (see the FBD). The friction force creates a torque about the mass-center which, in turn, starts the rolling motion of the ball. However, rolling and sliding coexist for a while, till the speed of the mass-center slows down enough to satisfy the pure rolling condition, $v = \omega r$. Let the instant of transition from the mixed motion to pure rolling be t^*. From linear momentum balance , we have

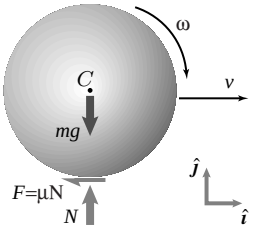


Figure 9.44: Free body diagram of the ball during sliding.
(Filename:sfig9.rollandslide.ball1a)

$$
\begin{aligned}
m\dot{v}\hat{\imath} &= -\mu N\hat{\imath} + (N - mg)\hat{\jmath} && (9.39)\\
\text{eqn. (9.39)} \cdot \hat{\jmath} \quad \Rightarrow \quad N &= mg\\
\text{eqn. (9.39)} \cdot \hat{\imath} \quad \Rightarrow \quad m\dot{v} &= -\mu N = -\mu mg\\
\Rightarrow \quad \dot{v} &= -\mu g\\
\Rightarrow \quad v &= v_0 - \mu g t && (9.40)
\end{aligned}
$$

Similarly, from angular momentum balance about the mass-center, we get

$$
\begin{aligned}
-I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} &= -\mu N r\hat{\boldsymbol{k}} = -\mu mgr\hat{\boldsymbol{k}}\\
\Rightarrow \quad \dot{\omega} &= \frac{\mu mgr}{I_{zz}^{\text{cm}}}\\
\Rightarrow \quad \omega &= \underbrace{\omega_0}_{0} + \frac{\mu mgr}{I_{zz}^{\text{cm}}}t && (9.41)
\end{aligned}
$$

At the instant of transition from mixed rolling and sliding to pure rolling, *i.e.*, at $t = t^*$, $v = \omega r$. Therefore, from eqn. (9.40) and eqn. (9.41), we get

$$
\begin{aligned}
v_0 - \mu g t^* &= \frac{\mu mgr^2}{I_{zz}^{\text{cm}}}t^*\\
\Rightarrow \quad v_0 &= \mu g t^*\left(1 + \frac{mr^2}{I_{zz}^{\text{cm}}}\right)\\
\Rightarrow \quad t^* &= \frac{v_0}{\mu g\left(1 + \frac{mr^2}{I_{zz}^{\text{cm}}}\right)}
\end{aligned}
$$

Now, for a sphere, $I_{zz}^{\text{cm}} = \frac{2}{5}mr^2$. Therefore,

$$
t^* = \frac{v_0}{\mu g\left(1 + \frac{mr^2}{\frac{2}{5}mr^2}\right)} = \frac{2v_0}{7\mu g}.
$$

Note that the expression for t^* is indepenent of mass and radius of the ball! Therefore, the bowling ball and the marble are going to change their mixed motion to pure rolling at exactly the same instant. This is not an intuitive result.

$\boxed{t^* = \frac{2v_0}{7\mu g} \text{ for both.}}$

SAMPLE 9.23 *Transition to pure rolling from sliding and rolling, using impulse-momentum.* Consider the problem in Sample 9.22 again: A ball of radius $r = 10\,\text{cm}$ and mass $m = 1\,\text{kg}$ is launched horizontally with initial velocity $v_0 = 5\,\text{m/s}$ on a surface with coefficient of friction $\mu = 0.12$. The ball starts sliding, rolls and slides simultaneously for a while, and then starts pure rolling. Find the time it takes to start pure rolling.

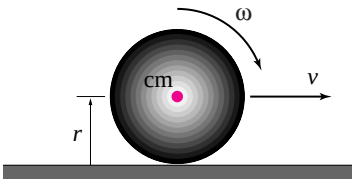


Figure 9.45: (Filename:sfig9.rollandslide.ball2)

Solution Let us denote the time of transition from mixed motion (rolling and sliding) to pure rolling by t^*. At $t = 0$, we know that $v_{\text{cm}} = v_0 = 5\,\text{m/s}$, and $\omega_0 = 0$. We also know that at $t = t^*$, $v_{\text{cm}} = v_{t^*}\omega_{t^*}r$, where r is the radius of the ball. We do not know t^* and v_{t^*}. However, we are considering a finite time event (during t^*) and the forces acting on the ball during this duration are known. Recall that impulse momentum equations involve the net force on the body, the time of impulse, and momenta of the body at the two instants. Momenta calculations involve velocities. Therefore, we should be able to use impulse-momentum equations here and find the desired unknowns. From linear impulse-momentum, we have

$$\begin{aligned} \sum \vec{\boldsymbol{F}} \cdot t^* &= m v_{t^*}\hat{\boldsymbol{\imath}} - m v_0 \hat{\boldsymbol{\imath}} \\ (-\mu N\hat{\boldsymbol{\imath}} + (N - mg)\hat{\boldsymbol{\jmath}})t^* &= m(v_{t^*} - v_0)\hat{\boldsymbol{\imath}} \end{aligned}$$

Dotting the above equation with $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\imath}}$, respectively, we get

$$\begin{aligned} N &= mg \\ -\mu \underbrace{N}_{mg} t^* &= m(v_{t^*} - v_0) \\ \Rightarrow \quad -\mu g t^* &= v_{t^*} - v_0 \end{aligned} \tag{9.42}$$

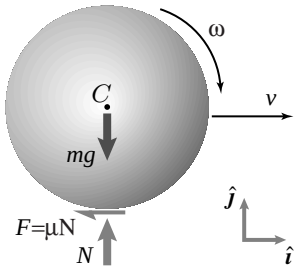


Figure 9.46: Free body diagram of the ball during sliding.
(Filename:sfig9.rollandslide.ball2a)

Similarly, from angular impulse-momentum relation about the mass-center, we get

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\text{cm}} t^* &= (\vec{\boldsymbol{H}}_{\text{cm}})_{t^*} - (\vec{\boldsymbol{H}}_{\text{cm}})_0 \\ (-\mu N r\hat{\boldsymbol{k}})t^* &= (I_{zz}^{\text{cm}}\omega_{t^*} - I_{zz}^{\text{cm}} \underbrace{\omega_0}_{0})(-\hat{\boldsymbol{k}}) \end{aligned}$$

$$\begin{aligned} \text{or} \qquad -\mu mgr t^* &= -I_{zz}^{\text{cm}}\omega_{t^*} \\ \Rightarrow \quad \omega_{t^*} &= \mu mgr t^*/I_{zz}^{\text{cm}} \\ \Rightarrow \quad v_{t^*} &\equiv \omega_{t^*} r = \mu mgr^2 t^*/I_{zz}^{\text{cm}} \end{aligned}$$

Substituting this expression for v_{t^*} in eqn. (9.42), we get

$$\begin{aligned} -\mu g t^* &= \mu mgr^2 t^*/I_{zz}^{\text{cm}} - v_0 \\ \Rightarrow \quad t^* &= \frac{v_0}{\mu g(1 + \frac{mr^2}{I_{zz}^{\text{cm}}})} \end{aligned}$$

which is, of course, the same expression we obtained for t^* in Sample 9.22. Again, noting that $I_{zz}^{\text{cm}} = \frac{2}{5}mr^2$ for a sphere, we calculate the time of transition as

$$t^* = \frac{2v_0}{7\mu g} = \frac{2 \cdot (5\,\text{m/s})}{7 \cdot (0.2) \cdot (9.8\,\text{m/s}^2)} = 0.73\,\text{s}.$$

$$\boxed{t^* = 0.73\,\text{s}}$$

9.5 Collisions

Sometimes when things interact, they do so in a sudden manner. The extreme cases of collisions are dramatic, like car and plane crashes. More esoteric 'sudden' interactions include those between subatomic particles in an accelerator and near passes of satellites with planets. The possible collisions of consumer products with floors is of obvious interest to a design engineer. But many collisions are simply part of the way things work. The collisions of racquets, bats, clubs, sticks, hands and legs with balls, pucks and bodies is a key part of some sports. The clicking of the ratchet in the winding mechanism of an old mechanical clock involves a collision, as does the click of a camera shutter, and the flip of an electric light switch.

When things touch suddenly the contact forces can be much larger than during more smooth motions. But the collision time is short so not much displacement happens while these forces are applied. Thus the elementary analysis of rigid body collisions is based on these ideas:

- Collision forces are big, so non-collisional forces are neglected in the free body diagram.
- Collision forces are of short duration, so the position and orientation of the colliding bodies does not change during the collision.

Because the motion is not tracked during a collision, the details of the acceleration time history is also not tracked. Attention is focussed on the net change in the velocities of the colliding bodies that the collision forces cause. Thus, instead of using the differential equation form of the linear momentum balance, angular momentum balance and energy equations (Ia, IIa, and IIIa from the inside front cover) we use the time integrated forms (Ib, IIb, and IIb). These latter are called the impulse-momentum equations and energy balance equations. Based on these assumptions one then uses linear and angular momentum balance in their time-integrated form (see inside front cover). If there is only one collisional force on a body, and often useful observation is that the angular momentum of the body about that point is conserved.

Two special limiting cases of collisions are the most common idealizations. First is a 'perfectly-plastic' sticking collision. In this case the relative velocities of the two contacting points are assumed to go suddenly to zero (and one is left to calculate the impulse that caused this stop).

The second special case is that of a 'perfectly elastic' frictionless collision. This is a collision in which the system energy is conserved and for which the tangential part of the collisional impulse is assumed to be zero. We state without proof the central theorem for such collisions: the normal component of the separation velocity is equal to the normal component of the approach velocity.

SAMPLE 9.24 The vector equation $m_1\vec{\boldsymbol{v}}_1 + \mathrm{m}_2\vec{\boldsymbol{v}}_2 = m_1\vec{\boldsymbol{v}}'_1 + \mathrm{m}_2\vec{\boldsymbol{v}}'_2$ expresses the conservation of linear momentum of two masses. Suppose $\vec{\boldsymbol{v}}_1 = \vec{\mathbf{0}}$, $\vec{\boldsymbol{v}}_2 = -v_0\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{v}}'_1 = v'_1\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{v}}'_2 = v'_{2_t}\hat{\boldsymbol{e}}_t + v'_{2_n}\hat{\boldsymbol{e}}_n$, where $\hat{\boldsymbol{e}}_t = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{e}}_n = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$. Obtain scalar equations corresponding to projection in the $\hat{\boldsymbol{e}}_n$ and $\hat{\boldsymbol{e}}_t$ directions from the momentum equation.

Solution

$$\begin{aligned} m_1\underbrace{\vec{\boldsymbol{v}}_1}_{\mathbf{0}} + \mathrm{m}_2\vec{\boldsymbol{v}}_2 &= m_1\vec{\boldsymbol{v}}'_1 + \mathrm{m}_2\vec{\boldsymbol{v}}'_2 \\ \text{or} \qquad -m_2v_0\hat{\boldsymbol{\jmath}} &= m_1v'_1\hat{\boldsymbol{\imath}} + m_2(v'_{2_t}\hat{\boldsymbol{e}}_t + v'_{2_n}\hat{\boldsymbol{e}}_n) \end{aligned} \tag{9.43}$$

The dot product $\hat{\boldsymbol{e}}_n\cdot$ [Eqn. (9.43)] gives

$$-m_2v_0(\overbrace{\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{\jmath}}}^{\cos\theta}) = m_1v'_1(\overbrace{\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{\imath}}}^{-\sin\theta}) + m_2(v'_{2_t}(\overbrace{\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{e}}_t}^{0}) + m_2v'_{2_n}(\overbrace{\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{e}}_n}^{1})$$

$$\text{or} \qquad -m_2v_0\cos\theta = -m_1v'_1\sin\theta + m_2v'_{2_n}. \tag{9.44}$$

Dotting both sides of [Eqn. (9.43)] with $\hat{\boldsymbol{e}}_t$ gives

$$-m_2v_0(\overbrace{\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\jmath}}}^{\sin\theta}) = m_1v'_1(\overbrace{\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\imath}}}^{\cos\theta}) + m_2(v'_{2_t}(\overbrace{\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{e}}_t}^{1}) + m_2v'_{2_n}(\overbrace{\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{e}}_n}^{0})$$

$$\text{or} \qquad -m_2v_0\sin\theta = m_1v'_1\cos\theta + m_2v'_{2_t}. \tag{9.45}$$

$$\boxed{-m_2v_0\cos\theta = -m_1v'_1\sin\theta + m_2v'_{2_n}, \quad -m_2v_0\sin\theta = m_1v'_1\cos\theta + m_2v'_{2_t}}$$

SAMPLE 9.25 *Be careful with units:* Two equations:

(a) $-m_2 v_0 \cos\theta = -m_1 v'_1 \sin\theta + m_2 v'_{2_n}$
(b) $-m_2 v_0 \sin\theta = m_1 v'_1 \cos\theta + m_2 v'_{2_t}$

were obtained in Sample 9.24. Assume another equation $v'_{2_t} = -v_0 \sin\theta$ is given. Set up the matrix equations to solve for three unknowns $v'_1,\ v'_{2_t},\ v'_{2_n}$ in terms of the known quantities $v_0 = 20\,\text{ft/s},\ \theta = 60^\text{o}, m_1 = 20\,\text{lbm}$ and $m_2 = 5\,\text{lbm}$.

Solution The equations are:

$$\begin{aligned} -m_1 v'_1 \sin\theta + m_2 v'_{2_n} &= -m_2 v_0 \cos\theta \\ m_1 v'_1 \cos\theta + m_2 v'_{2_t} &= -m_2 v_0 \sin\theta \\ v'_{2_t} &= -v_0 \sin\theta. \end{aligned}$$

They can be written in matrix form as:

$$\begin{bmatrix} -m_1 \sin\theta & 0 & m_2 \\ -m_1 \cos\theta & m_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} v'_1 \\ v'_{2_t} \\ v'_{2_n} \end{Bmatrix} = \begin{Bmatrix} -m_2 v_0 \cos\theta \\ -m_2 v_0 \sin\theta \\ -v_0 \sin\theta \end{Bmatrix}.$$

Although this equation is correct, the elements of the known vector on the right hand side have different units — the first two elements have the units of mass times speed (e.g. lbm- ft/s) and the third element has the units of speed. This situation is slightly awkward from the computational point of view and can be potentially confusing in more complex sets of equations. Therefore, we should aim to get the same units.

Dividing the first equation by m_1 and the second equation by m_2 on both sides we get

$$\begin{bmatrix} -\sin\theta & 0 & \frac{m_2}{m_1} \\ -\frac{m_1}{m_2}\cos\theta & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} v'_1 \\ v'_{2_t} \\ v'_{2_n} \end{Bmatrix} = \begin{Bmatrix} -\frac{m_2}{m_1} v_0 \cos\theta \\ -v_0 \sin\theta \\ -v_0 \sin\theta \end{Bmatrix}.$$

Now all the elements in the matrix on the left hand side are pure numbers without any units and the right hand side vector has the units of speed. If we choose to find the solution by inverting the matrix, the inverse of the matrix will have no units and therefore, we may write the solution as:

$$\begin{Bmatrix} v'_1 \\ v'_{2_t} \\ v'_{2_n} \end{Bmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1/4 \\ -4\cos\theta & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{Bmatrix} (-5\cos\theta)\,\text{ft/s} \\ (-20\sin\theta)\,\text{ft/s} \\ (-20\sin\theta)\,\text{ft/s} \end{Bmatrix}.$$

SAMPLE 9.26 *Cueing a billiard ball.* A billiard ball is cued by striking it horizontally at a distance $d = 10\,\text{mm}$ above the center of the ball. The ball has mass $m = 0.2\,\text{kg}$ and radius $r = 30\,\text{mm}$. Immediately after the strike, the center of mass of the ball moves with linear speed $v = 1\,\text{m/s}$. Find the angular speed of the ball immediately after the strike. Ignore friction between the ball and the table during the strike.

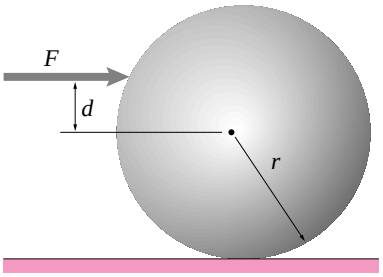


Figure 9.47: (Filename:sfig7.3.DH1)

Solution Let the force imparted during the strike be F. Since the ball is cued by giving a blow with the cue, F is an impulsive force. Impulsive forces, such as F, are in general so large that all non-impulsive forces are negligible in comparison during the time such forces act. Therefore, we can ignore all other forces (mg, N, f) acting on the ball from its free body diagram during the strike.

Now, from the linear momentum balance of the ball we get

$$F\hat{\boldsymbol{\imath}} = \dot{\vec{\boldsymbol{L}}} \quad \text{or} \quad (F\hat{\boldsymbol{\imath}})dt = d\vec{\boldsymbol{L}} \quad \Rightarrow \quad \int (F\hat{\boldsymbol{\imath}})dt = \vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1$$

where $L_2 - L_1 = \Delta\vec{\boldsymbol{L}}$ is the net change in the linear momentum of the ball during the strike. Since the ball is at rest before the strike, $\vec{\boldsymbol{L}}_1 = m\underbrace{v}_{0} = \vec{\boldsymbol{0}}$. Immediately after the strike, $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}} = 1\,\text{m/s}$.

$$\text{Thus} \quad \vec{\boldsymbol{L}}_2 = m\vec{\boldsymbol{v}} = 0.2\,\text{kg}\cdot 1\,\text{m/s}\hat{\boldsymbol{\imath}} = 0.2\,\text{N}\cdot\text{s}\hat{\boldsymbol{\imath}}.$$

$$\text{Hence} \quad \int (F\hat{\boldsymbol{\imath}})dt = 0.2\,\text{N}\cdot\text{s}\hat{\boldsymbol{\imath}} \quad \text{or} \quad \int F dt = 0.2\,\text{N}\cdot\text{s}. \tag{9.46}$$

To find the angular speed we apply the angular momentum balance. Let ω be the angular speed immediately after the strike and $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. Now,

$$\sum \vec{\boldsymbol{M}}_{\text{cm}} = \dot{\vec{\boldsymbol{H}}}_{\text{cm}} \quad \Rightarrow \quad \int \sum \vec{\boldsymbol{M}}_{\text{cm}}\, dt = \int d\vec{\boldsymbol{H}}_{\text{cm}} = (\vec{\boldsymbol{H}}_{\text{cm}})_2 - (\vec{\boldsymbol{H}}_{\text{cm}})_1.$$

$$\begin{aligned}
\text{Since} \quad \vec{\boldsymbol{H}}_{\text{cm}} &= I^{zz}_{cm}\vec{\boldsymbol{\omega}} \quad \text{and just before the strike, } \vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}},\\
(\vec{\boldsymbol{H}}_{\text{cm}})_1 &\equiv \text{angular momentum just before the strike } = \vec{\boldsymbol{0}}\\
(\vec{\boldsymbol{H}}_{\text{cm}})_2 &\equiv \text{angular momentum just after the strike } = I^{zz}_{cm}\omega\hat{\boldsymbol{k}},\\
\int \sum \vec{\boldsymbol{M}}_{\text{cm}}\, dt &= I^{zz}_{cm}\omega\hat{\boldsymbol{k}} = \frac{2}{5}mr^2\omega\hat{\boldsymbol{k}} \text{ (since for a sphere, } I^{zz}_{cm} = \tfrac{2}{5}mr^2).
\end{aligned}$$

$$\begin{aligned}
\text{But} \quad & \sum \vec{\boldsymbol{M}}_{\text{cm}} = -Fd\hat{\boldsymbol{k}},\\
\text{therefore} \quad & -\int (Fd)dt\hat{\boldsymbol{k}} = \frac{2}{5}mr^2\omega\hat{\boldsymbol{k}}\\
\text{or} \quad & -\underbrace{d}_{\text{constant}}\int F dt = \frac{2}{5}mr^2\omega \quad \Rightarrow \quad \omega = -\frac{5d}{2mr^2}\int F dt.
\end{aligned}$$

Substituting the given values and $\int F dt = 0.2\,\text{N}\cdot\text{s}$ from equation 9.46 we get

$$\omega = -\frac{5(0.01\,\text{m})}{2\cdot 0.2\,\text{kg}\cdot(0.03\,\text{m})^2}\cdot 0.2\,\text{N}\cdot\text{s} = -27.78\,\text{rad/s}$$

The negative value makes sense because the ball will spin clockwise after the strike, but we assumed that ω was anticlockwise.

$$\boxed{\omega = -27.78\,\text{rad/s}.}$$

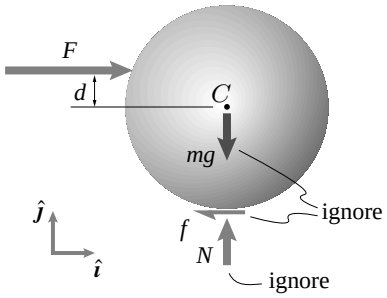


Figure 9.48: FBD of the ball during the strike. The nonimpulsive forces mg, N, and f can be ignored in comparison to the strike force F.

(Filename:sfig7.3.DH2)

10 Kinematics using time-varying base vectors

Many parts of practical machines and structures move in ways that can be idealized as straight-line motion (Chapter 6) or circular motion (Chapters 7 and 8). But often an engineer most analyze parts with more general motions (as in chapter 9).

The kinematics of rigid body motion is often discussed using base vectors which change with time. You have seen some of this concept with polar coordinate base vectors for circular motion. We now apply polar coordinates to more general motion and also introduce other time-varying vector concepts. Altogether we introduce 4 approaches: polar coordinates, path coordinates, direct differentiation of time-varying base vectors, and general relations based on moving frames.

There is no doubt that these concepts are confusing at first for a beginner. But in the end they simplify the description of complex motions.

10.1 Polar coordinates and path coordinates

As we consider more and more complex mechanisms and motions our mathematical tools to keep track of the motion also grow a little in complexity. In this section we extend the use of polar coordinates to include more than just circular motion. We also introduce path coordinates.

Our study of motion depends on being able to conveniently express the velocity

and acceleration of points. In principle, we could just use one fixed coordinate system with base vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ and write the velocity and acceleration of a point at position $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ as

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} \quad \text{and} \quad \vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}.$$

This was the approach used in sections 5.7-10. But, as you saw with circular motion, it is often more convenient to use polar coordinates, the first case we consider here.

Polar coordinates

You have already seen that for problems involving circular motion, polar coordinates can be useful. Polar coordinates, or their extension to 3 dimensions as cylindrical coordinates, are useful for many mechanical systems.

Rather than identifying the location of a point by its x, y and z coordinates, a point is located by its cylindrical coordinates (see figure 10.1):

- R, the distance to the point from the z axis.
- θ, the angle that the most direct line from the z axis to the point makes with the positive x direction, and
- z, the distance of the particle from the xy plane.

Writing the position as a vector we can write:

$$\vec{r} = R\hat{e}_R + z\hat{k}.$$

or, if only two-dimensional problems are being considered;

$$\vec{r} = R\hat{e}_R.$$

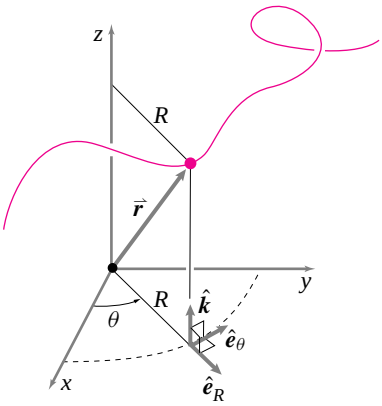


Figure 10.1: Polar coordinates.

(Filename:tfigure6.1)

As the particle moves, the values of its coordinates R, θ and z also change. The velocity and acceleration *cannot* be found by just differentiating R, θ and z with respect to time. Why not? Because the base vectors $\hat{e}_R$ and $\hat{e}_\theta$ also change with time. How do they change? The polar coordinate base vectors change the same way they did for circular motion:

$$\dot{\hat{e}}_R = \dot{\theta}\hat{e}_\theta \qquad \text{and} \qquad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_R.$$

Note that, since $\hat{e}_\theta$ is perpendicular to $\hat{e}_R$, $\hat{e}_\theta$ is *not* tangent to the particle's path in general. For example, consider a particle moving on an elliptical path in the plane shown in figure 10.2. The transverse vector $\hat{e}_\theta$ is only tangent to the path where the major and minor axes intersect it. For the special case of circular motion, as in Chapters 4 and 5, $\hat{e}_\theta$ is always tangent to the path.

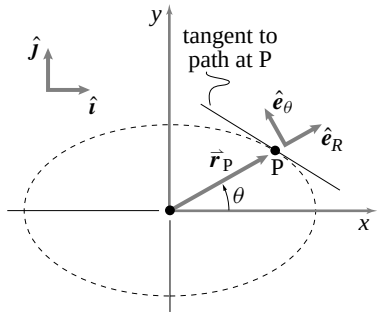


Figure 10.2: Particle P on an elliptical path

(Filename:tfigure6.polar.coord)

Now we can find the velocity and acceleration of a point whose polar coordinates we know as a function of time. We find these $\vec{v}$ and $\vec{a}$ by differentiation $\vec{r}$ using the product rule.

Velocity in polar coordinates

We find the velocity by taking the time derivative of the position:

$$\begin{aligned}
\vec{v} = \frac{d}{dt}\vec{r} &= \frac{d}{dt}[R\hat{e}_R + z\hat{k}] \\
&= \frac{d}{dt}(R\hat{e}_R) + \frac{d}{dt}(z\hat{k}) \\
&= (\dot{R}\hat{e}_R + R\underbrace{\dot{\hat{e}}_R}_{\dot{\theta}\hat{e}_\theta}) + (\dot{z}\hat{k}) \\
&= \underbrace{\dot{R}}_{v_R}\hat{e}_R + \underbrace{R\dot{\theta}}_{v_\theta}\hat{e}_\theta + \underbrace{\dot{z}}_{v_z}\hat{k}.
\end{aligned}$$

This formula is easy enough to understand: the velocity is the sum of three vectors: one due to moving towards or away from the z axis, $\dot{R}\hat{\boldsymbol{e}}_R$, one to motion perpendicular to the xy plane, $\dot{z}\hat{\boldsymbol{k}}$, and one having to do with the angle being swept, $\dot{\theta}R\hat{\boldsymbol{e}}_\theta$.

But, as has been mentioned before, $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}$. So, $v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}} = v_R\hat{\boldsymbol{e}}_R + v_\theta\hat{\boldsymbol{e}}_\theta + v_z\hat{\boldsymbol{k}}$. That is, the same vector $\vec{\boldsymbol{v}}$ can be represented in two different coordinate systems.

Note that this formula adds two terms to the circular motion case from chapter 5—one for variations in R and one for variations in z.

Acceleration in polar coordinates

To find the acceleration, we differentiate once again. The formula gets a little more complicated because there are more terms in the product rule of differentiation. Here is the calculation of acceleration:

$$\begin{aligned}
\vec{\boldsymbol{a}} &= \frac{d}{dt}\vec{\boldsymbol{v}} \\
&= \frac{d}{dt}(\dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}}) \\
&= (\ddot{R}\hat{\boldsymbol{e}}_R + \dot{R}\underbrace{\dot{\hat{\boldsymbol{e}}}_R}_{\dot{\theta}\hat{\boldsymbol{e}}_\theta}) + (\dot{R}\dot{\theta}\hat{\boldsymbol{e}}_\theta + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta + R\dot{\theta}\underbrace{\dot{\hat{\boldsymbol{e}}}_\theta}_{-\dot{\theta}\hat{\boldsymbol{e}}_R}) + (\ddot{z}\hat{\boldsymbol{k}}) \\
&= (\underbrace{\ddot{R} - \dot{\theta}^2 R}_{a_R})\hat{\boldsymbol{e}}_R + (\underbrace{2\dot{R}\dot{\theta} + R\ddot{\theta}}_{a_\theta})\hat{\boldsymbol{e}}_\theta + \underbrace{\ddot{z}}_{a_z}\hat{\boldsymbol{k}}.
\end{aligned}$$

The five terms comprising the polar coordinate formula for acceleration are mostly easy to understand. $\ddot{R}$ is just the acceleration due to the distance from the origin changing with time. $\dot{\theta}^2 R$ is the familiar *centripetal acceleration*. $R\ddot{\theta}\hat{\boldsymbol{e}}_\theta$ is just the acceleration due to rotation proceeding at a faster and faster rate. $\ddot{z}$ should need no comment.

The difficult term is the $2\dot{\theta}\dot{R}$, called the *Coriolis acceleration*. It is the acceleration that occurs even if $\dot{\theta}$ is constant and $\dot{R}$ is constant. The presence of the '2' in this term is due to the two effects which it encompasses. One is that if you think of something moving out a radial line, the orientation of that line changes with time due to $\dot{\theta}$. The other is that if something is, roughly, going in circles the radius of these circles increases with $\dot{R}$.

But one should not be confused by these terms, they are just a complex way of describing the same acceleration which is described with cartesian coordinates. Namely, $\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}$ and

$$\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}} = (\underbrace{\ddot{R} - \dot{\theta}^2 R}_{a_R})\hat{\boldsymbol{e}}_R + (\underbrace{2\dot{R}\dot{\theta} + R\ddot{\theta}}_{a_\theta})\hat{\boldsymbol{e}}_\theta + \underbrace{\ddot{z}}_{a_z}\hat{\boldsymbol{k}}.$$

Summary of polar coordinates

See table II, row 3:

$$\begin{aligned}
\vec{\boldsymbol{r}} &= R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{v}} &= \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{a}} &= (\ddot{R} - \dot{\theta}^2 R)\hat{\boldsymbol{e}}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}
\end{aligned}$$

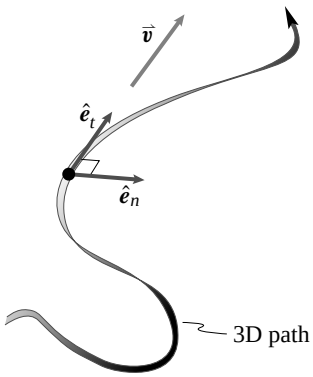


Figure 10.3: The base vectors for path coordinates are tangent and perpendicular to the path.

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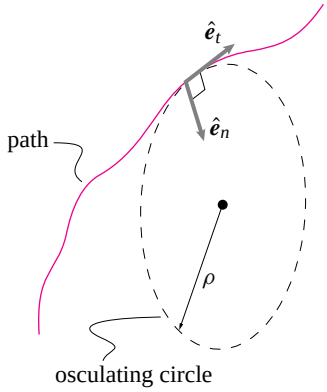


Figure 10.4: The osculating circle is a circle that is tangent to the path and has the same curvature as the path.

(Filename:tfigure6.4)

Path coordinates

Another way still to describe the velocity and acceleration is to use base vectors which are defined by the motion. In particular, the base vectors used are:

(a) the unit tangent to the path $\hat{e}_t$, and
(b) the unit normal to the path $\hat{e}_n$.

Somewhat surprisingly at first glance, only two base vectors are needed to define the velocity and acceleration, even in three dimensions.

The base vectors can be described geometrically and analytically. Let's start out with a geometric description.

The geometry of the path basis vectors.

As a particle moves through space it traces a path $\vec{r}(t)$. At the moment of interest the path has a unique tangent line. The unit tangent $\hat{e}_t$ is in the direction of this line, the direction of motion, as shown in figure 10.3.

Less clear is that the path has a unique 'kissing' *plane*. One line on this plane is the tangent line. The other line needed to define this plane is determined by the position of the particle just before and just after the time of interest. Just before and just after the time of interest the particle is a little off the tangent line (unless the motion happens to be a straight line and the tangent plane is not uniquely determined). Three points, the position of the particle just before, just at, and just after the moment of interest determine the tangent plane.

Another way to picture the tangent plane is to find the circle in space that is tangent to the path and which turns at the same rate and in the same direction as the path turns. This circle, which touches the path so intimately, is called the *osculating* or 'kissing' circle. The tangent plane is the plane of this circle. (See figure 10.4).

The unit normal $\hat{e}_n$ is the unit vector which is perpendicular to the unit tangent and is in the tangent plane. It is pointed in the direction from the edge of the osculating circle towards the center of the circle as shown in figure 10.4. For 2-D motion in the xy plane the osculating plane is the xy plane and the osculating circle is in the xy plane.

Formal definition of path basis vectors

The path of a particle is $\vec{r}(t)$. The path can also be parameterized by arc length s, as you can read about in any introductory calculus text. So the path in space is $\vec{r}(s)$. The unit tangent is:

$$\hat{e}_t \equiv \frac{d\vec{r}(s)}{ds}.$$

It is easy to show (use the chain rule with $\vec{r}(s(t))$) that this is also

$$\hat{e}_t = \frac{d\vec{r}(t)}{dt}\frac{dt}{ds} = \frac{\vec{v}}{v}.$$

To define the unit normal let's first define the *curvature* of the path $\vec{\kappa}$. The unit normal $\hat{e}_n$ is the unit vector in the direction of the curvature $\vec{\kappa} \equiv \frac{d\hat{e}_t}{ds}$:

$$\hat{e}_n = \frac{\vec{\kappa}}{|\vec{\kappa}|}.$$

Finally, the binormal e_b is the unit vector perpendicular to $\hat{e}_t$ and $\hat{e}_n$:

$$e_b \equiv \hat{e}_t \times \hat{e}_n.$$

The radius of the osculating circle ρ is

$$\rho = \frac{1}{|\vec{\kappa}|}.$$

Note that, in general, the polar and path coordinate basis vectors are not parallel; i.e., $\hat{e}_n$ is *not* parallel to to $\hat{e}_R$ and $\hat{e}_t$ is *not* parallel to $\hat{e}_\theta$ — it is easy to confuse these basis vectors! For example, consider a particle moving on an elliptical path in the plane shown in figure 10.5. In this case, the polar coordinate and path coordinate basis vectors are only parallel where the major and minor axes intersect the path. For the special case of circular motion, as in Chapters 4 and 5, the polar and path coordinate vectors are everywhere parallel on the path.

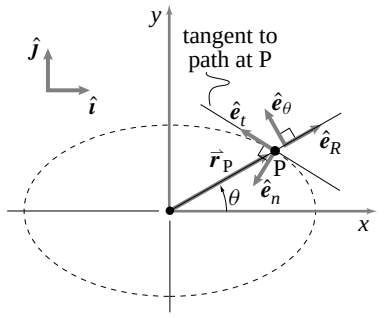


Figure 10.5: Particle P on an elliptical path

(Filename:tfigure6.path.coord)

Velocity and acceleration in path coordinates.

Though it is not necessarily easy to compute the path basis vectors $\hat{e}_t$ and $\hat{e}_n$, they lead to simple expressions for the velocity and acceleration. Namely:

$$\vec{v} = v\hat{e}_t = \frac{ds}{dt}\hat{e}_t, \quad \text{and} \tag{10.1}$$

$$\vec{a} = \underbrace{\dot{v}\hat{e}_t}_{\vec{a}_t} + \underbrace{\frac{v^2}{\rho}\hat{e}_n}_{\vec{a}_n} \qquad v^2\vec{\kappa} \tag{10.2}$$

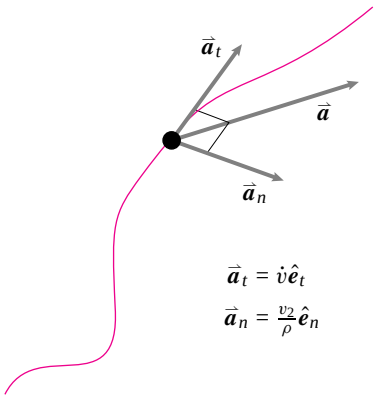


Figure 10.6: The acceleration of any particle can be broken into a part that is parallel to its path and a part perpendicular to its path.

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where s is arc length. This formula for velocity is obvious: velocity is speed times a unit vector in the direction of motion. The formula for acceleration is more interesting. It says that the acceleration of any particle at any time is given by the same formula as the formula for acceleration of a particle going around in circles at non-constant rate. There is a term directed towards the center of the osculating circle v^2/ρ and a second term tangent to the path (also tangent to the osculating circle), as shown in figure 10.6.

Another way to describe the acceleration formula is to say that the acceleration has two parts: one associated with change of direction, this normal acceleration does not vanish even if the *speed* is constant. This normal acceleration is perpendicular to the path. The second term is associated with the change of speed, this tangential acceleration does not vanish even if the particle moves in a straight line. The tangential acceleration is, like its name implies, tangent to the path.

Earlier we found the curvature by assuming the particle's path was parameterized by arc length s. A second way of calculating the curvature $\vec{\kappa}$ (and then the unit normal $\hat{e}_n$) is to calculate the normal part of the acceleration. First calculate the acceleration. Then subtract from the acceleration that part which is parallel to the velocity.

$$\vec{a}_n = \vec{a} - (\vec{a}\cdot\hat{e}_t)\hat{e}_t = \vec{a} - \frac{(\vec{a}\cdot\vec{v})\vec{v}}{v^2}$$

The curvature is $\vec{a}_n/v^2$ so

$$\vec{\kappa} = \frac{\vec{a}}{v^2} - \frac{(\vec{a}\cdot\vec{v})\vec{v}}{v^4}.$$

Recipes for path coordinates.

Assume that you know the position as a function of time in either cartesian or polar coordinates. Then, say, at a particular time of interest when the particle is at $\vec{\boldsymbol{r}}$, you can calculate the velocity of the particle using:

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}} \quad \text{or} \quad \vec{\boldsymbol{v}} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}}$$

and the acceleration using

$$\vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}} \quad \text{or} \quad \vec{\boldsymbol{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}.$$

From these expressions we can calculate all the quantities used in the path coordinate description. So we repeat what we have already said but in an algorithmic form. Here is one set of steps one can follow.

(a) Calculate $\hat{\boldsymbol{e}}_t = \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}|$.
(b) Calculate $a_t = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}}/v$.
(c) Calculate $\vec{\boldsymbol{a}}_n = \vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}})\vec{\boldsymbol{v}}/v^2$.
(d) Calculate $\hat{\boldsymbol{e}}_n = \frac{\vec{\boldsymbol{a}}_n}{|\vec{\boldsymbol{a}}_n|}$
(e) Calculate the radius of curvature as $\rho = \frac{|\vec{\boldsymbol{v}}|^2}{|\vec{\boldsymbol{a}}_n|}$.
(f) Though not often important to calculate explicitly, it is nice to know that the osculating circle does have explicit representation. A parametric equation for the osculating circle is given by

$$\vec{\boldsymbol{r}}_{\text{osculating}} = \vec{\boldsymbol{r}} + \rho(\hat{\boldsymbol{e}}_n + \cos\phi\hat{\boldsymbol{e}}_n + \sin\phi\hat{\boldsymbol{e}}_t)$$

where ϕ ($0 \leq \phi < 2\pi$) is the parameter used to parameterize the points on the circle $\vec{\boldsymbol{r}}_{\text{osculating}}$. $\vec{\boldsymbol{r}}$ is the point on the curve where the circle is tangent to the curve and has the same curvature. The plane of the osculating circle is the plane determined by the path of the particle and the direction in which it is turning. For strictly two-dimensional curves, the osculating circle is exactly in the plane of the curve.

SAMPLE 10.1 *Velocity in path coordinates.* The path of a particle, stuck at the edge of a disk rolling on a level ground with constant speed, is called a cycloid. The parametric equations of a cycloid described by a particle is $x = t - \sin t$, $y = 1 - \cos t$ where t is a dimensionless time. Find the velocity of the particle at

(a) $t = \frac{\pi}{2}$,
(b) $t = \pi$, and
(c) $t = 2\pi$

and express the velocity in terms of path basis vectors $(\hat{\boldsymbol{e}}_t, \hat{\boldsymbol{e}}_n)$.

Solution The position of the particle is given:

$$\begin{aligned} & & \vec{\boldsymbol{r}} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \\ & &&= (t - \sin t)\hat{\boldsymbol{\imath}} + (1 - \cos t)\hat{\boldsymbol{\jmath}} \\ \Rightarrow & & \vec{\boldsymbol{v}} &= \frac{d\vec{\boldsymbol{r}}}{dt} = (1 - \cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}, \\ \text{and} & & v &= |\vec{\boldsymbol{v}}| = \sqrt{(1 - \cos t)^2 + \sin^2 t} \\ & &&= \sqrt{2 - 2\cos t}. \end{aligned} \tag{10.3}$$

In terms of path basis vectors, the velocity is given by

$$\vec{\boldsymbol{v}} = v\hat{\boldsymbol{e}}_t \quad \text{where} \quad \hat{\boldsymbol{e}}_t = \vec{\boldsymbol{v}}/v.$$

Here,

$$\hat{\boldsymbol{e}}_t = \frac{(1 - \cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}}{\sqrt{2 - 2\cos t}} \tag{10.4}$$

Substituting the values of t in equations 10.3 and 10.4 we get

(a) at $t = \frac{\pi}{2}$:

$$v = \sqrt{2}, \quad \hat{\boldsymbol{e}}_t = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}), \quad \vec{\boldsymbol{v}} = \sqrt{2}\hat{\boldsymbol{e}}_t$$

$$\boxed{\vec{\boldsymbol{v}} = \sqrt{2}\hat{\boldsymbol{e}}_t, \quad \hat{\boldsymbol{e}}_t = \tfrac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

(b) at $t = \pi$:

$$v = 2, \quad \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}, \quad \vec{\boldsymbol{v}} = 2\hat{\boldsymbol{e}}_t$$

$$\boxed{\vec{\boldsymbol{v}} = 2\hat{\boldsymbol{e}}_t, \quad \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}}$$

(c) at $t = 2\pi$:

$$v = 0, \quad \hat{\boldsymbol{e}}_t = \text{undefined}, \quad \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

$$\boxed{\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}}$$

SAMPLE 10.2 *Path coordinates in 2-D.* A particle of mass $m = \frac{1}{2}$ lbm traverses a limacon $R = (1 + 2\cos\theta)$ ft, with constant angular speed $\dot{\theta} = 3$ rad/s.

(a) Find the normal and tangential accelerations (a_t and a_n) of the particle at $\theta = \frac{\pi}{2}$.

(b) Find the radius of the osculating circle and draw the circle at $\theta = \frac{\pi}{2}$.

Solution

(a) The equation of the path is

$$R = (1 + 2\cos\theta)\ \text{ft}$$

The path is shown in Fig. 10.7. Since the equation of the path is given in polar coordinates, we can calculate the velocity and acceleration using the polar coordinate formulae:

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta \tag{10.5}$$
$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta. \tag{10.6}$$

Thus we need to find $\dot{R}$, $\ddot{R}$, $\ddot{\theta}$ for computing $\vec{v}$ and $\vec{a}$. From the given equation for R

$$\begin{aligned}\dot{R} &= -(2\ \text{ft})\sin\theta\,\dot{\theta}\\ \ddot{R} &= -(2\ \text{ft})\sin\theta\,\underbrace{\ddot{\theta}}_{0} - (2\ \text{ft})\cos\theta\,\dot{\theta}^2\\ &= -(2\ \text{ft})\dot{\theta}^2\cos\theta\end{aligned}$$

where we set $\ddot{\theta} = 0$ because $\dot{\theta} =$ constant. Substituting these expressions in Eqn. (10.5) and (10.6), we get

$$\begin{aligned}\vec{v} &= -(2\ \text{ft})\dot{\theta}\sin\theta\hat{e}_R + \dot{\theta}(1 + \cos\theta)\ \text{ft}\hat{e}_\theta\\ \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\underbrace{\ddot{\theta}}_{0})\hat{e}_\theta\\ &= [-2\ \text{ft}\dot{\theta}^2\cos\theta - (1 + 2\cos\theta)\ \text{ft}\,\dot{\theta}^2]\hat{e}_R + (-2\ \text{ft})\dot{\theta}^2\sin\theta\hat{e}_\theta\\ &= -\dot{\theta}^2[(1 + 4\cos\theta)\hat{e}_R + 2\sin\theta\hat{e}_\theta]\ \text{ft}\end{aligned}$$

which give velocity and acceleration at any θ. Now substituting $\theta = \pi/2$ we get the velocity and acceleration at the desired point:

$$\begin{aligned}\vec{v}|_{\frac{\pi}{2}} &= \dot{\theta}[-2\sin\frac{\pi}{2}\hat{e}_R + (1 + 2\cos\frac{\pi}{2})\hat{e}_\theta]\ \text{ft}\\ &= 3\ \text{ft/s}(-2\hat{e}_R + \hat{e}_\theta)\\ \vec{a}|_{\frac{\pi}{2}} &= -9\ \text{ft/s}^2(\hat{e}_R + 2\hat{e}_\theta.)\end{aligned}$$

Thus we know the velocity and the acceleration of the particle in polar coordinates. Now we proceed to find the tangential and the normal components of acceleration (acceleration in path coordinates). In path coordinates

$$\vec{a} = \vec{a}_t + \vec{a}_n \equiv a_t\hat{e}_t + a_n\hat{e}_n$$

where $\hat{e}_t$ and $\hat{e}_n$ are unit vectors in the directions of the tangent and the principal normal of the path. We compute these unit vectors as follows.

$$\hat{e}_t = \frac{\vec{v}}{|\vec{v}|}$$

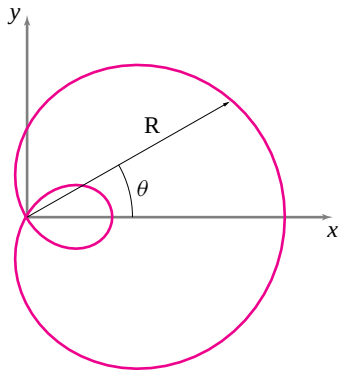


Figure 10.7: The limacon $R = (1 + 2\cos\theta)$ ft.

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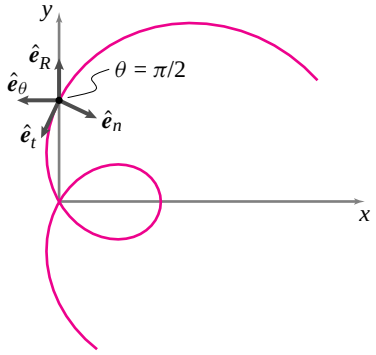


Figure 10.8: The unit vectors $\hat{e}_R, \hat{e}_\theta$, and $\hat{e}_t, \hat{e}_n$ at $\theta = \pi/2$.

(Filename:sfig6.3.1b)

$$
\begin{aligned}
&= \frac{-6\,\text{ft/s}\hat{\boldsymbol{e}}_R + 3\,\text{ft/s}\hat{\boldsymbol{e}}_\theta}{\sqrt{45}\ \text{ft/s}} \\
&= -\frac{2}{\sqrt{5}}\hat{\boldsymbol{e}}_R + \frac{1}{\sqrt{5}}\hat{\boldsymbol{e}}_\theta \\
\vec{\boldsymbol{a}}_t &= (\vec{\boldsymbol{a}}\cdot\hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t \\
&= 9\,\text{ft/s}^2(-\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}})\hat{\boldsymbol{e}}_t = \vec{\boldsymbol{0}} \\
\vec{\boldsymbol{a}}_n &= \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_t \\
&= -9\,\text{ft/s}^2(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta) \\
\hat{\boldsymbol{e}}_n &= \frac{\vec{\boldsymbol{a}}_n}{|\vec{\boldsymbol{a}}_n|} \\
&= -\frac{1}{\sqrt{5}}(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= 9\sqrt{5}\ \text{ft/s}^2\hat{\boldsymbol{e}}_n \\
\Rightarrow\quad a_t &= 0 \quad \text{and} \quad a_n = 20.12\,\text{ft/s}^2.
\end{aligned}
$$

$$\boxed{a_t = 0,\ a_n = 20.12\,\text{ft/s}^2}$$

(b) In path coordinates the acceleration is also expressed as

$$\vec{\boldsymbol{a}} = \dot{v}\hat{\boldsymbol{e}}_t + \frac{v^2}{\rho}\hat{\boldsymbol{e}}_n$$

where ρ is the radius of the osculating circle. Since we already know the speed v and the normal component of acceleration a_n we can easily compute the radius of the osculating circle.

$$
\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
\Rightarrow\quad \rho &= \frac{v^2}{a_n} \\
&= \frac{45(\,\text{ft/s})^2}{9\sqrt{5}\,\text{ft/s}^2} \\
&= \sqrt{5}\,\text{ft}.
\end{aligned}
$$

$$\boxed{\rho = 2.24\,\text{ft}}$$

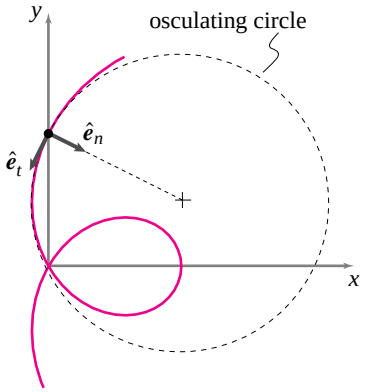


Figure 10.9: The osculating circle of radius $\rho = 5^{1/2}$ ft at $\theta = \pi/2$. Note that $\hat{\boldsymbol{e}}_n$ points to the center of the osculating circle.

(Filename:sfig6.3.1c)

SAMPLE 10.3 *Acceleration in polar coordinates.* A bug walks along the spiral section of a natural shell. The path of the bug is described by the equation $R = R_0 e^{a\theta}$ where $a = 0.182$ and $R_0 = 5\,\text{mm}$. The bug's radial distance from the center of the spiral is seen to be increasing at a constant rate of $2\,\text{mm/s}$. Find the x and y components of the acceleration of the bug at $\theta = \pi$.

Solution In polar coordinates the acceleration of a particle in planar motion is expressed as (see table II, row 3)

$$\vec{\boldsymbol{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta.$$

Since we know the position of the bug,

$$\begin{aligned} R &= R_0 e^{a\theta}, \\ \dot{R} &= R_0 a e^{a\theta}\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{\dot{R}}{R_0 a\, e^{a\theta}}, \\ \ddot{R} &= R_0 a e^{a\theta}\ddot{\theta} + R_0 a^2 e^{a\theta}\dot{\theta}^2. \end{aligned}$$

Since the radial distance R of the bug is increasing at a constant rate $\dot{R} = 2\ \text{mm/s}$, $\ddot{R} = 0$, that is,

$$R_0 a e^{a\theta}(\ddot{\theta} + a\dot{\theta}^2) = 0$$

$$\Rightarrow \qquad \ddot{\theta} = -a\dot{\theta}^2 = -\frac{a\dot{R}^2}{R_0^2 a^2\, e^{2a\theta}} = -\frac{\dot{R}^2}{R_0^2 a\, e^{2a\theta}}.$$

Therefore,

$$\begin{aligned} \vec{\boldsymbol{a}} &= (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta \\ &= \left(0 - R_0 e^{a\theta}\cdot\frac{\dot{R}^2}{R_0^2 a^2 e^{2a\theta}}\right)\hat{\boldsymbol{e}}_R + \left(\frac{2\dot{R}^2}{R_0 a\, e^{a\theta}} + \overbrace{R_0 e^{a\theta}}^{R}\cdot\overbrace{\frac{-\dot{R}^2}{R_0^2 a\, e^{2a\theta}}}^{\ddot{\theta}}\right)\hat{\boldsymbol{e}}_\theta \\ &= \frac{\dot{R}^2}{R_0 a\, e^{a\theta}}\left[-\frac{1}{a}\hat{\boldsymbol{e}}_R + (2-1)\hat{\boldsymbol{e}}_\theta\right]. \end{aligned}$$

Now substituting $R_0 = 5\,\text{mm}$, $a = 0.182$, $\dot{R} = 2\,\text{mm/s}$, and $\theta = \pi$ in the above expression, we get

$$\vec{\boldsymbol{a}} = -13.63\,\text{mm/s}^2\hat{\boldsymbol{e}}_R + 2.48\,\text{mm/s}^2\hat{\boldsymbol{e}}_\theta.$$

But, at $\theta = \pi$

$$\hat{\boldsymbol{e}}_R = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}} \quad \text{and} \quad \hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\jmath}},$$

therefore,

$$\begin{aligned} \vec{\boldsymbol{a}} &= 13.63\,\text{mm/s}^2\hat{\boldsymbol{\imath}} - 2.48\,\text{mm/s}^2\hat{\boldsymbol{\jmath}}, \\ \Rightarrow \quad a_x &= 13.63\,\text{mm/s}^2 \quad \text{and} \quad a_y = -2.48\,\text{mm/s}^2. \end{aligned}$$

$$\boxed{a_x = 13.63\,\text{mm/s}^2,\ a_y = -2.48\,\text{mm/s}^2}$$

10.2 Rotating reference frames

The direct differentiation method

An alternative approach to computing velocity, acceleration is to express the position of a particle in terms of a combination of based vectors, some of which change in time. Velocity and acceleration are then determined by directly differentiating the expression for position, taking account that the base vectors themselves are changing. This method is sometimes convenient for bodies connected in series, one body to the next, etc. The overall approach is as follows:

1) Glue a coordinate system to every moving body. If needed, also create moving frames that move independently of any particular body.
2) Call the basis vectors associated with these frames $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ for the fixed frame $\mathcal{F}$; $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ for the moving frame $\mathcal{B}$; and $\hat{\imath}''$, $\hat{\jmath}''$, $\hat{k}''$ for the moving frame $\mathcal{C}$, etc.
3) Evaluate all of the relative angular velocities; $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$, $\vec{\omega}_{\mathcal{C}/\mathcal{B}}$, etc. in terms of the the scalar angular rates $\dot{\theta}$, $\dot{\phi}$, etc. and the base vectors glued to the frames.
4) Express all of the absolute angular velocities in terms of the relative angular velocities.
5) Differentiate to get the angular accelerations using, for example,

$$\begin{aligned}\dot{\hat{\imath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\imath}' \text{ or}\\ \dot{\hat{\jmath}}'' &= \vec{\omega}_{\mathcal{C}} \times \hat{\jmath}''\end{aligned}$$

6) Write the position of all points of interest in terms of the various base vectors.
7) Differentiate the position to get the velocities (again using $\dot{\hat{\jmath}}'' = \vec{\omega}_{\mathcal{C}} \times \hat{\jmath}''$, etc.)
8) Differentiate again to get acceleration.

First, reconsider the bug crawling on the tire in figure 10.11.

Example: **Absolute velocity of a point moving relative to a moving frame (2-D), Alternative Method: Bug crawling on a tire, again**

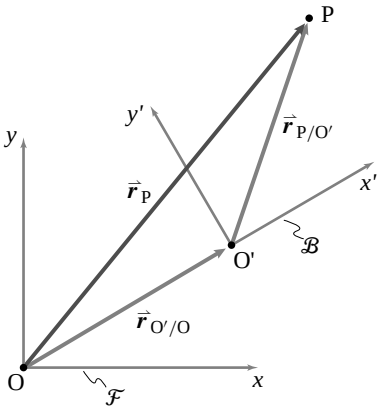
We write the position of the bug in terms of the various basis vectors as

$$\begin{aligned}\vec{r}_{P/O} &= \vec{r}_{O'/O} + \vec{r}_{P/O'}\\ &= \overbrace{d\hat{\imath} + R\hat{\jmath}}^{\vec{r}_{O'/O}} + \overbrace{s\hat{\imath}' + \ell\hat{\jmath}'}^{\vec{r}_{P/O'}}.\end{aligned}$$

To get the absolute velocity $\frac{\mathcal{F}d}{dt}(\vec{r}_{P/O})$ of the bug at the instant shown, we differentiate the position of the bug once, using the product rule and the rates of change of the rotating basis vectors with respect to the fixed frame, to get

$$\begin{aligned}\dot{\vec{r}}_{P/O} = \vec{v}_P &= \dot{d}\hat{\imath} + \underbrace{\dot{R}}_{0}\hat{\jmath} + \underbrace{\dot{s}}_{0}\hat{\imath}' + s\dot{\hat{\imath}}' + \dot{\ell}\hat{\jmath}' + \ell\dot{\hat{\jmath}}'\\ &= \dot{d}\hat{\imath} + s(\vec{\omega}_{\mathcal{B}} \times \hat{\imath}') + \dot{\ell}\hat{\jmath}' + \ell(\vec{\omega}_{\mathcal{B}} \times \hat{\jmath}')\\ &= \dot{d}\hat{\imath} + s(-\dot{\theta}\hat{k}' \times \hat{\imath}') + \dot{\ell}\hat{\jmath}' + \ell(-\dot{\theta}\hat{k}' \times \hat{\jmath}')\\ &= \dot{d}\hat{\imath} + \ell\dot{\theta}\hat{\imath}' + (\dot{\ell} - s\dot{\theta})\hat{\jmath}'. \qquad (10.7)\end{aligned}$$

□



$\vec{r}_{P'/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$
$= (x_O\hat{\imath} + y_O\hat{\jmath}) + (x_P\hat{\imath}' + y_P\hat{\jmath}')$

Figure 10.10: (Filename:tfigure8.alt.app)

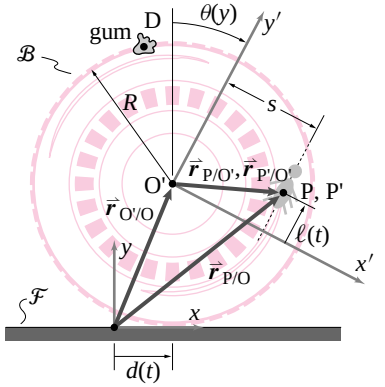


Figure 10.11: The xyz coordinate system is attached to the fixed frame with basis vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and the $x'y'z'$ coordinate system is attached to the rolling tire with basis vectors $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ .The absolute angular velocity of the tire is $\vec{\omega}_{\mathcal{B}/\mathcal{F}} = \vec{\omega}_{\mathcal{B}} = -\dot{\theta}\hat{k}' = -\dot{\theta}\hat{k}$.

(Filename:tfigure8.alt.meth1)

Example: **Absolute acceleration of a point moving relative to a moving frame (2-D), Alternative Method: Bug crawling on a tire, again**

Differentiating equation 10.7 from the example above again, we get the absolute acceleration $\frac{\mathcal{F}d^2}{dt^2}(\vec{\boldsymbol{r}}_{P/O}) = \frac{\mathcal{F}d}{dt}(\vec{\boldsymbol{v}}_P)$ of the bug at the instant shown,

$$\begin{aligned}\ddot{\vec{\boldsymbol{r}}}_{P/O} = \dot{\vec{\boldsymbol{v}}}_P = \vec{\boldsymbol{a}}_P &= \ddot{d}\hat{\boldsymbol{\imath}} + (\ell\ddot{\theta} + \dot{\ell}\dot{\theta})\hat{\boldsymbol{\imath}}' + \ell\dot{\theta}(-\dot{\theta}\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\imath}}') \\ &\quad +(\ddot{\ell} - s\ddot{\theta} - \underbrace{\dot{s}}_{0}\dot{\theta})\hat{\boldsymbol{\jmath}}' + (\dot{\ell} - s\dot{\theta})(-\dot{\theta}\hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\jmath}}') \\ &= \ddot{d}\hat{\boldsymbol{\imath}} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\ell}\dot{\theta})\hat{\boldsymbol{\imath}}' + (\ddot{\ell} - \ell\dot{\theta}^2 - s\ddot{\theta})\hat{\boldsymbol{\jmath}}'.\end{aligned}$$

□

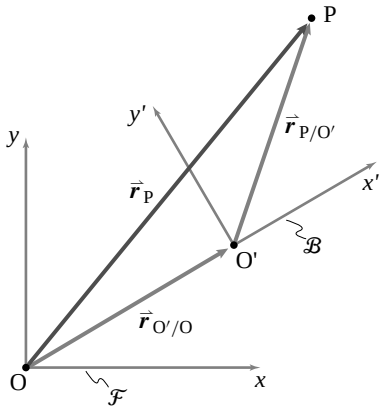


Figure 10.12: (Filename:tfigure8.alt.app2)

Summary of the direct differentiation method

In the direct differentiation method, we calculate

$$\begin{aligned}\vec{\boldsymbol{v}}_P &= \frac{d}{dt}\vec{\boldsymbol{r}}_P \\ &= \frac{d}{dt}\left[\vec{\boldsymbol{r}}_{O'/O} + \vec{\boldsymbol{r}}_{P/O'}\right] \\ &= \frac{d}{dt}\left[(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}) + (x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' + z'\hat{\boldsymbol{k}}')\right] \\ &= (\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}) + (\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}' + \dot{z}'\hat{\boldsymbol{k}}') + \\ &\quad \left[x'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}') + z'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{k}}')\right]\end{aligned}$$

We could calculate $\vec{\boldsymbol{a}}_P$ similarly using a combination of the product rule of differentiation and the facts that $\dot{\hat{\boldsymbol{\imath}}}' = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}'$, $\dot{\hat{\boldsymbol{\jmath}}}' = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}'$, and $\dot{\hat{\boldsymbol{k}}}' = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{k}}'$, and would get a formula with 15 non-zero terms.

SAMPLE 10.4 *Acceleration of a point moving in a rotating frame.* Consider the rotating tube of Sample 10.7 again. It is given that the arm OAB rotates with counterclockwise angular acceleration $\dot{\omega} = 3\,\text{rad/s}^2$ and at the instant shown the angular speed $\omega = 5\,\text{rad/s}$. Also, at the same instant, the particle P is falling down with speed $v_{/\text{tube}} = 4\,\text{ft/s}$ and acceleration $a_{/\text{tube}} = 2\,\text{ft/s}^2$. Find the absolute acceleration of the particle at the given instant. Take $L = 2\,\text{ft}$ in the figure.

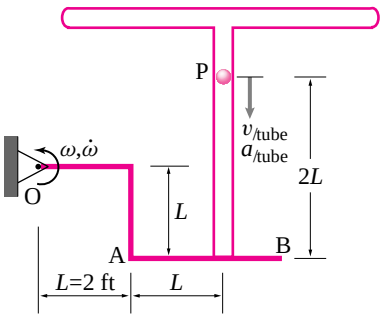


Figure 10.13: (Filename:sfig8.6.1)

Solution Let us attach a body frame $\mathcal{B}$ to the rigid arm OAB. For calculations we fix a coordinate system $x'y'z'$ in this frame such that the origin O' of the coordinate system coincides with O, and at the given instant, the axes are aligned with the inertial coordinate axes xyz. Since $x'y'z'$ is fixed in the frame $\mathcal{B}$ and $\mathcal{B}$ rotates with the rigid arm with $\dot{\vec{\omega}}_{\mathcal{B}} = 3\,\text{rad/s}^2\hat{k}$ and $\vec{\omega}_{\mathcal{B}} = 5\,\text{rad/s}\hat{k}$, the basis vectors $\hat{\imath}'$, $\hat{\jmath}'$ and $\hat{k}'$ rotate with the same $\dot{\vec{\omega}}_{\mathcal{B}}$ and $\vec{\omega}_{\mathcal{B}}$.

In the rotating (primed) coordinate system,

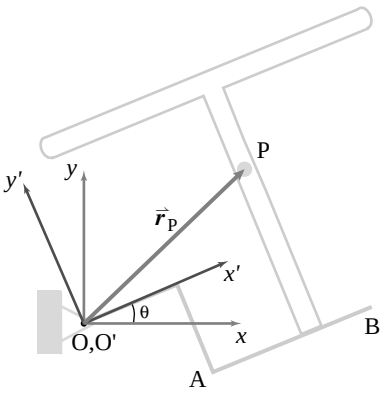
$$\begin{aligned}\vec{r}_P &= x'\hat{\imath}' + y'\hat{\jmath}' \\ \vec{v}_P &= \frac{d}{dt}(\vec{r}_P) = \frac{d}{dt}(x'\hat{\imath}' + y'\hat{\jmath}') \\ &= \dot{x}'\hat{\imath}' + x'\dot{\hat{\imath}}' + \dot{y}'\hat{\jmath}' + y'\dot{\hat{\jmath}}'\end{aligned}$$

Now, we use the $\dot{\vec{Q}}$ formula to evaluate $\dot{\hat{\imath}}'$ and $\dot{\hat{\jmath}}'$, *i.e.*,

$$\begin{aligned}\dot{\hat{\imath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\imath}' = \omega\hat{k}' \times \hat{\imath}' = \omega\hat{\jmath}' \\ \dot{\hat{\jmath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\jmath}' = \omega\hat{k}' \times \hat{\jmath}' = -\omega\hat{\imath}'\end{aligned}$$

Also, note that x' is constant since in frame $\mathcal{B}$, the motion of the particle is always along the tube, *i.e.*, along the negative y' axis (see Fig. 10.14). Thus, $x' = 2L$, $\dot{x}' = 0$, $y' = L$, and $\dot{y}' = -v_{/\text{tube}}$. Substituting these quantities in $\vec{v}_P$, we get:

$$\begin{aligned}\vec{v}_P &= x'\omega\hat{\jmath}' - v_{/\text{tube}}\hat{\jmath}' + y'(-\omega\hat{\imath}') \\ &= (x'\omega - v_{/\text{tube}})\hat{\jmath}' - \omega y'\hat{\imath}' \qquad (10.8)\end{aligned}$$



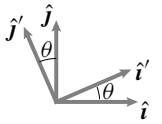


Figure 10.14: (Filename:sfig8.6.1a)

Now substituting $x' = 2L = 4\,\text{ft}$, $\omega = 5\,\text{rad/s}$, $y' = L = 2\,\text{ft}$, $v_{/\text{tube}} = 4\,\text{ft/s}$ and noting that $\hat{\imath}' = \hat{\imath}$, $\hat{\jmath}' = \hat{\jmath}$ at the given instant, we get:

$$\vec{v}_P = [(20-4)\hat{\jmath} - 10\hat{\imath}]\,\text{ft/s} = (-10\hat{\imath} + 16\hat{\jmath})\,\text{ft/s}$$

We can find $\vec{a}_P$ by differentiating Eq. (10.8) and noting again that $\hat{\imath}' = \hat{\imath}$, $\hat{\jmath}' = \hat{\jmath}$ at the given instant:

$$\begin{aligned}\vec{a}_P &= \frac{d}{dt}(\vec{v}_P) = \frac{d}{dt}[(x'\omega - v_{/\text{tube}})\hat{\jmath}' - \omega y'\hat{\imath}'] \\ &= (\underbrace{\dot{x}'}_{0}\omega + x'\dot{\omega} - \overbrace{\dot{v}_{/\text{tube}}}^{a_{/\text{tube}}})\hat{\jmath}' + (x'\omega - v_{/\text{tube}})\dot{\hat{\jmath}}' - (\dot{\omega}y' + \omega\overbrace{\dot{y}'}^{-v_{/\text{tube}}})\hat{\imath}' - \omega y'\dot{\hat{\imath}}' \\ &= (x'\dot{\omega} - a_{/\text{tube}})\hat{\jmath}' + (x'\omega - v_{/\text{tube}})(-\omega\hat{\imath}') - (\dot{\omega}y' - \omega v_{/\text{tube}})\hat{\imath}' - \omega y'(\omega\hat{\jmath}') \\ &= = a_{/\text{tube}}\hat{\jmath}' + 2\omega v_{/\text{tube}}\hat{\imath}' - \omega^2(x'\hat{\imath}' + y'\hat{\jmath}') + \dot{\omega}(x'\hat{\jmath}' - y'\hat{\imath}') \\ &= -2\,\text{ft/s}^2\hat{\jmath} + 40\,\text{ft/s}^2\hat{\imath} - 25(4\hat{\imath} + 2\hat{\jmath})\,\text{ft/s}^2 + 3(4\hat{\jmath} - 2\hat{\imath})\,\text{ft/s}^2 \\ &= -(66\hat{\imath} + 40\hat{\jmath})\,\text{ft/s}^2.\end{aligned}$$

$$\boxed{\vec{a}_P = -(66\hat{\imath} + 40\hat{\jmath})\,\text{ft/s}^2}$$

① This problem is the same as Sample 10.7 and has the same solution. Here we use the $\dot{\vec{Q}}$ formula on various base vectors instead of using the relative velocity and acceleration formulae.

①

10.3 General expressions for velocity and acceleration

Rate of change of a vector relative to a rotating frame: the $\dot{\vec{Q}}$ formula

Because dynamics involves the time derivatives of so many different vectors (e.g. $\vec{r}$, $\vec{v}$, $\vec{L}$, $\vec{H}_C$, and $\vec{\omega}$) it is easier to think about the derivative of some arbitrary vector, call it $\vec{Q}$, and then apply what we learn to these other vectors.

[$\vec{Q}(t)$ is an arbitrary vector ***not*** attached to $\mathcal{B}$ or $\mathcal{F}$.]

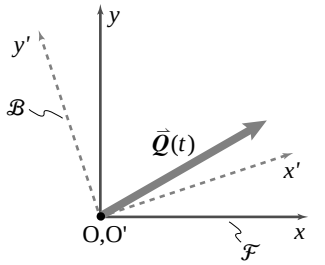


Figure 10.15: A fixed frame $\mathcal{F}$ with coordinate system $Oxyz$ attached, a moving frame $\mathcal{B}$ with coordinate system $O'x'y'z'$ attached, and an arbitrary vector $\vec{Q}$ which changes relative to both frames.

(Filename:tfigure8.qdot1)

So we think about some vector $\vec{Q}$ as a quantity that could be represented by an arrow. We can write $\vec{Q}$ using the coordinates of the fixed Newtonian frame with base vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$: $\vec{Q} = Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k}$. Similarly we could write $\vec{Q}$ in terms of the coordinates of some moving and rotating frame $\mathcal{B}$ with base vectors $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$: $\vec{Q} = Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'$. Now of course $\vec{Q} = \vec{Q}$ so

$$\vec{Q} \;=\; Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k} \;=\; Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}' \;=\; \vec{Q}.$$

Similarly, $\dot{\vec{Q}} = \dot{\vec{Q}}$. We can calculate $\dot{\vec{Q}}$ the same way we have from the start of the book, namely,

$$\dot{\vec{Q}} = \dot{Q}_x\hat{\imath} + \dot{Q}_y\hat{\jmath} + \dot{Q}_z\hat{k}.$$

We didn't have to use the product rule of differentiation because the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, associated with a fixed frame, are constant in time. Note that, to keep the notation simple, we have not used a superscript $\mathcal{F}$ to indicate that we have differentiated with respect to a Newtonian frame, but, by $\dot{\vec{Q}}$, we really mean ${}^{\mathcal{F}}\dot{\vec{Q}} = \frac{{}^{\mathcal{F}}d}{dt}(\vec{Q})$.

What if we wanted to use the coordinate information that was given to us by a person who was moving and rotating with the moving frame $\mathcal{B}$? Lets assume that base vectors $\hat{\imath}'$, ujp, and $\hat{k}'$ are glued to the frame $\mathcal{B}$. Now we calculate $\dot{\vec{Q}}$ using the moving frame representation:

$$\begin{aligned}\dot{\vec{Q}} &= \frac{d}{dt}[Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'] \\ &= \underbrace{[\dot{Q}_{x'}\hat{\imath}' + \dot{Q}_{y'}\hat{\jmath}' + \dot{Q}_{z'}\hat{k}']}_{{}^{\mathcal{B}}\dot{\vec{Q}}} \;+\; \underbrace{[Q_{x'}\dot{\hat{\imath}}' + Q_{y'}\dot{\hat{\jmath}}' + Q_{z'}\dot{\hat{k}}']}_{?}. \qquad (10.9)\end{aligned}$$

The first term in the product rule is just the derivative of $\vec{Q}$ in the moving frame ${}^{\mathcal{B}}\dot{\vec{Q}}$. To calculate ${}^{\mathcal{B}}\dot{\vec{Q}}$, you have to differentiate the components without worrying over the changing basis vectors. The second term requires a little thought. What are $\dot{\hat{\imath}}'$, $\dot{\hat{\jmath}}'$, and $\dot{\hat{k}}'$?

We know that the relative velocity of any two points A and B glued to a rigid body $\mathcal{B}$ is given by

$$\vec{v}_{B/A} = \vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A}.$$

So the rate of change of *any* vector which is fixed relative to the body is given by a similar formula because any vector can be written as a combination of relative positions of points on the body. In particular, this definition of body fixed vectors in terms of point on the body, works for the moving basis vectors $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$:

$$\dot{\hat{\imath}}' = \vec{\omega}_{\mathcal{B}} \times \hat{\imath}', \quad \dot{\hat{\jmath}}' = \vec{\omega}_{\mathcal{B}} \times \hat{\jmath}', \quad \text{and} \quad \dot{\hat{k}}' = \vec{\omega}_{\mathcal{B}} \times \hat{k}'.$$

Now we can go back to the expression for $\overset{\mathcal{B}}{\dot{\vec{Q}}}$ in equation 10.9. Since each of the terms in the second expression can be expressed in terms of a cross product with $\vec{\omega}_{\mathcal{B}}$ that factor can be pulled out and we get the desired result:

$$\overset{\mathcal{F}}{\dot{\vec{Q}}} = \overset{\mathcal{B}}{\dot{\vec{Q}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}. \tag{10.10}$$

or more simply, but less explicitly,

$$\dot{\vec{Q}} = \dot{\vec{Q}}_{rel} + \vec{\omega} \times \vec{Q}. \tag{10.11}$$

This formula says that

> *the derivative of a vector with respect to a Newtonian frame $\mathcal{F}$ (or 'absolute derivative') can be calculated as the derivative of the vector with respect to a moving frame $\mathcal{B}$, $\overset{\mathcal{B}}{\dot{\vec{Q}}}$, plus a term that corrects for the rotation of frame $\mathcal{B}$ relative to frame $\mathcal{F}$, $\vec{\omega}_{\mathcal{B}} \times \vec{Q}$.*

Also note that this equation equates vector expressions; its validity does not depend on a particular choice of coordinate system and base vectors.

This formula is useful for the derivation of a variety of formulas and is also useful in the solution of problems. For example, while we have shown how to use this formula to calculate the rate of change of a vector with respect to a Newtonian frame, the formula can be used to calculate its rate of change with respect to a non-Newtonian frame. Letting $\mathcal{A}$ and $\mathcal{B}$ be two possibly non-Newtonian frames, the $\dot{\vec{Q}}$ formula for the rate of change of $\vec{Q}$ with respect to frame $\mathcal{A}$ is

$$\overset{\mathcal{A}}{\dot{\vec{Q}}} = \overset{\mathcal{B}}{\dot{\vec{Q}}} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{Q}. \tag{10.12}$$

Here, we have explicitly denoted frames of reference since they are both possibly non-Newtonian.

See figure 10.16 for a geometrical 'derivation' of the $\dot{\vec{Q}}$ formula in two dimensions. Referring to this figure,

- Part (a) shows a vector $\vec{Q}$ at time t.
- Part (b) shows $\vec{Q}$ at time $t + \Delta t$ *and* the change in $\vec{Q}$, $\Delta\vec{Q} \approx \dot{\vec{Q}} \cdot \Delta t$.
- Part (c) is like (a) but shows a moving body or frame $\mathcal{A}$.
- Part (d) shows the change in $\vec{Q}$, $(\vec{\omega}_{\mathcal{A}} \times \vec{Q}) \cdot \Delta t$, that would occur if $\vec{Q}$ were fixed (constant) in $\mathcal{A}$.
- Part (e) shows the change in $\vec{Q}$ that would be observed in the moving frame $\mathcal{A}$.
- Part (f) shows the net change in $\vec{Q}$, $\Delta\vec{Q}$, that is the same as that in (b) above; here, it is shown as the sum of the two contributions from (d) and (e).

Thus, using $\mathcal{A}$, $\Delta\vec{Q}$ for small Δt is composed of two parts: (1) the $\Delta\vec{Q}$ observed in $\mathcal{A}(t)$, and (2) the change in $\vec{Q}$ which would occur if $\vec{Q}$ were constant in $\mathcal{A}(t)$ and thus rotating with it. Dividing $\Delta\vec{Q}$ by Δt gives the '$\dot{\vec{Q}}$ *formula*', $\dot{\vec{Q}} = \overset{\mathcal{A}}{\dot{\vec{Q}}} + \vec{\omega}_{\mathcal{A}} \times \vec{Q}$.

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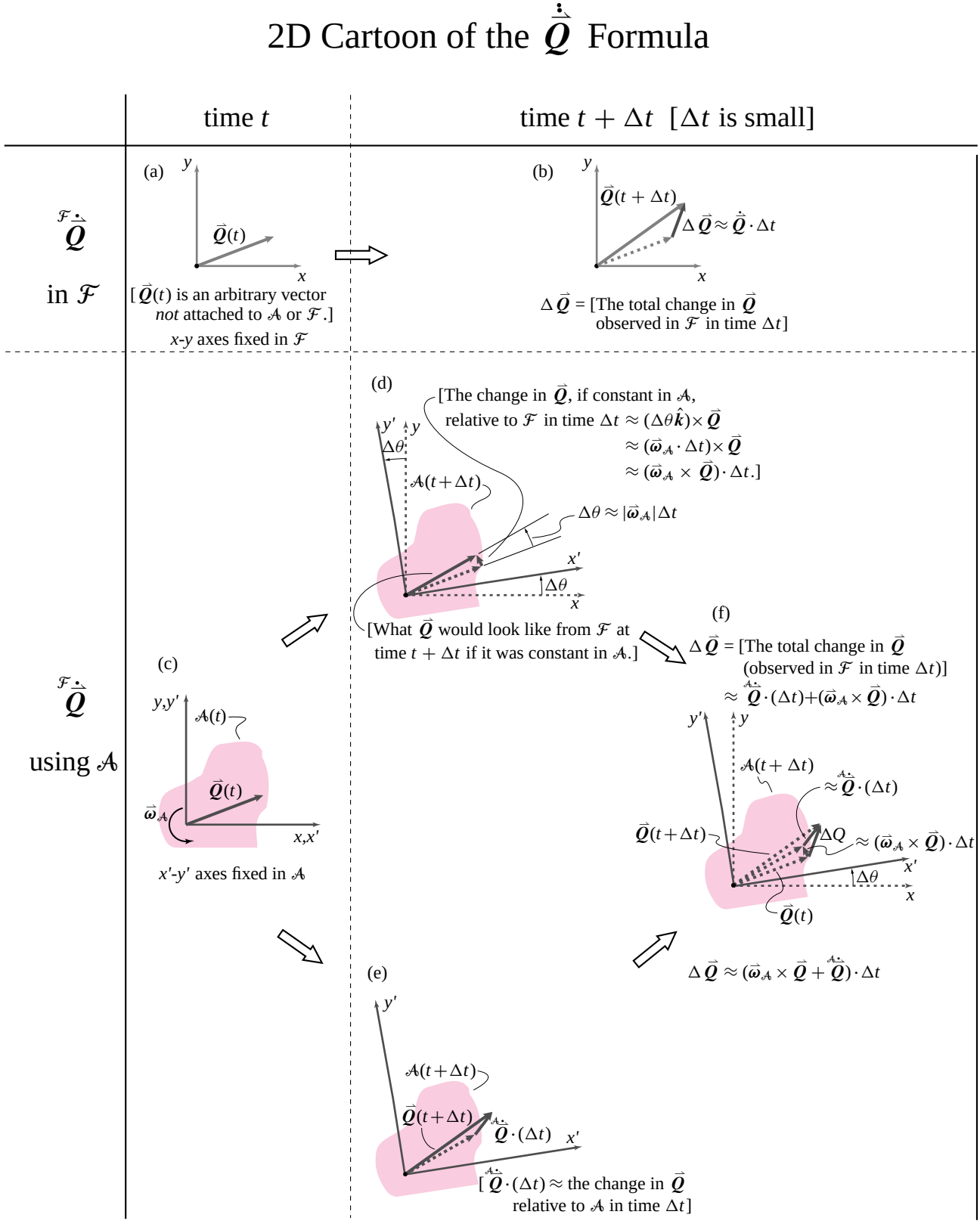


Figure 10.16: Two different looks at the change in the vector $\vec{Q}$, $\Delta\vec{Q}$, over a time interval Δt. See text on page 561 (Filename:tfigure8.qdotnewer.mike)

Summary of the $\dot{\vec{Q}}$ formula

For a vector $\vec{Q}$ fixed in $\mathcal{B}$,

$$\dot{\vec{Q}} = \vec{\omega}_{\mathcal{B}} \times \vec{Q} \qquad \text{or} \qquad {}^{\mathcal{F}}\dot{\vec{Q}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}.$$

For any vector $\vec{Q}$,

$$\dot{\vec{Q}} = {}^{\mathcal{B}}\dot{\vec{Q}} + \vec{\omega}_{\mathcal{B}} \times \vec{Q} \qquad \text{or} \qquad {}^{\mathcal{F}}\dot{\vec{Q}} = {}^{\mathcal{B}}\dot{\vec{Q}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}.$$

Some examples of applying the $\dot{\vec{Q}}$ formula are:

$$\begin{aligned} \dot{\vec{r}}_{P/O'} &= {}^{\mathcal{B}}\dot{\vec{r}}_{P/O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} && \text{(absolute velocity of a point } P \text{ relative to } O') \\ \dot{\hat{\imath}}' &= \underbrace{\dot{\hat{\imath}}'_{rel}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \hat{\imath}' && \text{(rate of change of a rotating unit vector in } \mathcal{B}) \end{aligned}$$

Absolute velocity of a point moving relative to a moving frame

Imagine that you know the absolute velocity of some point O' on a body $\mathcal{B}$, say the center of a car, tire and the angular velocity of the body, $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$. Finally, imagine you also know the relative velocity of point P, $\vec{v}_{P/\mathcal{B}}$, say of a bug crawling on the tire.

If the frame $\mathcal{B}$ is translating or rotating, the velocity of particle P relative to the frame $\vec{v}_{P/\mathcal{B}}$ is not the absolute velocity (the velocity relative to a Newtonian frame). The absolute velocity in this case is $\vec{v}_{P/\mathcal{F}}$, or more simply $\vec{v}_P$, or more simply still, just $\vec{v}$. The relationship between the absolute velocity $\vec{v}_{P/\mathcal{B}}$ and the relative velocity $\vec{v}_{P/\mathcal{F}} \equiv \vec{v}$ is of interest.

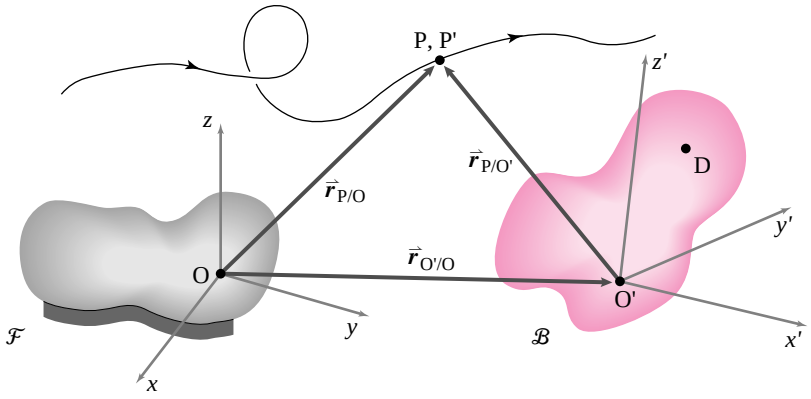


Figure 10.17: The position of point P relative to the origin O' of moving frame $\mathcal{B}$ is $\vec{r}_{P/O'}$. The position of the origin of the frame $\mathcal{B}$ relative to the origin O of the fixed frame $\mathcal{F}$ is $\vec{r}_{O'/O}$. The position of point P relative to O is the sum of $\vec{r}_{P/O'}$ and $\vec{r}_{O'/O}$. The motion of point P relative to the fixed frame may be complicated.

(Filename:tfigure8.v3)

Let's start by looking at the position. The position of a point P that is moving is:

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

where O' is the origin of a coordinate system which is glued to the rigid body, as shown in figure 10.17.

To find the absolute velocity of point P we will use the $\dot{\vec{Q}}$ *formula*, equation **??**, for computing the rate of change of a vector. The velocity of P is the rate of change of its position. Here, we use $\vec{Q} = \vec{r}_{P/O'}$

$$\begin{aligned}\vec{v}_{P/\mathcal{F}} &= {}^{\mathcal{F}}\dot{\vec{r}}_{P/O} \\ &= {}^{\mathcal{F}}\dot{\vec{r}}_{O'/O} + {}^{\mathcal{F}}\dot{\vec{r}}_{P/O'} \\ &= \vec{v}_{O'/\mathcal{F}} + \underbrace{{}^{\mathcal{B}}\dot{\vec{r}}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}} + \overbrace{\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P'/O'}}^{{}^{\mathcal{F}}\dot{\vec{r}}_{P/O'}}\end{aligned}$$

The $\dot{\vec{Q}}$ formula **??** was used in the calculation to compute ${}^{\mathcal{F}}\dot{\vec{r}}_{P/O'}$

$${}^{\mathcal{F}}\dot{\vec{r}}_{P/O'} = {}^{\mathcal{B}}\dot{\vec{r}}_{P/O'} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P'/O'}. \tag{10.13}$$

Thus, the 'three term velocity formula'.

3. velocity relative to point O' of point P' which is glued to body and coincident with point P

1. velocity of reference point O' on body $\mathcal{B}$

$$\vec{v}_P = \vec{v}_{O'} + \underbrace{\vec{v}_{P/\mathcal{B}}}_{\text{2. velocity of point P relative to moving frame } \mathcal{B}} + \overbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}}^{\text{3. velocity relative to point } O' \text{ of point } P' \text{ which is glued to body and coincident with point } P} \tag{10.14}$$

absolute velocity of point P

Another way to write the formula for absolute velocity is as

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathcal{B}}$$

where P' is a point glued to $\mathcal{B}$ which is instantaneously coincident with P and the absolute velocity of P' is

$$\vec{v}_{P'} = \vec{v}_{O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}.$$

Now, we consider simple 2-D examples of absolute velocity of a point moving relative to a body or frame. First, reconsider the bug crawling on the tire, body $\mathcal{B}$, in figure 10.18. To find the absolute velocity of the bug, we need be concerned with how the bug moves relative to the tire *and* how the tire moves relative to the ground.

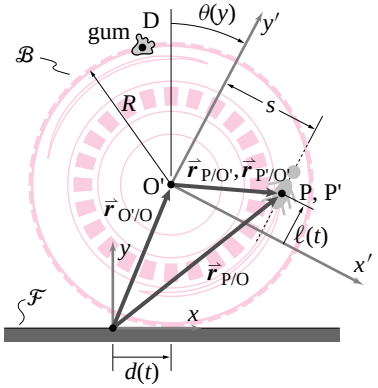


Figure 10.18: (Filename:tfigure8.velabsbug)

Example: **Absolute velocity of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again**

Referring to equation 10.13 on page 564, the absolute velocity of the bug is

$$\begin{aligned}\vec{v}_{P/\mathcal{F}} &= \vec{v}_{O'/\mathcal{F}} + \underbrace{{}^{\mathcal{B}}\dot{\vec{r}}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'} \\ &= R\dot{\theta}\hat{\imath} + \dot{\ell}\hat{\jmath}' + (-\dot{\theta}\hat{k}') \times (s\hat{\imath}' + \ell\hat{\jmath}') \\ &= R\dot{\theta}\hat{\imath} + \ell\dot{\theta}\hat{\imath}' + (\dot{\ell} - \dot{\theta})\hat{\jmath}'.\end{aligned}$$

At the instant of interest, the direction of the bug's absolute velocity depends upon the relative magnitudes of $\dot{\ell}$ and $\dot{\theta}$ as well as the orientation of $\hat{\imath}'$ and $\hat{\jmath}'$. □

As we noted earlier, another way to write the formula for absolute velocity is

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathcal{B}}$$

where, in the example above, $\vec{v}_{P'} = R\dot{\theta}\hat{\imath} + \dot{\theta}(\ell\hat{\imath}' - s\hat{\jmath}')$ and $\vec{v}_{P/\mathcal{B}} = \dot{\ell}\hat{\jmath}'$. At the instant of concern, we can think of the absolute velocity of the bug as the velocity of the mark labeled P' under the bug *plus* the velocity of the bug relative to the tire.

Acceleration

We would like to find acceleration of a point using information about its motion relative to a moving frame. The result, the 'five term acceleration formula' is the most complicated formula in this book. (For reference, it is in Table II, 5c).

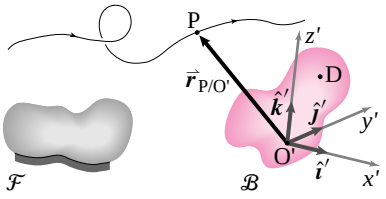


Figure 10.19: A body or reference frame $\mathcal{B}$ and a point P move around. We are interested in the acceleration of point P.

(Filename:tfigure8.a1)

Acceleration relative to a body or frame

The acceleration of a point relative to a body or frame is the acceleration you would calculate if you were looking at the particle while you translated and rotated with the frame and took no account of the outside world. That is, if the position of a particle P relative to the origin O' of a coordinate system in a moving frame $\mathcal{B}$ is given by:

$$\vec{r}_{P/O'} = \underbrace{r_{Px'/O'}}_{x'}\hat{\imath}' + \underbrace{r_{Py'/O'}}_{y'}\hat{\jmath}' + \underbrace{r_{Pz'/O'}}_{z'}\hat{k}',$$

then the acceleration of the particle P relative to the frame is:

$$\vec{a}_{P/\mathcal{B}} = \underbrace{\ddot{r}_{Px'/O'}}_{\ddot{x}'}\hat{\imath}' + \underbrace{\ddot{r}_{Py'/O'}}_{\ddot{y}'}\hat{\jmath}' + \underbrace{\ddot{r}_{Pz'/O'}}_{\ddot{z}'}\hat{k}'.$$

That is, the acceleration relative to the frame takes no account of (a) the motion of the frame or of (b) the rotation of the base vectors with the frame to which they are fixed.

We consider now simple examples in 2-D and 3-D of acceleration relative to a frame. First, reconsider the bug labeled point P crawling on the tire, body $\mathcal{B}$, in figure 10.20.

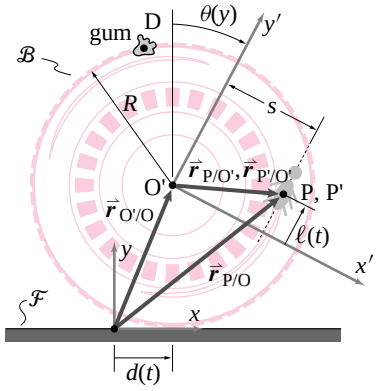


Figure 10.20: A bug crawling on a tire at point P. Point P' is on the tire and instantaneously corresponds to point P.

(Filename:tfigure8.accelrelbug)

Example: **Acceleration relative to a frame (2-D): Bug crawling on a tire, again**

If we are sitting on the tire, all that we see is the bug crawling in a straight line at non-constant rate relative to us. Thus, its acceleration relative to the tire is

$$\vec{a}_{P/\mathcal{B}} = \ddot{\ell}\hat{\jmath}'.$$

So, very simply, at the instant of interest, the bug has an acceleration relative to the tire parallel to the y'-axis, along the line scribed on the tire, at a position relative to the tire. □

Absolute acceleration of a point P glued to a moving frame

Imagine that you know the absolute acceleration of some point O' at the center of a frame $\mathcal{B}$, say the center of a car tire. Imagine you also know the angular velocity of the tire, $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$, and the angular acceleration, $\vec{\alpha}_{\mathcal{B}/\mathcal{F}}$. Then, you can find the absolute acceleration of a piece of gum labeled point D stuck to the sidewall. If we start with the equation **??** for the absolute velocity of a point glued to a moving frame on page **??** and differentiate with respect to time, we get the absolute acceleration of a point D fixed in a moving frame $\mathcal{B}$ as follows:

$$\begin{aligned}\vec{a}_D &= \frac{d}{dt}[\vec{v}_{O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{D/O'}] \\ &= \vec{a}_{O'/O} + [\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{D/O'} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{D/O'})] \\ &= \vec{a}_{O'/O} + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{D/O'} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{D/O'}) \qquad (10.15)\end{aligned}$$

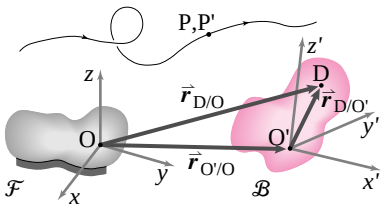


Figure 10.21: This figure is a copy of figure **??** on page **??**.

(Filename:tfigure8.a2)

Now, we present simple 2-D and 3-D examples of absolute acceleration of a point glued to a moving frame. First, reconsider the bug labeled point P crawling on the tire, body $\mathcal{B}$, in figure 10.22. Point P' is defined as before, coincident with the bug at the instant of interest, but fixed to the tire.

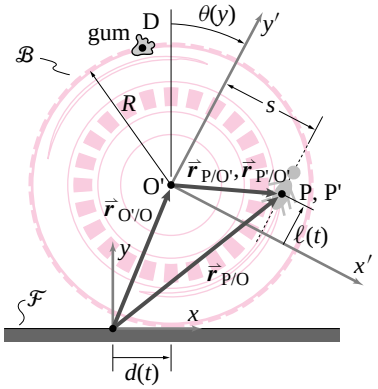


Figure 10.22: (Filename:tfigure8.accelgluebug)

Example: **Absolute acceleration of a point glued to a moving frame (2-D): Bug crawling on a tire, again**

Here, the acceleration of point P' relative to the tire is zero, $\vec{a}_{P'/\mathcal{B}} = 0$. The angular velocity of the wheel with respect to the ground is $\vec{\omega}_{\mathcal{B}/\mathcal{F}} = -\dot{\theta}\hat{k} = -\dot{\theta}\hat{k}'$. The angular speed is increasing at a rate $\ddot{\theta}$. Thus, $\vec{\alpha}_{\mathcal{B}/\mathcal{F}} = -\ddot{\theta}\hat{k} = -\ddot{\theta}\hat{k}'$. The position of P' relative to O' is $\vec{r}_{P'/O'} = s\hat{\imath}' + \ell\hat{\jmath}'$.

Using equation 10.15 on page 566, we get the absolute acceleration of point P' to be

$$\vec{a}_{P'/\mathcal{F}} = \vec{a}_{O'/\mathcal{F}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times (\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P'/O'}) + \vec{\alpha}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P'/O'}$$

centripetal term; tangential term

$$= \underbrace{R\ddot{\theta}\hat{\imath}}_{\text{acceleration of origin of moving frame}} - \overbrace{\dot{\theta}^2(s\hat{\imath}' + \ell\hat{\jmath}')}^{\text{centripetal term}} + \overbrace{R\ddot{\theta}(s\hat{\jmath}' - \ell\hat{\imath}')}^{\text{tangential term}}$$

In this example, the absolute acceleration P' is due to:

1. the increase in the translational speed of the tire relative to the ground (acceleration of origin of moving frame),
2. its going in circles at non-constant rate about point O' relative to the ground ('tangential term'), and
3. 'centripetal term' towards the origin of the moving frame. (In three-dimensional problems, this term is directed towards an axis through $\vec{\omega}$ that goes through O').

□

Absolute acceleration of a point moving relative to a moving frame

If we start with the equation for absolute velocity 10.13 on page 564 and differentiate with respect to time we get the absolute acceleration of a point P using a moving frame $\mathcal{B}$. To do this calculation we need to use the product rule of differentiation. Refer to the $\dot{\vec{Q}}$ *formula*, equation 10.10 on page 561 in section 10.3. Here is the calculation:

$$\begin{aligned}
\vec{a}_P &= \frac{d}{dt}[\vec{v}_{O'/O} + \vec{v}_{P/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}] \\
&= \vec{a}_{O'/O} + (\vec{a}_{P/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}) \\
&\quad + \ [\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{P/O} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'})] \\
&= \underbrace{\vec{a}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}) + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P/O'}}_{\vec{a}_{P'}} + \vec{a}_{P/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}.
\end{aligned}$$

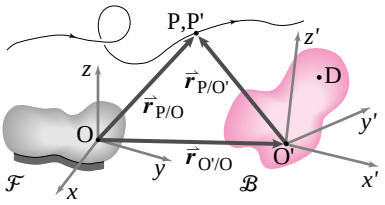


Figure 10.23: Keeping track of P which instantaneously coincides with point P' which is glued to frame $\mathcal{B}$.

(Filename:tfigure8.a3)

The collection of terms $\vec{a}_{P'}$ is the acceleration of a point P' which is glued to body $\mathcal{B}$ and is instantaneously coincident with P. It is the same as $\vec{a}_D$ using $D = P'$ in equation 10.15. To repeat, the result is

1. acceleration of reference point O' on body $\mathcal{B}$

4. acceleration of P relative to moving frame $\mathcal{B}$

$$\vec{a}_P = \vec{a}_{O'/O} + \underbrace{\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'})}_{\text{2. centripetal}} + \underbrace{\vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P/O'}}_{\text{3. tangential}} + \vec{a}_{P/\mathcal{B}} + \underbrace{2\,\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}}_{\text{5. Coriolis accel.}} \tag{10.16}$$

2. centripetal acceleration: acceleration relative to O' of point glued to body, coincident with P, and going in circles around O' at constant rate

3. tangential acceleration: acceleration relative to O' of point glued to body, coincident with P, due to non-constant $\vec{\omega}$

This expression is the famous and infamous 'five-term-acceleration' formula. Famous because it is given a lot of emphasis by some instructors. Infamous because it takes some getting used to.

The first three terms are acceleration of a point P' which is fixed relative to $\mathcal{B}$. One way to get used to this formula is to find situations where various of the terms drop out.

Reconsider the bug labeled point P crawling on the tire, body $\mathcal{B}$, in figure 10.24. To find the absolute acceleration of the bug we need be concerned with how the bug moves relative to the tire *and* how the tire moves relative to the ground.

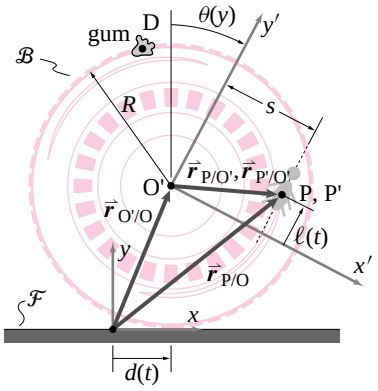


Figure 10.24: (Filename:tfigure8.accelabsbug)

Example: **Absolute acceleration of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again**

We have from the previous bug examples on page **??** and page 566,

$$\vec{\boldsymbol{v}}_{P/\mathcal{B}} = \dot{\ell}\hat{\boldsymbol{j}}', \quad \text{and } \vec{\boldsymbol{a}}_{P/\mathcal{B}} = \ddot{\ell}\hat{\boldsymbol{j}}'.$$

Referring to the five term acceleration formula, equation 10.16 on page 567, the absolute acceleration of the bug is

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P/\mathcal{F}} &= \vec{\boldsymbol{a}}_{O'/O} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'}) + \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'} + \vec{\boldsymbol{a}}_{P/\mathcal{B}} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}} \\
&= \underbrace{R\ddot{\theta}\hat{\boldsymbol{\imath}} - \dot{\theta}^2(s\hat{\boldsymbol{\imath}}' + \ell\hat{\boldsymbol{j}}') + \ddot{\theta}(\ell\hat{\boldsymbol{\imath}}' - s\hat{\boldsymbol{j}}')}_{\vec{\boldsymbol{a}}_{P'/\mathcal{F}}} + \ddot{\ell}\hat{\boldsymbol{j}}' + 2\dot{\theta}\dot{\ell}\hat{\boldsymbol{\imath}}' \\
&= R\ddot{\theta}\hat{\boldsymbol{\imath}} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\theta}\dot{\ell})\hat{\boldsymbol{\imath}}' + (-s\ddot{\theta} + \ddot{\ell} - \ell\dot{\theta}^2)\hat{\boldsymbol{j}}'.
\end{aligned}$$

So, at the instant of interest, the bug's absolute acceleration is due to:

1. the translational acceleration of the tire, $\vec{\boldsymbol{a}}_{O'/O} = R\ddot{\theta}\hat{\boldsymbol{\imath}}$
2. the centripetal acceleration of going in circles of radius $\sqrt{s^2+\ell^2}$ about the center of the tire as it rolls, $-\dot{\theta}^2(s\hat{\boldsymbol{\imath}}' + \ell\hat{\boldsymbol{j}}')$, pointing at the center of the tire,
3. the tangential acceleration of going in circles about the center of the tire as the tire rolls at non-constant rate, $\ddot{\theta}(\ell\hat{\boldsymbol{\imath}}' - s\hat{\boldsymbol{j}}')$,
4. the acceleration of the bug relative to the tire as it crawls on the line, $\vec{\boldsymbol{a}}_{P/\mathcal{B}} = \ddot{\ell}\hat{\boldsymbol{j}}'$, and
5. the Coriolis acceleration caused, in part, by the change in direction, relative to the ground, of the velocity of the bug relative to the tire, $2\dot{\theta}\dot{\ell}\hat{\boldsymbol{\imath}}'$.

Items 1, 2 and 3 sum to be the acceleration of point P' described previously in the bug example on page **??**. □

10.1 *THEORY*
Relation between moving frame formulae and polar coordinate formulae

A similarity exists between the polar coordinate velocity formula

$$\vec{\boldsymbol{v}} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta$$

and the second two terms in the 'three-term' velocity formula

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{O'/O} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'} + \vec{\boldsymbol{v}}_{P/\mathcal{B}}.$$

In fact, we have tried to build your understanding of moving frames by means of that connection.

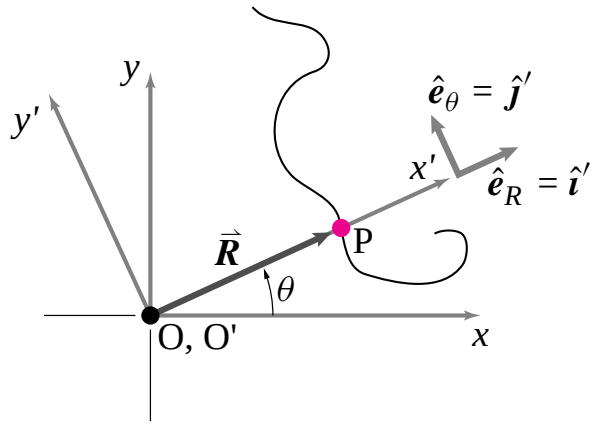
Similarly, the polar coordinate formula for acceleration

$$\vec{\boldsymbol{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta$$

is somehow closely linked to the last four terms of the '5-term' acceleration formula

$$\begin{aligned}\vec{\boldsymbol{a}}_P &= \vec{\boldsymbol{a}}_{O'/O} + \vec{\boldsymbol{a}}_{P/\mathcal{B}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{P/O'}) \\ &\quad + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{P/O'} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}}.\end{aligned}$$

Let's make these connections explicit. Imagine a particle P moving around on the xy-plane.



$\mathcal{F} \equiv$ fixed frame xy
$\mathcal{B} \equiv$ rotating frame $x'y'$

Let's create a moving frame $\mathcal{B}$ with rotating coordinate system $x'y'$ attached to it whose origin O' is coincident with origin O of a coordinate system xy attached to a fixed frame $\mathcal{F}$. Let this frame rotate in exactly such a way so that the particle is always on the x'-axis. So, in this frame, $\vec{\boldsymbol{r}}_{P/O'} = R\hat{\boldsymbol{i}}'$, $\vec{\boldsymbol{v}}_{P/\mathcal{B}} = \dot{R}\hat{\boldsymbol{i}}'$, and $\vec{\boldsymbol{a}}_{P/\mathcal{B}} = \ddot{R}\hat{\boldsymbol{i}}'$. Also, the frame motion is characterized by $\vec{\boldsymbol{v}}_{O'/O} = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{a}}_{O'/O} = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \dot{\theta}\hat{\boldsymbol{k}}$, and $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} = \ddot{\theta}\hat{\boldsymbol{k}}$. So, if we plug in the three-term velocity formula, we get

$$\begin{aligned}\vec{\boldsymbol{v}}_P &= \vec{\boldsymbol{v}}_{O'/O} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O'} + \vec{\boldsymbol{v}}_{P/\mathcal{B}} \\ &= \vec{\boldsymbol{0}} + (\dot{\theta}\hat{\boldsymbol{k}}) \times (R\hat{\boldsymbol{i}}') + \dot{R}\hat{\boldsymbol{i}}' \\ &= R\dot{\theta}\hat{\boldsymbol{j}}' + \dot{R}\hat{\boldsymbol{i}}' \\ &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{R}\hat{\boldsymbol{e}}_r \qquad (\text{since } \hat{\boldsymbol{i}}' \parallel \hat{\boldsymbol{e}}_r \text{ and } \hat{\boldsymbol{j}}' \parallel \hat{\boldsymbol{e}}_\theta)\end{aligned}$$

which is the polar coordinate velocity formula.

Similarly, if we plug into the five-term acceleration formula, we get

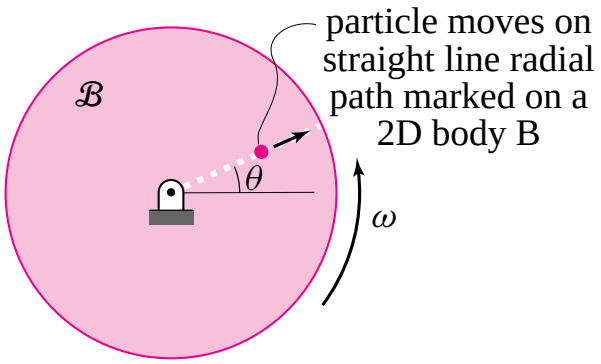
$$\begin{aligned}\vec{\boldsymbol{a}}_P &= \vec{\boldsymbol{a}}_{O'/O} + \vec{\boldsymbol{a}}_{P/\mathcal{B}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{P/O'}) \\ &\quad + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{P/O'} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}} \\ &= \vec{\boldsymbol{0}} + \ddot{R}\hat{\boldsymbol{i}}' + \dot{\theta}\hat{\boldsymbol{k}} \times (\dot{\theta}\hat{\boldsymbol{k}} \times R\hat{\boldsymbol{i}}') \\ &\quad + (\ddot{\theta}\hat{\boldsymbol{k}}) \times (R\hat{\boldsymbol{i}}') + 2(\dot{\theta}\hat{\boldsymbol{k}}) \times \dot{R}\hat{\boldsymbol{i}}' \\ &= (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta\end{aligned}$$

Again, we recover the appropriate polar coordinate formula.

We have just shown how the polar coordinate formulae are special cases of the relative motion formulae.

Warning!

In problems where we want the rotating frame to be a rotating body on which a particle moves, the polar coordinate formulae only correspond term by term with the relative motion formulae if the particle path is a straight radial line fixed on a $2D$ body, as in the example of a bug walking on a straight line scribed on the surface of a rotating CD or a bead sliding in a tube rotating about an axis perpendicular to the tube.



Since we are sometimes interested in more general relative motions, the polar coordinate formulae do not always apply and we must make use of the more general relative motion formulae.

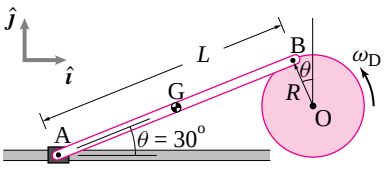


Figure 10.25: (Filename:sfig7.1.2a)

SAMPLE 10.5 *Motion of a rigid body with constrained ends.* In machines we often encounter mechanisms and links in which the ends of a link or a rod are constrained to move on a specified geometric path. A simplified typical link AB is shown in Fig. 10.25.

Link AB is a uniform rigid rod of length $L = 2\,\text{m}$. End A of the rod is attached to a collar which slides on a horizontal track. End B of the rod is attached to a uniform disk of radius $R = 0.5\,\text{m}$ which rotates about its center O. At the instant shown, when $\theta = 30^\text{o}$, end A is observed to move at 2 m/s to the left.

(a) Find the angular velocity of the rod.
(b) Find the angular velocity of the disk.
(c) Find the velocity of the center of mass of the rod.

Solution Let the angular velocities of the rod and the disk be $\vec{\boldsymbol{\omega}}_{AB} = \omega_{AB}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\omega}}_D = \omega_D\hat{\boldsymbol{k}}$ respectively, where ω_{AB} and ω_D are unknowns. We are given $\vec{\boldsymbol{v}}_A = -v_A\hat{\boldsymbol{\imath}}$ where $v_A = 2\,\text{m/s}$.

(a) Point B is on the rod as well as the disk. Hence, the velocity of point B can be found by considering either the motion of the rod or the disk. Considering the motion of the rod we write,

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{AB} \times \vec{\boldsymbol{r}}_{B/A} \\
&= -v_A\hat{\boldsymbol{\imath}} + \omega_{AB}\hat{\boldsymbol{k}} \times \overbrace{L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})}^{\vec{\boldsymbol{r}}_{B/A}} \\
&= -v_A\hat{\boldsymbol{\imath}} + \omega_{AB}L\cos\theta\hat{\boldsymbol{\jmath}} - \omega_{AB}L\sin\theta\hat{\boldsymbol{\imath}} \\
&= -(v_A + \omega_{AB}L\sin\theta)\hat{\boldsymbol{\imath}} + \omega_{AB}L\cos\theta\hat{\boldsymbol{\jmath}}. \qquad (10.17)
\end{aligned}$$

Now considering the motion of the disk we write,

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{\omega}}_D \times \vec{\boldsymbol{r}}_{B/O} \\
&= \omega_D\hat{\boldsymbol{k}} \times R(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) \\
&= -\omega_D R\sin\theta\hat{\boldsymbol{\jmath}} - \omega_D R\cos\theta\hat{\boldsymbol{\imath}}. \qquad (10.18)
\end{aligned}$$

But $\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_B$, therefore, from equations (10.17) and (10.18) we get

$$-(v_A + \omega_{AB}L\sin\theta)\hat{\boldsymbol{\imath}} + \omega_{AB}L\cos\theta\hat{\boldsymbol{\jmath}} = -\omega_D R\sin\theta)\hat{\boldsymbol{\jmath}} - \omega_D R\cos\theta\hat{\boldsymbol{\imath}}$$

By equating the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components of the above equation we get

$$\begin{aligned}
-(v_A + \omega_{AB}L\sin\theta) &= -\omega_D R\cos\theta, \qquad (10.19) \\
\text{and} \qquad \omega_{AB}L\cos\theta &= -\omega_D R\sin\theta \\
\Rightarrow \quad \omega_{AB} &= -\omega_D\frac{R}{L}\tan\theta. \qquad (10.20)
\end{aligned}$$

Dividing Eqn. (10.19) by (10.20) we get

$$\begin{aligned}
-\left(\frac{v_A}{\omega_{AB}} + L\sin\theta\right) &= \frac{L\cos\theta}{\tan\theta} = L\frac{\cos^2\theta}{\sin\theta} \\
\Rightarrow \quad -\frac{v_A}{\omega_{AB}} &= L\left(\frac{\cos^2\theta}{\sin\theta} + \sin\theta\right) = \frac{L}{\sin\theta} \\
\Rightarrow \quad \omega_{AB} &= -\frac{v_A}{L}\sin\theta = -\frac{v_A}{L}\sin 30^\text{o} \\
&= -\frac{2\,\text{m/s}}{2\,\text{m}}\cdot\frac{1}{2} = -0.5\,\text{rad/s}.
\end{aligned}$$

$$\boxed{\vec{\omega}_{AB} = -0.5\,\text{rad/s}\hat{k}}$$

(b) From Eqn. (10.20)

$$\begin{aligned}\omega_D &= -\frac{\omega_{AB}L}{R\tan\theta} = -\frac{\overbrace{-\frac{v_A}{L}\sin\theta\, L}^{\omega_{AB}}}{R\frac{\sin\theta}{\cos\theta}} = \frac{v_A}{R}\cos\theta \\ &= -\frac{2\,\text{m/s}}{0.5\,\text{m}}\frac{\sqrt{3}}{2} = 2\sqrt{3}\,\text{rad/s}.\end{aligned}$$

$$\boxed{\vec{\omega}_D = 3.46\,\text{rad/s}\hat{k}}$$

[At this point, it is a good idea to check our algebra by substituting the values of ω_{AB} and ω_D in equations (10.17) and (10.18) to calculate $\vec{v}_B$.] ①

(c) Now we can calculate the velocity of the center of mass of the rod by considering either point A or point B as a reference:

$$\begin{aligned}\vec{v}_G &= \vec{v}_A + \vec{v}_{G/A} \\ &= \vec{v}_A + \vec{\omega}_{AB}\times\vec{r}_{G/A} \\ &= -v_A\hat{\imath} + \omega_{AB}\hat{k}\times\frac{L}{2}(\cos\theta\hat{\imath}+\sin\theta\,\hat{\jmath}) \\ &= -(v_A + \omega_{AB}\frac{L}{2}\sin\theta)\hat{\imath} + \omega_{AB}\frac{L}{2}\cos\theta\,\hat{\jmath} \\ &= -(2\,\text{m/s} - 0.5\,\text{rad/s}\cdot\frac{2\,\text{m}}{2}\cdot\frac{1}{2})\hat{\imath} + (-0.5\,\text{rad/s}\cdot\frac{2\,\text{m}}{2}\cdot\frac{\sqrt{3}}{2})\hat{\jmath} \\ &= -\frac{7}{4}\,\text{m/s}\hat{\imath} - \frac{\sqrt{3}}{4}\,\text{m/s}\hat{\jmath}.\end{aligned}$$

$$\boxed{\vec{v}_G = -(1.75\hat{\imath} + 0.43\hat{\jmath})\,\text{m/s}}$$

We could easily check our calculation by taking point B as a reference and writing

$$\begin{aligned}\vec{v}_G &= \vec{v}_B + \vec{v}_{G/B} \\ &= \vec{v}_B + \vec{\omega}_{AB}\times\vec{r}_{G/B}\end{aligned}$$

By plugging in appropriate values we get, of course, the same value as above.

① Substituting $\omega_{AB} = -0.5\,\text{rad/s}$ in Eqn. (10.17) and plugging in the given values of other variables we get

$$\vec{v}_B = -(\frac{3}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})\,\text{m/s}.$$

Similarly, substituting $\omega_D = 2\sqrt{3}\,\text{rad/s}$ in Eqn. (10.17) and plugging in the other given values we get

$$\vec{v}_B = -(\frac{3}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})\,\text{m/s},$$

which checks with the $\vec{v}_B$ found above.

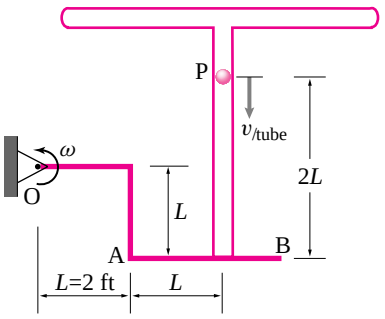


Figure 10.26: (Filename:sfig8.1.1a)

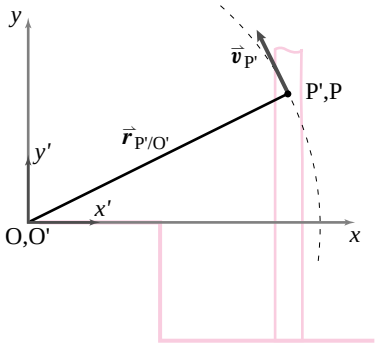
SAMPLE 10.6 A 'T' shaped tube is welded to a massless rigid arm OAB which rotates about O at a constant rate $\omega = 5\,\text{rad/s}$. At the instant shown a particle P is falling down in the vertical section of the tube with speed $v_{/\text{tube}} = 4\,\text{ft/s}$. Find the absolute velocity of the particle. Take $L = 2\,\text{ft}$ in the figure.

Solution Let us attach a frame $\mathcal{B}$ to arm OAB. Thus $\mathcal{B}$ rotates with OAB with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}}$ where $\omega = 5\,\text{rad/s}$. To do calculations in $\mathcal{B}$ we attach a coordinate system $x'y'z'$ to $\mathcal{B}$ at point O. At the instant of interest the rotating coordinate system $x'y'z'$ coincides with the fixed coordinate system xyz. (Since the entire motion is in the xy-plane, the z-axis is not shown in the figure). Let P' be a point coincident with P but fixed in $\mathcal{B}$. Now,

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\text{rel}}$$

where

$$\begin{aligned}\vec{\boldsymbol{v}}_{P'} &= \underbrace{\vec{\boldsymbol{v}}_{O'}}_{\vec{\boldsymbol{0}}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} \\ &= \omega\hat{\boldsymbol{k}} \times (2L\hat{\boldsymbol{\imath}} + L\hat{\boldsymbol{\jmath}}) \\ &= 2\omega L\hat{\boldsymbol{\jmath}} - \omega L\hat{\boldsymbol{\imath}},\end{aligned}$$



and

$$\begin{aligned}\vec{\boldsymbol{v}}_{\text{rel}} &= \text{Velocity relative to the frame } \mathcal{B} \\ &= -v_{/\text{tube}}\hat{\boldsymbol{\jmath}}.\end{aligned}$$

Thus,

$$\begin{aligned}\vec{\boldsymbol{v}}_P &= \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{rel} \\ &= -\omega L\hat{\boldsymbol{\imath}} + (2\omega L - v_{/\text{tube}})\hat{\boldsymbol{\jmath}} \\ &= -5\,\text{rad/s}\cdot 2\,\text{ft}\hat{\boldsymbol{\imath}} + (2\cdot 5\,\text{rad/s}\cdot 2\,\text{ft} - 4\,\text{ft/s})\hat{\boldsymbol{\jmath}} \\ &= -10\,\text{ft/s}\hat{\boldsymbol{\imath}} + 16\,\text{ft/s}\hat{\boldsymbol{\jmath}}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_P = -10\,\text{ft/s}\hat{\boldsymbol{\imath}} + 16\,\text{ft/s}\hat{\boldsymbol{\jmath}}}$$

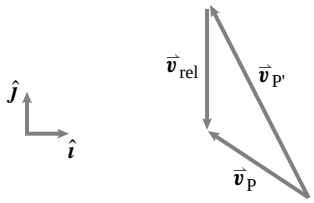


Figure 10.27: (Filename:sfig8.1.1b)

Comments: The kinematics calculation is equivalent to the vector addition shown in Figure 10.27. The velocity of P is the sum of $\vec{\boldsymbol{v}}_{P'}$ and $\vec{\boldsymbol{v}}_{\text{rel}} = \vec{\boldsymbol{v}}_{P/\mathcal{B}}$.

SAMPLE 10.7 *Acceleration of a point in a rotating frame.* Consider the rotating tube of Sample 10.6 again. The arm OAB rotates with counterclockwise angular acceleration $\dot{\omega} = 3\,\text{rad/s}^2$ and, at the instant shown, its angular speed $\omega = 5\,\text{rad/s}$. Also, at the same instant, the particle P falls down with speed $v_{/\text{tube}} = 4\,\text{ft/s}$ and acceleration $a_{/\text{tube}} = 2\,\text{ft/s}^2$. Find the absolute acceleration of the particle at the given instant. Take $L = 2\,\text{ft}$ in the figure.

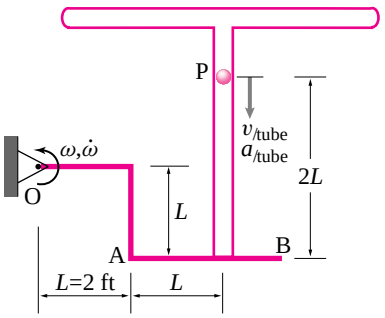


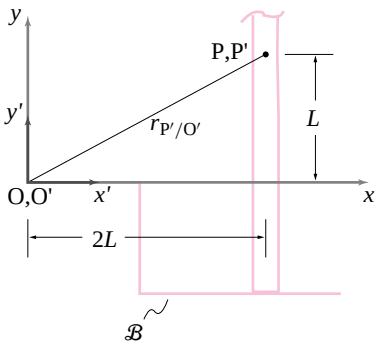
Figure 10.28: (Filename:sfig8.2.1)

Solution We consider a frame $\mathcal{B}$, with coordinate axes $x'y'z'$, fixed to the arm OAB and thus rotating with $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}} = 5\,\text{rad/s}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\alpha}}_{\mathcal{B}} = \dot{\omega}\hat{\boldsymbol{k}} = 3\,\text{rad/s}^2\hat{\boldsymbol{k}}$. The acceleration of point P is given by

$$\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{a}}_{P'} + \vec{\boldsymbol{a}}_{\text{cor}} + \vec{\boldsymbol{a}}_{\text{rel}}$$

where

$\vec{\boldsymbol{a}}_{P'}$ = acceleration of a point P′ that is fixed in $\mathcal{B}$ and at the moment coincides with P,
$\vec{\boldsymbol{a}}_{\text{cor}}$ = Coriolis acceleration, and
$\vec{\boldsymbol{a}}_{\text{rel}}$ = acceleration of P relative to frame $\mathcal{B}$.



Now we calculate each of these terms separately. For calculating $\vec{\boldsymbol{a}}_{P'}$, imagine a rigid rod from point O to point P, rotating with the frame $\mathcal{B}$. Mark the far end of the rod as P′ (same as point P). The acceleration of this end of the rod is $\vec{\boldsymbol{a}}_{P'}$. To find the relative terms $\vec{\boldsymbol{v}}_{\text{rel}}$ and $\vec{\boldsymbol{a}}_{\text{rel}}$, freeze the motion of the frame $\mathcal{B}$ at the given moment and watch the motion of point P. The non-intuitive term $\vec{\boldsymbol{a}}_{\text{cor}}$ has no such simple physical interpretation but has a simple formula. Thus,

$$\begin{aligned}
\vec{\boldsymbol{a}}_{P'} &= \underbrace{\vec{\boldsymbol{a}}_{O'}}_{\vec{\boldsymbol{0}}} + \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} + \overbrace{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'})}^{-\omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{P'/O'}} \\
&= \dot{\omega}\hat{\boldsymbol{k}} \times (2L\hat{\boldsymbol{\imath}} + L\hat{\boldsymbol{\jmath}}) - \omega^2(2L\hat{\boldsymbol{\imath}} + L\hat{\boldsymbol{\jmath}}) \\
&= 2\dot{\omega}L\hat{\boldsymbol{\jmath}} - \dot{\omega}L\hat{\boldsymbol{\imath}} - 2\omega^2 L\hat{\boldsymbol{\imath}} - \omega^2 L\hat{\boldsymbol{\jmath}} \\
&= -L(\dot{\omega} + 2\omega^2)\hat{\boldsymbol{\imath}} + L(2\dot{\omega} - \omega^2)\hat{\boldsymbol{\jmath}} \\
&= -(106\hat{\boldsymbol{\imath}} + 38\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2,
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{cor}} &= 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{\text{rel}} \\
&= 2\omega\hat{\boldsymbol{k}} \times v_{/\text{tube}}(-\hat{\boldsymbol{\jmath}}) \\
&= 2\omega v_{/\text{tube}}\hat{\boldsymbol{\imath}} = 2 \cdot 5\,\text{rad/s} \cdot 4\,\text{ft/s} \\
&= 40\,\text{ft/s}^2\hat{\boldsymbol{\imath}},
\end{aligned}$$

$$\vec{\boldsymbol{a}}_{\text{rel}} = a_{/\text{tube}}(-\hat{\boldsymbol{\jmath}}) = -2\,\text{ft/s}^2\hat{\boldsymbol{\jmath}}.$$

Adding the three terms together, we get

$$\begin{aligned}
\vec{\boldsymbol{a}}_P &= -106\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 38\,\text{ft/s}^2\hat{\boldsymbol{\jmath}} + 40\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 2\,\text{ft/s}^2\hat{\boldsymbol{\jmath}} \\
&= -66\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 40\,\text{ft/s}^2\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_P = -(66\hat{\boldsymbol{\imath}} + 40\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2}$$

Note that the single term $\vec{\boldsymbol{a}}_{P'}$ encompasses three terms of the five term acceleration formula.

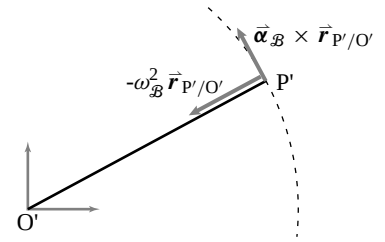


Figure 10.29: Acceleration of point P′. P′ is fixed to the frame $\mathcal{B}$ and at the moment coincides with point P. Therefore, $\vec{\boldsymbol{a}}_{P'} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} - \omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{P'/O'}$.
(Filename:sfig8.2.1a)

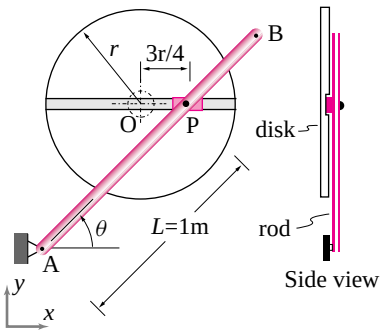
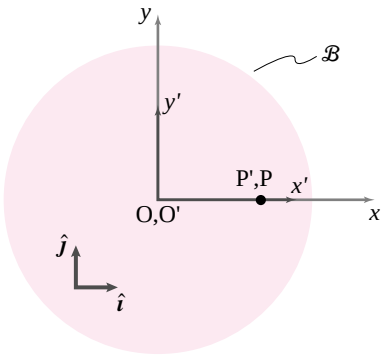


Figure 10.30: (Filename:sfig8.1.2a)



xy is the fixed frame. *x'y'* rotates with the disk with ω_B. P' is fixed in the rotating frame.

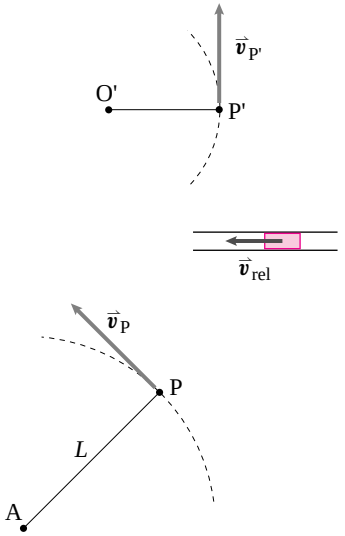


Figure 10.31: (Filename:sfig8.1.2b)

SAMPLE 10.8 A small collar P is pinned to a rigid rod AB at length $L = 1\,\text{m}$ along the rod. The collar is free to slide in a straight track on a disk of radius $r = 400\,\text{mm}$. The disk rotates about its center O at a constant $\omega = 2\,\text{rad/s}$. At the instant shown, when $\theta = 45^o$ and the collar is at a distance $\frac{3}{4}r$ in the track from the center O, find

(a) the angular velocity of the rod AB and
(b) the velocity of point P relative to the disk.

Solution We will think of P in two ways: one as attached to the rod and the other as sliding in the slot. First, let us attach a frame $\mathcal{B}$ to the disk. Thus $\mathcal{B}$ rotates with the disk with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}} = 2\,\text{rad/s}\hat{\boldsymbol{k}}$. We attach a coordinate system $x'y'z'$ to $\mathcal{B}$ at point O. At the instant of interest, the rotating coordinate system $x'y'z'$ coincides with the fixed coordinate system xyz. Now let us consider point P' which is fixed on the disk (and hence in $\mathcal{B}$) and coincides with point P at the moment of interest. We can write the velocity of P as:

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\text{rel}}$$

where

$$\begin{aligned}\vec{\boldsymbol{v}}_{P'} &= \overbrace{\vec{\boldsymbol{v}}_{O'}}^{\vec{\boldsymbol{0}}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} = \omega\hat{\boldsymbol{k}} \times \left(\frac{3}{4}r\hat{\boldsymbol{\imath}}'\right) = \frac{3}{4}\omega r\hat{\boldsymbol{\jmath}}' = \frac{3}{4}\omega r\hat{\boldsymbol{\jmath}},\\ \vec{\boldsymbol{v}}_{\text{rel}} &\equiv \vec{\boldsymbol{v}}_{P/\mathcal{B}} = v_{\text{rel}}\hat{\boldsymbol{\imath}}' = v_{\text{rel}}\hat{\boldsymbol{\imath}}.\end{aligned}$$

In the last expression, $\vec{\boldsymbol{v}}_{\text{rel}} = v_{\text{rel}}\hat{\boldsymbol{\imath}}$, we do not know the magnitude of $\vec{\boldsymbol{v}}_{\text{rel}}$ and hence have left it as an unknown v_{rel}, but its direction is known because $\vec{\boldsymbol{v}}_{\text{rel}}$ has to be along the track and the track at the given instant is along the x-axis. Thus,

$$\vec{\boldsymbol{v}}_P = \frac{3}{4}\omega r\hat{\boldsymbol{\jmath}} + v_{\text{rel}}\hat{\boldsymbol{\imath}}. \tag{10.21}$$

Now let us consider the motion of rod AB. Let $\vec{\boldsymbol{\Omega}} = \Omega\hat{\boldsymbol{k}}$ be the angular velocity of AB at the instant of interest where Ω is unknown. Since P is pinned to the rod, it executes circular motion about A with radius $AP = L$. Therefore,

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{r}}_{P/A} = \Omega\hat{\boldsymbol{k}} \times L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) = \Omega L(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) \tag{10.22}$$

But, and this trivial formula is the key, $\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_P$. Therefore, from Eqn. (10.21) and (10.22),

$$\frac{3}{4}\omega r\hat{\boldsymbol{\jmath}} + v_{\text{rel}}\hat{\boldsymbol{\imath}} = \Omega L(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}). \tag{10.23}$$

Taking dot product of both sides of the above equation with $\hat{\boldsymbol{\jmath}}$ we get

$$\begin{aligned}\frac{3}{4}\omega r &= \Omega L\cos\theta\\ \Rightarrow\quad \Omega &= \frac{3\omega r}{4L\cos\theta} = \frac{3\cdot 2\,\text{rad/s}\cdot 0.4\,\text{m}}{4\cdot 1\,\text{m}\cdot\frac{1}{\sqrt{2}}} = 0.85\,\text{rad/s}.\end{aligned}$$

Again taking the dot product of both sides of Eqn. (10.23) with $\hat{\boldsymbol{\imath}}$ we get

$$v_{\text{rel}} = -\Omega L\sin\theta = -0.85\,\text{rad/s}\cdot 1\,\text{m}\cdot\frac{1}{\sqrt{2}} = -0.6\,\text{m/s}.$$

$$\boxed{(i)\ \vec{\boldsymbol{\Omega}} = 0.85\,\text{rad/s}\hat{\boldsymbol{k}},\quad (ii)\ \vec{\boldsymbol{v}}_{\text{rel}} = -0.6\,\text{m/s}\hat{\boldsymbol{\imath}}}$$

SAMPLE 10.9 *Rate of change of unit vectors.* A circular disk $\mathcal{D}$ is welded to a rigid rod AB. The rod rotates about point A with angular velocity $\vec{\omega} = \omega\hat{k}$. A frame $\mathcal{B}$ is attached to the disk and therefore rotates with the same $\vec{\omega}$. Two coordinate systems, $(\hat{\imath}', \hat{\jmath}')$ and $(\hat{e}_R, \hat{e}_\theta)$ are fixed in frame $\mathcal{B}$ as shown in the figure.

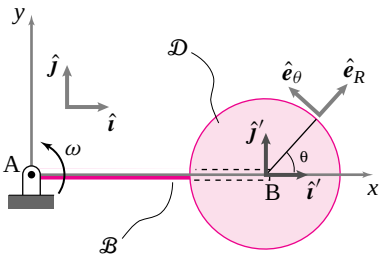


Figure 10.32: (Filename:sfig8.qdot.1)

(a) Find the rate of change of unit vectors $\hat{e}_R$, $\hat{e}_\theta$, $\hat{\imath}'$ and $\hat{\jmath}'$ using the $\dot{\vec{Q}}$ formula.
(b) Express the $\hat{e}_R$ and $\hat{e}_\theta$ vectors in terms of $\hat{\imath}'$ and $\hat{\jmath}'$ and verify the results obtained above for $\dot{\hat{e}}_R$ and $\dot{\hat{e}}_\theta$ by direct differentiation.

Solution Since the disk is welded to the rod and frame $\mathcal{B}$ is fixed in the disk, the frame rotates with $\vec{\omega}_\mathcal{B} = \omega\hat{k}$.

(a) To find the rate of change of the unit vectors using the $\dot{\vec{Q}}$ formula, we substitute the desired unit vector in place of $\vec{Q}$ in the formula (Eqn 8.12 of the Text). For example,

$$\dot{\hat{e}}_R = {}^\mathcal{B}\dot{\hat{e}}_R + \vec{\omega}_\mathcal{B} \times \hat{e}_R.$$

It should be clear that ${}^\mathcal{B}\dot{\hat{e}}_R = 0$, since $\hat{e}_R$ does not change with respect to an observer sitting in frame $\mathcal{B}$. Therefore,

$$\dot{\hat{e}}_R = \vec{\omega}_\mathcal{B} \times \hat{e}_R = \omega\hat{k} \times \hat{e}_R = \omega\hat{e}_\theta.$$

Similarly,

$$\begin{aligned}
\dot{\hat{e}}_\theta &= \underbrace{{}^\mathcal{B}\dot{\hat{e}}_\theta}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{e}_\theta = \omega\overbrace{\hat{k} \times \hat{e}_\theta}^{-\hat{e}_R} = -\omega\hat{e}_R. \\
\dot{\hat{\imath}}' &= \underbrace{{}^\mathcal{B}\dot{\hat{\imath}}'}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{\imath}' = \omega\hat{k} \times \hat{\imath}' = \omega\hat{\jmath}'. \\
\dot{\hat{\jmath}}' &= \underbrace{{}^\mathcal{B}\dot{\hat{\jmath}}'}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{\jmath}' = \omega\hat{k} \times \hat{\jmath}' = -\omega\hat{\imath}'.
\end{aligned}$$

(b) Since $\hat{e}_R = \cos\theta\hat{\imath}' + \sin\theta\hat{\jmath}'$ and $\hat{e}_\theta = -\sin\theta\hat{\imath}' + \cos\theta\hat{\jmath}'$, we get their rates of change by direct differentiation as

$$\begin{aligned}
\dot{\hat{e}}_R &= \cos\theta\dot{\hat{\imath}}' + \sin\theta\dot{\hat{\jmath}}' \\
&= \cos\theta(\omega\hat{\jmath}') + \sin\theta(-\omega\hat{\imath}') \\
&= \omega(-\sin\theta\hat{\imath}' + \cos\theta\hat{\jmath}') = \omega\hat{e}_\theta, \\
\dot{\hat{e}}_\theta &= -\sin\theta\dot{\hat{\imath}}' + \cos\theta\dot{\hat{\jmath}}' \\
&= -\sin\theta(\omega\hat{\jmath}') + \cos\theta(-\omega\hat{\imath}') \\
&= -\omega(\cos\theta\hat{\imath}' + \sin\theta\hat{\jmath}') = -\omega\hat{e}_R.
\end{aligned}$$

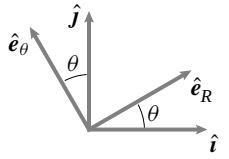


Figure 10.33: (Filename:sfig8.qdot.1a)

Here we have used the fact that θ, the angle between the unit vectors $\hat{e}_R$ and $\hat{\imath}'$, remains constant during the motion. The results obtained are the same as in part (a). ◁

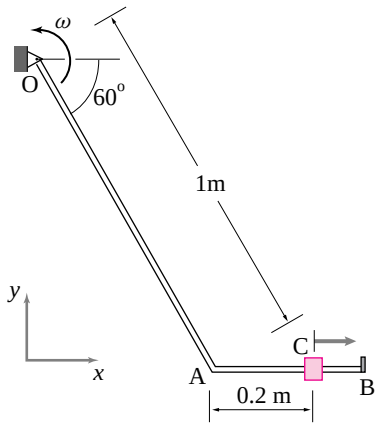


Figure 10.34: (Filename:sfig8.qdot.2)

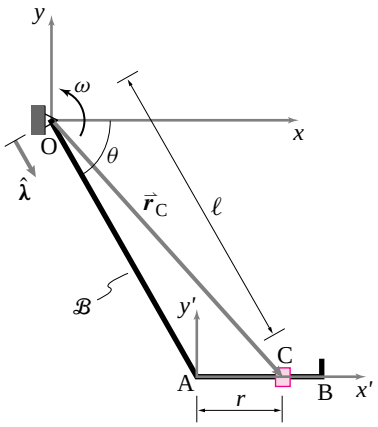


Figure 10.35: (Filename:sfig8.qdot.2a)

SAMPLE 10.10 *Rate of change of a position vector.* A rigid rod OAB rotates counterclockwise about point O with constant angular speed $\omega = 5\,\text{rad/s}$. A collar C slides out on the bent arm AB with constant speed $v = 0.5\,\text{m/s}$ with respect to the arm. Find the velocity of the collar using the $\vec{Q}$ formula.

Solution Let $\vec{r}_C$ be the position vector of the collar. Then the velocity of the collar is $\dot{\vec{r}}_C$. Let the rod OAB be the rotating frame $\mathcal{B}$. Now we can find $\dot{\vec{r}}_C$ using the $\vec{Q}$ formula:

$$\dot{\vec{r}}_C = {}^{\mathcal{B}}\dot{\vec{r}}_C + \vec{\omega}_{\mathcal{B}} \times \vec{r}_C$$

To compute $\dot{\vec{r}}_C$, let us first find ${}^{\mathcal{B}}\dot{\vec{r}}_C$, the rate of change of $\vec{r}_C$ as seem in frame $\mathcal{B}$ (this term represents the velocity of the collar you see if you sit on the rod and watch the collar; also called $\vec{v}_{rel}$).

$$\begin{aligned} \vec{r}_C &= \vec{r}_A + \vec{r}_{C/A} \\ {}^{\mathcal{B}}\dot{\vec{r}}_C &= {}^{\mathcal{B}}\dot{\vec{r}}_A + {}^{\mathcal{B}}\dot{\vec{r}}_{C/A} \end{aligned}$$

Note that the vector $\vec{r}_A = \ell\hat{\boldsymbol{\lambda}}$ does not change in frame $\mathcal{B}$ since both its magnitude, ℓ, and direction, $\hat{\boldsymbol{\lambda}}$, remain fixed in $\mathcal{B}$. Therefore,

$${}^{\mathcal{B}}\dot{\vec{r}}_A = 0$$

$$\text{Now} \quad \vec{r}_{\text{C/A}} = \text{r}\hat{\boldsymbol{\imath}}'. \quad \Rightarrow \quad {}^{\mathcal{B}}\dot{\vec{r}}_{\text{C/A}} = \text{r}\hat{\boldsymbol{\imath}}' = 0.5\,\text{m/s}\hat{\boldsymbol{\imath}}'$$

because $\hat{\boldsymbol{\imath}}'$ does not change in $\mathcal{B}$ and $\dot{r}$ = speed of the collar with respect to the arm. (see Figure 10.35) Thus,

$${}^{\mathcal{B}}\dot{\vec{r}}_C = 0.5\,\text{m/s}\hat{\boldsymbol{\imath}}'.$$

Hence,

$$\begin{aligned} \dot{\vec{r}}_C &= {}^{\mathcal{B}}\dot{\vec{r}}_C + \omega\hat{\boldsymbol{k}} \times (\ell\hat{\boldsymbol{\lambda}} + r\hat{\boldsymbol{\imath}}') \\ &= {}^{\mathcal{B}}\dot{\vec{r}}_C + \omega L(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}) + \omega r(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}}') \\ &= \dot{r}\hat{\boldsymbol{\imath}}' + \omega\ell(\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) + \omega r\hat{\boldsymbol{\jmath}}' \\ &= 0.5\,\text{m/s}\hat{\boldsymbol{\imath}}' + 5\,\text{m/s}\left(\frac{\sqrt{3}}{2}\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}}\right) + 1\,\text{m/s}\hat{\boldsymbol{\jmath}}' \\ &= 4.83\,\text{m/s}\hat{\boldsymbol{\imath}} + 3.5\,\text{m/s}\hat{\boldsymbol{\jmath}} \end{aligned}$$

where we have used the fact that at the given instant, $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$.

$$\boxed{\vec{v}_C = 4.83\,\text{m/s}\hat{\boldsymbol{\imath}} + 3.5\,\text{m/s}\hat{\boldsymbol{\jmath}}}$$

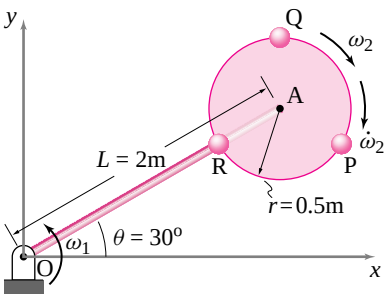


Figure 10.36: (Filename:sfig8.2.2)

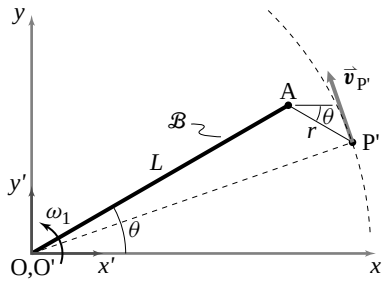


Figure 10.37: P′ goes in circles around point O with radius OP′. Therefore, $\vec{v}_{P'}$ is tangential to the circular path at P′.

(Filename:sfig8.2.2a)

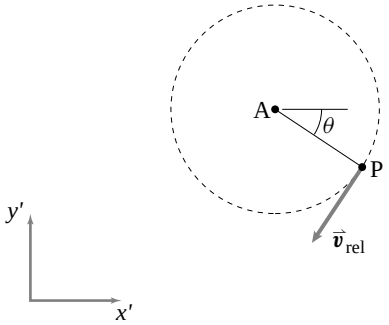


Figure 10.38: Velocity of point P with respect to the frame $\mathcal{B}$.

(Filename:sfig8.2.2b)

SAMPLE 10.11 *Spinning wheel on a rotating rod. 2-D* . A rigid body OA is attached to a wheel that is massless except for three point masses P, Q, and R, placed symmetrically on the wheel. Each of the three masses is $m = 0.5\,\text{kg}$. The rod OA rotates about point O in the counterclockwise direction at a constant rate $\omega_1 = 3\,\text{rad/s}$. The wheel rotates with respect to the arm about point A with angular acceleration $\dot{\omega}_2 = 1\,\text{rad/s}^2$ and at the instant shown it has angular speed $\omega_2 = 5\,\text{rad/s}$. Note that both ω_2 and $\dot{\omega}_2$ are given with respect to the arm.

Using a rotating frame $\mathcal{B}$ attached to the rod and a coordinate system attached to the frame with origin at O, find

(a) the velocity of the mass P and
(b) the acceleration of the mass P.

Solution Frame $\mathcal{B}$ is attached to the rod. We choose a coordinate system $x'y'z'$ in frame $\mathcal{B}$ with its origin at O and, at the instant, aligned with the fixed coordinate system xyz. We consider a point P′ momentarily coincident with point P but fixed in frame $\mathcal{B}$. Since P′ is fixed in $\mathcal{B}$, it rotates with $\mathcal{B}$ with $\vec{\omega}_{\mathcal{B}} = \omega_1\hat{k}$. To visualize the motion of P′ imagine a rigid rod from O to P′ (see Fig. 10.37). Now we can calculate the velocity and acceleration of point P′ as follows.

(a) **Velocity of point P:**

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}}.$$

Now we calculate the two terms separately:

$$\begin{aligned}
\vec{v}_{P'} &= \underbrace{\vec{v}_{O'}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'} \\
&= \omega_1\hat{k} \times (\vec{r}_{A/O'} + \vec{r}_{P'/A}) \\
&= \omega_1\hat{k} \times (\underbrace{L(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})}_{\vec{r}_{A/O'}} + \underbrace{r(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})}_{\vec{r}_{P'/A}} \\
&= \omega_1(L+r)\cos\theta\hat{\jmath} - \omega_1(L-r)\sin\theta\hat{\imath} \\
&= 3\,\text{rad/s}\cdot 2.5\,\text{m}\cdot\cos 30^\text{o}\hat{\jmath} - 3\,\text{rad/s}\cdot 1.5\,\text{m}\cdot\sin 30^\text{o}\hat{\imath} \\
&= (6.50\hat{\jmath} - 2.25\hat{\imath})\,\text{m/s}.
\end{aligned}$$

Since the wheel rotates with angular speed ω_2 with respect to the rod, an observer sitting in frame $\mathcal{B}$ would see a circular motion of point P about point A. Therefore,

$$\begin{aligned}
\vec{v}_{\text{rel}} &= \vec{\omega}_{\text{wheel}/\mathcal{B}} \times \vec{r}_{P/A} \\
&= -\omega_2\hat{k}' \times r(\cos\theta\hat{\imath}' - \sin\theta\hat{\jmath}') \\
&= -\omega_2 r(\cos\theta\hat{\jmath}' + \sin\theta\hat{\imath}') \\
&= -(2.16\hat{\jmath}' + 1.25\hat{\imath}')\,\text{m/s}.
\end{aligned}$$

But at the instant of interest

$$\hat{\imath}' = \hat{\imath}, \quad \hat{\jmath}' = \hat{\jmath}, \quad \text{and} \quad \hat{k}' = \hat{k}.$$
$$\Rightarrow \qquad \vec{v}_{\text{rel}} = -(2.16\hat{\jmath} + 1.25\hat{\imath})\,\text{m/s}$$

Therefore,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}} = 4.33\,\text{m/s}\hat{\jmath} - 3.50\,\text{m/s}\hat{\imath}.$$

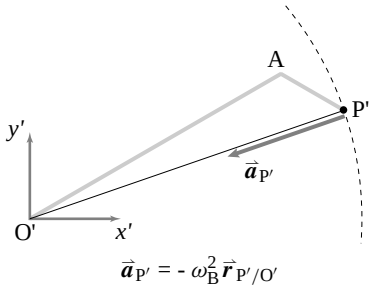
$$\boxed{\vec{v}_P = (-3.50\hat{\imath} + 4.33\hat{\jmath})\,\text{m/s}}$$

(b) **Acceleration of point P:** We can similarly find the acceleration of point P:

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{\text{cor}} + \vec{a}_{\text{rel}}$$

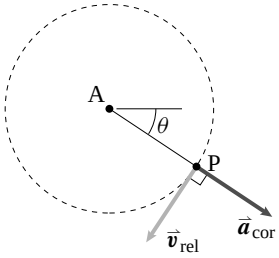
where

$$\begin{aligned}
\vec{a}_{P'} &= \text{acceleration of point P}' \\
&= \underbrace{\vec{a}_{O'}}_{\vec{0}} + \underbrace{\vec{\alpha}_{\mathcal{B}}}_{\vec{0}} \times \vec{r}_{P'/O'} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}) \\
&= -\omega_{\mathcal{B}}^2 \vec{r}_{P'/O'} \\
&= \ldots \\
&= \ldots \\
&= \ldots
\end{aligned}$$



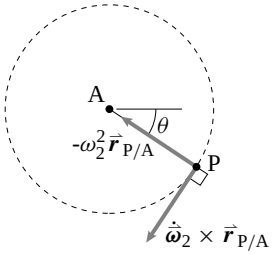
$\vec{a}_{P'} = -\omega_B^2 \vec{r}_{P'/O'}$

$$\begin{aligned}
\vec{a}_{\text{cor}} &= \text{Coriolis acceleration} \\
&= 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\text{rel}} \\
&= 2\omega_1 \hat{k} \times \vec{v}_{\text{rel}} \qquad \text{(see part (a) above for } \vec{v}_{\text{rel}}\text{)}. \\
&= 6\,\text{rad/s}\hat{k} \times (-2.16\hat{j} - 1.25\hat{i})\,\text{m/s} \\
&= 12.99\,\text{m/s}^2\hat{i} - 7.50\,\text{m/s}^2\hat{j},
\end{aligned}$$



$\vec{a}_{\text{cor}} = 2\vec{\omega}_B \times \vec{v}_{\text{rel}}$

$$\begin{aligned}
\vec{a}_{\text{rel}} &= \text{acceleration of P relative to frame } \mathcal{B} \\
&= \vec{a}_{P/\mathcal{B}} = \dot{\vec{\omega}}_2 \times \vec{r}_{P/A} - \omega_2^2 \vec{r}_{P/A} \\
&= -\dot{\omega}_2 \hat{k}' \times r(\cos\theta \hat{i}' - \sin\theta \hat{j}') - \omega_2^2 r(\cos\theta \hat{i}' - \sin\theta \hat{j}') \\
&= -r[(\dot{\omega}_2 \sin\theta + \omega_2^2 \cos\theta)\hat{i}' + (\dot{\omega}_2 \cos\theta - \omega_2^2 \sin\theta)\hat{j}'] \\
&= -0.5\,\text{m}[(1\,\text{rad/s}^2 \cdot \sin 30^\text{o} + 25(\,\text{rad/s})^2 \cdot \cos 30^\text{o})\hat{i}' \\
&\qquad +(1\,\text{rad/s}^2 \cdot \cos 30^\text{o} - 25(\,\text{rad/s})^2 \cdot \sin 30^\text{o})\hat{j}'] \\
&= (-11.08\hat{i}' + 5.82\hat{j}')\,\text{m/s}^2 \\
&= (-11.08\hat{i} + 5.82\hat{j})\,\text{m/s}^2.
\end{aligned}$$



$\vec{a}_{\text{rel}} = -\omega_2^2 \vec{r}_{P/A} + \dot{\vec{\omega}}_2 \times \vec{r}_{P/A}$

Figure 10.39: (Filename:sfig8.2.2c)

The term $\vec{a}_{P'}$ encompasses three terms of the five term acceleration formula. The last line in the calculation of $\vec{a}_{\text{rel}}$ follows from the fact that at the instant of interest $\hat{i}' = \hat{i}$ and $\hat{j}' = \hat{j}$.

Now adding the three parts of $\vec{a}_P$ we get

$$\begin{aligned}
\vec{a}_P &= \vec{a}_{P'} + \vec{a}_{\text{cor}} + \vec{a}_{\text{rel}} \\
&= -17.83\,\text{m/s}^2\hat{i} - 3.63\,\text{m/s}^2\hat{j}.
\end{aligned}$$

$$\boxed{\vec{a}_P = -(17.83\hat{i} + 3.63\hat{j})\,\text{m/s}^2}$$

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TABLE I

Momenta and energy

	Linear Momentum		**Angular Momentum**		**Kinetic Energy**	
What system	$\vec{L}$ (a)	$\dot{\vec{L}} = \frac{d}{dt}\vec{L}$ (b)	$\vec{H}_{\rm C}$ (c)	$\dot{\vec{H}}_{\rm C} = \frac{d}{dt}\vec{H}_{\rm C}$ (d)	$E_{\rm K}$ (e)	
In General	$L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k}$ $= m_{\rm tot}\vec{v}_{\rm cm}$ $= \frac{d}{dt}(m_{\rm tot}\vec{r}_{\rm cm})$	$\dot{L}_x\hat{\imath} + \dot{L}_y\hat{\jmath} + \dot{L}_z\hat{k}$ $= m_{\rm tot}\vec{a}_{\rm cm}$ "F = ma"	$H_{Cx}\hat{\imath} + H_{Cy}\hat{\jmath} + H_{Cz}\hat{k}$ $= \vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \vec{H}_{\rm cm}$ $= \frac{d}{dt}$(no such thing)	$\dot{H}_{Cx}\hat{\imath} + \dot{H}_{Cy}\hat{\jmath} + \dot{H}_{Cz}\hat{k}$ $= \vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot} + \dot{\vec{H}}_{\rm cm}$ $=$ (no simple general expression)	$\frac{1}{2} m_{\rm tot}\, v_{\rm cm}^2 + E_{{\rm K}/cm}$	(1)
One Particle P	$m_{\rm P}\vec{v}_{\rm P}$	$m_{\rm P}\vec{a}_{\rm P}$	$\vec{r}_{\rm P/C} \times \vec{v}_P m_{\rm P}$	$\vec{r}_{\rm P/C} \times \vec{a}_{\rm P} m_{\rm P}$	$\frac{1}{2} m_{\rm P} v_{\rm P}^2$	(2)
System of Particles	$\sum_{\text{all particles i}} m_i \vec{v}_i$	$\sum_{\text{all particles i}} m_i \vec{a}_i$	$\sum_{\text{all particles}} \vec{r}_{i/\rm C} \times \vec{v}_i m_i$	$\sum_{\text{all particles}} \vec{r}_{i/\rm C} \times \vec{a}_i m_i$	$\frac{1}{2} \sum_{\text{all particles}} v_i^2 m_i$	(3)
Continuum	$\int_{\text{all mass}} \vec{v}\, dm$	$\int_{\text{all mass}} \vec{a}\, dm$	$\int_{\text{all mass}} \vec{r}_{/\rm C} \times \vec{v}\, dm$	$\int_{\text{all mass}} \vec{r}_{/\rm C} \times \vec{a}\, dm$	$\frac{1}{2}\int_{\text{all mass}} v^2\, dm$	(4)
System of Systems (eg. rigid bodies)	$\sum_{\text{all sub-systems}} m_i \vec{v}_i$	$\sum_{\text{all sub-systems}} m_i \vec{a}_i$	$\sum_{\text{all sub-systems}} \vec{H}_{{\rm C}i}$	$\sum_{\text{all sub-systems}} \dot{\vec{H}}_{{\rm C}i}$	$\sum_{\text{all sub-systems}} E_{{\rm K}i}$	(5)
Rigid Bodies						
One rigid body (2D and 3D)	$m_{\rm tot}\vec{v}_{\rm cm}$	$m_{\rm tot}\vec{a}_{\rm cm}$	$\vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \underbrace{[\mathbf{I}^{cm}]\cdot\vec{\omega}}_{\vec{H}_{\rm cm}}$	$\vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot} + \underbrace{[\mathbf{I}^{cm}]\cdot\dot{\vec{\omega}} + \vec{\omega} \times \vec{H}_{\rm cm}}_{\dot{\vec{H}}_{\rm cm}}$	$\frac{1}{2} m_{\rm tot}\, v_{\rm cm}^2 + \underbrace{\frac{1}{2}\vec{\omega}\cdot[\mathbf{I}^{cm}]\cdot\vec{\omega}}_{E_{{\rm K}/cm}}$	(6)
2D rigid body in xy plane with $\vec{\omega} = \omega\hat{k}$	$m_{\rm tot}\vec{v}_{\rm cm}$	$m_{\rm tot}\vec{a}_{\rm cm}$	$\vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \underbrace{I_{zz}^{cm}\omega\hat{k}}_{\vec{H}_{\rm cm}}$	$\vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot} + \underbrace{I_{zz}^{cm}\dot{\omega}\hat{k}}_{\dot{\vec{H}}_{\rm cm}}$	$\frac{1}{2} m_{\rm tot}\, v_{\rm cm}^2 + \underbrace{\frac{1}{2} I_{zz}^{cm}\omega^2}_{E_{{\rm K}/cm}}$	(7)
One rigid body ***if*** C is a fixed point (2D and 3D)	$m_{\rm tot}\vec{v}_{\rm cm}$	$m_{\rm tot}\vec{a}_{\rm cm}$	$[\mathbf{I}^C]\cdot\vec{\omega} = \vec{H}_C$	$[\mathbf{I}^C]\cdot\dot{\vec{\omega}} + \vec{\omega} \times \vec{H}_C$	$\frac{1}{2}\vec{\omega}\cdot[\mathbf{I}^C]\cdot\vec{\omega}$	(8)
2D rigid body ***if*** C is a fixed point with $\vec{\omega} = \omega\hat{k}$	$m_{\rm tot}\vec{v}_{\rm cm}$	$m_{\rm tot}\vec{a}_{\rm cm}$	$I_{zz}^C\omega\hat{k}$	$I_{zz}^C\dot{\omega}\hat{k}$ "$M = I\alpha$"	$\frac{1}{2} I_{zz}^C\omega^2$	(9)

The table has used the following terms:

$m_{\rm tot}$ =total mass of system,

m_i = mass of body or subsystem i,

$\vec{r}_{\rm cm/C}$ = the position of the center of mass relative to point C,

$\vec{v}_i$ = velocity of the center of mass of sub-system or particle i,

$\vec{a}_i$ = acceleration of the center of mass of sub-system i,

$\vec{H}_{{\rm C}i}$ = angular momentum of subsystem i relative to point C.

$\dot{\vec{H}}_{{\rm C}i}$ = rate of change of angular momentum of sub-system i relative to point C.

$\vec{H}_{\rm cm} = \sum \vec{r}_{i/\rm cm} \times (m_i \vec{v}_i)$ angular momentum about the center of mass

$\dot{\vec{H}}_{\rm cm} = \sum \vec{r}_{i/\rm cm} \times (m_i \vec{a}_i)$ rate of change of angular momentum about the center of mass

$\vec{\omega}$ is the angular velocity of a rigid body,

$\dot{\vec{\omega}} = \vec{\alpha}$ is the angular acceleration of the rigid body,

$[\mathbf{I}^{cm}]$ is the moment of inertia matrix of the rigid body relative to the center of mass, and

$[\mathbf{I}^o]$ is the moment of inertia matrix of the rigid body relative to a fixed point (not moving point) on the body.

Table II

Summary of methods of calculating velocity and acceleration

Method	Position	Velocity	Acceleration
In general, as measured relative to the fixed frame $\mathcal{F}$.	$\vec{r}$ or $\vec{r}_P$ or $\vec{r}_{P/O}$	$\vec{v}$ or $\vec{v}_P$ or $\vec{v}_{P/\mathcal{F}}$	$\vec{a}$ or $\vec{a}_P$ or $\vec{a}_{P/\mathcal{F}}$
Cartesian Coordinates	$r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}$	$v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ $= \dot{r}_x\hat{\imath} + \dot{r}_y\hat{\jmath} + \dot{r}_z\hat{k}$	$a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ $= \dot{v}_x\hat{\imath} + \dot{v}_y\hat{\jmath} + \dot{z}_z\hat{k}$ $= \ddot{r}_x\hat{\imath} + \ddot{r}_y\hat{\jmath} + \ddot{r}_z\hat{k}$
Polar Coordinates/ Cylindrical Coordinates	$R\hat{e}_R + z\hat{k}$	$v_R\hat{e}_R + v_\theta\hat{e}_\theta + v_z\hat{k}$ $= \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}$	$a_R\hat{e}_R + a_\theta\hat{e}_\theta + a_z\hat{k}$ $= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}$
Path Coordinates	not used	$v\hat{e}_t$	$a_t\hat{e}_t + a_n\hat{e}_n$ $= \dot{v}\hat{e}_t + (v^2/\rho)\hat{e}_n$
Using data from a moving frame $\mathcal{B}$ with origin at O' and angular velocity relative to the fixed frame of $\vec{\omega}_\mathcal{B}$. The point P' is glued to $\mathcal{B}$ and instantaneously coincides with P.	$\vec{r}_{O'/O} + \vec{r}_{P/O'}$	$\vec{v}_{P'/\mathcal{F}} + \vec{v}_{P/\mathcal{B}} =$ $\underbrace{\dot{\vec{r}}_{O'/O} + \vec{\omega}_\mathcal{B} \times \vec{r}_{P'/O'}}_{\vec{v}_{P'/\mathcal{F}}} + \underbrace{{}^\mathcal{B}\dot{\vec{r}}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}}$	$\vec{a}_{P'/\mathcal{F}} + \vec{a}_{P/\mathcal{B}} + 2\vec{\omega}_\mathcal{B} \times \vec{v}_{P/\mathcal{B}} =$ $\overbrace{\ddot{\vec{r}}_{O'/O} + \dot{\vec{\omega}}_\mathcal{B} \times \vec{r}_{P'/O'} + \vec{\omega}_\mathcal{B} \times \vec{\omega}_\mathcal{B} \times \vec{r}_{P'/O'}}^{\vec{a}_{P'/\mathcal{F}}}$ $+ \underbrace{{}^\mathcal{B}\ddot{\vec{r}}_{P/O'}}_{\vec{a}_{P/\mathcal{B}}} + 2\vec{\omega}_\mathcal{B} \times \vec{v}_{P/\mathcal{B}}$ ‘the 5-term acceleration formula’

Some facts about path coordinates

The path of a particle is $\vec{r}(t)$.

$$\hat{e}_t \equiv \frac{d\vec{r}(s)}{ds}, \quad \hat{e}_t = \frac{d\vec{r}(t)}{dt}\frac{dt}{ds} = \frac{\vec{v}}{v}, \quad \vec{\kappa} \equiv \frac{d\hat{e}_t}{ds} = \frac{d\hat{e}_t}{dt}\frac{1}{v}, \quad \hat{e}_n = \frac{\vec{\kappa}}{|\vec{\kappa}|}, \quad e_b \equiv \hat{e}_t \times \hat{e}_n, \quad \rho = \frac{1}{|\vec{\kappa}|}.$$

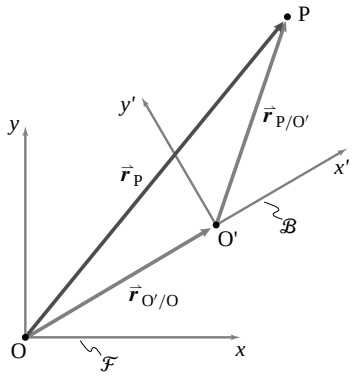
Summary of the direct differentiation method

In the direct differentiation method, using moving frame $\mathcal{B}$, we calculate $\vec{v}_P$ by using a combination of the product rule of differentiation and the facts that

$$\dot{\hat{\imath}}' = \vec{\omega}_\mathcal{B} \times \hat{\imath}', \quad \dot{\hat{\jmath}}' = \vec{\omega}_\mathcal{B} \times \hat{\jmath}', \text{ and } \dot{\hat{k}}' = \vec{\omega}_\mathcal{B} \times \hat{k}',$$

as follows:

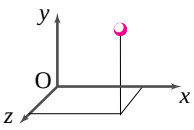
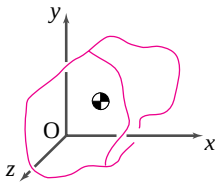
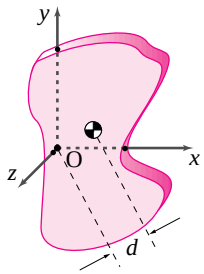
$$\begin{aligned}
\vec{v}_P &= \frac{d}{dt}\vec{r}_P \\
&= \frac{d}{dt}\left[\vec{r}_{O'/O} + \vec{r}_{P/O'}\right] \\
&= \frac{d}{dt}\left[(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) + (x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}')\right] \\
&= (\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}) + (\dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}') + \\
&\quad \left[x'(\vec{\omega}_\mathcal{B} \times \hat{\imath}') + y'(\vec{\omega}_\mathcal{B} \times \hat{\jmath}') + z'(\vec{\omega}_\mathcal{B} \times \hat{k}')\right]
\end{aligned}$$



but stop short of identifying these three groups of three terms as $\vec{v}_P = \vec{v}_{O'/O} + \dot{\vec{r}}_{rel} + \vec{\omega}_\mathcal{B} \times \vec{r}_{P/O}$.

We could calculate $\vec{a}_P$ similarly and would get a similar formula with 15 non-zero terms (3 for each term in the ‘five-term’ acceleration formula).

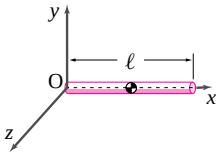
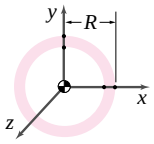
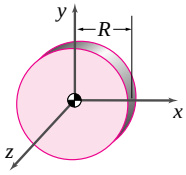
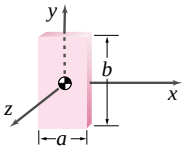
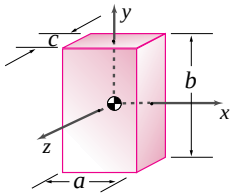
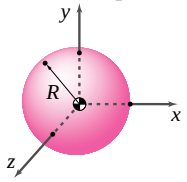
Table III

Object	$[I]$
Point mass 	$[\boldsymbol{I}^{cm}] = m\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $[\boldsymbol{I}^{O}] = m\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$
General 3D body 	$[\boldsymbol{I}^{cm}] = \int \begin{bmatrix} y_{/cm}^2+z_{/cm}^2 & -x_{/cm}y_{/cm} & -x_{/cm}z_{/cm} \\ -x_{/cm}y_{/cm} & x_{/cm}^2+z_{/cm}^2 & -y_{/cm}z_{/cm} \\ -x_{/cm}z_{/cm} & -y_{/cm}z_{/cm} & x_{/cm}^2+y_{/cm}^2 \end{bmatrix} d\,m$ If the axes are principal axes of the body. $\overbrace{[\boldsymbol{I}^{cm}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}}$ With $A, B, C \geq 0$ and $A+B \geq C$, $B+C \geq A$, and $A+C \geq B$. $[\boldsymbol{I}^{o}] = \int \begin{bmatrix} y_{/o}^2+z_{/o}^2 & -x_{/o}y_{/o} & -x_{/o}z_{/o} \\ -x_{/o}y_{/o} & x_{/o}^2+z_{/o}^2 & -y_{/o}z_{/o} \\ -x_{/o}z_{/o} & -y_{/o}z_{/o} & x_{/o}^2+y_{/o}^2 \end{bmatrix} d\,m$ $\underbrace{[\boldsymbol{I}^{o}] = [\boldsymbol{I}^{cm}] + m\begin{bmatrix} y_{cm/o}^2+z_{cm/o}^2 & -x_{cm/o}y_{cm/o} & -x_{cm/o}z_{cm/o} \\ -x_{cm/o}y_{cm/o} & x_{cm/o}^2+z_{cm/o}^2 & -y_{cm/o}z_{cm/o} \\ -x_{cm/o}z_{cm/o} & -y_{cm/o}z_{cm/o} & x_{cm/o}^2+y_{cm/o}^2 \end{bmatrix}}$ The 3D Parallel Axis Theorem
General 2D Body 	$[\boldsymbol{I}^{cm}] = \int \begin{bmatrix} y_{/cm}^2 & -x_{/cm}y_{/cm} & 0 \\ -x_{/cm}y_{/cm} & x_{/cm}^2 & 0 \\ 0 & 0 & \underbrace{x_{/cm}^2+y_{/cm}^2}_{I_{zz}^{cm}} \end{bmatrix} d\,m$ If the axes are principal axes of the body. $\overbrace{[\boldsymbol{I}^{cm}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}}$ With $A+B=C$ (The perpendicular axis theorem). Also, $A \geq 0$, $B \geq 0$. $[\boldsymbol{I}^{o}] = \int \begin{bmatrix} y_{/o}^2 & -x_{/o}y_{/o} & 0 \\ -x_{/o}y_{/o} & x_{/o}^2 & 0 \\ 0 & 0 & x_{/o}^2+y_{/o}^2 \end{bmatrix} d\,m$ $\underbrace{[\boldsymbol{I}^{o}] = [\boldsymbol{I}^{cm}] + m\begin{bmatrix} y_{cm/o}^2 & -x_{cm/o}y_{cm/o} & 0 \\ -x_{cm/o}y_{cm/o} & x_{cm/o}^2 & 0 \\ 0 & 0 & \underbrace{x_{cm/o}^2+y_{cm/o}^2}_{d^2} \end{bmatrix}}$ The 3D Parallel Axis Theorem. The 2D thm concerns the lower right terms of these 3 matrices.

General moments of inertia. The tableshows a point mass, a general 3-D body, and a general 2-D body. The most general cases of the perpeendicual axis theorem and the parallel axis theorem are also shown..

Table IV

Examples of Moment of Inertia

Object	$[\boldsymbol{I}]$
Uniform rod 	$I_{zz}^{\rm cm} = \frac{1}{12}m\ell^2, \quad [\boldsymbol{I}^{cm}] = \frac{1}{12}m\ell^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I_{zz}^{\rm O} = \frac{1}{3}m\ell^2, \quad [\boldsymbol{I}^{O}] = \frac{1}{3}m\ell^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Uniform hoop 	$I_{zz}^{\rm cm} = mR^2, \quad [\boldsymbol{I}^{cm}] = mR^2 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Uniform disk 	$I_{zz}^{\rm cm} = \frac{1}{2}mR^2, \quad [\boldsymbol{I}^{cm}] = mR^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$
Rectangular plate 	$I_{zz}^{\rm cm} = \frac{1}{12}m(a^2+b^2), \quad [\boldsymbol{I}^{cm}] = \frac{1}{12}m \begin{bmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$
Solid Box 	$[\boldsymbol{I}^{cm}] = \frac{1}{12}m \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$
Uniform sphere 	$[\boldsymbol{I}^{cm}] = \frac{2}{5}mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Moments of inertia of some simple objects. For the rod both the $[\boldsymbol{I}^{cm}]$ and $[\boldsymbol{I}^{O}]$ (for the end point at O) are shown. In the other cases only $[\boldsymbol{I}^{cm}]$ is shown. To calculate $[\boldsymbol{I}^{O}]$ relative to other points one has to use the parallel axis theorem. In all the cases shown the coordinate axes are principal axes of the objects.

11 Mechanics of planar mechanisms

Many parts of practical machines and structures move in ways that can be idealized as straight-line motion (Chapter 6) or circular motion (Chapters 7 and 8). But often an engineer most analyze parts with more general motions, like a plane in unsteady flight, and a connecting rod in a car engine. Of course, the same basic laws of mechanics still apply. The chapter starts with the kinematics of a rigid body in two dimensions and then progresses to the mechanics and analysis of motions of a planar body.

11.1 Dynamics of particles in the context of 2-D mechanisms

Now that we know more kinematics, we can deal with the mechanics of more mechanisms. Although it is not efficient for problem solving, we take a simple example to illustrate some comparison between some of the ways of keeping track of the motion. For one point mass it is easy to write balance of linear momentum. It is:

$$\vec{F} = m\vec{a}.$$

The mass of the particle m times its vector acceleration $\vec{a}$ is equal to the total force on the particle $\vec{F}$. Big deal.

Now, however, we can write this equation in four different ways.

(a) In general abstract vector form: $\vec{F} = m\vec{a}$.

(b) In cartesian coordinates: $F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} = m[\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}]$.

(c) In polar coordinates:
$$F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta + F_z\hat{\boldsymbol{k}} = m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}].$$
(d) In path coordinates: $F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n = m[\dot{v}\hat{\boldsymbol{e}}_t + (v^2/\rho)\hat{\boldsymbol{e}}_n]$

All of these equations are always right. They are summarized in the table on the inside cover. Additionally, for a given particle moving under the action of a given force there are many more correct equations that can be found by shifting the origin and orientation of the coordinate systems.

A particle that moves under the influence of no force.

In the special case that a particle has no force on it we know intuitively, or from the verbal statement of Newton's First Law, that the particle travels in a straight line at constant speed. As a first example, let's look at this result using the vector equations of motion four different ways: in the general abstract form, in cartesian coordinates, in polar coordinates, and in path coordinates.

General abstract form

The equation of linear momentum balance is $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ or, if there is no force, $\vec{\boldsymbol{a}} = \vec{\boldsymbol{0}}$, which means that $d\vec{\boldsymbol{v}}/dt = \vec{\boldsymbol{0}}$. So $\vec{\boldsymbol{v}}$ is a constant. We can call this constant $\vec{\boldsymbol{v}}_0$. So after some time the particle is where it was at $t = 0$, say, $\vec{\boldsymbol{r}}_0$, plus its velocity $\vec{\boldsymbol{v}}_0$ times time. That is:
$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_0 + \vec{\boldsymbol{v}}_0 t. \tag{11.1}$$

This vector relation is a parametric equation for a straight line. The particle moves in a straight line, as expected.

Cartesian coordinates

If instead we break the linear momentum balance equation into cartesian coordinates we get
$$F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}} = m(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}).$$

Because the net force is zero and the net mass is not negligible,
$$\ddot{x} = 0, \qquad \ddot{y} = 0, \text{ and} \qquad \ddot{z} = 0.$$

These equations imply that $\dot{x}$, $\dot{y}$, and $\dot{z}$ are all constants, lets call them v_{x0}, v_{y0}, v_{z0}. So x, y, and z are given by
$$x = x_0 + v_{x0}t, \quad y = y_0 + v_{y0}t, \quad \& \; z = z_0 + v_{z0}t.$$

We can put these components into their place in vector form to get:
$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} = (x_0 + v_{x0}t)\hat{\boldsymbol{\imath}} + (y_0 + v_{y0}t)\hat{\boldsymbol{\jmath}} + (z_0 + v_{z0}t)\hat{\boldsymbol{k}}. \tag{11.2}$$

Note that there are six free constants in this equation representing the initial position and velocity. Equation 11.2 is a cartesian representation of equation 11.1; it describes a straight line being traversed at constant rate.

Particle with no force: Polar/cylindrical coordinates

When there is no force, in polar coordinates we have:

$$\underbrace{F_R}_{0}\hat{\boldsymbol{e}}_R + \underbrace{F_\theta}_{0}\hat{\boldsymbol{e}}_\theta + \underbrace{F_z}_{0}\hat{\boldsymbol{k}} = m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}].$$

This vector equation leads to the following three scalar differential equations, the first two of which are coupled non-linear equations (neither can be solved without the other).

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ 2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0 \\ \ddot{z} &= 0 \end{aligned}$$

A tedious calculation will show that these equations are solved by the following functions of time:

$$\begin{aligned} R &= \sqrt{d^2 + [v_0(t - t_0)]^2} \\ \theta &= \theta_0 + \tan^{-1}[v_0(t - t_0)/d] \\ z &= z_0 + v_{z0}t, \end{aligned} \tag{11.3}$$

where $\theta_0, d, t_0, v_0, z_0$, and v_{z0} are constants. Note that, though eqn. 11.4 looks different than equation 11.2, there are still 6 free constants. From the physical interpretation you know that equation 11.4 must be the parametric equation of a straight line. And, indeed, you can verify that picking arbitrary constants and using a computer to make a polar plot of equation 11.4 does in fact show a straight line. From equation 11.4 it seems that polar coordinates' main function is to obfuscate rather than clarify. For the simple case that a particle moves with no force at all, we have to solve non-linear differential equations whereas using cartesian coordinates we get linear equations which are easy to solve and where the solution is easy to interpret.

But, if we add a central force, a force like earth's gravity acting on an orbiting satellite (the force on the satellite is directed towards the center of the earth), the equations become almost intolerable in cartesian coordinates. But, in polar coordinates, the solution is almost as easy (which is not all that easy for most of us) as the solution 11.4. So the classic solutions of celestial mechanics are usually expressed in terms of polar coordinates.

Particle with no force: Path coordinates

When there is no force, $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ is expressed in path coordinates as

$$\underbrace{F_t}_{0}\hat{\boldsymbol{e}}_t + \underbrace{F_n}_{0}\hat{\boldsymbol{e}}_n = m(\dot{v}\hat{\boldsymbol{e}}_t + \underbrace{(v^2/\rho)\hat{\boldsymbol{e}}_n}_{v^2\vec{\boldsymbol{\kappa}}}).$$

That is,

$$\dot{v} = 0 \quad \text{and} \quad v^2/\rho = 0.$$

So the speed v must be constant and the radius of curvature ρ of the path infinite. That is, the particle moves at constant speed in a straight line.

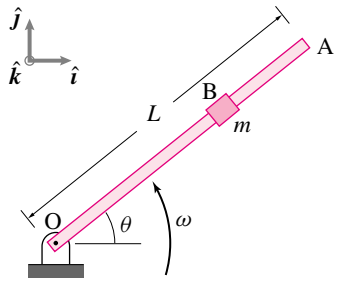


Figure 11.1: A collar slides on a rough bar and finally shoots off the end of the bar as the bar rotates with constant angular speed.

(Filename:sfig6.5.1)

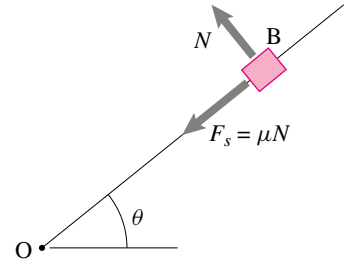


Figure 11.2: Free Body Diagram of the collar. The only forces on the collar are the interaction forces of the bar, which are the normal force N and the friction force $F_s = \mu N$.

(Filename:sfig6.5.1a)

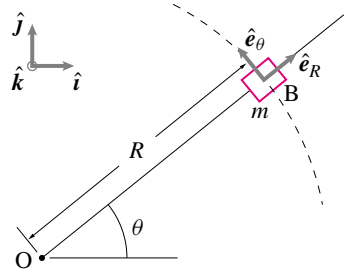


Figure 11.3: Geometry of the collar position at an arbitrary time during its slide on the rod.

(Filename:sfig6.5.1b)

SAMPLE 11.1 *A collar sliding on a rough rod.* A collar of mass $m = 0.5$ lb slides on a massless rigid rod OA of length $L = 8$ ft. The rod rotates counterclockwise with a constant angular speed $\dot{\theta} = 5$ rad/s. The coefficient of friction between the rod and the collar is $\mu = 0.3$. At time $t = 0$ s, the bar is horizontal and the collar is at rest at 1 ft from the center of rotation O. Ignore gravity.

(a) How does the position of the collar change with time (*i.e.*, what is the equation of motion of the rod)?

(b) Plot the path of the collar starting from $t = 0$ s till the collar shoots off the end of the bar.

(c) How long does it take for the collar to leave the bar?

Solution

(a) First, we draw a Free Body Diagram of the the collar at a general position $(R,\ \theta)$. The FBD is shown in Fig. 11.2 and the geometry of the position vector and basis vectors is shown in Fig. 11.3. In the Free Body Diagram there are only two forces acting on the collar (forces exerted by the bar) — the normal force $\vec{N} = N\hat{e}_\theta$ acting normal to the rod and the force of friction $\vec{F}_s = -\mu N\hat{e}_R$ acting along the rod. Now, we can write the linear momentum balance for the collar:

$$\begin{aligned} \sum \vec{F} &= m\vec{a} \qquad \text{or} \\ -\mu N\hat{e}_R + N\hat{e}_\theta &= m[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\underbrace{\ddot{\theta}}_{0})\hat{e}_\theta \quad (11.4) \end{aligned}$$

Note that $\ddot{\theta} = 0$ because the rod is rotating at a constant rate. Now dotting both sides of Eqn. (11.4) with $\hat{e}_R$ and $\hat{e}_\theta$ we get

$$\begin{aligned} [\text{Eqn. (11.4)}] \cdot \hat{e}_R \qquad &\Rightarrow \qquad -\mu N = m(\ddot{R} - R\dot{\theta}^2) \\ &\text{or} \qquad \ddot{R} - R\dot{\theta}^2 = -\frac{\mu N}{m}, \end{aligned}$$

$$[\text{Eqn. (11.4)}] \cdot \hat{e}_\theta \quad \Rightarrow \quad N = 2m\dot{R}\dot{\theta}.$$

Eliminating N from the last two equations we get

$$\ddot{R} + 2\mu\dot{\theta}\ \dot{R} - \dot{\theta}^2\ R = 0.$$

Since $\dot{\theta} = \omega$ is constant, the above equation is of the form

$$\ddot{R} + C\ \dot{R} - \omega^2\ R = 0 \qquad (11.5)$$

where $C = 2\mu\omega$ and $\omega = \dot{\theta}$.

Solution of equation (11.5): The characteristic equation associated with Eqn. (11.5) (time to pull out your math books and see the solution of ODEs) is

$$\begin{aligned} \lambda^2 + C\lambda - \omega^2 &= 0 \\ \Rightarrow \qquad \lambda &= \frac{-C \pm \sqrt{C^2 + 4\omega^2}}{2} \\ &= \omega(-\mu \pm \sqrt{\mu^2 + 1}). \end{aligned}$$

Therefore, the solution of Eqn. (11.5) is

$$\begin{aligned} R(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ &= Ae^{\omega(-\mu+\sqrt{\mu^2+1})t} + Be^{\omega(-\mu-\sqrt{\mu^2+1})t}. \end{aligned}$$

Substituting the given initial conditions: $R(0) = 1\,\text{ft}$ and $\dot{R}(0) = 0$ we get

$$R(t) = \frac{1\,\text{ft}}{2}\left[e^{\omega(-\mu+\sqrt{\mu^2+1})t} + e^{\omega(-\mu-\sqrt{\mu^2+1})t}\right]. \qquad (11.6)$$

$$\boxed{R(t) = \tfrac{1\,\text{ft}}{2}\left[e^{\omega(-\mu+\sqrt{\mu^2+1})t} + e^{\omega(-\mu-\sqrt{\mu^2+1})t}\right].}$$

(b) To draw the path of the collar we need both R and θ. SInce $\dot{\theta} = 5\,\text{rad/s} =$ constant,

$$\theta = \dot{\theta}\,t = (5\,\text{rad/s})\,t.$$

Now we can take various values of t from 0 s to, say, 5 s, and calculate values of θ and R. Plotting all these values of R and θ, however, does not give us an entirely correct path of the collar, since the equation for $R(t)$ is valid only till $R =$ length of the bar $= 8\,\text{ft}$. We, therefore, need to find the final time t_f such that $R(t_f) = 8\,\text{ft}$. Equation (11.6) is a nonlinear algebraic equation which is hard to solve for t. We can, however, solve the equation iteratively on a computer, or with some patience, even on a calculator using trial and error. Here is a MATLAB script which finds t_f and plots the path of the collar from $t = 0\,\text{s}$ to $t = t_f$:

```
tf = fzero('slidebar',2);   % run built-in function fzero
                            % to find a zero of function
                            % 'slidebar' near t = 2 sec.

t = 0:tf/100:tf;            % take 101 time steps from 0 to tf
r0 = 1; w = 5; mu = .3;     % initialize variables

f1 = -mu + sqrt(mu^2 +1);   % first partial exponent
f2 = -mu - sqrt(mu^2 +1);   % second partial exponent
r = 0.5*r0*(exp(f1*t) + exp(f2*t));  % calculate r for all t
theta = w*t;                 % calculate theta
polar(theta,r), grid         % make polar plot and put grid
title('Polar....')           % put a title
hold on                      % hold the current plot
polar(theta(1),r(1),'o')     % mark the first point by a 'o'
polar(theta(101),r(101),'*') % mark the last point by a '*'
```

The user written function `slidebar` is as follows.

```
function delr = slidebar(t);
% ------------------------------------------
%    function delr = slidebar(t);
% this function returns the difference between
% r and rf (=8 in problem) for any given t
% ------------------------------------------

r0 = 1; w = 5; mu = .3; rf = 8;
f1 = -mu + sqrt(mu^2 +1);
f2 = -mu - sqrt(mu^2 +1);
delr = 0.5*r0*(exp(f1*t) + exp(f2*t)) - rf;
```

The plot produced by MATLAB is shown in Fig. 11.4.

(c) The time computed by MATLAB's built-in function `fzero` was

$$t_f = 3.7259\,\text{s}.$$

By plugging this value in the expression for $R(t)$ (Eqn. (11.6) we get, indeed,

$$R = 8\,\text{ft}.$$

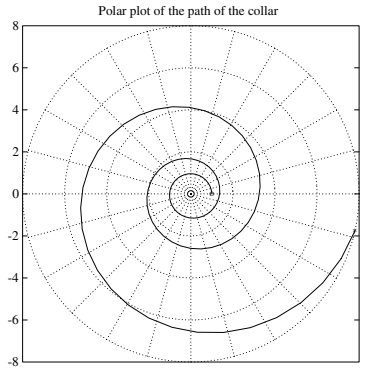


Figure 11.4: Plot of the path of the collar till it leaves the rod.

(Filename:slidebar)

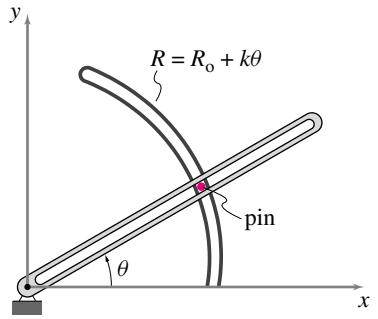


Figure 11.5: A pin is constrained to move in a groove and a slotted arm.

(Filename:sfig6.2.2)

SAMPLE 11.2 *Constrained motion of a pin.* During a small interval of its motion, a pin of 100 grams is constrained to move in a groove described by the equation $R = R_0 + k\theta$ where $R_0 = 0.3\,\text{m}$ and $k = 0.05\,\text{m}$. The pin is driven by a slotted arm AB and is free to slide along the arm in the slot. The arm rotates at a constant speed $\omega = 6\,\text{rad/s}$. Find the magnitude of the force on the pin at $\theta = 60^\text{o}$.

Solution Let $\vec{\boldsymbol{F}}$ denote the net force on the pin. Then from the linear momentum balance

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

where $\vec{\boldsymbol{a}}$ is the acceleration of the pin. Therefore, to find the force at $\theta = 60^\text{o}$ we need to find the acceleration at that position.

From the given figure, we assume that the pin is in the groove at $\theta = 60^\text{o}$. Since the equation of the groove (and hence the path of the pin) is given in polar coordinates, it seems natural to use polar coordinate formula for the acceleration. For planar motion, the acceleration is

$$a = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta.$$

We are given that $\dot{\theta} \equiv \omega = 6\,\text{rad/s}$ and the radial position of the pin $R = R_0 + k\,\theta$. Therefore, ①

$$\begin{aligned}
\ddot{\theta} &= \frac{d\dot{\theta}}{dt} = 0 \quad (\text{ since } \dot{\theta} = \text{constant}) \\
\dot{R} &= \frac{d}{dt}(R_0 + k\theta) = k\dot{\theta} \quad \text{and} \\
\ddot{R} &= k\ddot{\theta} = 0.
\end{aligned}$$

① Note that R is a function of θ and θ is a function of time, therefore R is a function of time. Although we are interested in finding $\dot{R}$ and $\ddot{R}$ at $\theta = 60^\text{o}$, we cannot first substitute $\theta = 60^\text{o}$ in the expression for R and then take its time derivatives (which will be zero).

Substituting these expressions in the acceleration formula and then substituting the numerical values at $\theta = 60^\text{o}$, (remember, θ must be in radians!), we get

$$\begin{aligned}
\vec{\boldsymbol{a}} &= -\overbrace{(R_0 + k\theta)\dot{\theta}^2}^{R\dot{\theta}^2}\,\hat{\boldsymbol{e}}_R + \overbrace{2k\dot{\theta}^2}^{2\dot{R}\dot{\theta}}\,\hat{\boldsymbol{e}}_\theta \\
&= -(0.3\,\text{m} + 0.05\,\text{m} \cdot \frac{\pi}{3}) \cdot (6\,\text{rad/s})^2\,\hat{\boldsymbol{e}}_R + 2 \cdot 0.3\,\text{m} \cdot (6\,\text{rad/s})^2\,\hat{\boldsymbol{e}}_\theta \\
&= -13.63\,\text{m/s}^2\hat{\boldsymbol{e}}_R + 21.60\,\text{m/s}^2\hat{\boldsymbol{e}}_\theta.
\end{aligned}$$

Therefore the net force on the pin is

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
&= 0.1\,\text{kg} \cdot (-13.63\hat{\boldsymbol{e}}_R + 21.60\hat{\boldsymbol{e}}_\theta)\,\text{m/s}^2 \\
&= (-1.36\hat{\boldsymbol{e}}_R + 2.16\hat{\boldsymbol{e}}_\theta)\,\text{N}
\end{aligned}$$

and the magnitude of the net force is

$$F = |\vec{\boldsymbol{F}}| = \sqrt{(1.36\,\text{N})^2 + (2.16\,\text{N})^2} = 2.55\,\text{N}.$$

$$\boxed{F = 2.55\,\text{N}}$$

11.2 Dynamics of rigid bodies in one-degree-of-freedom 2-D mechanisms

Energy method: single degree of freedom systems

The preponderance of systems where vibrations occur is not due to the fact that so many systems look like a spring connected to a mass, a simple pendulum, or a torsional oscillator. Instead there is a general class of systems which can be expected to vibrate sinusoidally near some equilibrium position. These systems are one-degree -of-freedom (one DOF) near an energy minimum.

Imagine a complex machine that only has one degree of freedom, meaning the position of the whole machine is determined by a single number q. Further assume that the machine has no motion when $dotq = 0$. The variable q could be, for example, the angle of one of the linked-together machine parts. Also, assume that the machine has no dissipative parts: no friction, no collisions, no inelastic deformation. Now because a single number q characterizes the position of all of the parts of the system we can calculate the potential energy of the system as a function of q,

$$E_{\mathrm{P}} = E_{\mathrm{P}}(q).$$

We find this function by adding up the potential energies of all the springs in the machine and the gravitational potential energy. Similarly we can write the system's kinetic energy in terms of q and it's rate of change $\dot{q}$. Because at any configuration the velocity of every point in the system is proportional to $\dot{q}$ we can write the kinetic energy as:

$$E_{\mathrm{K}} = M(q)\dot{q}^2/2$$

where $M(q)$ is a function that one can determine by calculating the machine's total kinetic energy in terms of q and $\dot{q}$ and then factoring $\dot{q}^2$ out of the resulting expression.

Now, if we accept the equation of mechanical energy conservation we have

$$\begin{aligned}
\text{constant} &= E_{\mathrm{T}} && \text{by conservation of energy,}\\
\Rightarrow\ 0 &= \frac{d}{dt}E_{\mathrm{T}} && \text{taking one time derivative,}\\
&= \frac{d}{dt}[E_{\mathrm{P}} + E_{\mathrm{K}}] && \text{breaking energy into total potential, plus kinetic}\\
&= \frac{d}{dt}[E_{\mathrm{P}}(q) + \frac{1}{2}M(q)\dot{q}^2] && \text{substituting from paragraphs above}
\end{aligned}$$

so,

$$0 = \frac{d}{dq}[E_{\mathrm{P}}(q)]\,\dot{q} + \frac{1}{2}\frac{d}{dq}[M(q)]\dot{q}\dot{q}^2 + M(q)\dot{q}\ddot{q} \qquad (11.7)$$

This expression is starting to get complicated because when we take the time derivative of a function of $M(q)$ and $E_{\mathrm{P}}(q)$ we have to use the chain rule. Also, because we have products of terms, we had to use the product rule. Equation(11.7) is the general equation of motion of a conservative one-degree-of-freedom system. It is really just a special case of the equation of motion for one-degree-of-freedom systems found from power balance. In order to specialize to the case of oscillations, we want to look at this system near a stable equilibrium point or potential energy minimum.

At a potential energy minimum we have, as you will recall from 'max-min' problems in calculus, that $dE_{\mathrm{P}}(q)/dq = 0$. To keep our notation simple, let's

assume that we have defined q so that $q = 0$ at this minimum. Physically this means that q measures how far the system is from its equilibrium position. That means that if we take a Taylor series approximation of the potential energy the expression for potential energy can be expressed as follows:

$$E_{\mathrm{P}} \approx \text{const} + \underbrace{\frac{dE_{\mathrm{P}}}{dq}}_{0} \cdot q + \frac{1}{2} \underbrace{\frac{d^2E_{\mathrm{P}}}{dq^2}}_{K_{\text{equiv}}} \cdot q^2 + \ldots \tag{11.8}$$

$$\Rightarrow \qquad \frac{dE_{\mathrm{P}}}{dq} \approx K_{\text{equiv}} \cdot q \tag{11.9}$$

Applying this result to equation(11.7), canceling a common factor of $\dot{q}$, we get: ①

$$0 = K_{\text{equiv}} q + \frac{1}{2}\frac{d}{dq}[M(q)]\dot{q}^2 + M(q)\ddot{q}. \tag{11.10}$$

We now write $M(q)$ in terms of its Taylor series. We have

$$M(q) = M(0) + dM/dq|_0 \cdot q + \ldots \tag{11.11}$$

and substitute this result into equation 11.10. We have not finished using our assumption that we are only going to look at motions that are close to the equilibrium position $q = 0$ where q is small. The nature of motion close to an equilibrium is that when the deflections are small, the rates and accelerations are also small. Thus, to be consistent in our approximation we should neglect any terms that involve products of $q, \dot{q},$ or $\ddot{q}$. Thus the middle term involving $\dot{q}^2$ is negligibly smaller than other terms. Similarly, using the Taylor series for $M(q)$, the last term is well approximated by $M(0)\ddot{q}$, where $M(0)$ is a constant which we will call M_{equiv}. Now we have for the equation of motion:

$$0 = \frac{d}{dt}E_{\mathrm{T}} \qquad \Rightarrow \qquad 0 = K_{\text{equiv}} q + M_{\text{equiv}}\ddot{q}, \tag{11.12}$$

which you should recognize as the harmonic oscillator equation. So we have found that for any energy conserving one degree of freedom system near a position of stable equilibrium, the equation governing small motions is the harmonic oscillator equation. The effective stiffness is found from the potential energy by $K_{\text{equiv}} = d^2E_{\mathrm{P}}/dq^2$ and the effective mass is the coefficient of $\dot{q}^2/2$ in the expansion for the kinetic energy E_{K}. The displacement of any part of the system from equilibrium will thus be given by

$$A\sin(\lambda t) + B\cos(\lambda t) \tag{11.13}$$

with $\lambda^2 = K_{\text{equiv}}/M_{\text{equiv}}$, and A and B determined by the initial conditions. So we have found that *all stable non-dissipative one-degree-of-freedom systems oscillate when disturbed slightly from equilibrium* and we have found how to calculate the frequency of vibration.

① The cancellation of the factor $\dot{q}$ from equation 11.7 depends on $\dot{q}$ being other than zero. During oscillatory motion $\dot{q}$ is generally not zero. Strictly we cannot cancel the $\dot{q}$ term from the equation at the instants when $\dot{q} = 0$. However, to say that a differential equation is true except for certain instants in time is, in practice to say that it is always true, at least if we make reasonable assumptions about the smoothness of the motions.

More examples of harmonic oscillators

In the previous section, we have shown that any non-dissipative one-degree-of-freedom system that is near a potential energy minimum can be expected to have simple harmonic motion. Besides the three examples we have given so far, namely,

- a spring and mass,
- a simple pendulum, and

- a rigid body and a torsional spring,

there are examples that are somewhat more complex, such as

- a cylinder rolling near the bottom of a valley,
- a cart rolling near the bottom of a valley, and a
- a four bar linkage swinging freely near its energy minimum.

The restriction of this theory to systems with only one-degree-of-freedom is not so bad as it seems at first sight. First of all, it turns out that simple harmonic motion is important for systems with multiple-degrees-of-freedom. We will discuss this generalization in more detail later with regard to normal modes. Secondly, one can also get a good understanding of a vibrating system with multiple-degrees-of-freedom by modeling it as if it has only one-degree-of-freedom.

Cylinder rolling in a valley

Consider the uniform cylinder with radius r rolling without slip in an cylindrical 'ideal' valley of radius R.

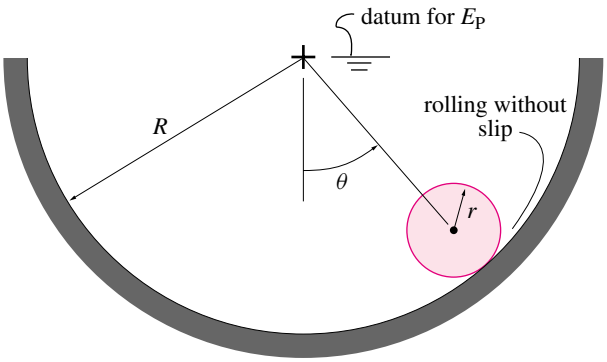


Figure 11.6: Cylinder rolling without slip in a cylinder. (Filename:tfigure12.bigcyl.smallcyl)

For this problem we can calculate E_k and E_p in terms of θ. Skipping the details,

$$\begin{aligned} E_p &= -mg(R-r)\cos\theta \\ E_k &= \frac{1}{2}\left(\frac{3}{2}mr^2\right)\left(\frac{\dot\theta(R-r)}{r}\right)^2 \\ &= \frac{3}{4}m(R-r)^2\dot\theta^2 \end{aligned}$$

So we can derive the equation of motion using the fact of constant total energy.

$$\begin{aligned} 0 &= \frac{d}{dt}(E_T) \\ &= \frac{d}{dt}(E_p + E_k) \\ &= \frac{d}{dt}\left(\underbrace{-mg(R-r)\cos\theta}_{E_p} + \underbrace{\frac{3}{4}m(R-r)^2\dot\theta^2}_{E_k}\right) \\ &= (mg(R-r)\sin\theta)\dot\theta + \frac{3}{2}(R-r)^2\dot\theta\ddot\theta \\ \Rightarrow \quad 0 &= mg(R-r)\sin\theta + \frac{3}{2}(R-r)^2 m\ddot\theta \end{aligned}$$

Now, assuming small angles, so $\theta \approx \sin\theta$, we get

$$g(R-r)\theta + \frac{3}{2}(R-r)^2\ddot{\theta} = 0 \tag{11.14}$$

$$\theta + \underbrace{\left(\frac{2}{3}\frac{g}{(R-r)}\right)}_{\lambda^2}\ddot{\theta} = 0 \tag{11.15}$$

This equation is our old friend the harmonic oscillator equation, as expected.

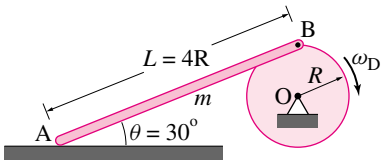


Figure 11.7: End A of bar AB is free to slide on the frictionless horizontal surface while end B is going in circles with a disk rotating at a constant rate.

(Filename:sfig7.3.2)

SAMPLE 11.3 *A zero degree of freedom system.* A uniform rigid rod AB of mass m and length $L = 4R$ has one of its ends pinned to the rim of a disk of radius R. The other end of the bar is free to slide on a frictionless horizontal surface. A motor, connected to the center of the disk at O, keeps the disk rotating at a constant angular speed ω_D. At the instant shown, end B of the rod is directly above the center of the disk making θ to be 30^o.

(a) Find all the forces acting on the rod.

(b) Is there a value of ω_D which makes end A of the rod lift off the horizontal surface when $\theta = 30^o$?

Solution The disk is rotating at constant speed. Since end B of the rod is pinned to the disk, end B is going in circles at constant rate. The motion of end B of the rod is completely prescribed. Since end A can only move horizontally (assuming it has not lifted off yet), the orientation (and hence the position of each point) of the rod is completely determined at any instant during the motion. Therefore, the rod represents a zero degree of freedom system.

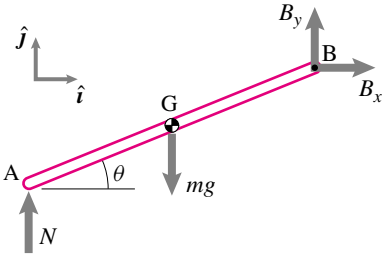


Figure 11.8: Free body diagram of the bar.

(Filename:sfig7.3.2a)

(a) **Forces on the rod:** The free body diagram of the rod is shown in Fig. 11.8. The pin at B exerts two forces B_x and B_y while the surface in contact at A exerts only a normal force N because there is no friction. Now, we can write the momentum balance equations for the rod. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the rod gives

$$B_x\hat{i} + (B_y + N - mg)\hat{j} = m\vec{a}_G. \tag{11.16}$$

The angular momentum balance about the center of mass G ($\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$) of the rod gives

$$\vec{r}_{A/G} \times N\hat{j} + \vec{r}_{B/G} \times (B_x\hat{i} + B_y\hat{j}) = I_{zz/G}\,\alpha_{rod}\hat{k}. \tag{11.17}$$

From these two vector equations we can get three scalar equations (the Angular Momentum Balance gives only one scalar equation in 2-D since the quantities on both sides of the equation are only in the $\hat{k}$ direction), but we have six unknowns — B_x, B_y, N, $\vec{a}_G$ (counts as two unknowns), and α_{rod}. Therefore, we need more equations. We have already used the momentum balance equations, hence, the extra equations have to come from kinematics.

$$\begin{aligned} \vec{v}_A &= \vec{v}_B + \overbrace{\vec{\omega}_{rod} \times \vec{r}_{A/B}}^{\vec{v}_{A/B}} \\ \text{or} \quad v_A\hat{i} &= \omega_D R\hat{i} + \omega_{rod}\hat{k} \times L(-\cos\theta\hat{i} - \sin\theta\hat{j}) \\ &= (\omega_D R + \omega_{rod} L\sin\theta)\hat{i} - \omega_{rod} L\cos\theta\,\hat{j} \end{aligned}$$

Dotting both sides of the equation with $\hat{j}$ we get

$$0 = \omega_{rod} L\cos\theta \qquad \Rightarrow \qquad \omega_{rod} = 0.$$

Also,

$$\begin{aligned} \vec{a}_A &= \vec{a}_B + \overbrace{\dot{\vec{\omega}} \times \vec{r}_{A/B} + \underbrace{\vec{\omega}_{rod}}_{\vec{0}} \times (\vec{\omega}_{rod} \times \vec{r}_{A/B})}^{\vec{a}_{A/B}} \\ \text{or} \quad a_A\hat{i} &= -\omega_D^2 R\hat{j} + \dot{\omega}_{rod}\hat{k} \times L(-\cos\theta\hat{i} - \sin\theta\hat{j}) \\ &= -(\omega_D^2 R + \dot{\omega}_{rod} L\cos\theta)\hat{j} + \dot{\omega}_{rod} L\sin\theta\hat{i}. \end{aligned}$$

Dotting both sides of this equation by $\hat{\boldsymbol{j}}$ we get

$$\dot{\omega}_{rod} = -\frac{\omega_D^2 R}{L\cos\theta}. \tag{11.18}$$

Now, we can find the acceleration of the center of mass:

$$\begin{aligned}
\vec{\boldsymbol{a}}_G &= \vec{\boldsymbol{a}}_B + \overbrace{\dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{G/B} + \underbrace{\vec{\boldsymbol{\omega}}_{rod}}_{\vec{\boldsymbol{0}}} \times (\vec{\boldsymbol{\omega}}_{rod} \times \vec{\boldsymbol{r}}_{G/B})}^{\vec{\boldsymbol{a}}_{G/B}} \\
&= -\omega_D^2 R\hat{\boldsymbol{j}} + \dot{\omega}_{rod}\hat{\boldsymbol{k}} \times \frac{1}{2}L(-\cos\theta\hat{\boldsymbol{i}} - \sin\theta\hat{\boldsymbol{j}}) \\
&= -(\omega_D^2 R + \frac{1}{2}\dot{\omega}_{rod}L\cos\theta)\hat{\boldsymbol{j}} + \frac{1}{2}\dot{\omega}_{rod}L\sin\theta\hat{\boldsymbol{i}}.
\end{aligned}$$

Substituting for $\dot{\omega}_{rod}$ from eqn. (11.18) and 30^o for θ above, we obtain

$$\vec{\boldsymbol{a}}_G = -\frac{1}{2}\omega_D^2 R(\frac{1}{\sqrt{3}}\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}).$$

Substituting this expression for $\vec{\boldsymbol{a}}_G$ in eqn. (11.16) and dotting both sides by $\hat{\boldsymbol{i}}$ and then by $\hat{\boldsymbol{j}}$ we get

$$\begin{aligned}
B_x &= -\frac{1}{2\sqrt{3}}m\omega_D^2 R, \\
B_y + N &= -\frac{1}{2}m\omega_D^2 R + mg
\end{aligned} \tag{11.19}$$

From eqn. (11.17)

$$\begin{aligned}
\frac{1}{2}L[(B_y - N)\cos\theta - B_x\sin\theta]\hat{\boldsymbol{k}} &= \frac{1}{12}mL^2(-\frac{\omega_D^2 R}{L\cos\theta})\hat{\boldsymbol{k}} \\
\text{or} \qquad B_y - N &= -\frac{1}{6}\frac{m\omega_D^2 R}{\cos^2\theta} + B_x\tan\theta \\
&= -\frac{2}{9}m\omega_D^2 R - \frac{1}{6}m\omega_D^2 R \\
&= -\frac{7}{18}m\omega_D^2 R
\end{aligned} \tag{11.20}$$

From eqns. (11.19) and (11.20)

$$B_y = \frac{1}{2}\left(mg - \frac{8}{9}m\omega_D^2 R\right)$$

and

$$N = \frac{1}{2}\left(mg - \frac{1}{9}m\omega_D^2 R\right).$$

(b) **Lift off of end A:** End A of the rod loses contact with the ground when normal force N becomes zero. From the expression for N from above, this condition is satisfied when

$$\begin{aligned}
\frac{2}{9}m\omega_D^2 R &= mg \\
\Rightarrow \qquad \omega_D &= 3\sqrt{\frac{g}{R}}.
\end{aligned}$$

11.3 Dynamics of rigid bodies in multi-degree-of-freedom 2-D mechanisms

To solve problems with multiple degrees of freedom the basic strategy is to

- draw FBDs of each body
- find simple variables to describe the configurations of the bodies
- write linear and angular momentum balance equations
- solve the equations for variables of interest (forces, second derivatives of the configuration variables).
- set up and solve the resulting differential equations if you are trying to find the motion.

Basically, however, the skills are the same as for simpler systems, the execution is just more complex.

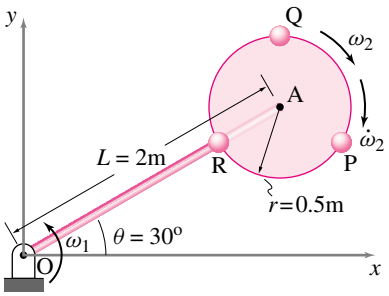
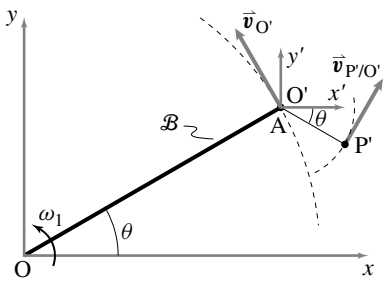


Figure 11.9: (Filename:sfig8.2.2again)



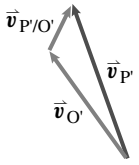


Figure 11.10: The velocity of point P' is the sum of two terms: the velocity of O' and the velocity of P' relative to O'.

(Filename:sfig8.2.2d)

SAMPLE 11.4 *Dynamics using a rotating and translating coordinate system.* Consider the rotating wheel of Sample 10.11 which is shown here again in Figure 11.9. At the instant shown in the figure find

(a) the linear momentum of the mass P and

(b) the net force on the mass P.

For calculations, use a frame $\mathcal{B}$ attached to the rod and a coordinate system in $\mathcal{B}$ with origin at point A of the rod OA.

Solution We attach a frame $\mathcal{B}$ to the rod. We choose a coordinate system $x'y'z'$ in this frame with its origin O′ at point A. We also choose the orientation of the primed coordinate system to be parallel to the fixed coordinate system xyz (see Fig. 11.10), *i.e.*, $\hat{\imath}' = \hat{\imath}, \quad \hat{\jmath}' = \hat{\jmath}, \quad$ and $\quad \hat{k}' = \hat{k}$.

(a) **Linear momentum of P:** The linear momentum of the mass P is given by

$$\vec{L} = m\vec{v}_P.$$

Clearly, we need to calculate the velocity of point P to find $\vec{L}$. Now,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}} = \underbrace{\vec{v}_{O'} + \vec{v}_{P'/O'}}_{\vec{v}_{P'}} + \vec{v}_{\text{rel}}.$$

Note that O′ and P′ are two points on the same (imaginary) rigid body OAP′. Therefore, we can find $\vec{v}_{P'}$ as follows:

$$\begin{aligned}
\vec{v}_{P'} &= \overbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}}^{\vec{v}_{O'}} + \overbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}}^{\vec{v}_{P'/O'}} \\
&= \omega_1\hat{k} \times L(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) + \omega_1\hat{k} \times r(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= \omega_1[(L+r)\cos\theta\hat{\jmath} - (L-r)\sin\theta\hat{\imath}] \\
&= 3\,\text{rad/s} \cdot [2.5\,\text{m} \cdot \cos 30^o\hat{\jmath} - 1.5\,\text{m} \cdot \sin 30^o\hat{\imath}] \\
&= (6.50\hat{\jmath} - 2.25\hat{\imath})\,\text{m/s} \quad \text{(same as in Sample 10.11.)},
\end{aligned}$$

$$\begin{aligned}
\vec{v}_{\text{rel}} &= \vec{v}_{P/\mathcal{B}} \\
&= -\omega_2\hat{k}' \times r(\cos\theta\hat{\imath}' - \sin\theta\hat{\jmath}') \\
&= -\omega_2 r(\cos\theta\hat{\jmath}' + \sin\theta\hat{\imath}') \\
&= -(2.16\hat{\jmath}' + 1.25\hat{\imath}')\,\text{m/s} \\
&= -(2.16\hat{\jmath} + 1.25\hat{\imath})\,\text{m/s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{v}_P &= \vec{v}_{P'} + \vec{v}_{\text{rel}} \\
&= (4.34\hat{\jmath} - 3.50\hat{\imath})\,\text{m/s} \qquad \text{and} \\
\vec{L} &= m\vec{v}_P \\
&= 0.5\,\text{kg} \cdot (4.34\hat{\jmath} - 3.50\hat{\imath})\,\text{m/s} \\
&= (-1.75\hat{\imath} + 2.17\hat{\jmath})\,\text{kg·m/s}.
\end{aligned}$$

$$\boxed{\vec{L} = (-1.75\hat{\imath} + 2.17\hat{\jmath})\,\text{kg·m/s}}$$

(b) **Net force on P:** From the

$$\sum \vec{F} = m\vec{a}$$

for the mass P we get $\sum \vec{F} = m\vec{a}_P$. Thus to find the net force $\sum \vec{F}$ we need to find $\vec{a}_P$. The calculation of $\vec{a}_P$ is the same as in Sample 10.11 except that $\vec{a}_{P'}$ is now calculated from

$$\vec{a}_{P'} = \vec{a}_{O'} + \vec{a}_{P'/O'}$$

where

$$\begin{aligned}
\vec{a}_{O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}) \\
&= -\omega_1^2 \vec{r}_{O'/O} \\
&= -\omega_1^2 L(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
&= -(3\,\text{rad/s})^2 \cdot 2\,\text{m} \cdot (\cos 30^\text{o}\hat{\imath} + \sin 30^\text{o}\hat{\jmath}) \\
&= -(15.59\hat{\imath} + 9.00\hat{\jmath})\,\text{m/s}^2,
\end{aligned}$$

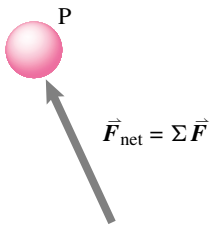


Figure 11.11: (Filename:sfig8.2.2again1)

$$\begin{aligned}
\vec{a}_{P'/O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}) \\
&= -\omega_1^2 \vec{r}_{P'/O'} \\
&= -\omega_1^2 r(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= -(3\,\text{rad/s})^2 \cdot 0.5\,\text{m} \cdot (\cos 30^\text{o}\hat{\imath} - \sin 30^\text{o}\hat{\jmath}) \\
&= -(3.90\hat{\imath} - 2.25\hat{\jmath})\,\text{m/s}^2.
\end{aligned}$$

Thus,

$$\vec{a}_{P'} = -(19.49\hat{\imath} + 6.75\hat{\jmath})\,\text{m/s}^2$$

which, of course, is the same as calculated in Sample 10.11. The other two terms, $\vec{a}_{\text{cor}}$ and $\vec{a}_{\text{rel}}$, are exactly the same as in Sample 10.11. Therefore, we get the same value for $\vec{a}_P$ by adding the three terms:

$$\vec{a}_P = -(17.83\hat{\imath} + 3.63\hat{\jmath})\,\text{m/s}^2.$$

The net force on P is

$$\begin{aligned}
\sum \vec{F} &= m\vec{a}_P \\
&= 0.5\,\text{kg} \cdot (-17.83\hat{\imath} - 3.63\hat{\jmath})\,\text{m/s}^2 \\
&= -(8.92\hat{\imath} + 1.81\hat{\jmath})\,\text{N}.
\end{aligned}$$

$$\boxed{\sum \vec{F} = -(8.92\hat{\imath} + 1.81\hat{\jmath})\,\text{N}}$$

11.4 Advance dynamics of planar motion

For more and more complex problems no new equations or principles are needed. There is a separate section here because less advanced classes may want to skip it.

A separate batch of harder problems goes with this section.

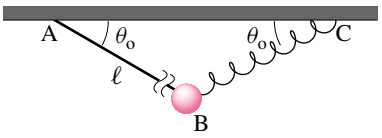
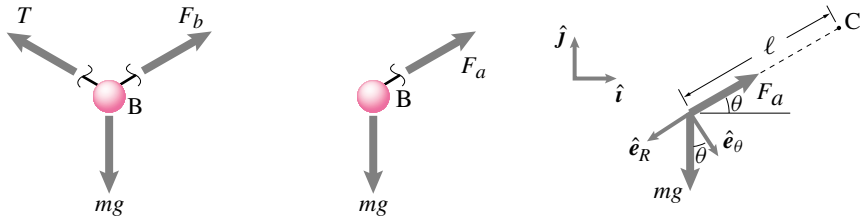


Figure 11.12: A small ball of mass m is supported by a string and a zero length (in relaxed position) spring. The string is suddenly cut.

(Filename:sfig6.4.2)

SAMPLE 11.5 *Zero length springs do interesting things.* A small ball of mass m is supported by a string AB of length ℓ and a spring BC with spring stiffness k. The spring is relaxed when the mass is at C (BC is a *zero length* spring). The spring and the string make the same angle θ with the horizontal in the static equilibrium of the mass. At this position, the string is suddenly cut near the mass point B. Find the resulting motion of the mass.

Solution The Free Body Diagrams of the mass are shown in Fig. 11.13(a) and (b) before and after the string is cut. Since the stretch in the spring, in the static equilibrium position of the mass, is equal to the length of the string, $\vec{\boldsymbol{F}}_b = k\ell\hat{\boldsymbol{e}}_R$. linear momentum balance ($\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$) for the mass in the static position gives



(a) FBD before AB is cut (b) FBD after AB is cut (c) geometry

Figure 11.13: Free Body Diagram of the ball (a) before the string is cut, (b) sometime after the string is cut, and (c) the geometry at the moment of interest.

(Filename:sfig6.4.2a)

$$(F_b \cos\theta_o - T\cos\theta_o)\hat{\boldsymbol{\imath}} + (F_b \sin\theta_o + T\sin\theta_o - mg)\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}.$$

The x and y components of this equation give

$$\begin{aligned} F_b &= T, \\ (F_b + T)\sin\theta_o &= mg. \end{aligned}$$

Substituting $F_b = T$ in the second equation and replacing F_b by $k\ell$ we get

$$\begin{aligned} 2kl\,\sin\theta_o &= mg \\ \text{or} \qquad \theta_o &= \sin^{-1}\frac{mg}{2k\ell}. \end{aligned} \tag{11.21}$$

After the string is cut, let the mass be at some general angular position θ. Let the stretch in the spring at this position be ℓ. Then, the Linear Momentum Balance for the mass may be written as

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

where

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= (-F_a + mg\,\sin\theta)\hat{\boldsymbol{e}}_R + mg\,\cos\theta\hat{\boldsymbol{e}}_\theta, \\ \vec{\boldsymbol{a}} &= (\ddot{\ell} - l\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{\ell}\dot{\theta} + l\ddot{\theta})\hat{\boldsymbol{e}}_\theta. \end{aligned}$$

Substituting these expressions in $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and dotting the resulting equation with $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ we get

$$\begin{aligned} m(\ddot{\ell} - l\dot{\theta}^2) &= mg\,\sin\theta - \overbrace{k\ell}^{F_a}, \\ m(2\dot{\ell}\dot{\theta} + \ell\ddot{\theta}) &= mg\,cos\theta, \end{aligned}$$

or

$$\ddot{\ell} - \ell\dot{\theta}^2 - g\,\sin\theta = \frac{-k\ell}{m}, \tag{11.22}$$

$$2\dot{\ell}\dot{\theta} + \ell\ddot{\theta} - g\,cos\theta = 0. \tag{11.23}$$

These equations are coupled, nonlinear ordinary differential equations! They look hopelessly difficult to solve. So, what should we do? How about trying to write the equations of motion in cartesian coordinates? Let us try. Referring to Fig. 11.13(c)

$$\begin{aligned}
\sum \vec{\boldsymbol{F}} &= F_a \cos\theta\hat{\boldsymbol{\imath}} + (F_a \sin\theta - mg)\hat{\boldsymbol{\jmath}} \\
&= k\overbrace{\ell\cos\theta}^{-x}\hat{\boldsymbol{\imath}} + (k\overbrace{l\sin\theta}^{-y} - mg)\hat{\boldsymbol{\jmath}} \\
&= -kx\hat{\boldsymbol{\imath}} + (-ky - mg)\hat{\boldsymbol{\jmath}}, \\
\vec{\boldsymbol{a}} &= \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

Now, substituting the expressions for $\sum \vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{a}}$ in the Linear Momentum Balance equation and dotting both sides with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ we get

$$(\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}) \cdot \hat{\boldsymbol{\imath}} \qquad \Rightarrow \qquad \ddot{x} + \frac{k}{m}x = 0 \tag{11.24}$$

$$(\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}) \cdot \hat{\boldsymbol{\jmath}} \qquad \Rightarrow \qquad \ddot{y} + \frac{k}{m}y = -g. \tag{11.25}$$

Unbelievable!! Two such nasty looking nonlinear, coupled equations (11.22) and (11.23) in polar coordinates become so simple, friendly looking linear, uncoupled equations (11.24) and (11.25) in cartesian coordinates. We can now write the solutions of these second order ODE's:

$$\begin{aligned}
x(t) &= A\sin(\lambda t) + B\cos(\lambda t), \\
y(t) &= C\sin(\lambda t) + D\cos(\lambda t) - \frac{mg}{k},
\end{aligned}$$

where A B C and D are constants and $\lambda = \sqrt{k/m}$. We need initial conditions to evaluate the constants A B C and D. Since the mass starts at $t = 0$ from the rest position when $\theta = \theta_o$,

$$\begin{aligned}
x(0) &= -\ell\cos\theta_o \quad \text{and} \quad \dot{x}(0) = 0, \\
y(0) &= -\ell\sin\theta_o \quad \text{and} \quad \dot{y}(0) = 0.
\end{aligned}$$

Substituting these initial conditions in the solutions above, we get

$$\begin{aligned}
x(t) &= -(\ell\cos\theta_o)\ \cos(\sqrt{(k/m)}t), \\
y(t) &= -(\ell\sin\theta_o - \frac{mg}{k})\cos(\sqrt{(k/m)}t) - \frac{mg}{k}.
\end{aligned}$$

From these equations, we can relate x and y by eliminating the cosine term, *i.e.*,

$$\begin{aligned}
\frac{x(t)}{\ell\cos\theta_o} &= \frac{y(t) + (mg/k)}{\ell\sin\theta_o - (mg/k)} \\
\text{or} \qquad y(t) &= \frac{\ell\sin\theta_o - (mg/k)}{\ell\cos\theta_o}\,x(t) - \frac{mg}{k}
\end{aligned}$$

which is the equation of a straight line passing through the vertical equilibrium position $y = -mg/k$. Thus the mass moves along a straight line! ①

① By choosing appropriate initial conditions, you can show that there are other straight line motions (for example, just horizontal or vertical motions) and motions on elliptic paths.

Introduction to

STATICS
and
DYNAMICS

Problem Book

Rudra Pratap and **Andy Ruina**

Spring 2001

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Answers to *'d questions

Problems for Chapter 1

Introduction to mechanics Because no mathematical skills have been taught so far, the questions below just demonstrate the ideas and vocabulary you should have gained from the reading.

1.1 What is mechanics?

1.2 Briefly define each of the words below (using rough English, not precise mathematical language):

a) Statics,

b) Dynamics,

c) Kinematics,

d) Strength of materials,

e) Force,

f) Motion,

g) Linear momentum,

h) Angular momentum,

i) A rigid body.

1.3 This chapter says there are three "pillars" of mechanics of which the third is 'Newton's' laws, what are the other two?

1.4 This book orgainzes the laws of mechanics into 4 basic laws numberred 0-III, not the standard 'Newton's three laws'. What are these four laws (in English, no equations needed)?

1.5 Describe, as precisely as possible, a problem that is not mentionned in the book but which is a mechanics problem. State which quantities are given and what is to be determined by the mechanics solution.

1.6 Describe an engineering problem which is *not* a mechanics problem.

1.7 About how old are Newton's laws?

1.8 Relativity and quantum mechanics have overthrown Newton's laws. Why are engineers still using them?

1.9 Computation is part of modern engineering.

a) What are the three primary computer skills you will need for doing problems in this book?

b) Give examples of each (different thatn the examples given).

c) (optional) Do an example of each on a computer.

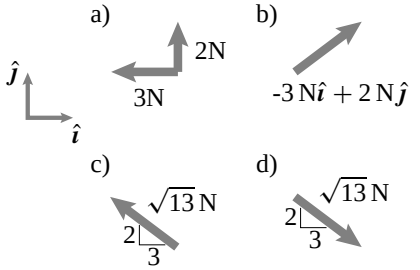
Problems for Chapter 2

Vector skills for mechanics

2.1 Vector notation and vector addition

2.1 Represent the vector $\vec{r} = 5\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}$ in three different ways.

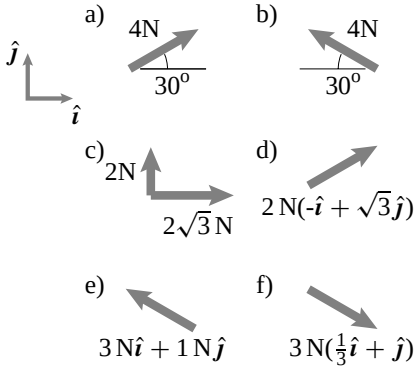
2.2 Which one of the following representations of the same vector $\vec{F}$ is wrong and why?



problem 2.2:

(Filename:pfigure2.vec1.2)

2.3 There are exactly two representations that describe the same vector in the following pictures. Match the correct pictures into pairs.

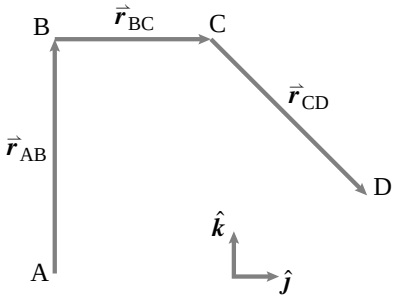


problem 2.3:

(Filename:pfigure2.vec1.3)

2.4 Find the sum of forces $\vec{F}_1 = 20\,\text{N}\hat{\imath} - 2\,\text{N}\hat{\jmath}$, $\vec{F}_2 = 30\,\text{N}(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$, and $\vec{F}_3 = -20\,\text{N}(-\hat{\imath} + \sqrt{3}\hat{\jmath})$.

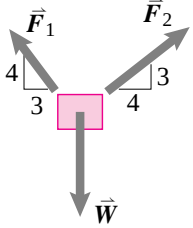
2.5 In the figure shown below, the position vectors are $\vec{r}_{\text{AB}} = 3\,\text{ft}\hat{k}$, $\vec{r}_{\text{BC}} = 2\,\text{ft}\hat{\jmath}$, and $\vec{r}_{\text{CD}} = 2\,\text{ft}(\hat{\jmath} + \hat{k})$. Find the position vector $\vec{r}_{\text{AD}}$.



problem 2.5:

(Filename:pfigure2.vec1.5)

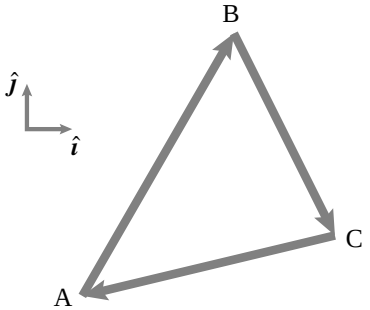
2.6 The forces acting on a block of mass $m = 5\,\text{kg}$ are shown in the figure, where $F_1 = 20\,\text{N}$, $F_2 = 50\,\text{N}$, and $W = mg$. Find the sum $\vec{F}$ ($= \vec{F}_1 + \vec{F}_2 + \vec{W}$)?



problem 2.6:

(Filename:pfigure2.vec1.6)

2.7 Three position vectors are shown in the figure below. Given that $\vec{r}_{\text{B/A}} = 3\,\text{m}(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$ and $\vec{r}_{\text{C/B}} = 1\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}$, find $\vec{r}_{\text{A/C}}$.



problem 2.7:

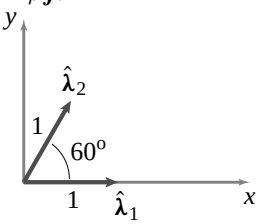
(Filename:pfigure2.vec1.7)

2.8 Given that the sum of four vectors $\vec{F}_i$, $i = 1$ to 4, is zero, where $\vec{F}_1 = 20\,\text{N}\hat{\imath}$, $\vec{F}_2 = -50\,\text{N}\hat{\jmath}$, $\vec{F}_3 = 10\,\text{N}(-\hat{\imath} + \hat{\jmath})$, find $\vec{F}_4$.

2.9 Three forces $\vec{F} = 2\,\text{N}\hat{\imath} - 5\,\text{N}\hat{\jmath}$, $\vec{R} = 10\,\text{N}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$ and $\vec{W} = -20\,\text{N}\hat{\jmath}$, sum up to zero. Determine the angle θ and draw the force vector $\vec{R}$ clearly showing its direction.

2.10 Given that $\vec{R}_1 = 1\,\text{N}\hat{\imath} + 1.5\,\text{N}\hat{\jmath}$ and $\vec{R}_2 = 3.2\,\text{N}\hat{\imath} - 0.4\,\text{N}\hat{\jmath}$, find $2\vec{R}_1 + 5\vec{R}_2$.

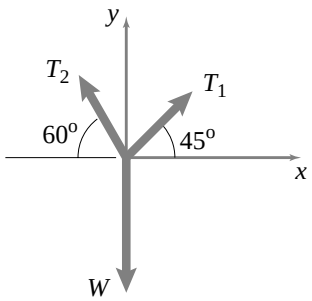
2.11 For the unit vectors $\hat{\boldsymbol{\lambda}}_1$ and $\hat{\boldsymbol{\lambda}}_2$ shown below, find the scalars α and β such that $\alpha\hat{\boldsymbol{\lambda}}_2 - 3\hat{\boldsymbol{\lambda}}_2 = \beta\hat{\jmath}$.



problem 2.11:

(Filename:pfigure2.vec1.11)

2.12 In the figure shown, $T_1 = 20\sqrt{2}\,\text{N}$, $T_2 = 40\,\text{N}$, and W is such that the sum of the three forces equals zero. If W is doubled, find α and β such that $\alpha\vec{T}_1$, $\beta\vec{T}_2$, and $2\vec{W}$ still sum up to zero.



problem 2.12:

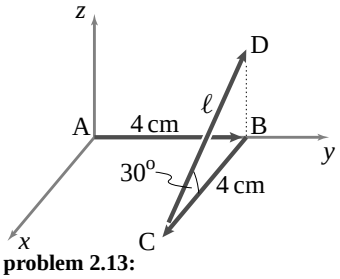
(Filename:pfigure2.vec1.12)

2.13 In the figure shown, rods AB and BC are each 4 cm long and lie along y and x axes, respectively. Rod CD is in the xz plane and makes an angle $\theta = 30$o with the x-axis.

(a) Find $\vec{r}_{\text{AD}}$ in terms of the variable length ℓ.

(b) Find ℓ and α such that

$$\vec{r}_{\text{AD}} = \vec{r}_{\text{AB}} - \vec{r}_{\text{BC}} + \alpha\hat{k}.$$



problem 2.13:

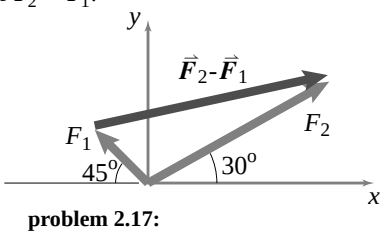
(Filename:pfigure2.vec1.13)

2.14 Find the magnitudes of the forces $\vec{F}_1 = 30\,\text{N}\hat{\imath} - 40\,\text{N}\hat{\jmath}$ and $\vec{F}_2 = 30\,\text{N}\hat{\imath} + 40\,\text{N}\hat{\jmath}$. Draw the two forces, representing them with their magnitudes.

2.15 Two forces $\vec{\boldsymbol{R}} = 2\,\text{N}(0.16\hat{\boldsymbol{\imath}} + 0.80\hat{\boldsymbol{\jmath}})$ and $\vec{\boldsymbol{W}} = -36\,\text{N}\hat{\boldsymbol{\jmath}}$ act on a particle. Find the magnitude of the net force. What is the direction of this force?

2.16 In Problem 2.13, find ℓ such that the length of the position vector $\vec{\boldsymbol{r}}_{\text{AD}}$ is 6 cm.

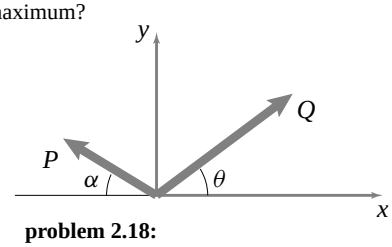
2.17 In the figure shown, $F_1 = 100\,\text{N}$ and $F_2 = 300\,\text{N}$. Find the magnitude and direction of $\vec{\boldsymbol{F}}_2 - \vec{\boldsymbol{F}}_1$.



problem 2.17:

(Filename:pfigure2.vec1.17)

2.18 Let two forces $\vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{Q}}$ act in the direction shown in the figure. You are allowed to change the direction of the forces by changing the angles α and θ while keeping the magnitudes fixed. What should be the values of α and θ if the magnitude of $\vec{\boldsymbol{P}} + \vec{\boldsymbol{Q}}$ has to be the maximum?



problem 2.18:

(Filename:pfigure2.vec1.18)

2.19 Two points A and B are located in the xy plane. The coordinates of A and B are (4 mm, 8 mm) and (90 mm, 6 mm), respectively.

(a) Draw position vectors $\vec{\boldsymbol{r}}_{\text{A}}$ and $\vec{\boldsymbol{r}}_{\text{B}}$.

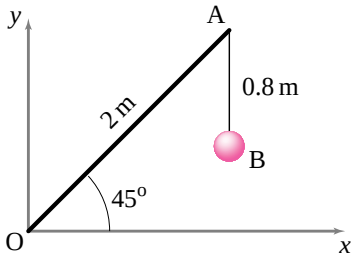
(b) Find the magnitude of $\vec{\boldsymbol{r}}_{\text{A}}$ and $\vec{\boldsymbol{r}}_{\text{B}}$.

(c) How far is A from B?

2.20 In the figure shown, a ball is suspended with a 0.8 mlong cord from a 2 mlong hoist OA.

(a) Find the position vector $\vec{\boldsymbol{r}}_{\text{B}}$ of the ball.

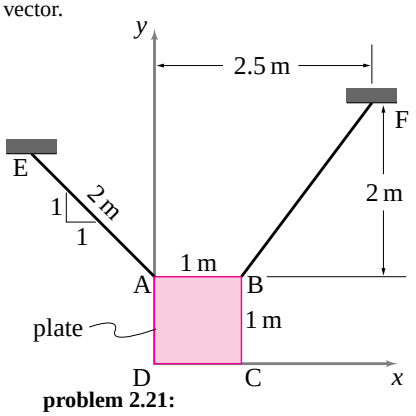
(b) Find the distance of the ball from the origin.



problem 2.20:

(Filename:pfigure2.vec1.20)

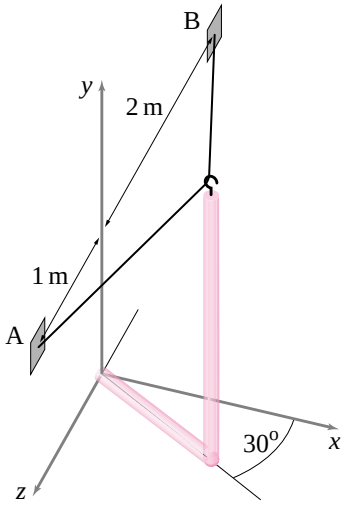
2.21 A 1 m × 1 m square board is supported by two strings AE and BF. The tension in the string BF is 20 N. Express this tension as a vector.



problem 2.21:

(Filename:pfigure2.vec1.21)

2.22 The top of an L-shaped bar, shown in the figure, is to be tied by strings AD and BD to the points A and B in the yz plane. Find the length of the strings AD and BD using vectors $\vec{\boldsymbol{r}}_{\text{AD}}$ and $\vec{\boldsymbol{r}}_{\text{BD}}$.



problem 2.22:

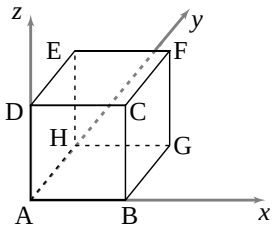
(Filename:pfigure2.vec1.22)

2.23 A cube of side 6 inis shown in the figure.

(a) Find the position vector of point F, $\vec{\boldsymbol{r}}_{\text{F}}$, from the vector sum $\vec{\boldsymbol{r}}_{\text{F}} = \vec{\boldsymbol{r}}_{\text{D}} + \vec{\boldsymbol{r}}_{\text{C/D}} + \vec{\boldsymbol{r}}_{\text{F/c}}$.

(b) Calculate $|\vec{\boldsymbol{r}}_{\text{F}}|$.

(c) Find $\vec{\boldsymbol{r}}_{\text{G}}$ using $\vec{\boldsymbol{r}}_{\text{F}}$.



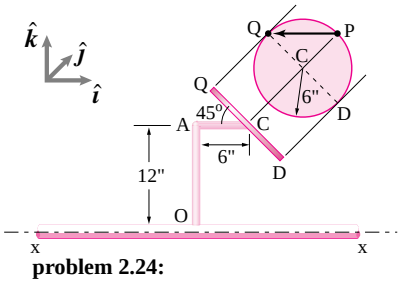
problem 2.23:

(Filename:pfigure2.vec1.23)

2.24 A circular disk of radius 6 inis mounted on axle x-x at the end an L-shaped bar as shown in the figure. The disk is tipped 45o with the horizontal bar AC. Two points, P and Q, are marked on the rim of the plate; P directly parallel to the center C into the page, and Q at the highest point above the center C. Taking the base vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ as shown in the figure, find

(a) the relative position vector $\vec{\boldsymbol{r}}_{\text{Q/P}}$,

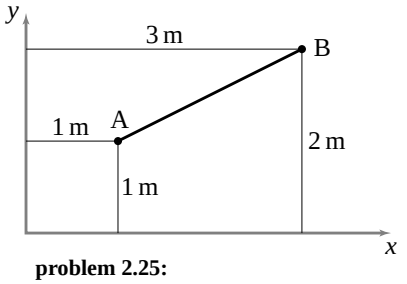
(b) the magnitude $|\vec{\boldsymbol{r}}_{\text{Q/P}}|$.



problem 2.24:

(Filename:pfigure2.vec1.24)

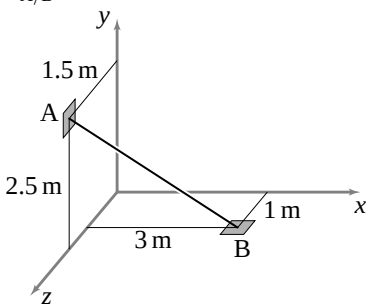
2.25 Find the unit vector $\hat{\boldsymbol{\lambda}}_{\text{AB}}$, directed from point A to point B shown in the figure.



problem 2.25:

(Filename:pfigure2.vec1.25)

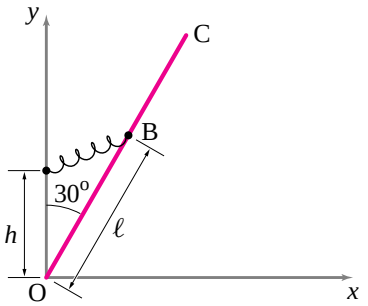
2.26 Find a unit vector along string BA and express the position vector of A with respect to B, $\vec{\boldsymbol{r}}_{\text{A/B}}$, in terms of the unit vector.



problem 2.26:

(Filename:pfigure2.vec1.26)

2.27 In the structure shown in the figure, $\ell = 2\,\text{ft}$, $h = 1.5\,\text{ft}$. The force in the spring is $\vec{\boldsymbol{F}} = k\vec{\boldsymbol{r}}_{\text{AB}}$, where $k = 100\,\text{lbf/ ft}$. Find a unit vector $\hat{\boldsymbol{\lambda}}_{\text{AB}}$ along AB and calculate the spring force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\lambda}}_{\text{AB}}$.



problem 2.27:

(Filename:pfigure2.vec1.27)

2.28 Express the vector $\vec{r}_A = 2\,\text{m}\hat{\imath} - 3\,\text{m}\hat{\jmath} + 5\,\text{m}\hat{k}$ in terms of its magnitude and a unit vector indicating its direction.

2.29 Let $\vec{F} = 10\,\text{lbf}\hat{\imath} + 30\,\text{lbf}\hat{\jmath}$ and $\vec{W} = -20\,\text{lbf}\hat{\jmath}$. Find a unit vector in the direction of the net force $\vec{F} + \vec{W}$, and express the the net force in terms of the unit vector.

2.30 Let $\hat{\lambda}_1 = 0.80\hat{\imath} + 0.60\hat{\jmath}$ and $\hat{\lambda}_2 = 0.5\hat{\imath} + 0.866\hat{\jmath}$.

(a) Show that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are unit vectors.

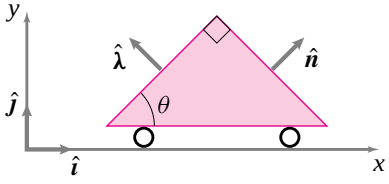
(b) Is the sum of these two unit vectors also a unit vector? If not, then find a unit vector along the sum of $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

2.31 If a mass slides from point A towards point B along a straight path and the coordinates of points A and B are (0 in, 5 in, 0 in) and (10 in, 0 in, 10 in), respectively, find the unit vector $\hat{\lambda}_{AB}$ directed from A to B along the path.

2.32 Write the vectors $\vec{F}_1 = 30\,\text{N}\hat{\imath} + 40\,\text{N}\hat{\jmath} - 10\,\text{N}\hat{k}$, $\vec{F}_2 = -20\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$, and $\vec{F}_3 = -10\,\text{N}\hat{\imath} - 100\,\text{N}\hat{k}$ as a list of numbers (rows or columns). Find the sum of the forces using a computer.

2.2 The dot product of two vectors

2.33 Express the unit vectors $\hat{n}$ and $\hat{\lambda}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$ shown in the figure. What are the x and y components of $\vec{r} = 3.0\,\text{ft}\hat{n} - 1.5\,\text{ft}\hat{\lambda}$? *



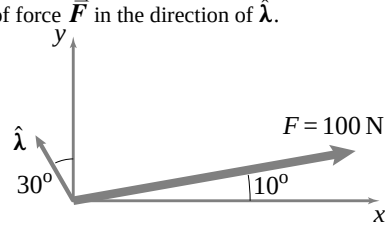
problem 2.33:

(Filename:efig1.2.27)

2.34 Find the dot product of two vectors $\vec{F} = 10\,\text{lbf}\hat{\imath} - 20\,\text{lbf}\hat{\jmath}$ and $\hat{\lambda} = 0.8\hat{\imath} + 0.6\hat{\jmath}$. Sketch $\vec{F}$ and $\hat{\lambda}$ and show what their dot product represents.

2.35 The position vector of a point A is $\vec{r}_A = 30\,\text{cm}\hat{\imath}$. Find the dot product of $\vec{r}_A$ with $\hat{\lambda} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$.

2.36 From the figure below, find the component of force $\vec{F}$ in the direction of $\hat{\lambda}$.



problem 2.36:

(Filename:pfigure2.vec1.33)

2.37 Find the angle between $\vec{F}_1 = 2\,\text{N}\hat{\imath} + 5\,\text{N}\hat{\jmath}$ and $\vec{F}_2 = -2\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath}$.

2.38 A force $\vec{F}$ is directed from point A(3,2,0) to point B(0,2,4). If the x-component of the force is 120 N, find the y- and z-components of $\vec{F}$.

2.39 A force acting on a bead of mass m is given as $\vec{F} = -20\,\text{lbf}\hat{\imath} + 22\,\text{lbf}\hat{\jmath} + 12\,\text{lbf}\hat{k}$. What is the angle between the force and the z-axis?

2.40 Given $\vec{\omega} = 2\,\text{rad/s}\hat{\imath} + 3\,\text{rad/s}\hat{\jmath}$, $\vec{H}_1 = (20\hat{\imath} + 30\hat{\jmath})\,\text{kg m}^2/\text{s}$ and $\vec{H}_2 = (10\hat{\imath} + 15\hat{\jmath} + 6\hat{k})\,\text{kg m}^2/\text{s}$, find (a) the angle between $\vec{\omega}$ and $\vec{H}_1$ and (b) the angle between $\vec{\omega}$ and $\vec{H}_2$.

2.41 The unit normal to a surface is given as $\hat{n} = 0.74\hat{\imath} + 0.67\hat{\jmath}$. If the weight of a block on this surface acts in the $-\hat{\jmath}$ direction, find the angle that a 1000 N normal force makes with the direction of weight of the block.

2.42 Vector algebra. For each equation below state whether:

(a) The equation is nonsense. If so, why?

(b) Is always true. Why? Give an example.

(c) Is never true. Why? Give an example.

(d) Is sometimes true. Give examples both ways.

You may use trivial examples.

a) $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

b) $\vec{A} + b = b + \vec{A}$

c) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

d) $\vec{B}/\vec{C} = B/C$

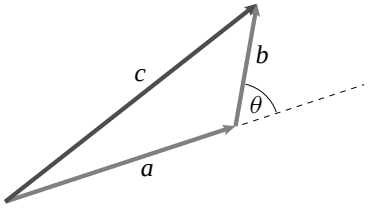
e) $b/\vec{A} = b/A$

f) $\vec{A} = (\vec{A}\cdot\vec{B})\vec{B} + (\vec{A}\cdot\vec{C})\vec{C} + (\vec{A}\cdot\vec{D})\vec{D}$

2.43 Use the dot product to show 'the law of cosines'; i. e.,

$$c^2 = a^2 + b^2 + 2ab\cos\theta.$$

(Hint: $\vec{c} = \vec{a} + \vec{b}$; also, $\vec{c}\cdot\vec{c} = \vec{c}\cdot\vec{c}$)

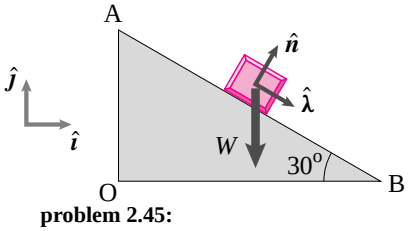


problem 2.43:

(Filename:pfigure.blue.2.1)

2.44 (a) Draw the vector $\vec{r} = 3.5\,\text{in}\hat{\imath} + 3.5\,\text{in}\hat{\jmath} - 4.95\,\text{in}\hat{k}$. (b) Find the angle this vector makes with the z-axis. (c) Find the angle this vector makes with the x-y plane.

2.45 In the figure shown, $\hat{\lambda}$ and $\hat{n}$ are unit vectors parallel and perpendicular to the surface AB, respectively. A force $\vec{W} = -50\,\text{N}\hat{\jmath}$ acts on the block. Find the components of $\vec{W}$ along $\hat{\lambda}$ and $\hat{n}$.



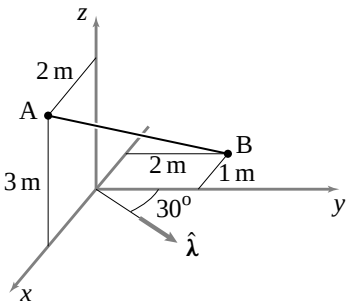
problem 2.45:

(Filename:pfigure2.vec1.41)

2.46 From the figure shown, find the components of vector $\vec{r}_{AB}$ (you have to first find this position vector) along

(a) the y-axis, and

(b) along $\hat{\lambda}$.



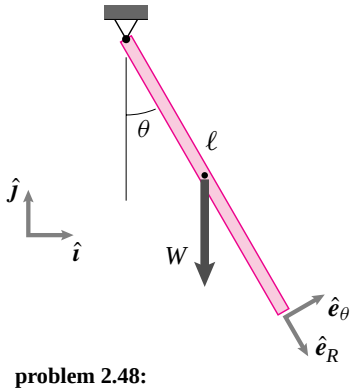
problem 2.46:

(Filename:pfigure2.vec1.42)

2.47 The net force acting on a particle is $\vec{F} = 2\,\text{N}\hat{\imath} + 10\,\text{N}\hat{\jmath}$. Find the components of this force in another coordinate system with basis vectors $\hat{\imath}' = -\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{\jmath}' = -\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}$. For $\theta = 30$o, sketch the vector $\vec{F}$ and show its components in the two coordinate systems.

2.48 Find the unit vectors $\hat{e}_R$ and $\hat{e}_\theta$ in terms of $\hat{\imath}$ and $\hat{\jmath}$ with the geometry shown in figure.

What are the componets of $\vec{W}$ along $\hat{e}_R$ and $\hat{e}_\theta$?



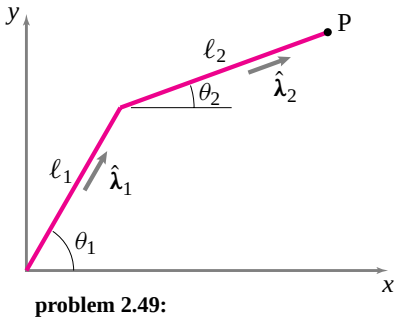
problem 2.48:

(Filename:pfigure2.vec1.44)

2.49 Write the position vector of point P in terms of $\hat{\boldsymbol{\lambda}}_1$ and $\hat{\boldsymbol{\lambda}}_2$ and

(a) find the y-component of $\vec{r}_P$,

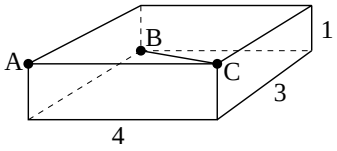
(b) find the component of $\vec{r}_P$ alon $\hat{\boldsymbol{\lambda}}_1$.



problem 2.49:

(Filename:pfigure2.vec1.45)

2.50 What is the distance between the point A and the diagonal BC of the parallelepiped shown? (Use vector methods.)



problem 2.50:

(Filename:pfigure.blue.2.3)

2.51 Let $\vec{F}_1 = 30\,\text{N}\hat{\imath} + 40\,\text{N}\hat{\jmath} - 10\,\text{N}\hat{k}$, $\vec{F}_2 = -20\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$, and $\vec{F}_3 = F_{3_x}\hat{\imath} + F_{3_y}\hat{\jmath} - F_{3_z}\hat{k}$. If the sum of all these forces must equal zero, find the required scalar equations to solve for the components of $\vec{F}_3$.

2.52 A vector equation for the sum of forces results into the following equation:

$$\frac{F}{2}(\hat{\imath} - \sqrt{3}\hat{\jmath}) + \frac{R}{5}(3\hat{\imath} + 6\hat{\jmath}) = 25\,\text{N}\hat{\boldsymbol{\lambda}}$$

where $\hat{\boldsymbol{\lambda}} = 0.30\hat{\imath} - 0.954\hat{\jmath}$. Find the scalar equations parallel and perpendicular to $\hat{\boldsymbol{\lambda}}$.

2.53 Let $\alpha\vec{F}_1 + \beta\vec{F}_2 + \gamma\vec{F}_3 = \vec{0}$, where $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ are as given in Problem 2.32. Solve for α, β, and γ using a computer.

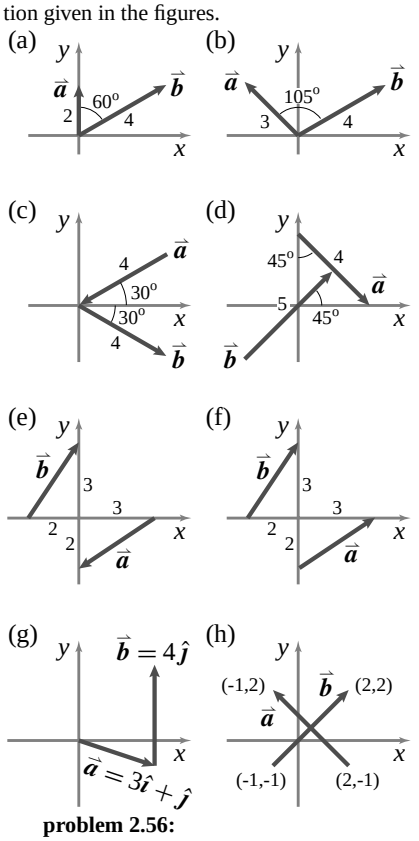
2.54 Write a computer program (or use a canned program) to find the dot product of two 3-D vectors. Test the program by computing the dot products $\hat{\imath}\cdot\hat{\imath}$, $\hat{\imath}\cdot\hat{\jmath}$, and $\hat{\jmath}\cdot\hat{k}$. Now use the program to find the components of $\vec{F} = (2\hat{\imath} + 2\hat{\jmath} - 3\hat{k})\,\text{N}$ along the line $\vec{r}_{AB} = (0.5\hat{\imath} - 0.2\hat{\jmath} + 0.1\hat{k})\,\text{m}$.

2.55 Let $\vec{r}_n = 1\,\text{m}(\cos\theta_n\hat{\imath} + \sin\theta_n\hat{\jmath})$, where $\theta_n = \theta_0 - n\Delta\theta$. Using a computer generate the required vectors and find the sum

$$\sum_{n=0}^{44}\vec{r}_i, \quad \text{with } \Delta\theta = 1^o \text{ and } \theta_0 = 45^o.$$

2.3 Cross product, moment, and moment about an axis

2.56 Find the cross product of the two vectors shown in the figures below from the information given in the figures.



problem 2.56:

(Filename:pfigure2.vec2.1)

2.57 Vector algebra. For each equation below state whether:

(a) The equation is nonsense. If so, why?

(b) Is always true. Why? Give an example.

(c) Is never true. Why? Give an example.

(d) Is sometimes true. Give examples both ways.

You may use trivial examples.

a) $\vec{B}\times\vec{C} = \vec{C}\times\vec{B}$

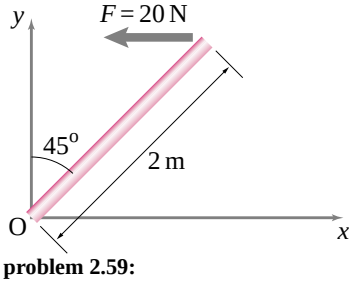
b) $\vec{B}\times\vec{C} = \vec{C}\cdot\vec{B}$

c) $\vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$

d) $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

2.58 What is the moment $\vec{M}$ produced by a 20 N force F acting in the x direction with a lever arm of $\vec{r} = (16\,\text{mm})\hat{\jmath}$?

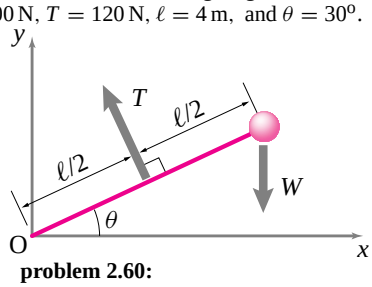
2.59 Find the moment of the force shown on the rod about point O.



problem 2.59:

(Filename:pfigure2.vec2.2)

2.60 Find the sum of moments of forces $\vec{W}$ and $\vec{T}$ about the origin, given that $W = 100\,\text{N}$, $T = 120\,\text{N}$, $\ell = 4\,\text{m}$, and $\theta = 30^o$.



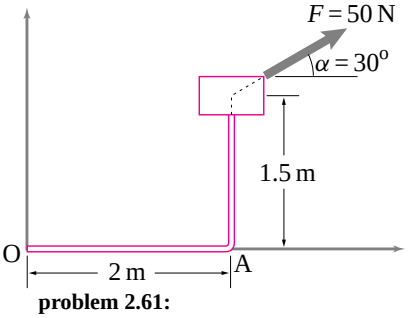
problem 2.60:

(Filename:pfigure2.vec2.3)

2.61 Find the moment of the force

a) about point A

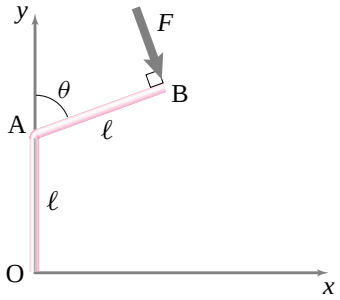
b) about point O.



problem 2.61:

(Filename:pfigure2.vec2.4)

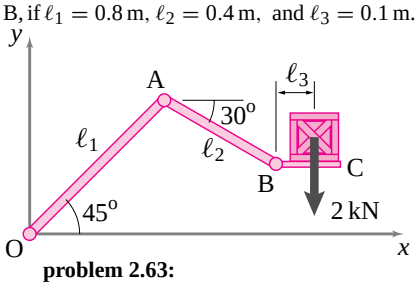
2.62 In the figure shown, OA = AB = 2 m. The force $F = 40\,\text{N}$ acts perpendicular to the arm AB. Find the moment of $\vec{F}$ about O, given that $\theta = 45^o$. If $\vec{F}$ always acts normal to the arm AB, would increasing θ increase the magnitude of the moment? In particular, what value of θ will give the largest moment?



problem 2.62:

(Filename:pfigure2.vec2.5)

2.63 Calculate the moment of the 2 kNpayload on the robot arm about (i) joint A, and (ii) joint B, if $\ell_1 = 0.8\,\text{m}$, $\ell_2 = 0.4\,\text{m}$, and $\ell_3 = 0.1\,\text{m}$.



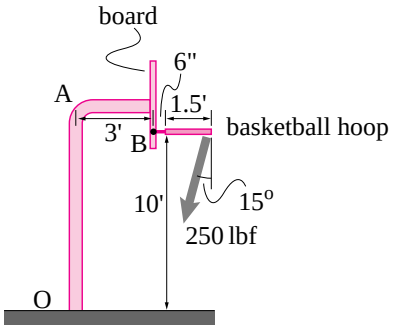
problem 2.63:

(Filename:pfigure2.vec2.6)

2.64 During a slam-dunk, a basketball player pulls on the hoop with a 250 lbf at point C of the ring as shown in the figure. Find the moment of the force about

a) the point of the ring attachment to the board (point B), and

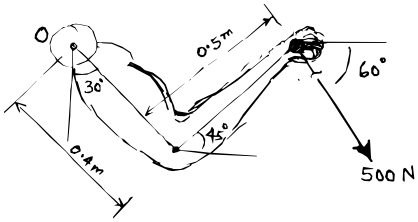
b) the root of the pole, point O.



problem 2.64:

(Filename:pfigure2.vec2.7)

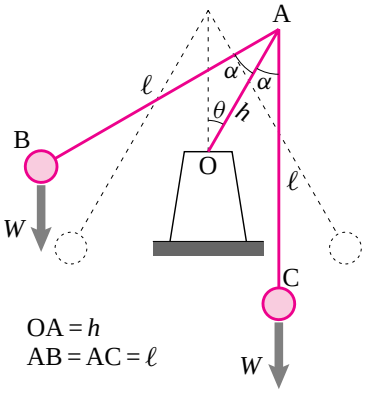
2.65 During weight training, an athelete pulls a weight of 500 Nwith his arms pulling on a hadlebar connected to a universal machine by a cable. Find the moment of the force about the shoulder joint O in the configuration shown.



problem 2.65:

(Filename:pfigure2.vec2.8)

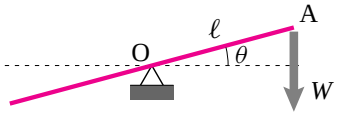
2.66 Find the sum of moments due to the two weights of the teeter-totter when the teeter-totter is tipped at an angle θ from its vertical position. Give your answer in terms of the variables shown in the figure.



problem 2.66:

(Filename:pfigure2.vec2.9)

2.67 Find the percentage error in computing the moment of $\vec{W}$ about the pivot point O as a function of θ, if the weight is assumed to act normal to the arm OA (a good approximation when θ is very small).



problem 2.67:

(Filename:pfigure2.vec2.10)

2.68 What do you get when you cross a vector and a scalar? *

2.69 Why did the chicken cross the road? *

2.70 Carry out the following cross products in different ways and determine which method takes the least amount of time for you.

a) $\vec{r} = 2.0\,\text{ft}\hat{\imath} + 3.0\,\text{ft}\hat{\jmath} - 1.5\,\text{ft}\hat{k}$; $\vec{F} = -0.3\,\text{lbf}\hat{\imath} - 1.0\,\text{lbf}\hat{k}$; $\vec{r} \times \vec{F} =$?

b) $\vec{r} = (-\hat{\imath} + 2.0\hat{\jmath} + 0.4\hat{k})\,\text{m}$; $\vec{L} = (3.5\hat{\jmath} - 2.0\hat{k})\,\text{kg m/s}$; $\vec{r} \times \vec{L} =$?

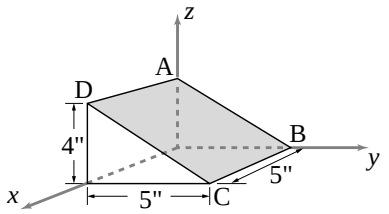
c) $\vec{\omega} = (\hat{\imath} - 1.5\hat{\jmath})\,\text{rad/s}$; $\vec{r} = (10\hat{\imath} - 2\hat{\jmath} + 3\hat{k})\,\text{in}$; $\vec{\omega} \times \vec{r} =$?

2.71 A force $\vec{F} = 20\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}$ acts through a point A with coordinates (200 mm, 300 mm, -100 mm). What is the moment $\vec{M}(= \vec{r} \times \vec{F})$ of the force about the origin?

2.72 Cross Product program Write a program that will calculate cross products. The input to the function should be the components of the two vectors and the output should be the components of the cross product. As a model, here is a function file that calculates dot products in pseudo code.

```
%program definition
z(1)=a(1)*b(1);
z(2)=a(2)*b(2);
z(3)=a(3)*b(3);
w=z(1)+z(2)+z(3);
```

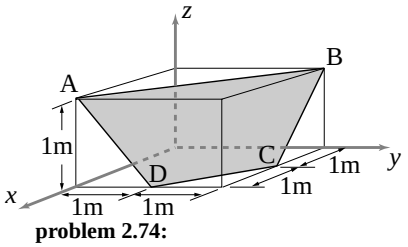
2.73 Find a unit vector normal to the surface ABCD shown in the figure.



problem 2.73:

(Filename:efig1.2.11)

2.74 If the magnitude of a force $\vec{N}$ normal to the surface ABCD in the figure is 1000 N, write $\vec{N}$ as a vector. *

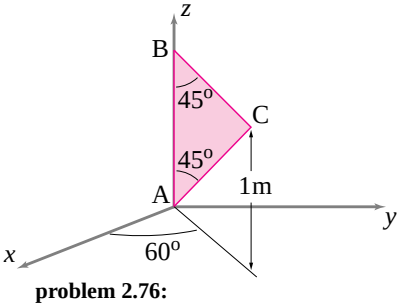


problem 2.74:

(Filename:efig1.2.12)

2.75 The equation of a surface is given as $z = 2x - y$. Find a unit vector $\hat{n}$ normal to the surface.

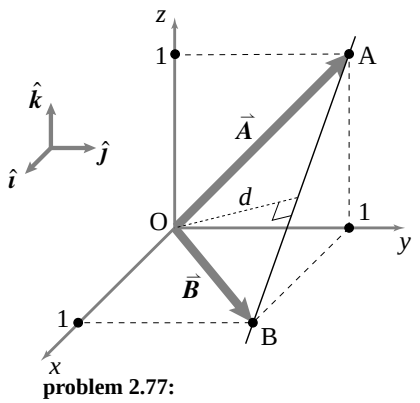
2.76 In the figure, a triangular plate ACB, attached to rod AB, rotates about the z-axis. At the instant shown, the plate makes an angle of 60^o with the x-axis. Find and draw a vector normal to the surface ACB.



problem 2.76:

(Filename:efig1.2.14)

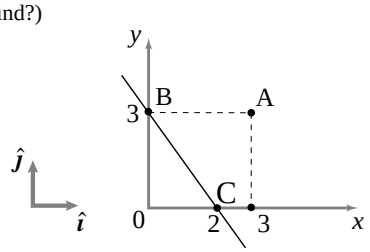
2.77 What is the distance d between the origin and the line AB shown? (You may write your solution in terms of $\vec{A}$ and $\vec{B}$ before doing any arithmetic). *



problem 2.77:

(Filename:pfigure.blue.1.3)

2.78 What is the perpendicular distance between the point A and the line BC shown? (There are at least 3 ways to do this using various vector products, how many ways can you find?)



problem 2.78:

(Filename:pfigure.blue.2.2)

2.79 Given a force, $\vec{F}_1 = (-3\hat{i} + 2\hat{j} + 5\hat{k})$ N acting at a point P whose position is given by $\vec{r}_{P/O} = (4\hat{i} - 2\hat{j} + 7\hat{k})$ m, what is the moment about an axis through the origin O with direction $\hat{\lambda} = \frac{2}{\sqrt{5}}\hat{j} + \frac{1}{\sqrt{5}}\hat{j}$?

2.80 Drawing vectors and computing with vectors. The point O is the origin. Point A has xyz coordinates (0, 5, 12)m. Point B has xyz coordinates (4, 5, 12)m.

a) Make a neat sketch of the vectors OA, OB, and AB.

b) Find a unit vector in the direction of OA, call it $\hat{\lambda}_{OA}$.

c) Find the force $\vec{F}$ which is 5N in size and is in the direction of OA.

d) What is the angle between OA and OB?

e) What is $\vec{r}_{BO} \times \vec{F}$?

f) What is the moment of $\vec{F}$ about a line parallel to the z axis that goes through the point B?

2.81 Vector Calculations and Geometry. The 5 N force $\vec{F}_1$ is along the line OA. The 7 N force $\vec{F}_2$ is along the line OB.

a) Find a unit vector in the direction OB. *

b) Find a unit vector in the direction OA. *

c) Write both $\vec{F}_1$ and $\vec{F}_2$ as the product of their magnitudes and unit vectors in their directions. *

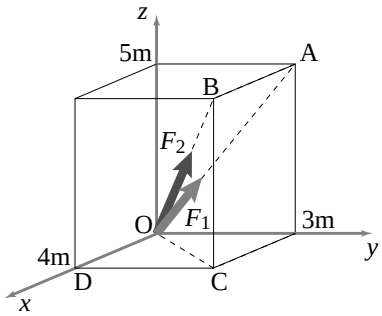
d) What is the angle AOB? *

e) What is the component of $\vec{F}_1$ in the x-direction? *

f) What is $\vec{r}_{DO} \times \vec{F}_1$? ($\vec{r}_{DO} \equiv \vec{r}_{O/D}$ is the position of O relative to D.) *

g) What is the moment of $\vec{F}_2$ about the axis DC? (The moment of a force about an axis parallel to the unit vector $\hat{\lambda}$ is defined as $M_\lambda = \hat{\lambda}\cdot(\vec{r}\times\vec{F})$ where $\vec{r}$ is the position of the point of application of the force relative to some point on the axis. The result does not depend on which point on the axis is used or which point on the line of action of $\vec{F}$ is used.). *

h) Repeat the last problem using either a different reference point on the axis DC or the line of action OB. Does the solution agree? [Hint: it should.] *



problem 2.81:

(Filename:p1sp92)

2.82 A, B, and C are located by position vectors $\vec{r}_A = (1, 2, 3)$, $\vec{r}_B = (4, 5, 6)$, and $\vec{r}_C = (7, 8, 9)$.

a) Use the vector dot product to find the angle BAC (A is at the vertex of this angle).

b) Use the vector cross product to find the angle BCA (C is at the vertex of this angle).

c) Find a unit vector perpendicular to the plane ABC.

d) How far is the infinite line defined by AB from the origin? (That is, how close is the closest point on this line to the origin?)

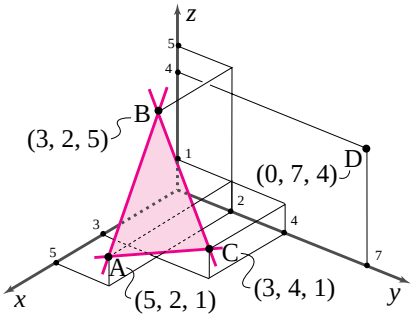
e) Is the origin co-planar with the points A, B, and C?

2.83 Points A, B, and C in the figure define a plane.

a) Find a unit normal vector to the plane. *

b) Find the distance from this infinite plane to the point D. *

c) What are the coordinates of the point on the plane closest to point D? *

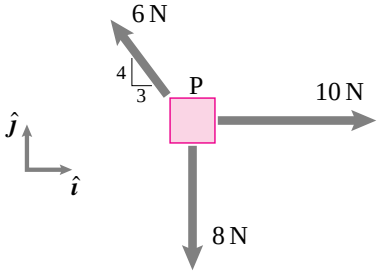


problem 2.83:

(Filename:pfigure.s95q2)

2.4 Equivalant force systems and couples

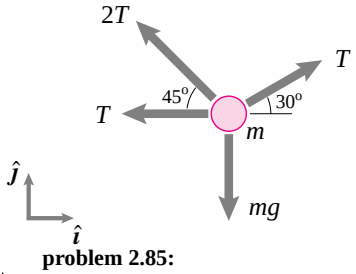
2.84 Find the net force on the particle shown in the figure.



problem 2.84:

(Filename:pfigure2.3.rp1)

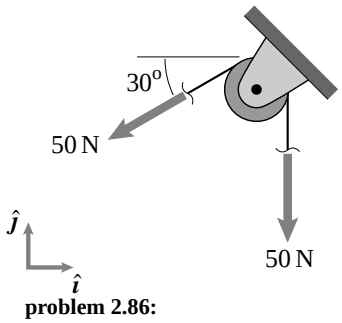
2.85 Replace the forces acting on the particle of mass m shown in the figure by a single equivalent force.



problem 2.85:

(Filename:pfigure2.3.rp2)

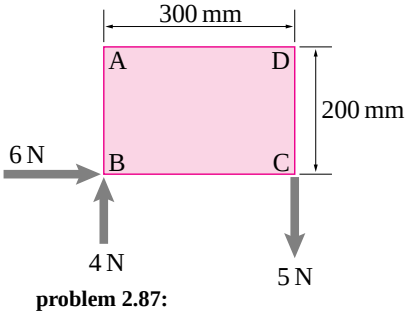
2.86 Find the net force on the pulley due to the belt tensions shown in the figure.



problem 2.86:

(Filename:pfigure2.3.rp3)

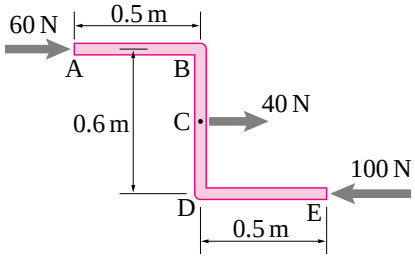
2.87 Replace the forces shown on the rectangular plate by a single equivalent force. Where should this equivalent force act on the plate and why?



problem 2.87:

(Filename:pfigure2.3.rp4)

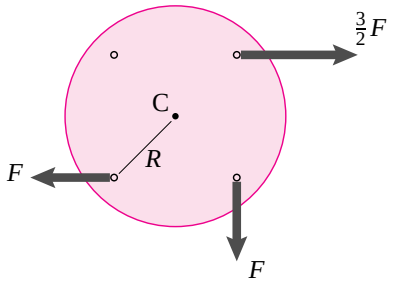
2.88 Three forces act on a Z-section ABCDE as shown in the figure. Point C lies in the middle of the vertical section BD. Find an equivalent force-couple system acting on the structure and make a sketch to show where it acts.



problem 2.88:

(Filename:pfigure2.3.rp5)

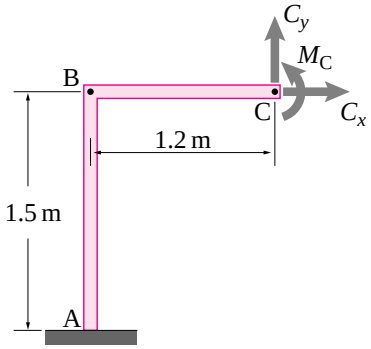
2.89 The three forces acting on the circular plate shown in the figure are equidistant from the center C. Find an equivalent force-couple system acting at point C.



problem 2.89:

(Filename:pfigure2.3.rp6)

2.90 The forces and the moment acting on point C of the frame ABC shown in the figure are $C_x = 48\,\text{N}$, $C_y = 40\,\text{N}$, and $M_c = 20\,\text{N·m}$. Find an equivalent force couple system at point B.



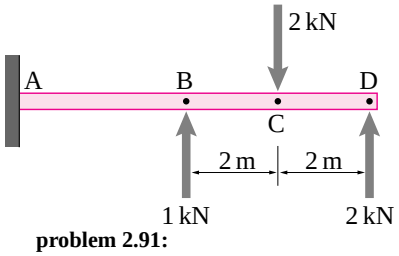
problem 2.90:

(Filename:pfigure2.3.rp7)

2.91 Find an equivalent force-couple system for the forces acting on the beam in Fig. **??**, if the equivalent system is to act at

a) point B,

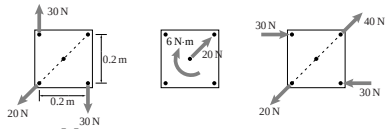
b) point D.



problem 2.91:

(Filename:pfigure2.3.rp8)

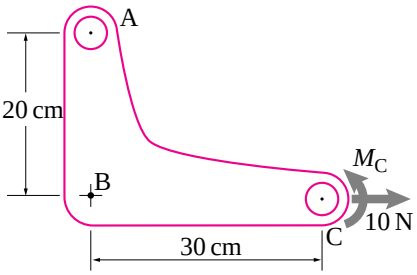
2.92 In Fig. **??**, three different force-couple systems are shown acting on a square plate. Identify which force-couple systems are equivalent.



problem 2.92:

(Filename:pfigure2.3.rp9)

2.93 The force and moment acting at point C of a machine part are shown in the figure where M_c is not known. It is found that if the given force-couple system is replaced by a single horizontal force of magnitude 10 N acting at point A then the net effect on the machine part is the same. What is the magnitude of the moment M_c?



problem 2.93:

(Filename:pfigure2.3.rp10)

2.5 Center of mass and center of gravity

2.94 An otherwise massless structure is made of four point masses, m, $2m$, $3m$ and $4m$, located at coordinates (0, 1 m), (1 m, 1 m), (1 m, −1 m), and (0, −1 m), respectively. Locate the center of mass of the structure. *

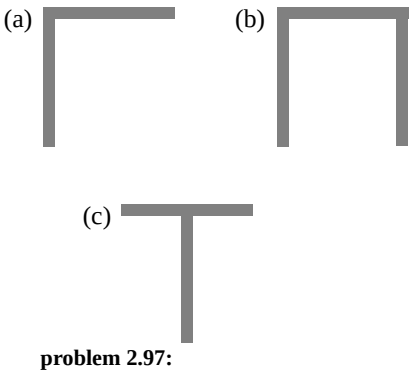
2.95 3-D: The following data is given for a structural system modeled with five point masses in 3-D-space:

mass	**coordinates (in m)**
0.4 kg	(1,0,0)
0.4 kg	(1,1,0)
0.4 kg	(2,1,0)
0.4 kg	(2,0,0)
1.0 kg	(1.5,1.5,3)

Locate the center of mass of the system.

2.96 Write a computer program to find the center of mass of a point-mass-system. The input to the program should be a table (or matrix) containing individual masses and their coordinates. (It is possible to write a single program for both 2-D and 3-D cases, write separate programs for the two cases if that is easier for you.) Check your program on Problems 2.94 and 2.95.

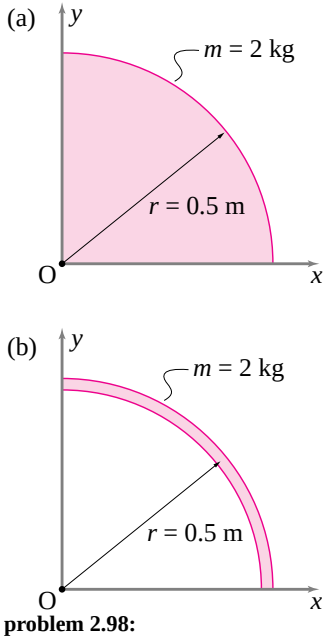
2.97 Find the center of mass of the following composite bars. Each composite shape is made of two or more uniform bars of length 0.2 m and mass 0.5 kg.



problem 2.97:

(Filename:pfigure3.cm.rp7)

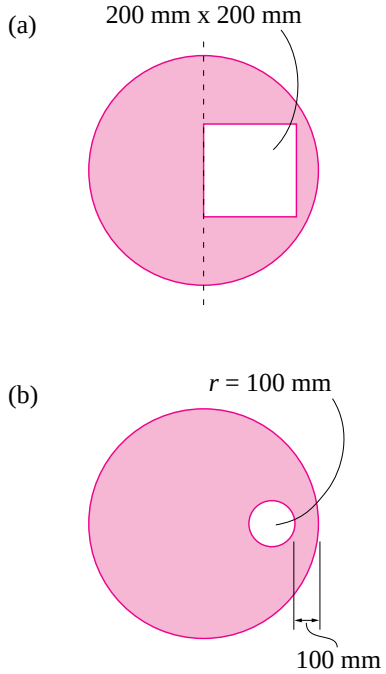
2.98 Find the center of mass of the following two objects [Hint: set up and evaluate the needed integrals.]



problem 2.98:

(Filename:pfigure3.cm.rp8)

2.99 Find the center of mass of the following plates obtained from cutting out a small section from a uniform circular plate of mass 1 kg (prior to removing the cutout) and radius 1/4 m.



problem 2.99:

(Filename:pfigure3.cm.rp9)

Problems for Chapter 3

Free body diagrams

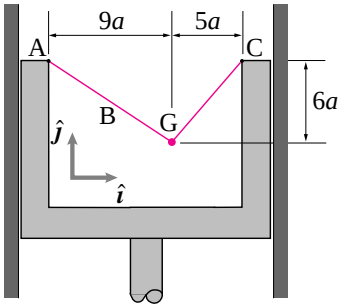
3.1 Free body diagrams

3.1 How does one know what forces and moments to use in

a) the statics force balance and moment balance equations?

b) the dynamics linear momentum balance and angular momentum balance equations?

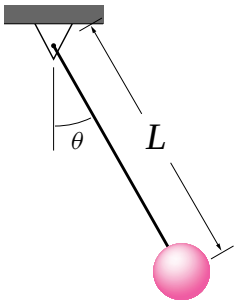
3.2 A point mass m is attached to a piston by two inextensible cables. There is gravity. Draw a free body diagram of the mass with a little bit of the cables.



problem 3.2:

(Filename:pfigure2.1.suspended.mass)

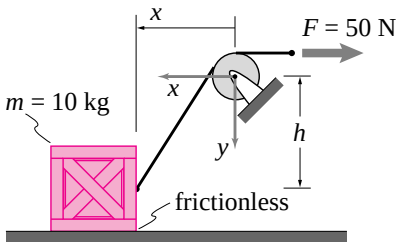
3.3 Simple pendulum. For the simple pendulum shown the "body"— the system of interest — is the mass and a little bit of the string. Draw a free body diagram of the system.



problem 3.3:

(Filename:pfigure.s94h2p1)

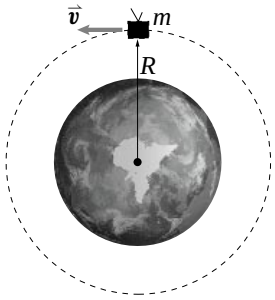
3.4 Draw a free body diagram of mass m at the instant shown in the figure. Evaluate the left hand side of the linear momentum balance equation ($\sum \vec{F} = m\vec{a}$) as explicitly as possible. Identify the unknowns in the expression.



problem 3.4:

(Filename:pfig2.2.rp1)

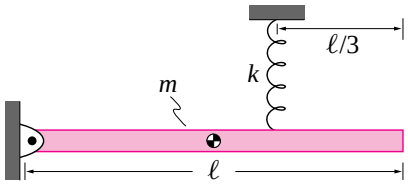
3.5 A 1000 kg satellite is in orbit. Its speed is v and its distance from the center of the earth is R. Draw a free body diagram of the satellite. Draw another that takes account of the slight drag force of the earth's atmosphere on the satellite.



problem 3.5:

(Filename:pfigure.s94h2p5)

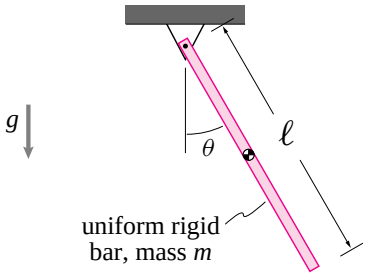
3.6 The uniform rigid rod shown in the figure hangs in the vertical plane with the support of the spring shown. Draw a free body diagram of the rod.



problem 3.6:

(Filename:pfig2.1.rp1)

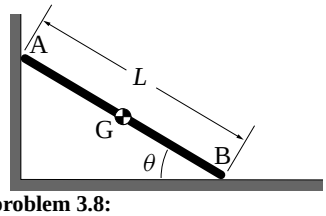
3.7 FBD of rigid body pendulum. The rigid body pendulum in the figure is a uniform rod of mass m. Draw a free body diagram of the rod.



problem 3.7:

(Filename:pfigure2.rod.pend.fbd)

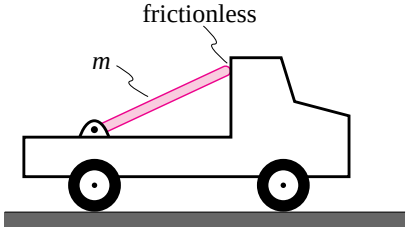
3.8 A thin rod of mass m rests against a frictionless wall and on a frictionless floor. There is gravity. Draw a free body diagram of the rod.



problem 3.8:

(Filename:ch2.6)

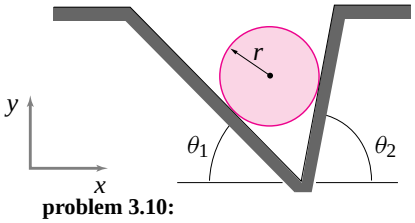
3.9 A uniform rod of mass m rests in the back of a flatbed truck as shown in the figure. Draw a free body diagram of the rod, set up a suitable coordinate system, and evaluate $\sum \vec{F}$ for the rod.



problem 3.9:

(Filename:pfig2.2.rp5)

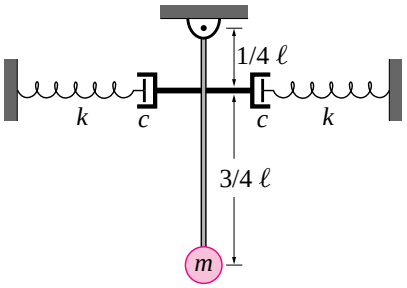
3.10 A disc of mass m sits in a wedge shaped groove. There is gravity and negligible friction. The groove that the disk sits in is part of an assembly that is still. Draw a free body diagram of the disk. (See also problems 4.15 and 6.47.)



problem 3.10:

(Filename:ch2.5)

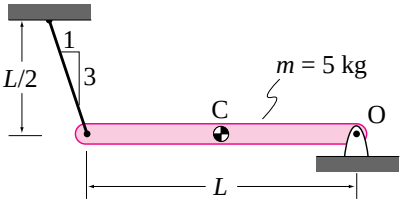
3.11 A pendulum, made up of a mass m attached at the end of a rigid massless rod of length ℓ, hangs in the vertical plane from a hinge. The pendulum is attached to a spring and a dashpot on each side at a point $\ell/4$ from the hinge point. Draw a free body diagram of the pendulum (mass and rod system) when the pendulum is slightly away from the vertical equilibrium position.



problem 3.11:

(Filename:pfig2.1.rp5)

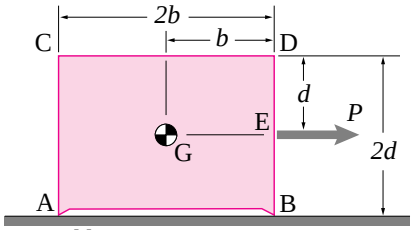
3.12 The left hand side of the angular momentum balance (Torque balance in statics) equation requires the evaluation of the sum of moments about some point. Draw a free body diagram of the rod shown in the figure and compute $\sum \vec{M}_O$ as explicitly as possible. Now compute $\sum \vec{M}_C$. How many unknown forces does each equation contain?



problem 3.12:

(Filename:pfig2.2.rp3)

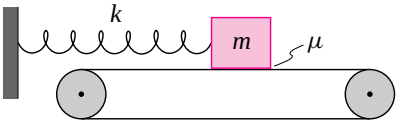
3.13 A block of mass m is sitting on a frictionless surface at points A and B and acted upon at point E by the force P. There is gravity. Draw a free body diagram of the block.



problem 3.13:

(Filename:ch2.1)

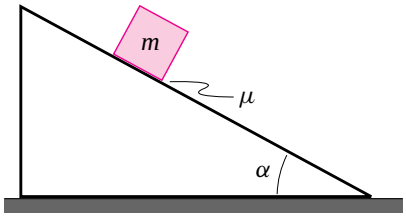
3.14 A mass-spring system sits on a conveyer belt. The spring is fixed to the wall on one end. The belt moves to the right at a constant speed v_0. The coefficient of friction between the mass and the belt is μ. Draw a free body diagram of the mass assuming it is moving to the left at the time of interest.



problem 3.14:

(Filename:pfig2.1.rp6)

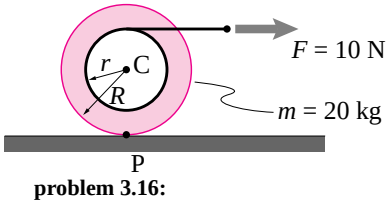
3.15 A small block of mass m slides down an incline with coefficient of friction μ. At an instant in time t during the motion, the block has speed v. Draw a free body diagram of the block.



problem 3.15:

(Filename:pfig2.3.rp5)

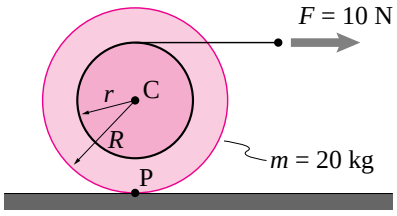
3.16 Assume that the wheel shown in the figure rolls without slipping. Draw a free body diagram of the wheel and evaluate $\sum \vec{F}$ and $\sum \vec{M}_C$. What would be different in the expressions obtained if the wheel were slipping?



problem 3.16:

(Filename:pfig2.2.rp4)

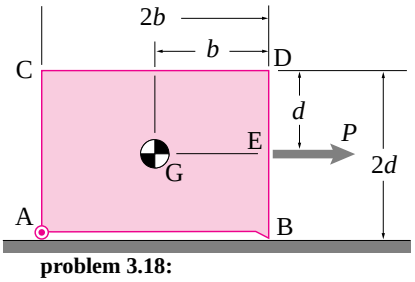
3.17 A compound wheel with inner radius r and outer radius R is pulled to the right by a 10 N force applied through a string wound around the inner wheel. Assume that the wheel rolls to the right without slipping. Draw a free body diagram of the wheel.



problem 3.17:

(Filename:pfig2.1.rp8)

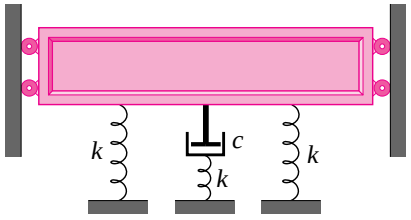
3.18 A block of mass m is sitting on a frictional surface and acted upon at point E by the horizontal force P through the center of mass. The block is resting on sharp edge at point B and is supported by a small ideal wheel at point A. There is gravity. Draw a free body diagram of the block including the wheel, assuming the block is sliding to the right with coefficient of friction μ at point B.



problem 3.18:

(Filename:ch2.2)

3.19 A spring-mass model of a mechanical system consists of a mass connected to three springs and a dashpot as shown in the figure. The wheels against the wall are in tracks (not shown) that do not let the wheels lift off the wall so the mass is constrained to move only in the vertical direction. Draw a free body diagram of the system.



problem 3.19:

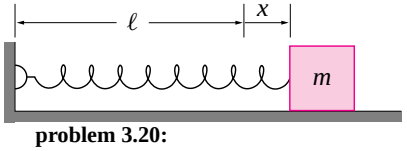
(Filename:pfig2.1.rp2)

3.20 A point mass of mass m moves on a frictionless surface and is connected to a spring with constant k and unstretched length ℓ. There is gravity. At the instant of interest, the mass has just been released at a distance x to the right from its position where the spring is unstretched.

a) Draw a free body diagram of the of the mass and spring together at the instant of interest.

b) Draw free body diagrams of the mass and spring separately at the instant of interest.

(See also problem 5.32.)



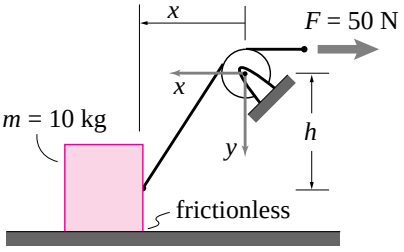
problem 3.20:

(Filename:ch2.10)

3.21 FBD of a block. The block of mass 10 kg is pulled by an inextensible cable over the pulley.

a) Assuming the block remains on the floor, draw a free diagram of the block.

b) Draw a free body diagram of the pulley and a little bit of the cable that rides over it.



problem 3.21:

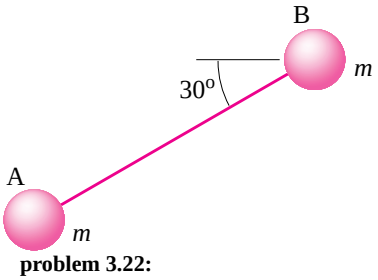
(Filename:pfigure2.1.block.pulley)

3.22 A pair of falling masses. Two masses A & B are spinning around each other and falling towards the ground. A string, which you can assume to be taught, connects the two masses. A snapshot of the system is shown in the figure. Draw free body diagrams of

a) mass A with a little bit of string,

b) mass B with a little bit of string, and

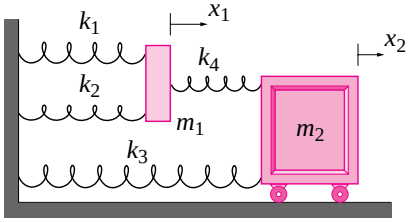
c) the whole system.



problem 3.22:

(Filename:pfigure.s94h2p4)

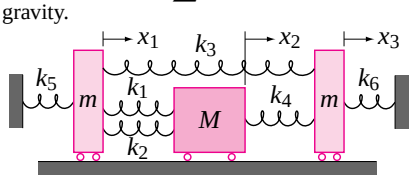
3.23 A two-degree of freedom spring-mass system is shown in the figure. Draw free body diagrams of each mass separately and then the two masses together.



problem 3.23:

(Filename:pfig2.1.rp4)

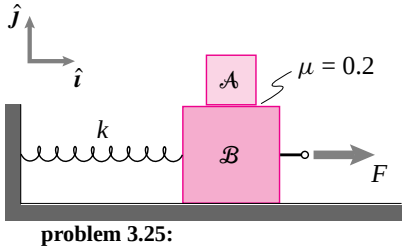
3.24 The figure shows a spring-mass model of a structure. Assume that the three masses are displaced to the right by x_1, x_2 and x_3 from the static equilibrium configuration such that $x_1 < x_2 < x_3$. Draw free body diagrams of each mass and evaluate $\sum \vec{F}$ in each case. Ignore gravity.



problem 3.24:

(Filename:pfig2.2.rp10)

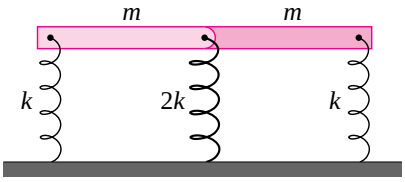
3.25 In the system shown, assume that the two masses A and B move together (i.e., no relative slip). Draw a free body diagram of mass A and evaluate the left hand side of the linear momentum balance equation. Repeat the procedure for the system consisting of both masses.



problem 3.25:

(Filename:pfig2.2.rp2)

3.26 Two identical rigid rods are connected together by a pin. The vertical stiffness of the system is modeled by three springs as shown in the figure. Draw free body diagrams of each rod separately. [This problem is a little tricky and there is more than one reasonable answer.]



problem 3.26:

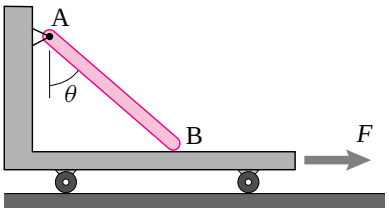
(Filename:pfig2.1.rp3)

3.27 A uniform rod rests on a cart which is being pulled to the right. The rod is hinged at one end (with a frictionless hinge) and has no friction at the contact with the cart. The cart rolls on massless wheels that have no bearing friction (ideal massless wheels). Draw FBD's of

a) the rod,

b) the cart, and

c) the whole system.



problem 3.27:

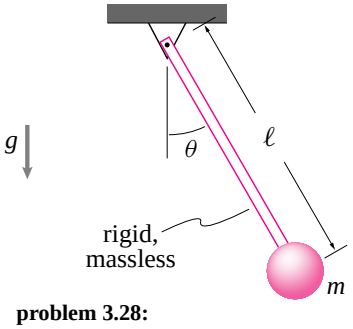
(Filename:pfigure.s94h2p6)

3.28 FBD's of simple pendulum and its parts. The simple pendulum in the figure is composed of a rod of negligible mass and a pendulum bob of mass m.

a) Draw a free body diagram of the pendulum bob.

b) Draw a free body diagram of the rod.

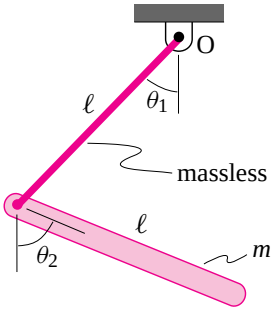
c) Draw a free body diagram of the rod and pendulum bob together.



problem 3.28:

(Filename:pfigure2.simp.pend.fbd)

3.29 For the double pendulum shown in the figure, evaluate $\sum \vec{F}$ and $\sum \vec{M}_O$ at the instant shown in terms of the given quantities and unknown forces (if any) on the bar.

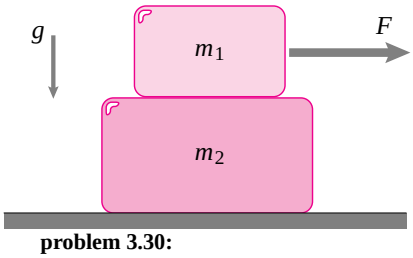


problem 3.29:

(Filename:pfig2.2.rp6)

3.30 See also problem 11.4. Two frictionless blocks sit stacked on a frictionless surface. A force F is applied to the top block. There is gravity.

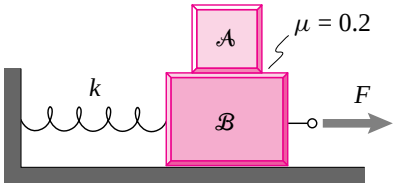
a) Draw a free body diagram of the two blocks together and a free body diagram of each block separately.



problem 3.30:

(Filename:ch2.3)

3.31 For the system shown in the figure draw free body diagrams of each mass separately assuming that there is no relative slip between the two masses.



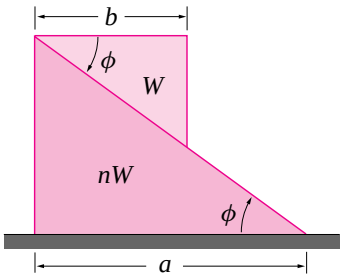
problem 3.31:

(Filename:pfig2.1.rp7)

3.32 Two frictionless prisms of similar right triangular sections are placed on a frictionless horizontal plane. The top prism weighs W and the lower one, nW. Draw free body diagrams of

a) the system of prisms and

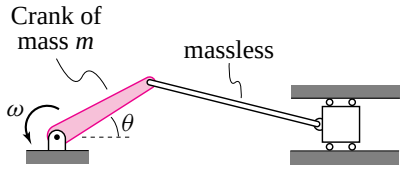
b) each prism separately.



problem 3.32:

(Filename:pfigure.blue.28.2)

3.33 In the slider crank mechanism shown, draw a free body diagram of the crank and evaluate $\sum \vec{F}$ and $\sum \vec{M}_O$ as explicitly as possible.



problem 3.33:

(Filename:pfig2.2.rp9)

3.34 FBD of an arm throwing a ball. An arm throws a ball up. A crude model of an arm is that it is made of four rigid bodies (shoulder, upper arm, forearm and a hand) that are connected with hinges. At each hinge there are muscles that apply torques between the links. Draw a FBD of

a) the ball, the shoulder (fixed to the wall),

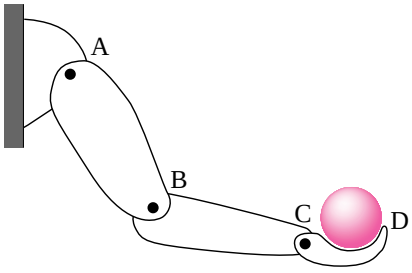
b) the upper arm,

c) the fore-arm,

d) the hand, and

e) the whole arm (all four parts) including the ball.

Write the equation of angular momentum balance about the shoulder joint A, evaluating the left-hand-side as explicitly as possible.



problem 3.34:

(Filename:efig2.1.23)

3.35 An imagined testing machine consists of a box fastened to a wheel as shown. The box always moves so that its floor is parallel to the ground (like an empty car on a Ferris Wheel). Two identical masses, A and B are connected together by cords 1 and 2 as shown. The floor of the box is frictionless. The machine and blocks are set in motion when $\theta = 0^o$, with constant $\dot{\theta} = 3$ rad/s. Draw free body diagrams of:

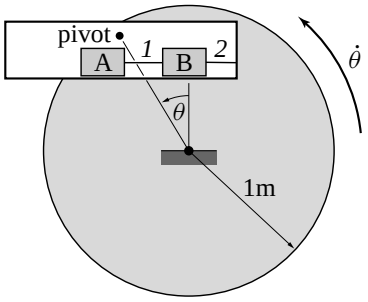
a) the system consisting of the box, blocks, and wheel,

b) the system of box and blocks,

c) the system of blocks and cords,

d) the system of box, block B, cord 2, and a portion of cord 1 and,

e) the box and blocks separately.



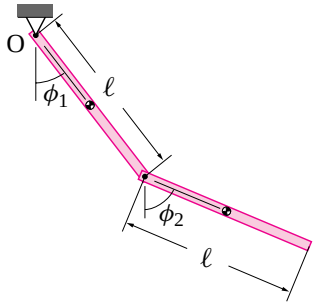
problem 3.35:

(Filename:pfigure.blue.52.1)

3.36 Free body diagrams of a double 'physical' pendulum. A double pendulum is a system where one pendulum hangs from another. Draw free body diagrams of various subsystems in a typical configuration.

a) Draw a free body diagram of the lower stick, the upper stick, and both sticks in arbitrary configurations.

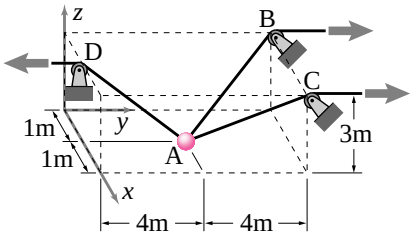
b) Repeat part (a) but use the simplifying assumption that the upper bar has negligible mass.



problem 3.36:

(Filename:pfigure.s94h2p3)

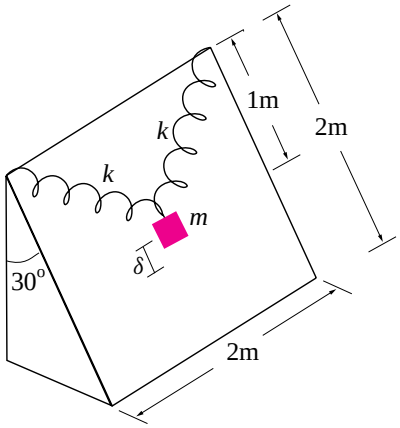
3.37 The strings hold up the mass $m = 3$ kg. There is gravity. Draw a free body diagram of the mass.



problem 3.37:

(Filename:pfigure2.1.3D.pulley.fbd)

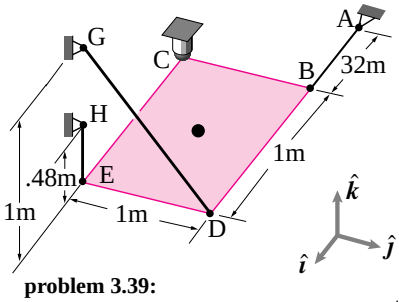
3.38 Mass on inclined plane. A block of mass m rests on a frictionless inclined plane. It is supported by two stretched springs. The mass is pulled down the plane by an amount δ and released. Draw a FBD of the mass just after it is released.



problem 3.38:

(Filename:efig2.1.24)

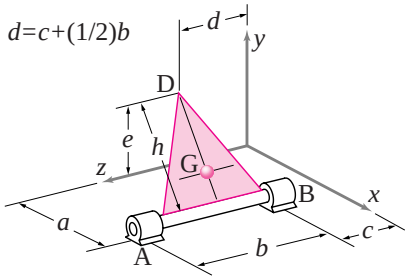
3.39 Hanging a shelf. A shelf with negligible mass supports a 0.5 kgmass at its center. The shelf is supported at one corner with a ball and socket joint and the other three corners with strings. At the moment of interest the shelf is in a rocket in outer space and accelerating at $10\,\mathrm{m/s^2}$ in the $\boldsymbol{k}$ direction. The shelf is in the xy plane. Draw a FBD of the shelf.



problem 3.39:

(Filename:ch3.14)

3.40 A massless triangular plate rests against a frictionless wall at point D and is rigidly attached to a massless rod supported by two ideal bearings. A ball of mass m is fixed to the centroid of the plate. There is gravity. Draw a free body diagram of the plate, ball, and rod as a system.



problem 3.40:

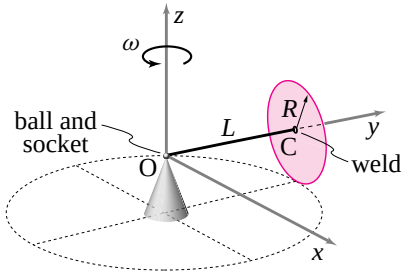
(Filename:ch2.9)

3.41 An undriven massless disc rests on its edge on a frictional surface and is attached rigidly by a weld at point C to the end of a rod that pivots at its other end about a ball-and-socket joint at point O. There is gravity.

a) Draw a free body diagram of the disk and rod together.

b) Draw free body diagrams of the disc and rod separately.

c) What would be different in the free body diagram of the rod if the ball-and-socket was rusty (not ideal)?



problem 3.41:

(Filename:ch2.7)

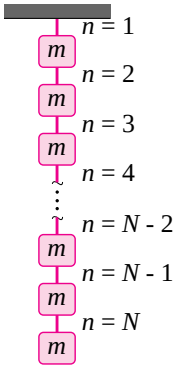
Problems for Chapter 4

Statics

4.1 Static equilibrium of one body

4.1 To evaluate the equation $\sum \vec{F} = m\vec{a}$ for some problem, a student writes $\sum \vec{F} = F_x\hat{\imath} - (F_y - 30\,\text{N})\hat{\jmath} + 50\,\text{N}\hat{k}$ in the xyz coordinate system, but $\vec{a} = 2.5\,\text{m/s}^2\hat{\imath}' + 1.8\,\text{m/s}^2\hat{\jmath}' - a_z\hat{k}'$ in a rotated $x'y'z'$ coordinate system. If $\hat{\imath}' = \cos 60^\text{o}\hat{\imath} + \sin 60^\text{o}\hat{\jmath}$, $\hat{\jmath}' = -\sin 60^\text{o}\hat{\imath} + \cos 60^\text{o}\hat{\jmath}$ and $\hat{k}' = \hat{k}$, find the scalar equations for the x', y', and z' directions.

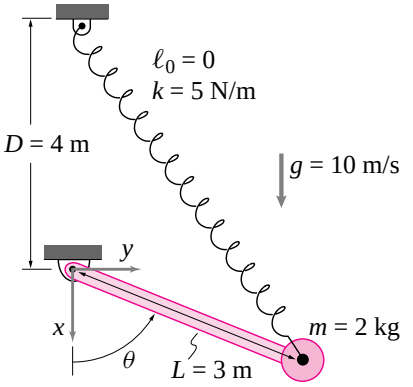
4.2 N small blocks each of mass m hang vertically as shown, connected by N inextensible strings. Find the tension T_n in string n. *



problem 4.2:

(Filename:pfigure2.hanging.masses)

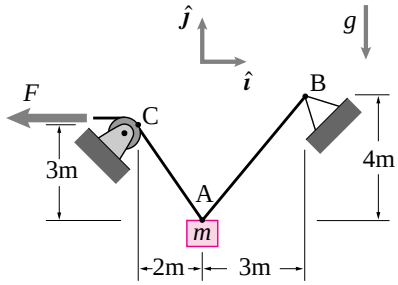
4.3 See also problem 7.98. A zero length spring (relaxed length $\ell_0 = 0$) with stiffness $k = 5\,\text{N/m}$ supports the pendulum shown. Assume $g = 10\,\text{N/m}$. Find θ for static equilibrium. *



problem 4.3:

(Filename:pfigure2.blue.80.2.a)

4.4 What force should be applied to the end of the string over the pulley at C so that the mass at A is at rest?



problem 4.4:

(Filename:f92h1p1.a)

4.2 Elementary truss analysis

The first set of problems concerns math skills that can be used to help solve truss problems. If computer solution is not going to be used, the following problems can be skipped.

4.5 Write the following equations in matrix form to solve for x, y, and z:

$$\begin{aligned} 2x - 3y + 5 &= 0, \\ y + 2\pi z &= 21, \\ \frac{1}{3}x - 2y + \pi z - 11 &= 0. \end{aligned}$$

4.6 Are the following equations linearly independent?

a) $x_1 + 2x_2 + x_3 = 30$

b) $3x_1 + 6x_2 + 9x_3 = 4.5$

c) $2x_1 + 4x_2 + 15x_3 = 7.5$.

4.7 Write computer commands (or a program) to solve for x, y and z from the following equations with r as an input variable. Your program should display an error message if, for a particular r, the equations are not linearly independent.

a) $5x + 2r\,y + z = 2$

b) $3x + 6y + (2r - 1)z = 3$

c) $2x + (r - 1)y + 3r\,z = 5$.

Find the solutions for $r = 3,\ 4.99$, and 5.

4.8 An exam problem in statics has three unknown forces. A student writes the following three equations (he knows that he needs three equations for three unknowns!) — one for the force balance in the x-direction and the other two for the moment balance about two different points.

a) $F_1 - \frac{1}{2}F_2 + \frac{1}{\sqrt{2}}F_3 = 0$

b) $2F_1 + \frac{3}{2}F_2 = 0$

c) $\frac{5}{2}F_2 + \sqrt{2}F_3 = 0$.

Can the student solve for F_1, F_2, and F_3 uniquely from these equations? *

4.9 What is the solution to the set of equations:

$$\begin{aligned} x + y + z + w &= 0 \\ x - y + z - w &= 0 \\ x + y - z - w &= 0 \\ x + y + z - w &= 2? \end{aligned}$$

4.3 Advanced truss analysis: determinacy, rigidity, and redundancy

4.4 Internal forces

4.5 Springs

4.10 What is the stiffness of two springs in parallel?

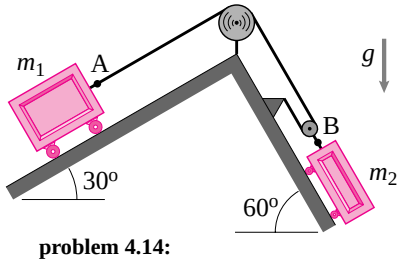
4.11 What is the stiffness of two springs in series?

4.12 What is the apparant stiffness of a pendulum when pushed sideways.

4.13 Optimize a triangular truss for stiffness and for strength and show that the resulting design is not the same.

4.6 Structures and machines

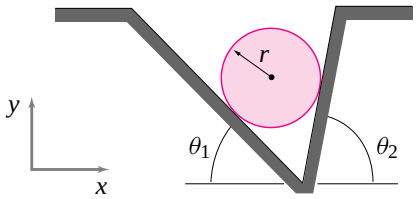
4.14 See also problems 6.18 and 6.19. Find the ratio of the masses m_1 and m_2 so that the system is at rest.



problem 4.14:

(Filename:pulley4.c)

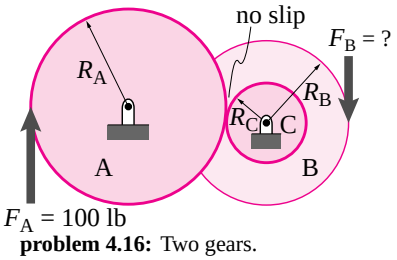
4.15 (See also problem 6.47.) What are the forces on the disk due to the groove? Define any variables you need.



problem 4.15:

(Filename:ch2.5.b)

4.16 Two gears at rest. See also problems 7.77 and **??**. At the input to a gear box, a 100 lbf force is applied to gear A. At the output, the machinery (not shown) applies a force of F_B to the output gear. Assume the system of gears is at rest. What is F_B?



problem 4.16: Two gears.

(Filename:pg131.3.a)

4.17 See also problem 4.18. A reel of mass M and outer radius R is connected by a horizontal string from point P across a pulley to a hanging object of mass m. The inner cylinder of the reel has radius $r = \frac{1}{2}R$. The slope has angle θ. There is no slip between the reel and the slope. There is gravity.

a) Find the ratio of the masses so that the system is at rest. *

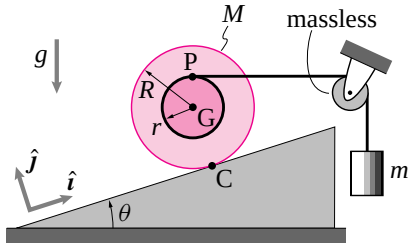
b) Find the corresponding tension in the string, in terms of M, g, R, and θ. *

c) Find the corresponding force on the reel at its point of contact with the slope, point C, in terms of M, g, R, and θ. *

d) **Another look at equilibrium.** [Harder] Draw a careful sketch and find a point where the lines of action of the gravity force and string tension intersect. For the reel to be in static equilibrium, the line of action of the reaction force at C must pass through this point. Using this information, what must the tangent of the angle ϕ of the reaction force at C be, measured with respect to the normal to the slope? Does this answer agree with that you would obtain from your answer in part(c)? *

e) What is the relationship between the angle ψ of the reaction at C, measured with respect to the normal to the ground, and the mass ratio required for static equilibrium of the reel? *

Check that for $\theta = 0$, your solution gives $\frac{m}{M} = 0$ and $\vec{F}_C = Mg\hat{j}$ and for $\theta = \frac{\pi}{2}$, it gives $\frac{m}{M} = 2$ and $\vec{F}_C = Mg(\hat{i} + 2\hat{j})$.



problem 4.17:

(Filename:pfigure2.blue.47.3.a)

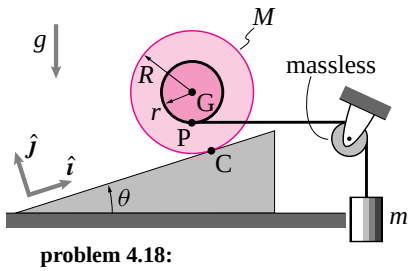
4.18 This problem is identical to problem 4.17 except for the location of the connection point of the string to the reel, point P. A reel of mass M and outer radius R is connected by an inextensible string from point P across a pulley to a hanging object of mass m. The inner cylinder of the reel has radius $r = \frac{1}{2}R$. The slope has angle θ. There is no slip between the reel and the slope. There is gravity. In terms of M, m, R, and θ, find:

a) the ratio of the masses so that the system is at rest, *

b) the corresponding tension in the string, and *

c) the corresponding force on the reel at its point of contact with the slope, point C. *

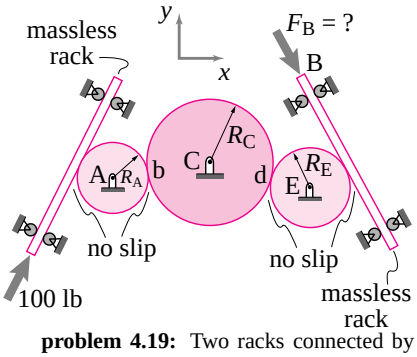
Check that for $\theta = 0$, your solution gives $\frac{m}{M} = 0$ and $\vec{F}_C = Mg\hat{j}$ and for $\theta = \frac{\pi}{2}$, it gives $\frac{m}{M} = -2$ and $\vec{F}_C = Mg(\hat{i} - 2\hat{j})$.The negative mass ratio is impossible since mass cannot be negative and the negative normal force is impossible unless the wall or the reel or both can 'suck' or they can 'stick' to each other (that is, provide some sort of suction, adhesion, or magnetic attraction).



problem 4.18:

(Filename:pfigure2.blue.47.3.b)

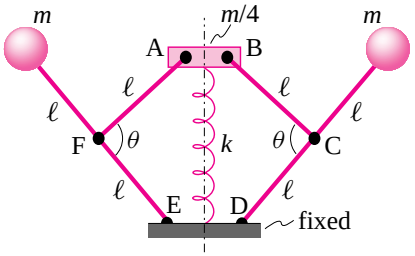
4.19 Two racks connected by three gears at rest. See also problem 7.86. A 100 lbf force is applied to one rack. At the output, the machinery (not shown) applies a force of F_B to the other rack. Assume the gear-train is at rest. What is F_B?



problem 4.19: Two racks connected by three gears.

(Filename:ch4.5.a)

4.20 In the flyball governor shown, the mass of each ball is $m = 5$ kg, and the length of each link is $\ell = 0.25$ m. There are frictionless hinges at points A, B, C, D, E, F where the links are connected. The central collar has mass $m/4$. Assuming that the spring of constant $k = 500$ N/m is uncompressed when $\theta = \pi$ radians, what is the compression of the spring?



problem 4.20:

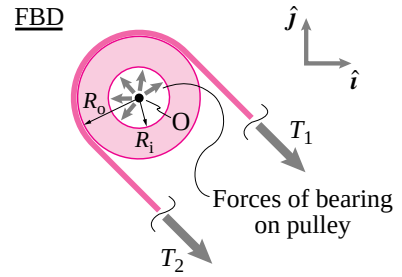
(Filename:summer95p2.2.a)

4.21 Assume a massless pulley is round and has outer radius R_2. It slides on a shaft that has radius R_i. Assume there is friction between the shaft and the pulley with coefficient of friction μ, and friction angle ϕ defined by $\mu = \tan(\phi)$. Assume the two ends of the line that are wrapped around the pulley are parallel.

a) What is the relation between the two tensions when the pulley is turning? You may assume that the bearing shaft touches the hole in the pulley at only one point. * .

b) Plug in some reasonable numbers for R_i, R_o and μ (or ϕ) to see one reason why wheels (say pulleys) are such a good idea even when the bearings are not all that well lubricated. *

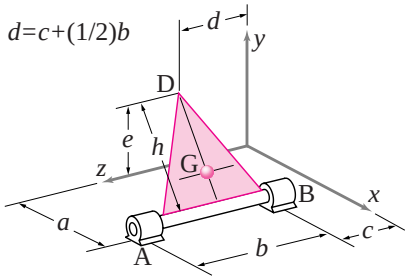
c) (optional) To further emphasize the point look at the relation between the two string tensions when the bearing is locked (frozen, welded) and the string slides on the pulley with same coefficient of friction μ (see, for example, Beer and Johnston Statics section 8.10). Look at the force ratios from parts (a) and (b) for a reasonable value of μ, say $\mu = 0.2$. *



problem 4.21:

(Filename:pfigure.blue.20.2)

4.22 A massless triangular plate rests against a frictionless wall at point D and is rigidly attached to a massless rod supported by two ideal bearings fixed to the floor. A ball of mass m is fixed to the centroid of the plate. There is gravity and the system is at rest What is the reaction at point D on the plate?



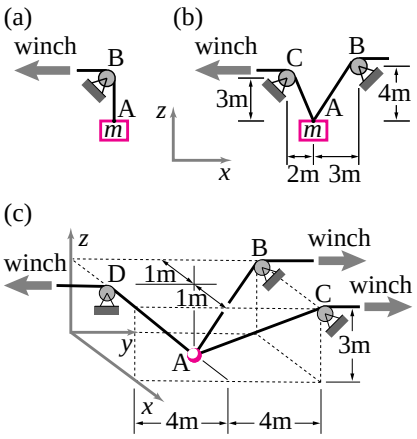
problem 4.22:

(Filename:ch3.1b)

4.7 Hydrostatics

4.8 Advanced statics

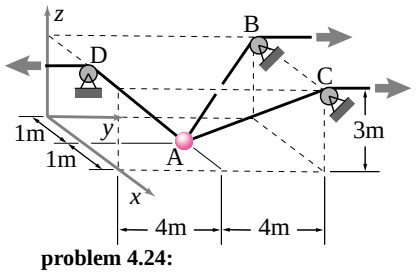
4.23 See also problem 5.119. For the three cases (a), (b), and (c), below, find the tension in the string AB. In all cases the strings hold up the mass $m = 3\,\text{kg}$. You may assume the local gravitational constant is $g = 10\,\text{m/s}^2$. In all cases the winches are pulling in the string so that the velocity of the mass is a constant 4 m/s upwards (in the $\hat{\boldsymbol{k}}$ direction). [Note that in problems (b) and (c), in order to pull the mass up at constant rate the winches must pull in the strings at an unsteady speed.] *



problem 4.23:

(Filename:f92h1p1)

4.24 The strings hold up the mass $m = 3\,\text{kg}$. You may assume the local gravitational constant is $g = 10\,\text{m/s}^2$. Find the tensions in the strings if the mass is at rest.



problem 4.24:

(Filename:f92h1p1.b)

4.25 Hanging a shelf. A uniform 5 kg shelf is supported at one corner with a ball and socket joint and the other three corners with strings. At the moment of interest the shelf is at rest. Gravity acts in the $-\hat{\boldsymbol{k}}$ direction. The shelf is in the xy plane.

a) Draw a FBD of the shelf.

b) Challenge: without doing any calculations on paper can you find one of the reaction force components or the tension in any of the cables? Give yourself a few minutes of staring to try to find this force. If you can't, then come back to this question after you have done all the calculations.

c) Write down the equation of force equilibrium.

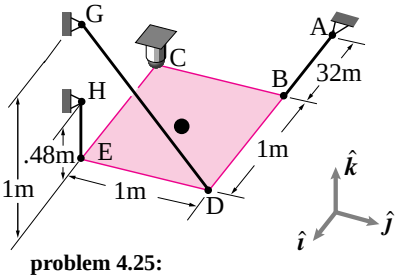
d) Write down the moment balance equation using the center of mass as a reference point.

e) By taking components, turn (b) and (c) into six scalar equations in six unknowns.

f) Solve these equations by hand or on the computer.

g) Instead of using a system of equations try to find a single equation which can be solved for T_{EH}. Solve it and compare to your result from before. *

h) Challenge: For how many of the reactions can you find one equation which will tell you that particular reaction without knowing any of the other reactions? [Hint, try moment balance about various axes as well as aforce balance in an appropriate direction. It is possible to find five of the six unknown reaction components this way.] Must these solutions agree with (d)? Do they?



problem 4.25:

(Filename:pfigure.s94h2p10.a)

Problems for Chapter 5

Unconstrained motion of particles

5.1 Force and motion in 1D

5.1 In elementary physics, people say "$F = ma$" What is a more precise statement of an equation we use here that reduces to $F = ma$ for one-dimensional motion of a particle?

5.2 Does linear momentum depend on reference point? (Assume all candidate points are fixed in the same Newtonian reference frame.)

5.3 The distance between two points in a bicycle race is 10 km. How many minutes does a bicyclist take to cover this distance if he/she maintains a constant speed of 15 mph.

5.4 Given that $\dot{x} = k_1 + k_2 t$, $k_1 = 1\,\text{ft/s}$, $k_2 = 1\,\text{ft/s}^2$, and $x(0) = 1\,\text{ft}$, what is $x(10\,\text{s})$?

5.5 Find $x(3\,\text{s})$ given that

$$\dot{x} = x/(1\,\text{s}) \quad \text{and} \quad x(0) = 1\,\text{m}$$

or, expressed slightly differently,

$$\dot{x} = cx \quad \text{and} \quad x(0) = x_0,$$

where $c = 1\,\text{s}^{-1}$ and $x_0 = 1\,\text{m}$. Make a sketch of x versus t. *

5.6 Given that $\dot{x} = A\sin[(3\,\text{rad/s})t]$, $A = 0.5\,\text{m/s}$, and $x(0) = 0\,\text{m}$, what is $x(\pi/2\,\text{s})$?

5.7 Given that $\dot{x} = x/\,\text{s}$ and $x(0\,\text{s}) = 1\,\text{m}$, find $x(5\,\text{s})$.

5.8 Let $a = \frac{dv}{dt} = -kv^2$ and $v(0) = v_0$. Find t such that $v(t) = \frac{1}{2}v_0$.

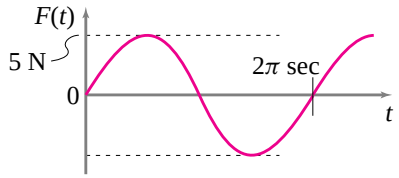
5.9 A sinusoidal force acts on a 1 kg mass as shown in the figure and graph below. The mass is initially still; i. e.,

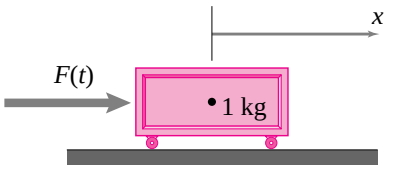
$$x(0) = v(0) = 0.$$

a) What is the velocity of the mass after 2π seconds?

b) What is the position of the mass after 2π seconds?

c) Plot position x versus time t for the motion.





problem 5.9:

(Filename:pfigure.blue.6.1)

5.10 A motorcycle accelerates from 0 mph to 60 mph in 5 seconds. Find the average acceleration in m/s^2. How does this acceleration compare with g, the acceleration of an object falling near the earth's surface?

5.11 A particle moves along the x-axis with an initial velocity $v_x = 60\,\text{m/s}$ at the origin when $t = 0$. For the first 5 s it has no acceleration, and thereafter it is acted upon by a retarding force which gives it a constant acceleration $a_x = -10m/s^2$. Calculate the velocity and the x-coordinate of the particle when $t = 8\,\text{s}$ and when $t = 12\,\text{s}$, and find the maximum positive x coordinate reached by the particle.

5.12 The linear speed of a particle is given as $v = v_0 + at$, where v is in m/s, $v_0 = 20\,\text{m/s}$, $a = 2\,\text{m/s}^2$, and t is in seconds. Define appropriate dimensionless variables and write a dimensionless equation that describes the relation of v and t.

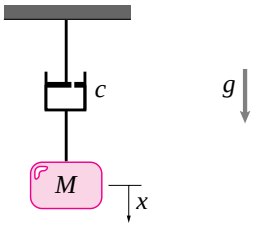
5.13 A ball of mass m has an acceleration $\vec{a} = cv^2\hat{\imath}$. Find the position of the ball as a function of velocity.

5.14 A ball of mass m is dropped from rest at a height h above the ground. Find the position and velocity as a function of time. Neglect air friction. When does the ball hit the ground? What is the velocity of the ball just before it hits?

5.15 A ball of mass m is dropped vertically from rest at a height h above the ground. Air resistance causes a drag force on the ball directly proportional to the speed v of the ball, $F_d = bv$. The drag force acts in a direction opposite to the direction of motion. Find the velocity and position of the ball as a function of time. Find the velocity as a function of position. Gravity is non-negligible, of course.

5.16 A grain of sugar falling through honey has a negative acceleration proportional to the difference between its velocity and its 'terminal' velocity (which is a known constant v_t). Write this sentence as a differential equation, defining any constants you need. Solve the equation assuming some given initial velocity v_0. [hint: acceleration is the time-derivative of velocity]

5.17 The mass-dashpot system shown below is released from rest at $x = 0$. Determine an equation of motion for the particle of mass m that involves only $\dot{x}$ and x (a first-order ordinary differential equation). The damping coefficient of the dashpot is c.



problem 5.17:

(Filename:pfigure.blue.151.2)

5.18 Due to gravity, a particle falls in air with a drag force proportional to the speed squared.

(a) Write $\sum \vec{F} = m\vec{a}$ in terms of variables you clearly define,

(b) find a constant speed motion that satisfies your differential equation,

(c) pick numerical values for your constants and for the initial height. Assume the initial speed is zero

(i) set up the equation for numerical solution,

(ii) solve the equation on the computer,

(iii) make a plot with your computer solution and show how that plot supports your answer to (b).

5.19 A ball of mass m is dropped vertically from rest at a height h above the ground. Air resistance causes a drag force on the ball proportional to the speed of the ball squared, $F_d = cv^2$. The drag force acts in a direction opposite to the direction of motion. Find the velocity as a function of position.

5.20 A force pulls a particle of mass m towards the origin according to the law (assume same equation works for $x > 0, x < 0$)

$$F = Ax + Bx^2 + C\dot{x}$$

Assume $\dot{x}(0) = 0$.
Using numerical solution, find values of A, B, C, m, and x_0 so that

(a) the mass never crosses the origin,

(b) the mass crosses the origin once,

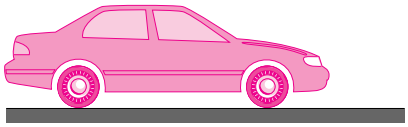
(c) the mass crosses the origin many times.

5.21 A car accelerates to the right with constant acceleration starting from a stop. There is wind resistance force proportional to the square of the speed of the car. Define all constants that you use.

a) What is its position as a function of time?

b) What is the total force (sum of *all* forces) on the car as a function of time?

c) How much power P is required of the engine to accelerate the car in this manner (as a function of time)?



problem 5.21: Car.

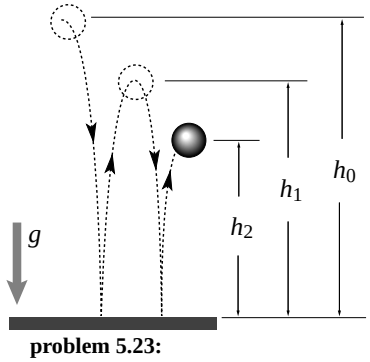
(Filename:s97p1.2)

5.22 A ball of mass m is dropped vertically from a height h. The only force acting on the ball in its flight is gravity. The ball strikes the ground with speed v^- and after collision it rebounds vertically with reduced speed v^+ directly proportional to the incoming speed, $v^+ = ev^-$, where $0 < e < 1$. What is the maximum height the ball reaches after one bounce, in terms of h, e, and g. *

a) Do this problem using linear momentum balance and setting up and solving the related differential equations and "jump" conditions at collision.

b) Do this problem again using energy balance.

5.23 A ball is dropped from a height of $h_0 = 10\,\text{m}$ onto a hard surface. After the first bounce, it reaches a height of $h_1 = 6.4\,\text{m}$. What is the vertical coefficient of restitution, assuming it is decoupled from tangential motion? What is the height of the second bounce, h_2?



problem 5.23:

(Filename:Danef94s1q7)

5.24 In problem 5.23, show that the number of bounces goes to infinity in finite time, assuming that the vertical coefficient is fixed. Find the time in terms of the initial height h_0, the coefficient of restitution, e, and the gravitational constant, g.

5.2 Energy methods in 1D

5.25 The power available to a very strong accelerating cyclist is about 1 horsepower. Assume a rider starts from rest and uses this constant power. Assume a mass (bike + rider) of 150 lbm, a realistic drag force of $.006\,\text{lbf}/(\text{ft}/\text{s})^2 v^2$. Neglect other drag forces.

(a) What is the peak speed of the cyclist?

(b) Using analytic or numerical methods make a plot of speed vs. time.

(c) What is the acceleration as $t \to \infty$ in this solution?

(d) What is the acceleration as $t \to 0$ in your solution?

5.26 Given $\dot{v} = -\frac{gR^2}{r^2}$, where g and R are constants and $v = \frac{dr}{dt}$. Solve for v as a function of r if $v(r = R) = v_0$. [Hint: Use the chain rule of differentiation to eliminate t, i.e., $\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = \frac{dv}{dr} \cdot v$. Or find a related dynamics problem and use conservation of energy.]

• • •

Also see several problems in the harmonic oscillator section.

5.3 The harmonic oscillator

The first set of problems are entirely about the harmonic oscillator governing differential equation, with no mechanics content or context.

5.27 Given that $\ddot{x} = -(1/\text{s}^2)x$, $x(0) = 1\,\text{m}$, and $\dot{x}(0) = 0$ find:

a) $x(\pi\ \text{s}) =$?

b) $\dot{x}(\pi\ \text{s}) =$?

5.28 Given that $\ddot{x} + x = 0$, $x(0) = 1$, and $\dot{x}(0) = 0$, find the value of x at $t = \pi/2$ s.

5.29 Given that $\ddot{x} + \lambda^2 x = C_0$, $x(0) = x_0$, and $\dot{x}(0) = 0$, find the value of x at $t = \pi/\lambda$ s.

• • •

The next set of problems concern one mass connected to one or more springs and possibly with a constant force applied.

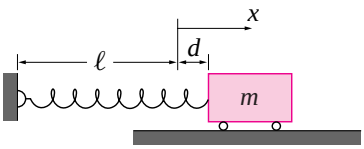
5.30 Consider a mass m on frictionless rollers. The mass is held in place by a spring with stiffness k and rest length ℓ. When the spring is relaxed the position of the mass is $x = 0$. At times $t = 0$ the mass is at $x = d$ and is let go with no velocity. The gravitational constant is g. In terms of the quantities above,

a) What is the acceleration of the block at $t = 0^+$?

b) What is the differential equation governing $x(t)$?

c) What is the position of the mass at an arbitrary time t?

d) What is the speed of the mass when it passes through $x = 0$?



problem 5.30:

(Filename:pfigure.blue.25.1)

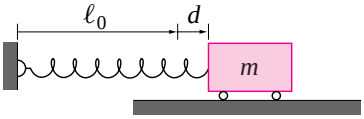
5.31 Spring and mass. A spring with rest length ℓ_0 is attached to a mass m which slides frictionlessly on a horizontal ground as shown. At time $t = 0$ the mass is released with no initial speed with the spring stretched a distance d. *[Remember to define any coordinates or base vectors you use.]*

a) What is the acceleration of the mass just after release?

b) Find a differential equation which describes the horizontal motion of the mass.

c) What is the position of the mass at an arbitrary time t?

d) What is the speed of the mass when it passes through the position where the spring is relaxed?



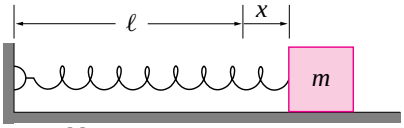
problem 5.31:

(Filename:s97f1)

5.32 Reconsider the spring-mass system in problem 3.20. Let $m = 2\,\text{kg}$ and $k = 5\,\text{N/m}$. The mass is pulled to the right a distance $x = x_0 = 0.5\,\text{m}$ from the unstretched position and released from rest. At the instant of release, no external forces act on the mass other than the spring force and gravity.

a) What is the initial potential and kinetic energy of the system?

b) What is the potential and kinetic energy of the system as the mass passes through the static equilibrium (unstretched spring) position?



problem 5.32:

(Filename:ch2.10.a)

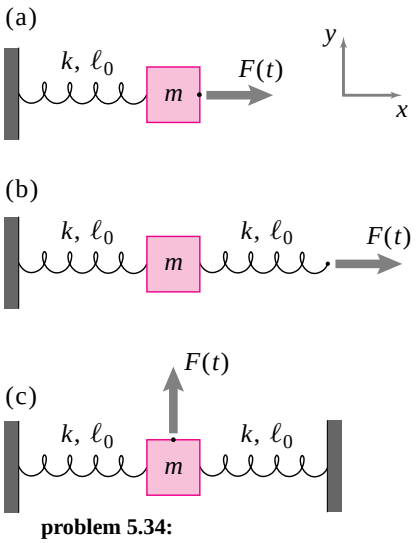
5.33 Reconsider the spring-mass system from problem 5.30.

a) Find the potential and kinetic energy of the spring mass system as functions of time.

b) Using the computer, make a plot of the potential and kinetic energy as a function of time for several periods of oscillation. Are the potential and kinetic energy ever equal at the same time? If so, at what position $x(t)$?

c) Make a plot of kinetic energy versus potential energy. What is the phase relationship between the kinetic and potential energy?

5.34 For the three spring-mass systems shown in the figure, find the equation of motion of the mass in each case. All springs are massless and are shown in their relaxed states. Ignore gravity. (In problem (c) assume vertical motion.) *



problem 5.34:

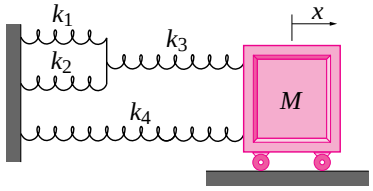
(Filename:summer95f.3)

5.35 A spring and mass system is shown in the figure.

a) First, as a review, let k_1, k_2, and k_3 equal zero and k_4 be nonzero. What is the natural frequency of this system?

b) Now, let all the springs have non-zero stiffness. What is the stiffness of a single spring equivalent to the combination of k_1, k_2, k_3, k_4? What is the frequency of oscillation of mass M?

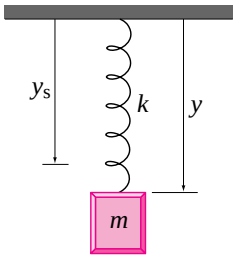
c) What is the equivalent stiffness, k_{eq}, of all of the springs together. That is, if you replace all of the springs with one spring, what would its stiffness have to be such that the system has the same natural frequency of vibration?



problem 5.35:

(Filename:pfigure.blue.159.3)

5.36 The mass shown in the figure oscillates in the vertical direction once set in motion by displacing it from its static equilibrium position. The position $y(t)$ of the mass is measured from the fixed support, taking downwards as positive. The static equilibrium position is y_s and the relaxed length of the spring is ℓ_0. At the instant shown, the position of the mass is y and its velocity $\dot{y}$, directed downwards. Draw a free body diagram of the mass at the instant of interest and evaluate the left hand side of the energy balance equation ($P = \dot{E}_K$).



problem 5.36:

(Filename:pfig2.3.rp1)

5.37 Mass hanging from a spring. A mass m is hanging from a spring with constant k which has the length l_0 when it is relaxed (i. e., when no mass is attached). It only moves vertically.

a) Draw a Free Body Diagram of the mass.

b) Write the equation of linear momentum balance. *

c) Reduce this equation to a standard differential equation in x, the position of the mass. *

d) Verify that one solution is that $x(t)$ is constant at $x = l_0 + mg/k$.

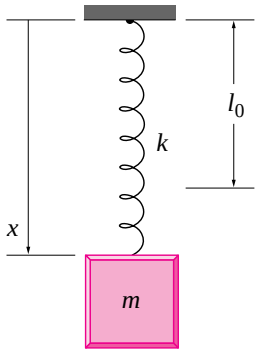
e) What is the meaning of that solution? (That is, describe in words what is going on.) *

f) Define a new variable $\hat{x} = x - (l_0 + mg/k)$. Substitute $x = \hat{x} + (l_0 + mg/k)$ into your differential equation and note that the equation is simpler in terms of the variable $\hat{x}$. *

g) Assume that the mass is released from an an initial position of $x = D$. What is the motion of the mass? *

h) What is the period of oscillation of this oscillating mass? *

i) Why might this solution not make physical sense for a long, soft spring if $D > \ell_0 + 2mg/k$)? *

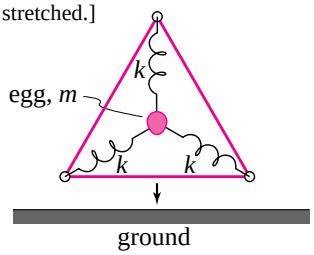


problem 5.37:

(Filename:pg141.1)

The following problem concerns simple harmonic motion for part of the motion. It involves pasting together solutions.

5.38 One of the winners in the egg-drop contest sponsored by a local chapter of ASME each spring, was a structure in which rubber bands held the egg at the center of it. In this problem, we will consider the simpler case of the egg to be a particle of mass m and the springs to be linear devices of spring constant k. We will also consider only a two-dimensional version of the winning design as shown in the figure. If the frame hits the ground on one of the straight sections, what will be the frequency of vibration of the egg after impact? [Assume small oscillations and that the springs are initially stretched.]



problem 5.38:

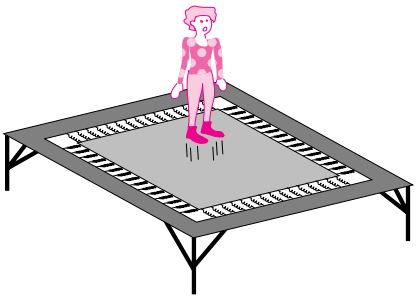
(Filename:pfigure.blue.149.1)

5.39 A person jumps on a trampoline. The trampoline is modeled as having an effective vertical undamped linear spring with stiffness $k = 200\,\text{lbf/ft}$. The person is modeled as a rigid mass $m = 150\,\text{lbm}$. $g = 32.2\,\text{ft/s}^2$.

a) What is the period of motion if the person's motion is so small that her feet never leave the trampoline? *

b) What is the maximum amplitude of motion for which her feet never leave the trampoline? *

c) (harder) If she repeatedly jumps so that her feet clear the trampoline by a height $h = 5\,\text{ft}$, what is the period of this motion? *



problem 5.39: A person jumps on a trampoline.

(Filename:pfigure3.trampoline)

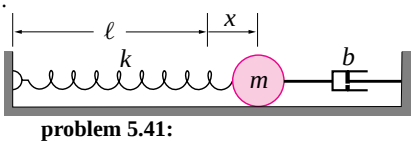
5.4 More on vibrations: damping

5.40 If $\ddot{x} + c\dot{x} + kx = 0$, $x(0) = x_0$, and $\dot{x}(0) = 0$, find $x(t)$.

5.41 A mass moves on a frictionless surface. It is connected to a dashpot with damping coefficient b to its right and a spring with constant k and rest length ℓ to its left. At the instant of interest, the mass is moving to the right and the spring is stretched a distance x from its position where the spring is unstretched. There is gravity.

a) Draw a free body diagram of the mass at the instant of interest.

b) Evaluate the left hand side of the equation of linear momentum balance as explicitly as possible. *



problem 5.41:

(Filename:ch2.11)

5.5 Forced oscillations and resonance

5.42 A 3 kg mass is suspended by a spring ($k = 10$ N/m) and forced by a 5 N sinusoidally oscillating force with a period of 1 s. What is the amplitude of the steady-state oscillations (ignore the "homogeneous" solution)

5.43 Given that $\ddot{\theta} + k^2\theta = \beta \sin \omega t$, $\theta(0) = 0$, and $\dot{\theta}(0) = \dot{\theta}_0$, find $\theta(t)$.

5.44 A machine produces a steady-state vibration due to a forcing function described by $Q(t) = Q_0 \sin \omega t$, where $Q_0 = 5000N$. The machine rests on a circular concrete foundation. The foundation rests on an isotropic, elastic half-space. The equivalent spring constant of the half-space is $k = 2,000,000$ N·m and has a damping ratio $d = c/c_c = 0.125$. The machine operates at a frequency of $\omega = 4$ Hz.

(a) What is the natural frequency of the system?

(b) If the system were undamped, what would the steady-state displacement be?

(c) What is the steady-state displacement given that $d = 0.125$?

(d) How much additional thickness of concrete should be added to the footing to reduce the damped steady-state amplitude by 50%? (The diameter must be held constant.)

5.6 Coupled motions in 1D

The primary emphasis of this section is setting up correct differential equations (without sign errors) and solving these equations on the computer. Experts note: normal modes are coverred in the vibrations chapter. These first problems are just math problems, using some of the skills that are needed for the later problems.

5.45 Write the following set of coupled second order ODE's as a system of first order ODE's.

$$\begin{aligned} \ddot{x}_1 &= k_2(x_2 - x_1) - k_1 x_1 \\ \ddot{x}_2 &= k_3 x_2 - k_2(x_2 - x_1) \end{aligned}$$

5.46 See also problem 5.47. The solution of a set of a second order differential equations is:

$$\begin{aligned} \xi(t) &= A \sin \omega t + B \cos \omega t + \xi^* \\ \dot{\xi}(t) &= A\omega \cos \omega t - B\omega \sin \omega t, \end{aligned}$$

where A and B are constants to be determined from initial conditions. Assume A and B are the only unknowns and write the equations in matrix form to solve for A and B in terms of $\xi(0)$ and $\dot{\xi}(0)$.

5.47 Solve for the constants A and B in Problem 5.46 using the matrix form, if $\xi(0) = 0$, $\dot{\xi}(0) = 0.5$, $\omega = 0.5$ rad/s and $\xi^* = 0.2$.

5.48 A set of first order linear differential equations is given:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 + kx_1 + cx_2 &= 0 \end{aligned}$$

Write these equations in the form $\dot{\vec{x}} = [\boldsymbol{A}]\vec{x}$, where $\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$.

5.49 Write the following pair of coupled ODE's as a set of first order ODE's.

$$\begin{aligned} \ddot{x}_1 + x_1 &= \dot{x}_2 \sin t \\ \ddot{x}_2 + x_2 &= \dot{x}_1 \cos t \end{aligned}$$

5.50 The following set of differential equations can not only be written in first order form but in matrix form $\dot{\vec{x}} = [\boldsymbol{A}]\vec{x} + \vec{c}$. In general things are not so simple, but this *linear* case is prevalant in the analytic study of dynamical systems.

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 + 5\Omega^2 x_1 - 4\Omega^2 x_2 &= 2\Omega^2 v_1^* \\ \dot{x}_4 - 4\Omega^2 x_1 + 5\Omega^2 x_2 &= -\Omega^2 v_1^* \end{aligned}$$

5.51 Write each of the following equations as a system of first order ODE's.

a) $\ddot{\theta} + \lambda^2\theta = \cos t$,

b) $\ddot{x} + 2p\dot{x} + kx = 0$,

c) $\ddot{x} + 2c\dot{x} + k\sin x = 0$.

5.52 A train is moving at constant absolute velocity $v\hat{\boldsymbol{\imath}}$. A passenger, idealized as a point mass, is walking at an absolute absolute velocity $u\hat{\boldsymbol{\imath}}$, where $u > v$. What is the velocity of the passenger relative to the train?

• • •

5.53 Two equal masses, each denoted by the letter m, are on an air track. One mass is connected by a spring to the end of the track. The other mass is connected by a spring to the first mass. The two spring constants are equal and represented by the letter k. In the rest (springs are relaxed) configuration, the masses are a distance ℓ apart. Motion of the two masses x_1 and x_2 is measured relative to this configuration.

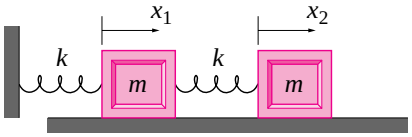
a) Draw a free body diagram for each mass.

b) Write the equation of linear momentum balance for each mass.

c) Write the equations as a system of first order ODEs.

d) Pick parameter values and initial conditions of your choice and simulate a motion of this system. Make a plot of the motion of, say, one of the masses vs time,

e) Explain how your plot does or does not make sense in terms of your understanding of this system. Is the initial motion in the right direction? Are the solutions periodic? Bounded? etc.



problem 5.53:

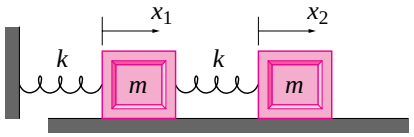
(Filename:pfigure.s94f1p4)

5.54 Two equal masses, each denoted by the letter m, are on an air track. One mass is connected by a spring to the end of the track. The other mass is connected by a spring to the first mass. The two spring constants are equal and represented by the letter k. In the rest configuration (springs are relaxed) the masses are a distance ℓ apart. Motion of the two masses x_1 and x_2 is measured relative to this configuration.

a) Write the potential energy of the system for arbitrary displacements x_1 and x_2 at some time t.

b) Write the kinetic energy of the system at the same time t in terms of $\dot{x}_1$, $\dot{x}_2$, m, and k.

c) Write the total energy of the system.



problem 5.54:

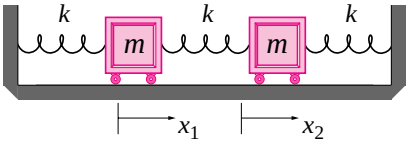
(Filename:pfigure.twomassenergy)

5.55 Normal Modes. Three equal springs (k) hold two equal masses (m) in place. There is no friction. x_1 and x_2 are the displacements of the masses from their equilibrium positions.

a) How many independent normal modes of vibration are there for this system? *

b) Assume the system is in a normal mode of vibration and it is observed that $x_1 = A\sin(ct) + B\cos(ct)$ where A, B, and c are constants. What is $x_2(t)$? (The answer is not unique. You may express your answer in terms of any of A, B, c, m and k.) *

c) Find all of the frequencies of normal-mode-vibration for this system in terms of m and k. *



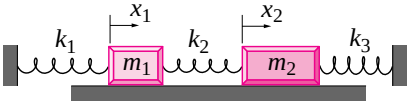
problem 5.55:

(Filename:pfigure.f93f2)

5.56 A two degree of freedom spring-mass system. A two degree of freedom mass-spring system, made up of two unequal masses m_1 and m_2 and three springs with unequal stiffnesses k_1, k_2 and k_3, is shown in the figure. All three springs are relaxed in the configuration shown. Neglect friction.

a) Derive the equations of motion for the two masses. *

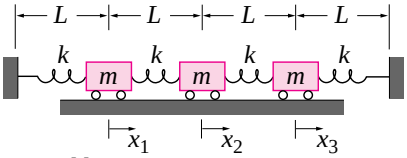
b) Does each mass undergo simple harmonic motion? *



problem 5.56:

(Filename:pfigure.s94h5p1)

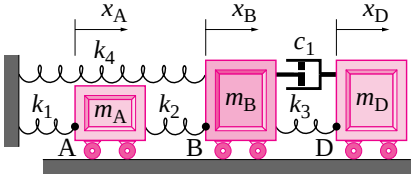
5.57 For the three-mass system shown, draw a free body diagram of each mass. Write the spring forces in terms of the displacements x_1, x_2, and x_3.



problem 5.57:

(Filename:s92f1p1)

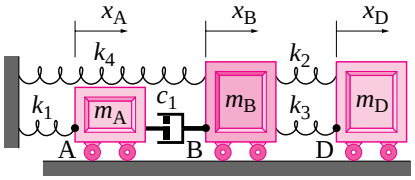
5.58 The springs shown are relaxed when $x_A = x_B = x_D = 0$. In terms of some or all of $m_A, m_B, m_D, x_A, x_B, x_D, \dot{x}_A, \dot{x}_B, \dot{x}_C$, and k_1, k_2, k_3, k_4, c_1, and F, find the acceleration of block B. *



problem 5.58:

(Filename:pfigure.s95q3)

5.59 A system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$. Given k_1, k_2, k_3, k_4, c_1, $m_A, m_B, m_D, x_A, x_B, x_D, \dot{x}_A, \dot{x}_B$, and $\dot{x}_D$, find the acceleration of mass B, $\vec{\boldsymbol{a}}_B = \ddot{x}_B\hat{\boldsymbol{\imath}}$. *



problem 5.59:

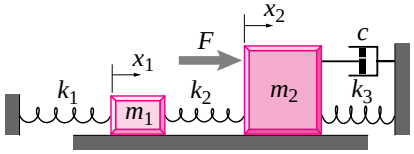
(Filename:pfigure.s95f1)

5.60 Equations of motion. Two masses are connected to fixed supports and each other with the three springs and dashpot shown. The force F acts on mass 2. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when the springs are unstretched. The ground is frictionless. The governing equations for the system shown can be writen in first order form if we define $v_1 \equiv \dot{x}_1$ and $v_2 \equiv \dot{x}_2$.

a) Write the governing equations in a neat first order form. Your equations should be in terms of any or all of the constants $m_1, m_2, k_1, k_2, k_3, C$, the constant force F, and t. Getting the signs right is important.

b) Write computer commands to find and plot $v_1(t)$ for 10 units of time. Make up appropriate initial conditions.

c) For constants and initial conditions of your choosing, plot x_1 vs t for enough time so that decaying erratic oscillations can be observed.

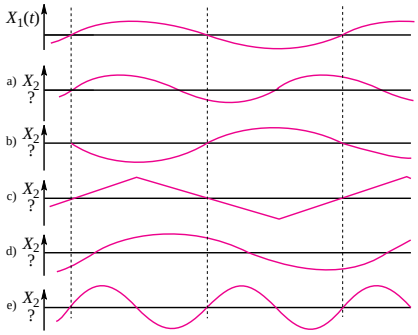


problem 5.60:

(Filename:p.f96.f.3)

5.61 $x_1(t)$ and $x_2(t)$ are measured positions on two points of a vibrating structure. $x_1(t)$ is shown. Some candidates for $x_2(t)$ are shown. Which of the $x_2(t)$ could possibly be associated with a normal mode vibration of the structure? Answer "could" or "could not" next to each

choice. (If a curve looks like it is meant to be a sine/cosine curve, it is.)



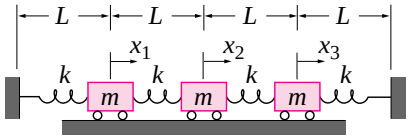
problem 5.61:

(Filename:pfigure.blue.144.1)

5.62 For the three-mass system shown, one of the normal modes is described with the eigenvector (1, 0, -1). Assume $x_1 = x_2 = x_3 = 0$ when all the springs are fully relaxed.

a) What is the angular frequency ω for this mode? Answer in terms of L, m, k, and g. (Hint: Note that in this mode of vibration the middle mass does not move.) *

b) Make a neat plot of x_2 versus x_1 for one cycle of vibration with this mode.



problem 5.62:

(Filename:pfigure.blue.160.2)

5.63 The three beads of masses m, $2m$, and m connected by massless linear springs of constant k slide freely on a straight rod. Let x_i denote the displacement of the i^{th} bead from its equilibrium position at rest.

a) Write expressions for the total kinetic and potential energies.

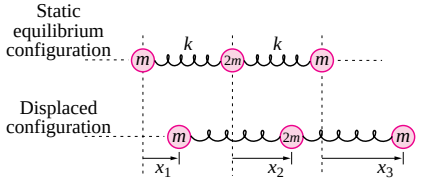
b) Write an expression for the total linear momentum.

c) Draw free body diagrams for the beads and use Newton's second law to derive the equations for motion for the system.

d) Verify that total energy and linear momentum are both conserved.

e) Show that the center of mass must either remain at rest or move at constant velocity.

f) What can you say about vibratory (sinusoidal) motions of the system?



problem 5.63:

(Filename:pfigure.blue.161.1)

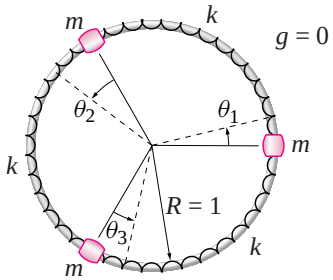
5.64 The system shown below comprises three identical beads of mass m that can slide frictionlessly on the rigid, immobile, circular hoop. The beads are connected by three identical linear springs of stiffness k, wound around the hoop as shown and equally spaced when the springs are unstretched (the strings are unstretched when $\theta_1 = \theta_2 = \theta_3 = 0$.)

a) Determine the natural frequencies and associated mode shapes for the system. (Hint: you should be able to deduce a 'rigid-body' mode by inspection.)

b) If your calculations in (a) are correct, then you should have also obtained the mode shape $(0, 1, -1)^T$. Write down the most general set of initial conditions so that the ensuing motion of the system is simple harmonic in that mode shape.

c) Since $(0, 1, -1)^T$ is a mode shape, then by "symmetry", $(-1, 0, 1)^T$ and $(1, -1, 0)^T$ are also mode shapes (draw a picture). Explain how we can have three mode shapes associated with the same frequency.

d) Without doing any calculations, compare the frequencies of the constrained system to those of the unconstrained system, obtained in (a).

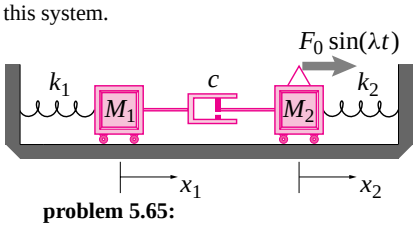


problem 5.64:

(Filename:pfigure.blue.158.1)

5.65 Equations of motion. Two masses are connected to fixed supports and each other with the two springs and dashpot shown. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when both springs are unstretched.

For the special case that $C = 0$ and $F_0 = 0$ clearly define two different set of initial conditions that lead to normal mode vibrations of this system.



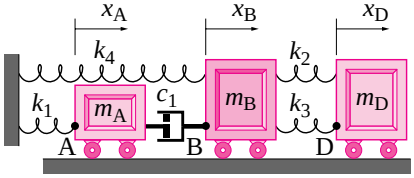
problem 5.65:

(Filename:p.s96.p3.1)

5.66 As in problem 5.59, a system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$.

a) In the special case when $k_1 = k_2 = k_3 = k_4 = k$, $c_1 = 0$, and $m_A = m_B = m_D = m$, find a normal mode of vibration. Define it in any clear way and explain or show why it is a normal mode in any clear way. *

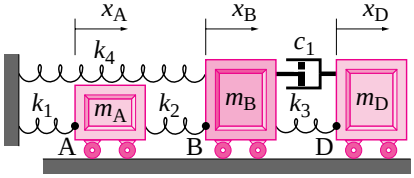
b) In the same special case as in (a) above, find another normal mode of vibration. *



problem 5.66:

(Filename:pfigure.s95f1a)

5.67 As in problem 5.143, a system of three masses, four springs, and one damper are connected as shown. In the special case when $c_1 = 0$, find the normal modes of vibration.



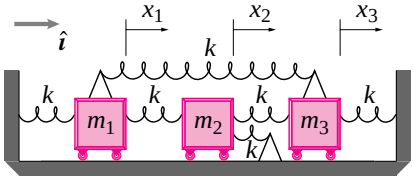
problem 5.67:

(Filename:pfigure.s95q3a)

5.68 Normal modes. All three masses have $m = 1\,\text{kg}$ and all 6 springs are $k = 1\,\text{N/m}$. The system is at rest when $x_1 = x_2 = x_3 = 0$.

a) Find as many different initial conditions as you can for which normal mode vibrations result. In each case, find the associated natural frequency. (we will call two initial conditions $[v]$ and $[w]$ different if there is no constant c so that $[v_1\ v_2\ v_3] = c[w_1\ w_2\ w_3]$. Assume the initial velocities are zero.)

b) For the initial condition
$[x_0] = [\ 0.1\,\text{m}\ \ 0\ \ 0\ ]$,
$[\dot{x}_0] = [\ 0\ \ 2\,\text{m/s}\ \ 0\ ]$
what is the initial (immediately after the start) acceleration of mass 2?



problem 5.68:

(Filename:pfigure3.f95p2p2)

5.7 Time derivative of a vector: position, velocity and acceleration

5.69 The position vector of a particle in the xy-plane is given as $\vec{r} = 3.0\,\text{m}\hat{\imath} + 2.5\,\text{m}\hat{\jmath}$. Find (a) the distance of the particle from the origin and (b) a unit vector in the direction of $\vec{r}$.

5.70 Given $\vec{r}(t) = A\sin(\omega t)\hat{\imath} + Bt\,\hat{\jmath} + C\hat{k}$, find

(a) $\vec{v}(t)$

(b) $\vec{a}(t)$

(c) $\vec{r}(t) \times \vec{a}(t)$.

5.71 A particle of mass =3 kg travels in space with its position known as a function of time, $\vec{r} = (\sin\frac{t}{\text{s}})\,\text{m}\hat{\imath} + (\cos\frac{t}{\text{s}})\,\text{m}\,\hat{\jmath} + te^{\frac{t}{\text{s}}}\,\text{m/s}\hat{k}$. At $t = 3$ s, find the particle's

a) velocity and

b) acceleration.

5.72 A particle of mass $m = 2$ kg travels in the xy-plane with its position known as a function of time, $\vec{r} = 3t^2\,\text{m/s}^2\hat{\imath} + 4t^3\frac{\text{m}}{s^3}\hat{\jmath}$. At $t = 5$ s, find the particle's

a) velocity and *

b) acceleration. *

5.73 If $\dot{\vec{r}} = (u_0\sin\Omega t)\hat{\imath} + v_0\hat{\jmath}$ and $\vec{r}(0) = x_0\hat{\imath} + y_0\hat{\jmath}$, with u_0, v_0, and Ω, find $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. *

5.74 The velocity of a particle of mass m on a frictionless surface is given as $\vec{v} = (0.5\,\text{m/s})\hat{\imath} - (1.5\,\text{m/s})\hat{\jmath}$. If the displacement is given by $\Delta\vec{r} = \vec{v}t$, find (a) the distance traveled by the mass in 2 seconds and (b) a unit vector along the displacement.

5.75 For $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{a} = 2\,\text{m/s}^2\hat{\imath} - (3\,\text{m/s}^2 - 1\,\text{m/s}^3\,t)\hat{\jmath} - 5\,\text{m/s}^4\,t^2\hat{k}$, write the vector equation $\vec{v} = \int \vec{a}\,dt$ as three scalar equations.

5.76 Find $\vec{r}\,(5\,\text{s})$ given that

$$\dot{\vec{r}} = v_1\sin(ct)\hat{\imath} + v_2\hat{\jmath} \quad \text{and} \quad \vec{r}(0) = 2\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}$$

and that v_1 is a constant 4 m/s, v_2 is a constant 5 m/s, and c is a constant $4\,\text{s}^{-1}$. Assume $\hat{\imath}$ and $\hat{\jmath}$ are constant.

5.77 Let $\dot{\vec{r}} = v_0\cos\alpha\hat{\imath} + v_0\sin\alpha\,\hat{\jmath} + (v_0\tan\theta - gt)\hat{k}$, where v_0, α, θ, and g are constants. If $\vec{r}(0) = \vec{0}$, find $\vec{r}(t)$.

5.78 On a smooth circular helical path the velocity of a particle is $\dot{\vec{r}} = -R\sin t\,\hat{\imath} + R\cos t\,\hat{\jmath} + gt\hat{k}$. If $\vec{r}(0) = R\hat{\imath}$, find $\vec{r}((\pi/3)\,\text{s})$.

5.79 Draw unit vectors along $\vec{\omega} = (4.33\,\text{rad/s})\hat{\imath} + (2.50\,\text{rad/s})\hat{\jmath}$ and $\vec{r} = (0.50\,\text{ft})\hat{\imath} - (0.87\,\text{ft})\hat{\jmath}$ and find the angle between the two unit vectors.

5.80 What is the angle between the x-axis and the vector $\vec{v} = (0.3\hat{\imath} - 2.0\,\hat{\jmath} + 2.2\hat{k})\,\text{m/s}$?

5.81 The position of a particle is given by $\vec{r}(t) = (t^2\,\text{m/s}^2\hat{\imath} + e^{\frac{t}{\text{s}}}\,\text{m}\hat{\jmath})$. What are the velocity and acceleration of the particle? *

5.82 A particle travels on a path in the xy-plane given by $y(x) = \sin^2(\frac{x}{\text{m}})\,\text{m}$. where $x(t) = t^3(\frac{\text{m}}{\text{s}^3})$. What are the velocity and acceleration of the particle in cartesian coordinates when $t = (\pi)^{\frac{1}{3}}$ s?

5.83 A particle travels on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$ with constant speed v. Find the velocity of the particle when $x = a/2$ and $y > 0$ in terms of a, b, and v.

5.84 A particle travels on a path in the xy-plane given by $y(x) = (1 - e^{-\frac{x}{\text{m}}})\,\text{m}$. Make a plot of the path. It is known that the x coordinate of the particle is given by $x(t) = t^2\,\text{m/s}^2$. What is the rate of change of speed of the particle? What angle does the velocity vector make with the positive x axis when $t = 3$ s?

5.85 A particle starts at the origin in the xy-plane, ($x_0 = 0$, $y_0 = 0$) and travels only in the positive xy quadrant. Its speed and x coordinate are known to be $v(t) = \sqrt{1 + (\frac{4}{s^2})t^2}$ m/s and $x(t) = t$ m/s, respectively. What is $\vec{r}(t)$ in cartesian coordinates? What are the velocity, acceleration, and rate of change of speed of the particle as functions of time? What kind of path is the particle on? What are the distance of the particle from the origin and its velocity and acceleration when $x = 3$ m?

5.8 Spatial dynamics of a particle

5.86 What symbols do we use for the following quantities? What are the definitions of these quantities? Which are vectors and which are scalars? What are the SI and US standard units for the following quantities?

a) linear momentum

b) rate of change of linear momentum

c) angular momentum

d) rate of change of angular momentum

e) kinetic energy

f) rate of change of kinetic energy

g) moment

h) work

i) power

5.87 Does angular momentum depend on reference point? (Assume that all candidate points are fixed in the same Newtonian reference frame.)

5.88 Does kinetic energy depend on reference point? (Assume that all candidate points are fixed in the same Newtonian reference frame.)

5.89 What is the relation between the dynamics 'Linear Momentum Balance' equation and the statics 'Force Balance' equation?

5.90 What is the relation between the dynamics 'Angular Momentum Balance' equation and the statics 'Moment Balance' equation?

5.91 A ball of mass $m = 0.1$ kg is thrown from a height of $h = 10$ m above the ground with velocity $\vec{v} = 120\,\text{km/h}\hat{\imath} - 120\,\text{km/h}\hat{\jmath}$. What is the kinetic energy of the ball at its release?

5.92 A ball of mass $m = 0.2$ kg is thrown from a height of $h = 20$ m above the ground with velocity $\vec{v} = 120\,\text{km/h}\hat{\imath} - 120\,\text{km/h}\,\hat{\jmath} - 10\,\text{km/h}\hat{k}$. What is the kinetic energy of the ball at its release?

5.93 How do you calculate P, the power of all external forces acting on a particle, from the forces $\vec{F}_i$ and the velocity $\vec{v}$ of the particle?

5.94 A particle A has velocity $\vec{v}_A$ and mass m_A. A particle B has velocity $\vec{v}_B = 2\,\vec{v}_A$ and mass equal to the other $m_B = m_A$. What is the relationship between:

a) $\vec{L}_\text{A}$ and $\vec{L}_\text{B}$,

b) $\vec{H}_{\text{A/C}}$ and $\vec{H}_{\text{B/C}}$, and

c) E_KA and E_KB?

5.95 A bullet of mass 50 g travels with a velocity $\vec{v} = 0.8\,\text{km/s}\hat{\imath} + 0.6\,\text{km/s}\hat{\jmath}$. (a) What is the linear momentum of the bullet? (Answer in consistent units.)

5.96 A particle has position $\vec{r} = 4\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$, velocity $\vec{v} = 6\,\text{m/s}\hat{\imath} - 3\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a} = -2\,\text{m/s}^2\hat{\imath} + 9\,\text{m/s}^2\hat{\jmath}$. For each position of a point P defined below, find $\vec{H}_P$, the angular momentum of the particle with respect to the point P.

a) $\vec{r}_P = 4\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$,

b) $\vec{r}_P = -2\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$, and

c) $\vec{r}_P = 0\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$,

d) $\vec{r}_P = \vec{0}$

5.97 The position vector of a particle of mass 1 kg at an instant t is $\vec{r} = 2\,\text{m}\hat{\imath} - 0.5\,\text{m}\hat{\jmath}$. If the velocity of the particle at this instant is $\vec{v} = -4\,\text{m/s}\hat{\imath} + 3\,\text{m/s}\hat{\jmath}$, compute (a) the linear momentum $\vec{L} = m\,\vec{v}$ and (b) the angular momentum ($\vec{H}_\text{O} = \vec{r}_{/\text{O}} \times (m\,\vec{v})$).

5.98 The position of a particle of mass $m = 0.5$ kg is $\vec{r}(t) = \ell\,\sin(\omega t)\hat{\imath} + h\,\hat{\jmath}$; where $\omega = 2$ rad/s, $h = 2$ m, $\ell = 2$ m, and $\vec{r}$ is measured from the origin.

a) Find the kinetic energy of the particle at $t = 0\,\text{s}$ and $t = 5\,\text{s}$.

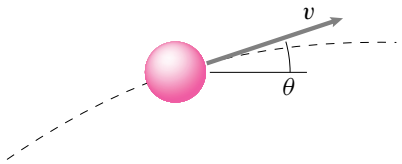
b) Find the rate of change of kinetic energy at $t = 0\,\text{s}$ and $t = 5\,\text{s}$.

5.99 For a particle

$$E_\text{K} = \frac{1}{2}\, m\, v^2.$$

Why does it follow that $\dot{E_\text{K}} = m\, \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}}$? [hint: write v^2 as $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}$ and then use the product rule of differentiation.]

5.100 Consider a projectile of mass m at some instant in time t during its flight. Let $\vec{\boldsymbol{v}}$ be the velocity of the projectile at this instant (see the figure). In addition to the force of gravity, a drag force acts on the projectile. The drag force is proportional to the square of the speed (speed$\equiv |\vec{\boldsymbol{v}}| = v$) and acts in the opposite direction. Find an expression for the net power of these forces ($P = \sum \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$) on the particle.



problem 5.100:

(Filename:pfig2.3.rp3)

5.101 A 10 gm wad of paper is tossed in the air (in a strong turbulent wind). The position, velocity, and acceleration of its center of mass are $\vec{\boldsymbol{r}} = 3\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}} + 6\,\text{m}\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{v}} = -9\,\text{m/s}\hat{\boldsymbol{\imath}} + 24\,\text{m/s}\hat{\boldsymbol{\jmath}} + 30\,\text{m/s}\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{a}} = -10\,\text{m/s}^2\hat{\boldsymbol{\imath}} + 24\,\text{m/s}^2\hat{\boldsymbol{\jmath}} + 32\,\text{m/s}^2\hat{\boldsymbol{k}}$, respectively. Find the linear momentum $\vec{\boldsymbol{L}}$ and its rate of change $\dot{\vec{\boldsymbol{L}}}$ of the the potato's center of mass.

5.102 A 10 gm wad of paper is tossed into the air. At a particular instant of interest, the position, velocity, and acceleration of its center of mass are $\vec{\boldsymbol{r}} = 3\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}} + 6\,\text{m}\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{v}} = -9\,\text{m/s}\hat{\boldsymbol{\imath}} + 24\,\text{m/s}\hat{\boldsymbol{\jmath}} + 30\,\text{m/s}\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{a}} = -10\,\text{m/s}^2\hat{\boldsymbol{\imath}} + 24\,\text{m/s}^2\hat{\boldsymbol{\jmath}} + 32\,\text{m/s}^2\hat{\boldsymbol{k}}$, respectively. What is the translational kinetic energy of the wad at the instant of interest?

5.103 A 2 kg particle moves so that its position $\vec{\boldsymbol{r}}$ is given by

$$\vec{\boldsymbol{r}}(t) = [5\sin(at)\hat{\boldsymbol{\imath}} + bt^2\hat{\boldsymbol{\jmath}} + ct\hat{\boldsymbol{k}}]\,\text{m}$$

where a $= \pi/\,\text{sec}$, $b = .25/\,\text{sec}^2$, $c = 2/\,\text{sec}$.

a) What is the linear momentum of the particle at $t = 1\,\text{sec}$?

b) What is the force acting on the particle at $t = 1\,\text{sec}$?

5.104 A particle A has mass m_A and velocity $\vec{\boldsymbol{v}}_\text{A}$. A particle B at the same location has mass $m_B = 2\, m_A$ and velocity equal to the other $\vec{\boldsymbol{v}}_\text{B} = \vec{\boldsymbol{v}}_\text{A}$. Point C is a reference point. What is the relationship between:

a) $\vec{\boldsymbol{L}}_\text{A}$ and $\vec{\boldsymbol{L}}_\text{B}$,

b) $\vec{\boldsymbol{H}}_{\text{A/C}}$ and $\vec{\boldsymbol{H}}_{\text{B/C}}$, and

c) E_KA and E_KB?

5.105 A particle of mass $m = 3\,\text{kg}$ moves in space. Its position, velocity, and acceleration at a particular instant in time are $\vec{\boldsymbol{r}} = 2\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}} + 5\,\text{m}\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{v}} = -3\,\text{m/s}\hat{\boldsymbol{\imath}} + 8\,\text{m/s}\hat{\boldsymbol{\jmath}} + 10\,\text{m/s}\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{a}} = -5\,\text{m/s}^2\hat{\boldsymbol{\imath}} + 12\,\text{m/s}^2\hat{\boldsymbol{\jmath}} + 16\,\text{m/s}^2\hat{\boldsymbol{k}}$, respectively. For this particle at the instant of interest, find its:

a) linear momentum $\vec{\boldsymbol{L}}$,

b) rate of change of linear momentum $\dot{\vec{\boldsymbol{L}}}$,

c) angular momentum about the origin $\vec{\boldsymbol{H}}_\text{O}$,

d) rate of change of angular momentum about the origin $\dot{\vec{\boldsymbol{H}}}_\text{O}$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy $\dot{E_\text{K}}$.

5.106 A particle has position $\vec{\boldsymbol{r}} = 3\,\text{m}\hat{\boldsymbol{\imath}} - 2\,\text{m}\hat{\boldsymbol{\jmath}} + 4\,\text{m}\hat{\boldsymbol{k}}$, velocity $\vec{\boldsymbol{v}} = 2\,\text{m/s}\hat{\boldsymbol{\imath}} - 3\,\text{m/s}\hat{\boldsymbol{\jmath}} + 7\,\text{m/s}\hat{\boldsymbol{k}}$, and acceleration $\vec{\boldsymbol{a}} = 1\,\text{m/s}^2\hat{\boldsymbol{\imath}} - 8\,\text{m/s}^2\hat{\boldsymbol{\jmath}} + 3\,\text{m/s}^2\hat{\boldsymbol{k}}$. For each position of a point P defined below, find the rate of change of angular momentum of the particle with respect to the point P, $\dot{\vec{\boldsymbol{H}}}_P$.

a) $\vec{\boldsymbol{r}}_P = 3\,\text{m}\hat{\boldsymbol{\imath}} - 2\,\text{m}\hat{\boldsymbol{\jmath}} + 4\,\text{m}\hat{\boldsymbol{k}}$,

b) $\vec{\boldsymbol{r}}_P = 6\,\text{m}\hat{\boldsymbol{\imath}} - 4\,\text{m}\hat{\boldsymbol{\jmath}} + 8\,\text{m}\hat{\boldsymbol{k}}$,

c) $\vec{\boldsymbol{r}}_P = -9\,\text{m}\hat{\boldsymbol{\imath}} + 6\,\text{m}\hat{\boldsymbol{\jmath}} - 12\,\text{m}\hat{\boldsymbol{k}}$, and

d) $\vec{\boldsymbol{r}}_P = \vec{\boldsymbol{0}}$

5.107 A particle of mass $m = 5\,\text{kg}$ has position, velocity, and acceleration $\vec{\boldsymbol{r}} = 2\,\text{m}\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{v}} = 3\,\text{m/s}\hat{\boldsymbol{\imath}}$, and $\vec{\boldsymbol{a}} = -7\,\text{m/s}^2\hat{\boldsymbol{\imath}}$, respectively, at a particular instant of interest. At the instant of interest find its:

a) linear momentum $\vec{\boldsymbol{L}}$,

b) rate of change of linear momentum $\dot{\vec{\boldsymbol{L}}}$,

c) angular momentum about the origin $\vec{\boldsymbol{H}}_\text{O}$,

d) rate of change of angular momentum about the origin $\dot{\vec{\boldsymbol{H}}}_\text{O}$,

e) kinetic energy E_K, and

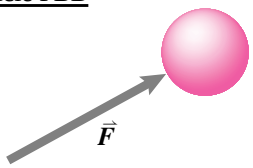
f) rate of change of kinetic energy $\dot{E_\text{K}}$.

g) the net force $\sum \vec{\boldsymbol{F}}$ on the particle,

h) the net moment on the particle about the origin $\sum \vec{\boldsymbol{M}}_\text{O}$ due to the applied forces, and

i) rate of change of work $\dot{W} = P$ done on the particle by the applied forces.

Particle FBD



problem 5.107: FBD of the particle

(Filename:pfigure1.1.part.fbd)

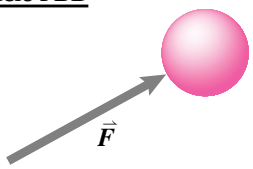
5.108 A particle of mass $m = 6\,\text{kg}$ is moving in space. Its position, velocity, and acceleration at a particular instant in time are $\vec{\boldsymbol{r}} = 1\,\text{m}\hat{\boldsymbol{\imath}} - 2\,\text{m}\hat{\boldsymbol{\jmath}} + 4\,\text{m}\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{v}} = 3\,\text{m/s}\hat{\boldsymbol{\imath}} + 4\,\text{m/s}\hat{\boldsymbol{\jmath}} - 7\,\text{m/s}\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{a}} = 5\,\text{m/s}^2\hat{\boldsymbol{\imath}} + 11\,\text{m/s}^2\hat{\boldsymbol{\jmath}} - 9\,\text{m/s}^2\hat{\boldsymbol{k}}$, respectively. For this particle at the instant of interest, find its:

a) the net force $\sum \vec{\boldsymbol{F}}$ on the particle,

b) the net moment on the particle about the origin $\sum \vec{\boldsymbol{M}}_\text{O}$ due to the applied forces, and

c) the power P of the applied forces.

Particle FBD



problem 5.108: FBD of the particle

(Filename:pfigure1.1.part.fbdb)

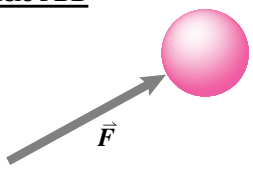
5.109 A particle of mass $m = 3\,\text{kg}$ is moving in the xz-plane. Its position, velocity, and acceleration at a particular instant of interest are $\vec{\boldsymbol{r}} = 4\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{k}}$, $\vec{\boldsymbol{v}} = 3\,\text{m/s}\hat{\boldsymbol{\imath}} - 7\,\text{m/s}\hat{\boldsymbol{k}}$, and $\vec{\boldsymbol{a}} = 3\,\text{m/s}^2\hat{\boldsymbol{\imath}} - 4\,\text{m/s}^2\hat{\boldsymbol{k}}$, respectively. For this particle at the instant of interest, find:

a) the net force $\sum \vec{\boldsymbol{F}}$ on the particle,

b) the net moment on the particle about the origin $\sum \vec{\boldsymbol{M}}_\text{O}$ due to the applied forces, and

c) rate of change of work $\dot{W} = P$ done on the particle by the applied forces.

Particle FBD



problem 5.109: FBD of the particle

(Filename:pfigure1.1.part.fbda)

5.110 A particle of mass $m = 3\,\text{kg}$ is moving in the xy-plane. Its position, velocity, and acceleration at a particular instant of interest are $\vec{\boldsymbol{r}} = 2\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{v}} = -3\,\text{m/s}\hat{\boldsymbol{\imath}} + 8\,\text{m/s}\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{a}} = -5\,\text{m/s}^2\hat{\boldsymbol{\imath}} + 12\,\text{m/s}^2\hat{\boldsymbol{\jmath}}$, respectively. For this particle at the instant of interest, find its:

a) linear momentum $\vec{\boldsymbol{L}}$,

b) rate of change of linear momentum $\dot{\vec{L}}$,

c) angular momentum about the origin $\vec{H}_O$,

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_O$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy $\dot{E}_K$.

5.111 At a particular instant of interest, a particle of mass $m_1 = 5\,\text{kg}$ has position, velocity, and acceleration $\vec{r}_1 = 3\,\text{m}\hat{\imath}$, $\vec{v}_1 = -4\,\text{m/s}\hat{\jmath}$, and $\vec{a}_1 = 6\,\text{m/s}^2\hat{\jmath}$, respectively, and a particle of mass $m_2 = 5\,\text{kg}$ has position, velocity, and acceleration $\vec{r}_2 = -6\,\text{m}\hat{\imath}$, $\vec{v}_2 = 5\,\text{m/s}\hat{\jmath}$, and $\vec{a}_2 = -4\,\text{m/s}^2\hat{\jmath}$, respectively. For the system of particles, find its

a) linear momentum $\vec{L}$,

b) rate of change of linear momentum $\dot{\vec{L}}$

c) angular momentum about the origin $\vec{H}_O$,

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_O$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy $\dot{E}_K$.

5.112 A particle of mass $m = 250$ gm is shot straight up (parallel to the y-axis) from the x-axis at a distance $d = 2\,\text{m}$ from the origin. The velocity of the particle is given by $\vec{v} = v\hat{\jmath}$ where $v^2 = v_0^2 - 2ah$, $v_0 = 100\,\text{m/s}$, $a = 10\,\text{m/s}^2$ and h is the height of the particle from the x-axis.

a) Find the linear momentum of the particle at the outset of motion ($h = 0$).

b) Find the angular momentum of the particle about the origin at the outset of motion ($h = 0$).

c) Find the linear momentum of the particle when the particle is 20 m above the x-axis.

d) Find the angular momentum of the particle about the origin when the particle is 20 m above the x-axis.

5.113 For a particle, $\sum \vec{F} = m\vec{a}$. Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a mass P as shown in the figure. P has mass 2 lbm. The acceleration of the mass is somehow measured to be $\vec{a} = -2\,\text{ft/s}^2\hat{\imath} + 5\,\text{ft/s}^2\hat{\jmath}$.

a) Write the equation

$$\sum \vec{F} = m\vec{a}$$

in vector form (evaluating each side as much as possible).

b) Write the equation in scalar form (use any method you like to get two scalar equations in the two unknowns F_1 and F_2).

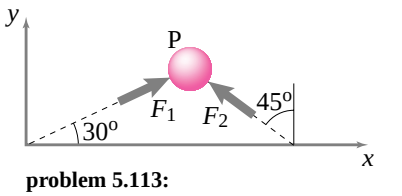
c) Write the equation in matrix form.

d) Find $F_1 = |\vec{F}_1|$ and $F_2 = |\vec{F}_2|$ by the following methods:

(a) from the scalar equations using hand algebra,

(b) from the matrix equation using a computer, and

(c) from the vector equation using a cross product.



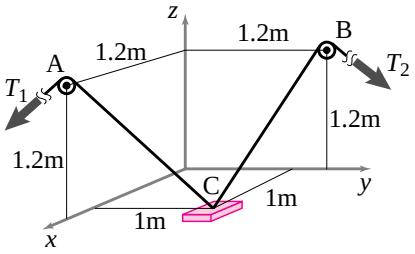
problem 5.113:

(Filename:h1.63a)

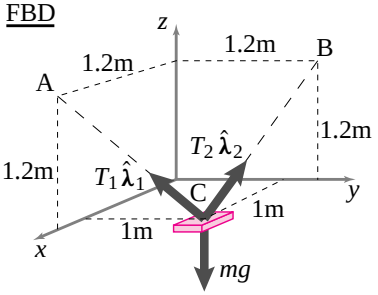
5.114 Three forces, $\vec{F}_1 = 20\,\text{N}\hat{\imath} - 5\,\text{N}\hat{\jmath}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2z}\hat{k}$, and $\vec{F}_3 = F_3\hat{\lambda}$, where $\hat{\lambda} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$, act on a body with mass 2 kg. The acceleration of the body is $\vec{a} = -0.2\,\text{m/s}^2\hat{\imath} + 2.2\,\text{m/s}^2\hat{\jmath} + 1.7\,\text{m/s}^2\hat{k}$. Write the equation $\sum \vec{F} = m\vec{a}$ as scalar equations and solve them (most conveniently on a computer) for F_{2x}, F_{2z}, and F_3.

5.115 The rate of change of linear momentum of a particle is known in two directions: $\dot{L}_x = 20\,\text{kg m/s}^2$, $\dot{L}_y = -18\,\text{kg m/s}^2$ and unknown in the z direction. The forces acting on the particle are $\vec{F}_1 = 25\,\text{N}\hat{\imath} + 32\,\text{N}\hat{\jmath} + 75\,\text{N}\hat{k}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2y}\hat{\jmath}$ and $\vec{F}_3 = -F_3\hat{k}$. Using $\sum \vec{F} = \dot{\vec{L}}$, separate the vector equation into scalar equations in the x, y, and z directions. Solve these equations (maybe with the help of a computer) to find F_{2x}, F_{2y}, and F_3.

5.116 A block of mass 100 kg is pulled with two strings AC and BC. Given that the tensions $T_1 = 1200\,\text{N}$ and $T_2 = 1500\,\text{N}$, neglecting gravity, find the magnitude and direction of the acceleration of the block. [$\sum \vec{F} = m\vec{a}$]



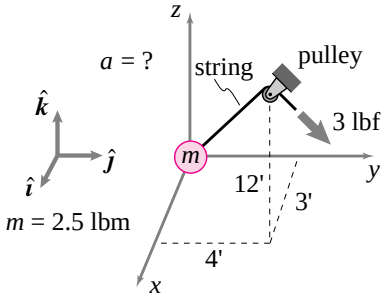
FBD



problem 5.116:

(Filename:efig1.2.7)

5.117 Neglecting gravity, the only force acting on the mass shown in the figure is from the string. Find the acceleration of the mass. Use the dimensions and quantities given. Recall that lbf is a pound force, lbm is a pound mass, and lbf/lbm $= g$. Use $g = 32\,\text{ft/s}^2$. Note also that $3^2 + 4^2 + 12^2 = 13^2$.



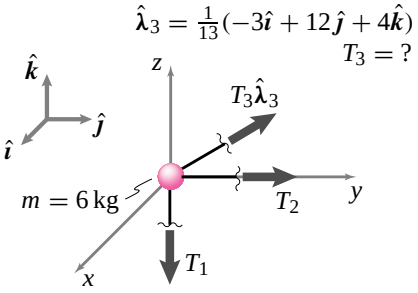
problem 5.117:

(Filename:pfigure.blue.4.1)

5.118 Three strings are tied to the mass shown with the directions indicated in the figure. They have unknown tensions T_1, T_2, and T_3. There is no gravity. The acceleration of the mass is given as $\vec{a} = (-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\,\text{m/s}^2$.

a) Given the free body diagram in the figure, write the equations of linear momentum balance for the mass.

b) Find the tension T_3. *



$\vec{\boldsymbol{a}} = (-.5\hat{\boldsymbol{\imath}} + 2.5\hat{\boldsymbol{\jmath}} + \frac{1}{3}\hat{\boldsymbol{k}})\frac{m}{s^2}$

problem 5.118:

(Filename:pfigure.blue.1.2)

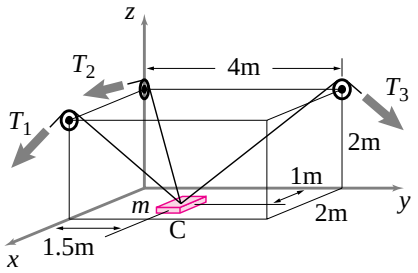
5.119 For part(c) of problem 4.23, assume now that the mass at A has non-zero acceleration of $(1m/s^2)\hat{\boldsymbol{\imath}} + (2m/s^2)\hat{\boldsymbol{\jmath}} + (3m/s^2)\hat{\boldsymbol{k}}$. Find the tension in the three ropes at the instant shown.

5.120 A small object (mass= 2 kg) is being pulled by three strings as shown. The acceleration of the object at the position shown is $\boldsymbol{a} = \left(-0.6\hat{\boldsymbol{\imath}} - 0.2\hat{\boldsymbol{\jmath}} + 2.0\hat{\boldsymbol{k}}\right)\text{m/s}^2$.

a) Draw a free body diagram of the mass.

b) Write the equation of linear momentum balance for the mass. Use $\hat{\boldsymbol{\lambda}}$'s as unit vectors along the strings.

c) Find the three tensions T_1, T_2, and T_3 at the instant shown. You may find these tensions by using hand algebra with the scalar equations, using a computer with the matrix equation, or by using a cross product on the vector equation.



problem 5.120:

(Filename:pfigure.s94h2p9)

5.121 Use a computer to draw a square with corners at (1, 0), (0, −1), (−1, 0), (0, 1). This must be done with scientific software and not with a purely graphics program.

5.122 Draw a Circle on the Computer. We will be interested in keeping track of the motions of systems. A simple example is that of a particle going in circles at a constant rate. One can draw a circle quite well with a compass or with simple drawing programs. *But*, more complicated motions will be more difficult. Draw a circle on the computer and label the drawing (using computer generated lettering) with your name and the date.

a) You can program the circular shape any way that you think is fun (or any other way if you don't feel like having a good time). Your circle should be round. Measure its length and width with a ruler, they should be within 10% of each other (mark the dimensions by hand on your drawing).

b) A good solution will clearly document and explain the computer methodology.

5.123 What curve is defined by $x = \cos(t)^2$ and $y = \sin(t) * \cos(t)$ for $0 \le t \le \pi$? Try to figure it out without a computer. Make a computer plot.

5.124 Particle moves on a strange path. Given that a particle moves in the xy plane for 1.77 s obeying

$$\boldsymbol{r} = (5\,\text{m})\cos^2(t^2/\,\text{s}^2)\hat{\boldsymbol{\imath}} + (5\,\text{m})\sin(t^2/\,\text{s}^2)\cos(t^2/\,\text{s}^2)\,\hat{\boldsymbol{\jmath}}$$

where x and y are the horizontal distance in meters and t is measured in seconds.

a) Accurately plot the trajectory of the particle.

b) Mark on your plot where the particle is going fast and where it is going slow. Explain how you know these points are the fast and slow places.

5.125 COMPUTER QUESTION: What's the plot? What's the mechanics question? Shown are shown some pseudo computer commands that are not commented adequately, unfortunately, and no computer is available at the moment.

a) Draw as accurately as you can, assigning numbers etc, the plot that results from running these commands.

b) See if you can guess a mechanical situation that is described by this program. Sketch the system and define the variables to make the script file agree with the problem stated.

```
ODEs = {z1dot = z2
        z2dot = 0}
ICs = {z1 =1, z2=1}

Solve ODEs with ICs from t=0 to t=5

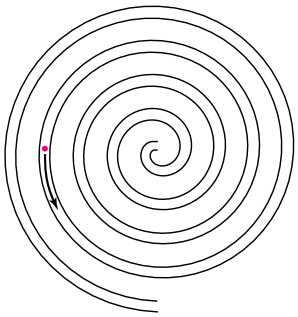
plot z2 and z1 vs t on the same plot
```

problem 5.125:

(Filename:pfigure.s94f1p3)

Here begins a series of problems concerning the motion of particles over non-infinitesimal time.

5.126 A particle is blown out through the uniform spiral tube shown, which lies flat on a horizontal frictionless table. Draw the particle's path after it is expelled from the tube. Defend your answer.



problem 5.126:

(Filename:pfigure.blue.29.1)

5.127 A ball going to the left with speed v_0 bounces against a frictionless rigid ramp which is sloped at an angle θ from the horizontal. The collision is completely elastic (the coefficient of restitution $e = 1$). Neglect gravity.

a) Find the velocity of the ball after the collision. You may express your answer in terms of any combination of m, θ, v_0, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{n}}$, and $\hat{\boldsymbol{\lambda}}$.

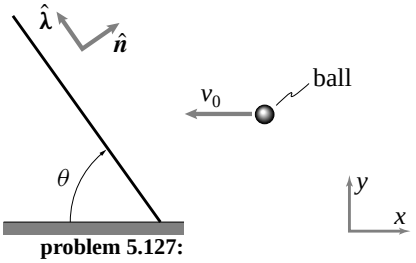
$$\hat{\boldsymbol{n}} = \sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$$

$$\hat{\boldsymbol{\lambda}} = -\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$$

$$\hat{\boldsymbol{\imath}} = \sin\theta\hat{\boldsymbol{n}} + -\cos\theta\hat{\boldsymbol{\lambda}}$$

$$\hat{\boldsymbol{\jmath}} = \cos\theta\hat{\boldsymbol{n}} + \sin\theta\hat{\boldsymbol{\lambda}}$$

b) For what value of θ would the vertical component of the speed be maximized?



problem 5.127:

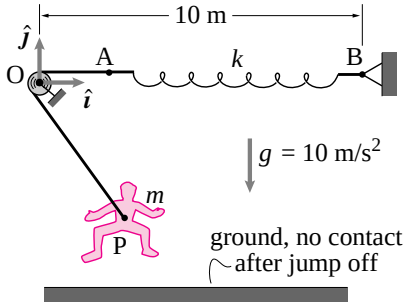
(Filename:ballramp)

5.128 Bungy Jumping. In a new safer bungy jumping system, people jump up from the ground while suspended from a rope that runs over a pulley at O and is connected to a stretched spring anchored at B. The pulley has negligible size, mass, and friction. For the situation shown the spring AB has rest length $\ell_0 = 2\,\text{m}$ and a stiffness of $k = 200\,\text{N/m}$. The inextensible massless rope from A to P has length $\ell_r = 8\,\text{m}$, the person has a mass of 100 kg. Take O to be the origin of an xy coordinate system aligned with the unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$

a) Assume you are given the position of the person $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$ and the velocity of the person $\boldsymbol{v} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\,\hat{\boldsymbol{\jmath}}$. Find her acceleration in terms of some or all of her position, her velocity, and the other parameters given. Use the numbers given, where supplied, in your final answer.

b) Given that bungy jumper's initial position and velocity are $\boldsymbol{r}_0 = 1\,\mathrm{m}\hat{\boldsymbol{\imath}} - 5\,\mathrm{m}\hat{\boldsymbol{\jmath}}$ and $\boldsymbol{v}_0 = \boldsymbol{0}$ write MATLAB commands to find her position at $t = \pi/\sqrt{2}\,\mathrm{s}$.

c) Find the answer to part (b) with pencil and paper (a final numerical answer is desired).



problem 5.128:

(Filename:s97p1.3)

5.129 A softball pitcher releases a ball of mass m upwards from her hand with speed v_0 and angle θ_0 from the horizontal. The only external force acting on the ball after its release is gravity.

a) What is the equation of motion for the ball after its release?

b) What are the position, velocity, and acceleration of the ball?

c) What is its maximum height?

d) At what distance does the ball return to the elevation of release?

e) What kind of path does the ball follow and what is its equation y as a function of x?

5.130 Find the trajectory of a not-vertically-fired cannon ball assuming the air drag is proportional to the speed. Assume the mass is 10 kg, $g = 10\,\mathrm{m/s}$, the drag proportionality constant is $C = 5\,\mathrm{N/(m/s)}$. The cannon ball is launched at 100 m/s at a 45 degree angle.

- Draw a free body diagram of the mass.
- Write linear momentum balance in vector form.
- Solve the equations on the computer and plot the trajectory.
- Solve the equations by hand and then use the computer to plot your solution.

5.131 See also problem 5.132. A baseball pitching machine releases a baseball of mass m from its barrel with speed v_0 and angle θ_0 from the horizontal. The only external forces acting on the ball after its release are gravity and air resistance. The speed of the ball is given by $v^2 = \dot{x}^2 + \dot{y}^2$. Taking into account air resistance on the ball proportional to its speed squared, $\underline{F}_d = -bv^2\hat{\boldsymbol{e}}_t$, find the equation of motion for the ball, after its release, in cartesian coordinates. *

5.132 The equations of motion from problem 5.131 are nonlinear and cannot be solved in closed form for the position of the baseball. Instead, solve the equations numerically. Make a computer simulation of the flight of the baseball, as follows.

a) Convert the equation of motion into a system of first order differential equations. *

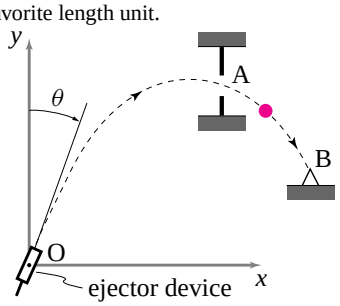
b) Pick values for the gravitational constant g, the coefficient of resistance b, and initial speed v_0, solve for the x and y coordinates of the ball and make a plots its trajectory for various initial angles θ_0.

c) Use Euler's, Runge-Kutta, or other suitable method to numerically integrate the system of equations.

d) Use your simulation to find the initial angle that maximizes the distance of travel for ball, with and without air resistance.

e) If the air resistance is very high, what is a qualitative description for the curve described by the path of the ball?

5.133 In the arcade game shown, the object of the game is to propel the small ball from the ejector device at O in such a way that is passes through the small aperture at A and strikes the contact point at B. The player controls the angle θ at which the ball is ejected and the initial velocity v_o. The trajectory is confined to the frictionless xy-plane, which may or may not be vertical. Find the value of θ that gives success. The coordinates of A and B are $(2\ell,\ 2\ell)$ and $(3\ell,\ \ell)$, respectively, where ℓ is your favorite length unit.



problem 5.133:

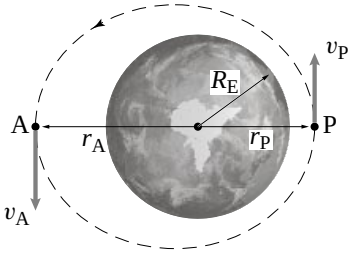
(Filename:pfigure.blue.32.1)

5.9 Central force motion and celestial mechanics

Experts note that none of these problems use polar or other fancy coordinates. Such descriptions come later in the text. At this point we want to lay out the basic equations and the qualitative features that can be found by numerical integration of the equations.

5.134 Under what circumstances is the angular momentum of a system, calculated relative to a point C which is fixed in a Newtonian frame, conserved?

5.135 A satellite is put into an elliptical orbit around the earth (that is, you can assume the orbit is closed) and has a velocity $\vec{\boldsymbol{v}}_P$ at position P. Find an expression for the velocity $\vec{\boldsymbol{v}}_A$ at position A. The radii to A and P are, respectively, r_A and r_P. [Hint: both total energy and angular momentum are conserved.]



problem 5.135:

(Filename:pfigure.blue.64.2)

5.136 The mechanics of nuclear war. A missile, modelled as a point, is launched on a ballistic trajectory from the surface of the earth. The force on the missile from the earth's gravity is $F = mgR^2/r^2$ and is directed towards the center of the earth. When it is launched from the equator it has speed v_0 and in the direction shown, 45° from horizontal. For the purposes of this calculation ignore the earth's rotation. That is, you can think of this problem as two-dimensional in the plane shown. If you need numbers, use the following values:

$m = 1000\,\mathrm{kg}$ is the mass of the missile,

$g = 10\,\mathrm{m/s^2}$ is earth's gravitational constant at the earth's surface,

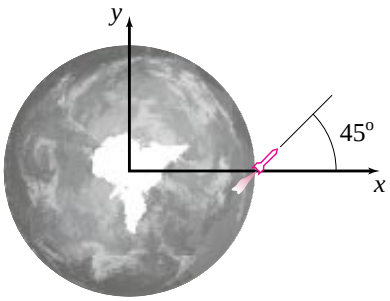
$R = 6,400,000\,\mathrm{m}$ is the radius of the earth, and

$v_0 = 9000\,\mathrm{m/s}$

$r(t)$ is the distance of the missile from the center of the earth.

a) Draw a free body diagram of the missile. Write the linear momentum balance equation. Break this equation into x and y components. Rewrite these equations as a system of 4 first order ODE's suitable for computer solution. Write appropriate initial conditions for the ODE's.

b) Using the computer (or any other means) plot the trajectory of the rocket after it is launched for a time of 6670 seconds. [Use a much shorter time when debugging your program.] On the same plot draw a (round) circle for the earth.



problem 5.136:

(Filename:pfigure.s94q12p1)

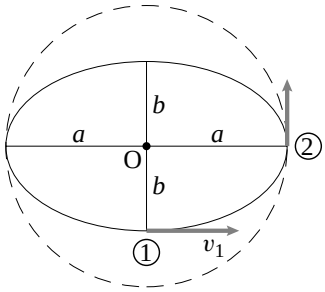
5.137 A particle of mass 2 kg moves in the horizontal xy-plane under the influence of a central force $\vec{F} = -k\vec{r}$ (attraction force proportional to distance from the origin), where $k = 200\,\mathrm{N/m}$ and $\vec{r}$ is the position of the particle relative to the force center. Neglect all other forces.

a) Show that circular trajectories are possible, and determine the relation between speed v and circular radius r_o which must hold on a circular trajectory. [hint: Write $\vec{F} = m\vec{a}$, break into x and y components, solve the separate scalar equations, pick fortuitous values for the free constants in your solutions.]

b) It turns out that trajectories are in general elliptical, as depicted in the diagram.

For a particular elliptical trajectory with $a = 1\,\mathrm{m}$ and $b = 0.8\,\mathrm{m}$, the velocity of the particle at point 1 is observed to be perpendicular to the radial direction, with magnitude v_1, as shown. When the particle reaches point 2, its velocity is again perpendicular to the radial direction.

Determine the speed increment Δv which would have to be added (instantaneously) to the particle's speed at point 2 to transfer it to the circular trajectory through point 2 (the dotted curve). Express your answer in terms of v_1.



problem 5.137:

(Filename:pfigure.blue.72.1)

5.10 Coupled motions of particles in space

5.138 Linear momentum balance for general systems with multiple interacting parts moving more or less independently reduces to $\vec{F} = m\vec{a}$ *if you interpret the terms correctly.* What does this mean? What is $\vec{F}$? What is m? What is $\vec{a}$?

5.139 A particle of mass $m_1 = 6\,\mathrm{kg}$ and a particle of mass $m_2 = 10\,\mathrm{kg}$ are moving in the xy-plane. At a particular instant of interest, particle 1 has position, velocity, and acceleration $\vec{r}_1 = 3\,\mathrm{m}\hat{\imath} + 2\,\mathrm{m}\hat{\jmath}$, $\vec{v}_1 = -16\,\mathrm{m/s}\hat{\imath} + 6\,\mathrm{m/s}\hat{\jmath}$, and $\vec{a}_1 = 10\,\mathrm{m/s^2}\hat{\imath} - 24\,\mathrm{m/s^2}\,\hat{\jmath}$, respectively, and particle 2 has position, velocity, and acceleration $\vec{r}_2 = -6\,\mathrm{m}\hat{\imath} - 4\,\mathrm{m}\hat{\jmath}$, $\vec{v}_2 = 8\,\mathrm{m/s}\hat{\imath} + 4\,\mathrm{m/s}\hat{\jmath}$, and $\vec{a}_2 = 5\,\mathrm{m/s^2}\hat{\imath} - 16\,\mathrm{m/s^2}\,\hat{\jmath}$, respectively.

a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of each particle at the instant of interest.

b) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of the system of the two particles at the instant of interest.

c) Find the center of mass of the system at the instant of interest.

d) Find the velocity and acceleration of the center of mass.

5.140 A particle of mass $m_1 = 5\,\mathrm{kg}$ and a particle of mass $m_2 = 10\,\mathrm{kg}$ are moving in space. At a particular instant of interest, particle 1 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_1 &= 1\,\mathrm{m}\hat{\imath} + 1\,\mathrm{m}\hat{\jmath}\\ \vec{v}_1 &= 2\,\mathrm{m/s}\hat{\jmath}\\ \vec{a}_1 &= 3\,\mathrm{m/s^2}\hat{k}\end{aligned}$$

respectively, and particle 2 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_2 &= 2\,\mathrm{m}\hat{\imath}\\ \vec{v}_2 &= 1\,\mathrm{m/s}\hat{k}\\ \vec{a}_2 &= 1\,\mathrm{m/s^2}\,\hat{\jmath}\end{aligned}$$

respectively. For the system of particles at the instant of interest, find its

a) linear momentum $\vec{L}$,

b) rate of change of linear momentum $\dot{\vec{L}}$,

c) angular momentum about the origin $\vec{H}_O$,

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_O$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy.

5.141 Two particles each of mass m are connected by a massless elastic spring of spring constant k and unextended length $2R$. The system slides without friction on a horizontal table, so that no net external forces act.

a) Is the total linear momentum conserved? Justify your answer.

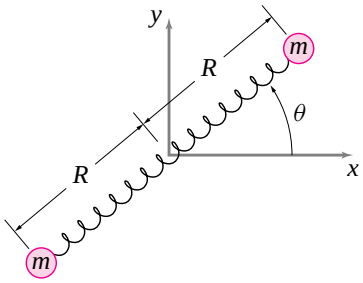
b) Can the center of mass accelerate? Justify your answer.

c) Draw free body diagrams for each mass.

d) Derive the equations of motion for each mass in terms of cartesian coordinates.

e) What are the total kinetic and potential energies of the system?

f) For constant values and initial conditions of your choosing plot the trajectories of the two particles and of the center of mass (on the same plot).



problem 5.141:

(Filename:pfigure.blue.65.2)

5.142 Two ice skaters whirl around one another. They are connected by a linear elastic cord whose center is stationary in space. We wish to consider the motion of one of the skaters by modeling her as a mass m held by a cord that exerts k Newtons for each meter it is extended from the central position.

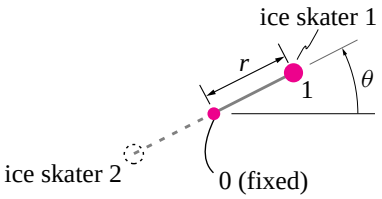
a) Draw a free body diagram showing the forces that act on the mass is at an arbitrary position.

b) Write the differential equations that describe the motion.

c) Describe in physical and mathematical terms the nature of the motion for the three cases

a) $\omega < \sqrt{k/m}$;
b) $\omega = \sqrt{k/m}$;
c) $\omega > \sqrt{k/m}$.

(You are not asked to solve the equation of motion.)



problem 5.142:

(Filename:pfigure.blue.154.1)

5.143 Theory question. If you are given the total mass, the position, the velocity, and the acceleration of the center of mass of a system of particles can you find the angular momentum $\vec{\boldsymbol{H}}_O$ of the system, where O is not at the center of mass? If so, how and why? If not, then give a reason and/or a counter example. *

5.144 The equation $(\vec{\boldsymbol{v}}'_1 - \vec{\boldsymbol{v}}'_2) \cdot \hat{\boldsymbol{n}} = e(\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1) \cdot \hat{\boldsymbol{n}}$ relates relative velocities of two point masses before and after frictionless impact in the normal direction $\hat{\boldsymbol{n}}$ of the impact. If $\vec{\boldsymbol{v}}'_1 = v_{1_x}\hat{\boldsymbol{\imath}} + v_{1_y}\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{v}}'_2 = -v_0\hat{\boldsymbol{\imath}}$, $e = 0.5$, $\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{v}}_1 = 2\,\text{ft/s}\hat{\boldsymbol{\imath}} - 5\,\text{ft/s}\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{n}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$, find the scalar equation relating the velocities in the normal direction.

5.145 See also problem 5.150. Assuming θ, v_0, and e to be known quantities, write the following equations in matrix form set up to solve for v'_{Ax} and v'_{Ay}:

$$\sin\theta v'_{Ax} + \cos\theta v'_{Ay} = ev_0\cos\theta$$
$$\cos\theta v'_{Ax} - \sin\theta v'_{Ay} = v_0\sin\theta.$$

5.146 Set up the following equations in matrix form and solve for v_A and v_B, if $v_0 = 2.6\,\text{m/s}$, $e = 0.8$, $m_A = 2\,\text{kg}$, and $m_B = 500\text{g}$:

$$m_A v_0 = m_A v_A + m_B v_B$$
$$-ev_0 = v_A - v_B.$$

5.147 The following three equations are obtained by applying the principle of conservation of linear momentum on some system.

$$m_0 v_0 = 24.0\,\text{m/s}\; m_a - 0.67 m_B v_B - 0.58 m_C v_C$$
$$0 = 36.0\,\text{m/s}\; m_A + 0.33 m_B v_B + 0.3 m_C v_C$$
$$0 = 23.3\,\text{m/s}\; m_A - 0.67 m_B v_B - 0.58 m_C v_C.$$

Assume v_0, v_B, and v_C are the only unknowns. Write the equations in matrix form set up to solve for the unknowns.

5.148 See also problem 5.149. The following three equations are obtained to solve for v'_{Ax}, v'_{Ay}, and v'_{Bx}:

$$(v'_{Bx} - v'_{Ax})\cos\theta = v'_{Ay}\sin\theta - 10\,\text{m/s}$$
$$v'_{Ax}\sin\theta = v'_{Ay}\cos\theta - 36\,\text{m/s}$$
$$m_B v'_{Bx} + m_A v'_{Ax} = (-60\,\text{m/s})\, m_A.$$

Set up these equations in matrix form.

5.149 Solve for the unknowns v'_{A_x}, v'_{A_y}, and v'_{B_x} in problem 5.148 taking $\theta = 50^o$, $m_A = 1.5\,m_B$ and $m_B = 0.8\,\text{kg}$. Use any computer program.

5.150 Using the matrix form of equations in Problem 5.145, solve for v'_{Ax} and v'_{Ay} if $\theta = 20^o$ and $v_0 = 5\,\text{ft/s}$.

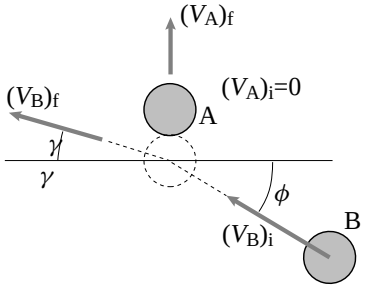
5.151 Two frictionless masses $m_A = 2\,\text{kg}$ and mass $m_B = 5\,\text{kg}$ travel on straight collinear paths with speeds $V_A = 5\,\text{m/s}$ and $V_B = 1\,\text{m/s}$, respectively. The masses collide since $V_A > V_B$. Find the amount of energy lost in the collision assuming normal motion is decoupled from tangential motion. The coefficient of restitution is $e = 0.5$.



problem 5.151:

(Filename:Danef94s3q7)

5.152 Two frictionless pucks sliding on a plane collide as shown in the figure. Puck A is initially at rest. Given that $(V_B)_i = 1.0\,\text{m/s}$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\text{m/s}$, find the approach angle ϕ and rebound angle γ. The coefficient of restitution is $e = 0.9$.



problem 5.152:

(Filename:Danef94s2q8)

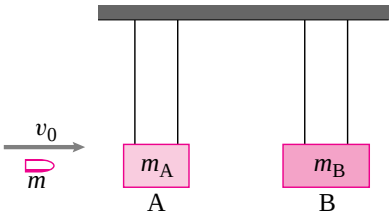
5.153 Reconsider problem 5.152. Given instead that $\gamma = 30^\circ$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\text{m/s}$, find the initial velocity of puck B.

5.154 A bullet of mass m with initial speed v_0 is fired in the horizontal direction through block A of mass m_A and becomes embedded in block B of mass m_B. Each block is suspended by thin wires. The bullet causes A and B to start moving with speed of v_A and v_B respectively. Determine

a) the initial speed v_0 of the bullet in terms of v_A and v_B, *

b) the velocity of the bullet as it travels from block A to block B,and *

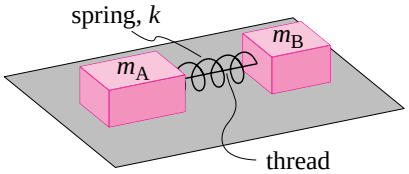
c) the energy loss due to friction as the bullet (1) moves through block A and (2) penetrates block B. *



problem 5.154:

(Filename:pfigure.blue.23.1)

5.155 A massless spring with constant k is held compressed a distance δ from its relaxed length by a thread connecting blocks A and B which are still on a frictionless table. The blocks have mass m_A and m_B, respectively. The thread is suddenly but gently cut, the blocks fly apart and the spring falls to the ground. Find the speed of block A as it slides away. *



problem 5.155:

(Filename:pfigure.blue.32.2)

Problems for Chapter 6

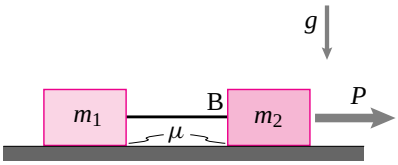
Constrained straight line motion

6.1 1-D constrained motion and pulleys

6.1 Write the following equations in matrix form for a_B, a_C, and T:

$$4a_B + a_C = 0$$
$$2T = -(25\,\text{kg})\ a_B$$
$$64\,\text{kg m/s}^2 = T + (10\,\text{kg})\ a_c.$$

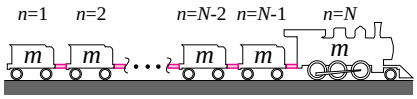
6.2 The two blocks, $m_1 = m_2 = m$, are connected by an inextensible string AB. The string can only withstand a tension T_{cr}. Find the maximum value of the applied force P so that the string does not break. The sliding coefficient of friction between the blocks and the ground is μ.



problem 6.2:

(Filename:Danef94s3q5)

6.3 A train engine of mass m pulls and accelerates on level ground N cars each of mass m. The power of the engine is P_t and its speed is v_t. Find the tension T_n between car n and car n+1. Assume there is no resistance to rolling for all of the cars. Assume the cars are connected with rigid links. *



problem 6.3:

(Filename:pfigure.newtrain)

6.4 Two blocks, each of mass m, are connected together across their tops by a massless string of length S; the blocks' dimensions are small compared to S. They slide down a slope of angle θ. Do not neglect gravity but do neglect friction.

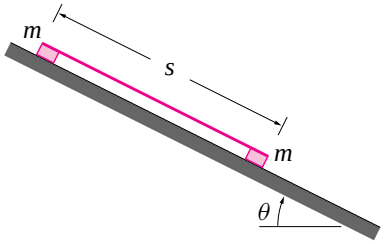
a) Draw separate free body diagrams of each block, the string, and the system of the two blocks and string.

b) Write separate equations for linear momentum balance for each block, the string, and the system of blocks and string.

c) What is the acceleration of the center of mass of the two blocks?

d) What is the force in the string?

e) What is the speed of the center of mass for the two blocks after they have traveled a distance d down the slope, having started from rest. [Hint: You need to dot your momentum balance equations with a unit vector along the ramp in order to reduce this problem to a problem in one dimensional mechanics.]



problem 6.4:

(Filename:pfigure.blue.27.1)

6.5 Two blocks, each of mass m, are connected together across their tops by a massless string of length S; the blocks' dimensions are small compared to S. They slide down a slope of angle θ. The materials are such that the coefficient of dynamic friction on the top block is μ and on the bottom block is $\mu/2$.

a) Draw separate free body diagrams of each block, the string, and the system of the two blocks and string.

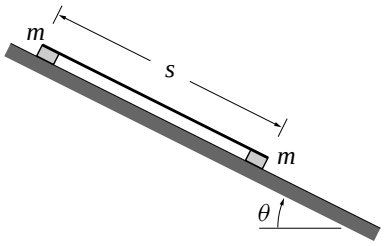
b) Write separate equations for linear momentum balance for each block, the string, and the system of blocks and string.

c) What is the acceleration of the center of mass of the two blocks?

d) What is the force in the string?

e) What is the speed of the center of mass for the two blocks after they have traveled a distance d down the slope, having started from rest.

f) How would your solutions to parts (a) and (c) differ in the following two variations: i.) If the two blocks were interchanged with the slippery one on top or ii.) if the string were replaced by a massless rod? Qualitative responses to this part are sufficient.



problem 6.5:

(Filename:pfigure.blue.27.1a)

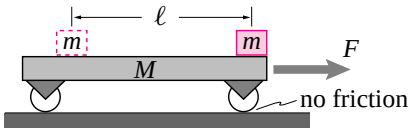
6.6 A cart of mass M, initially at rest, can move horizontally along a frictionless track. When $t = 0$, a force F is applied as shown to the cart. During the acceleration of M by the force F, a small box of mass m slides along the cart from the front to the rear. The coefficient of friction between the cart and box is μ, and it is assumed that the acceleration of the cart is sufficient to cause sliding.

a) Draw free body diagrams of the cart, the box, and the cart and box together.

b) Write the equation of linear momentum balance for the cart, the box, and the system of cart and box.

c) Show that the equations of motion for the cart and box can be combined to give the equation of motion of the mass center of the system of two bodies.

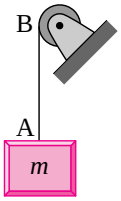
d) Find the displacement of the cart at the time when the box has moved a distance ℓ along the cart.



problem 6.6:

(Filename:pfigure.blue.28.1)

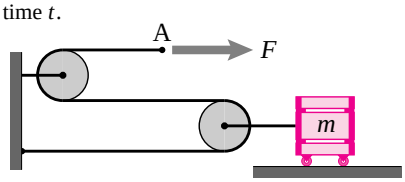
6.7 A motor at B allows the block of mass $m = 3\,\text{kg}$ shown in the figure to accelerate downwards at $2\,\text{m/s}^2$. There is gravity. What is the tension in the string AB?



problem 6.7:

(Filename:pfigure.blue.12.2)

6.8 For the mass and pulley system shown in the figure, the point of application A of the force moves twice as fast as the mass. At some instant in time t, the speed of the mass is $\dot{x}$ to the left. Find the input power to the system at time t.



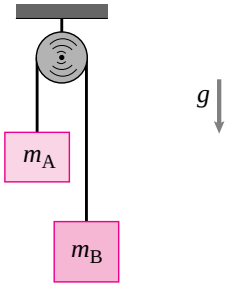
problem 6.8:

(Filename:pfig2.3.rp8)

6.9 Pulley and masses. Two masses connected by an inextensible string hang from an ideal pulley.

a) Find the downward acceleration of mass B. Answer in terms of any or all of m_A, m_B, g, and the present velocities of the blocks. As a check, your answer should give $a_B = g$ when $m_A = 0$ and $a_B = 0$ when $m_A = m_B$. *.

b) Find the tension in the string. As a check, your answer should give $T = m_B g = m_A g$ when $m_A = m_B$ and $T = 0$ when $m_A = 0$. *



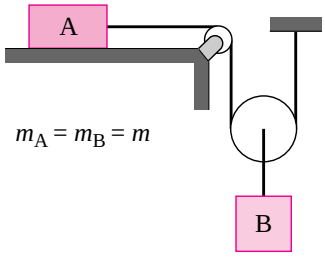
problem 6.9:

(Filename:pfigure3.f95p1p2)

6.10 The blocks shown are released from rest. Make reasonable assumptions about strings, pulleys, string lengths, and gravity.

a) What is the acceleration of block A at $t = 0^+$ (just after release)?

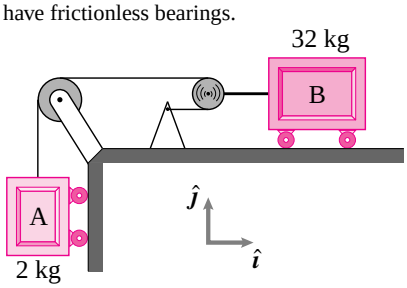
b) What is the speed of block B after it has fallen 2 meters?



problem 6.10:

(Filename:pfigure.blue.29.2)

6.11 What is the acceleration of block A? Use $g = 10\,\mathrm{m/s^2}$. Assume the string is massless and that the pulleys are massless, round, and have frictionless bearings.

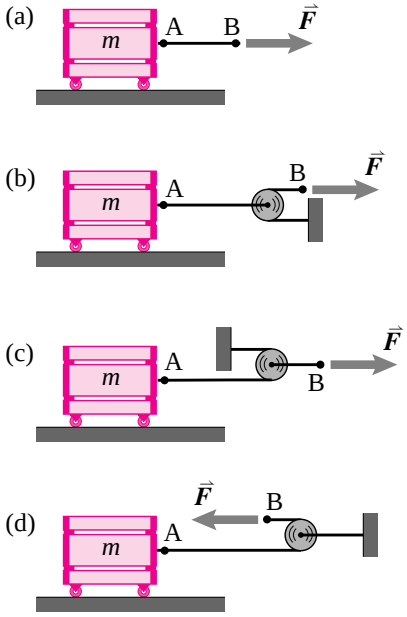


problem 6.11:

(Filename:pfigure.f93q4)

6.12 For the system shown in problem 6.9, find the acceleration of mass B using energy balance ($P = \dot{E}_K$).

6.13 For the various situations pictured, find the acceleration of the mass A and the point B shown using balance of linear momentum ($\sum \vec{F} = m\vec{a}$). Define any variables, coordinates or sign conventions that you need to do your calculations and to define your solution. *

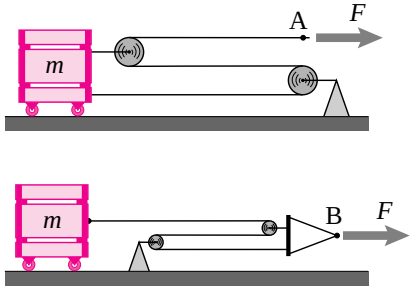


problem 6.13:

(Filename:pulley1)

6.14 For each of the various situations pictured in problem 6.13 find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

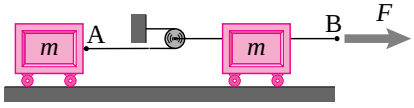
6.15 What is the ratio of the acceleration of point A to that of point B in each configuration? In both cases, the strings are inextensible, the pulleys massless, and the mass and force the same. *



problem 6.15:

(Filename:sum95.p1.p3)

6.16 See also problem 6.17. Find the acceleration of points A and B in terms of F and m. Assume that the carts stay on the ground, have good (frictionless) bearings, and have wheels of negligible mass.



problem 6.16:

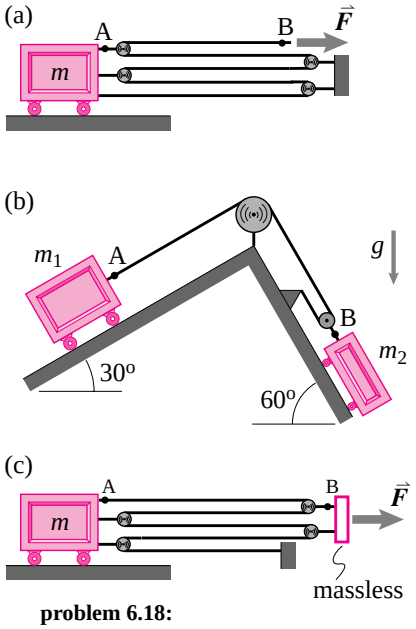
(Filename:pfigure.s94q5p1)

6.17 For the situation pictured in problem 6.16 find the accelerations of the two masses using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

6.18 See also problem 6.19. For the various situations pictured, find the acceleration of the mass A and the point B shown using balance of linear momentum ($\sum \vec{F} = m\vec{a}$). Define any variables, coordinates or sign conventions that you need to do your calculations and to define your solution.

a) *

b) *



problem 6.18:

(Filename:pulley4)

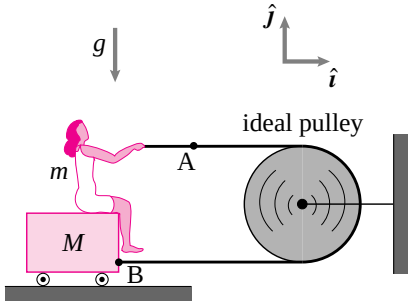
6.19 For the various situations pictured in problem 6.18 find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

6.20 A person of mass m, modeled as a rigid body is sitting on a cart of mass $M > m$ and pulling the massless inextensible string towards herself. The coefficient of friction between her seat and the cart is μ. All wheels and pulleys are massless and frictionless. Point B is attached to the cart and point A is attached to the rope.

a) If you are given that she is pulling rope in with acceleration a_0 relative to herself (that is, $\boldsymbol{a}_{A/B} \equiv \boldsymbol{a}_A - \boldsymbol{a}_B = -a_0\hat{\boldsymbol{\imath}}$) and that she is not slipping relative to the cart, find $\boldsymbol{a}_A$. (Answer in terms of some or all of $m, M, g, \mu, \hat{\boldsymbol{\imath}}$ and a_0.)

b) Find the largest possible value of a_0 without the person slipping off the cart? (Answer in terms of some or all of m, M, g and μ. You may assume her legs get out of the way if she slips backwards.)

c) If instead, $m < M$, what is the largest possible value of a_0 without the person slipping off the cart? (Answer in terms of some or all of m, M, g and μ. You may assume her legs get out of the way if she slips backwards.)



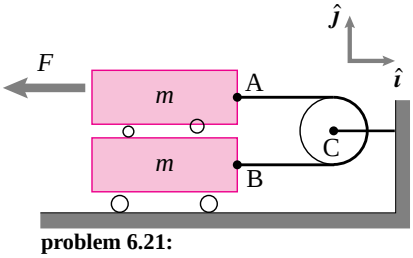
problem 6.20: Pulley.

(Filename:s97p2.2)

6.21 Two blocks and a pulley. Two equal masses are stacked and tied together by the pulley as shown. All bearings are frictionless. All rotating parts have negligible mass. Find

a) the acceleration of point A, and

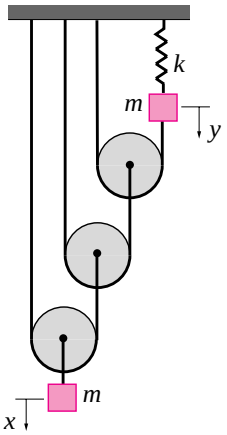
b) the tension in the line.



problem 6.21:

(Filename:p.s96.p1.1)

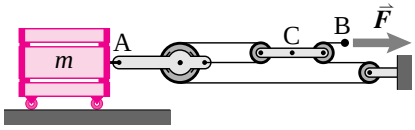
6.22 The pulleys are massless and frictionless. Neglect air friction. Include gravity. x measures the vertical position of the lower mass from equilibrium. y measures the vertical position of the upper mass from equilibrium. What is the natural frequency of vibration of this system? *



problem 6.22:

(Filename:pfigure.s95q4)

6.23 See also problem 6.24. For the situation pictured, find the acceleration of the mass A and the points B and C shown using balance of linear momentum ($\sum \vec{F} = m\vec{a}$). Define any variables, coordinates or sign conventions that you need to do your calculations and to define your solution. [Hint: the situation with point C is tricky and the answer is subtle.]

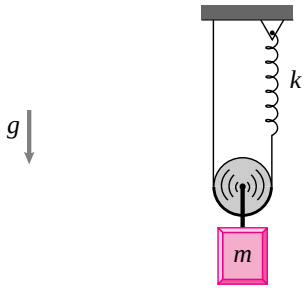


problem 6.23:

(Filename:pulley2)

6.24 For situation pictured in problem 6.23, find the acceleration of point A using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

6.25 Pulley and spring. For the mass hanging at the right, find the period of oscillation. Assume a massless pulley with good bearings. The massless string is inextensible. Only vertical motion is of interest. There is gravity. [Hint: Draw FBD, carefully keep track of string length to figure spring stretch, set up equations of motion and solve them.]



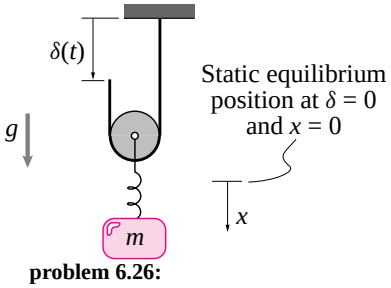
problem 6.25:

(Filename:pfigure.s94h4p4)

6.26 The spring-mass system shown (m = 10 slugs ($\equiv$ lb·sec^2/ft), $k = 10$ lb/ft) is excited by moving the free end of the cable vertically according to $\delta(t) = 4\sin(\omega t)$ in, as shown in the figure. Assuming that the cable is inextensible and massless and that the pulley is massless, do the following.

a) Derive the equation of motion for the block in terms of the displacement x from the static equilibrium position, as shown in the figure.

b) If $\omega = 0.9$ rad/s, check to see if the pulley is always in contact with the cable (ignore the transient solution).



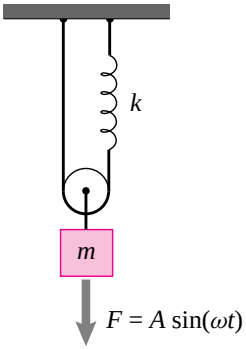
problem 6.26:

(Filename:pfigure.blue.151.1)

6.27 The block of mass m hanging on the spring with constant k and a string shown in the figure is forced (by an unseen agent) with the force $F = A\sin(\omega t)$. (Do not neglect gravity). The pulley is massless.

a) What is the differential equation governing the motion of the block? You may assume that the only motion is vertical motion. *

b) Given A, m and k, for what values of ω would the string go slack at some point in the cyclical motion? (You should neglect the homogeneous solution to the differential equation.) *



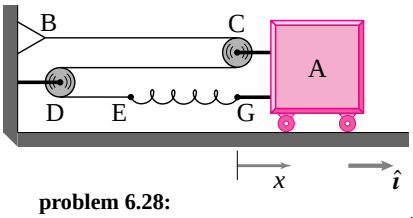
problem 6.27:

(Filename:pfigure.blue.155.1)

6.28 Block A, with mass m_A, is pulled to the right a distance d from the position it would have if the spring were relaxed. It is then released from rest. Assume ideal string, pulleys and wheels. The spring has constant k.

a) What is the acceleration of block A just after it is released (in terms of k, m_A, and d)? *

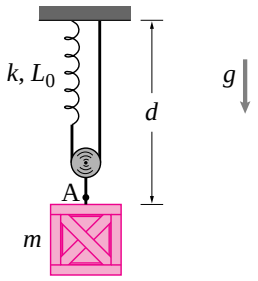
b) What is the speed of the mass when the mass passes through the position where the spring is relaxed? * .



problem 6.28:

(Filename:pfigure.f93q5)

6.29 What is the static displacement of the mass from the position where the spring is just relaxed?

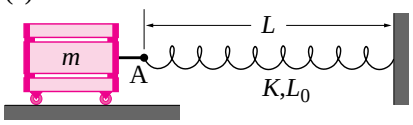


problem 6.29:

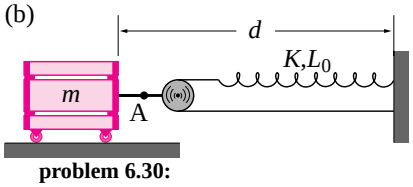
(Filename:pulley3.a)

6.30 For the two situations pictured, find the acceleration of point A shown using balance of linear momentum ($\sum \vec{F} = m\vec{a}$). Assuming both masses are deflected an equal distance from the position where the spring is just relaxed, how much smaller or bigger is the acceleration of block (b) than of block(a). Define any variables, coordinate system origins, coordinates or sign conventions that you need to do your calculations and to define your solution.

(a)



(b)



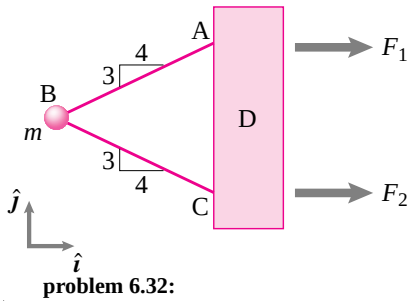
problem 6.30:

(Filename:pulley3)

6.31 For each of the various situations pictured in problem 6.30, find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

6.2 2D and 3D forces even though the motionis straight

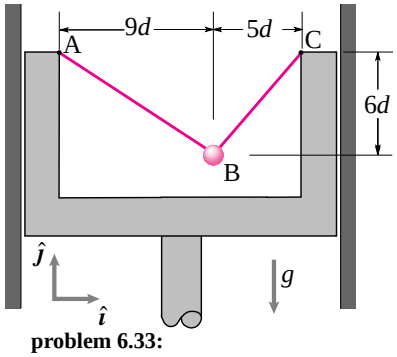
6.32 Mass pulled by two strings. F_1 and F_2 are applied so that the system shown accelerates to the right at 5 m/s^2 (*i.e.*, $\boldsymbol{a} = 5\,\text{m/s}^2\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}}$) and has no rotation. The mass of D and forces F_1 and F_2 are unknown. What is the tension in string AB?



problem 6.32:

(Filename:pg9.1)

6.33 A point mass m is attached to a piston by two inextensible cables. The piston has upwards acceleration $a_y\hat{\boldsymbol{\jmath}}$. There is gravity. In terms of some or all of m, g, d, and a_y find the tension in cable AB. *



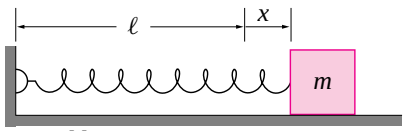
problem 6.33:

(Filename:ch3.13)

6.34 A point mass of mass m moves on a frictional surface with coefficient of friction μ and is connected to a spring with constant k and unstretched length ℓ. There is gravity. At the instant of interest, the mass is at a distance x to the right from its position where the spring is unstretched and is moving with $\dot{x} > 0$ to the right.

a) Draw a free body diagram of the mass at the instant of interest.

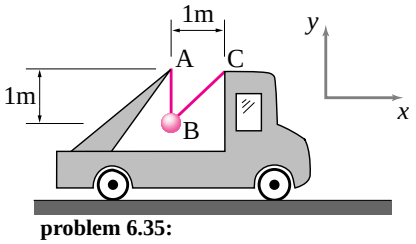
b) At the instant of interest, write the equation of linear momentum balance for the block evaluating the left hand side as explicitly as possible. Let the acceleration of the block be $\vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}}$.



problem 6.34:

(Filename:ch2.10a)

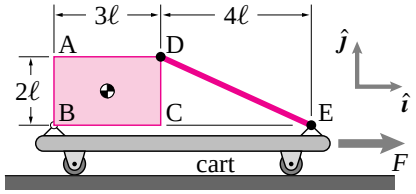
6.35 Find the tension in two strings. Consider the mass at B (2 kg) supported by two strings in the back of a truck which has acceleration of 3 m/s^2. Use your favorite value for the gravitational constant. What is the tension T_{AB} in the string AB in Newtons?



problem 6.35:

(Filename:pfigure.s94h2p8)

6.36 Guyed plate on a cart A uniform rectangular plate $ABCD$ of mass m is supported by a rod DE and a hinge joint at point B. The dimensions are as shown. The cart has acceleration $a_x\hat{\boldsymbol{\imath}}$ due to a force F. There is gravity. What must the acceleration of the cart be in order for the rod DE to be in tension? *



problem 6.36: Uniform plate supported by a hinge and a cable on an accelerating cart.

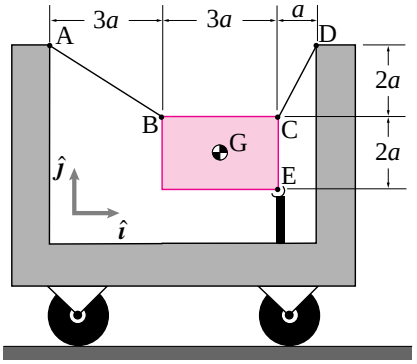
(Filename:tfigure3.2D.a.guyed)

6.37 A uniform rectangular plate of mass m is supported by two inextensible cables AB and CD and by a hinge at point E on the cart as shown. The cart has acceleration $a_x\hat{\boldsymbol{\imath}}$ due to a force not shown. There is gravity.

a) Draw a free body diagram of the plate.

b) Write the equation of linear momentum balance for the plate and evaluate the left hand side as explicitly as possible.

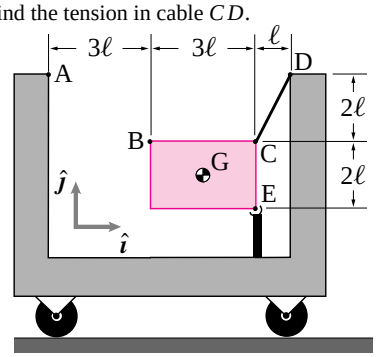
c) Write the equation for angular momentum balance about point E and evaluate the left hand side as explicitly as possible.



problem 6.37:

(Filename:ch2.12)

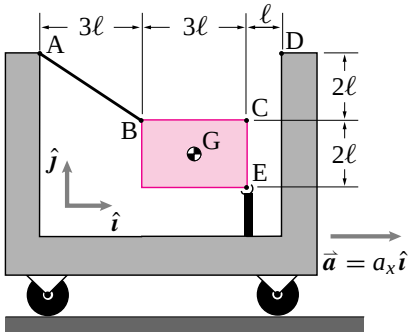
6.38 See also problem 6.67. A uniform rectangular plate of mass m is supported by an inextensible cable CD and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart is at rest. There is gravity. Find the tension in cable CD.



problem 6.38:

(Filename:ch3.11.b)

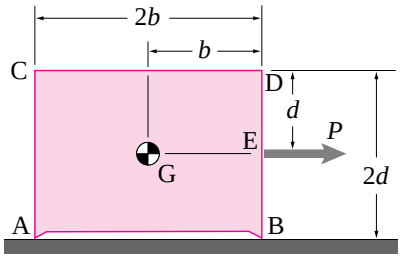
6.39 A uniform rectangular plate of mass m is supported by an inextensible cable AB and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart has acceleration $a_x\hat{\boldsymbol{\imath}}$. There is gravity. Find the tension in cable AB. (What's 'wrong' with this problem? What if instead point B was at the bottom left hand corner of the plate?) *



problem 6.39:

(Filename:ch3.11a)

6.40 See also 6.41. A block of mass m is sitting on a frictionless surface and acted upon at point E by the horizontal force P through the center of mass. Draw a free body diagram of the block. There is gravity. Find the acceleration of the block and reactions on the block at points A and B.

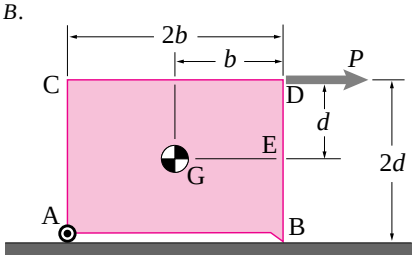


problem 6.40:

(Filename:ch3.9)

6.41 Reconsider the block in problem 6.40. This time, find the acceleration of the block and the reactions at A and B if the force P is applied instead at point D. Are the acceleration and the reactions on the block different from those found if P is applied at point E?

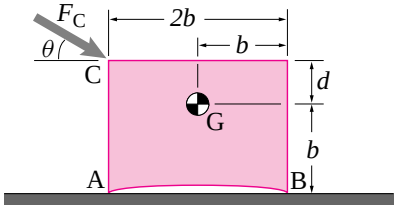
6.42 A block of mass m is sitting on a frictional surface and acted upon at point D by the horizontal force P. The block is resting on a sharp edge at point B and is supported by an ideal wheel at point A. There is gravity. Assuming the block is sliding with coefficient of friction μ at point B, find the acceleration of the block and the reactions on the block at points A and B.



problem 6.42:

(Filename:ch3.12)

6.43 A force F_C is applied to the corner C of a box of weight W with dimensions and center of gravity at G as shown in the figure. The coefficient of sliding friction between the floor and the points of contact A and B is μ. Assuming that the box slides when F_C is applied, find the acceleration of the box and the reactions at A and B in terms of W, F_C, θ, b, and d.



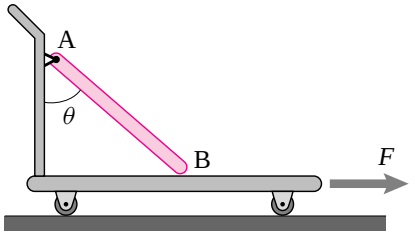
problem 6.43:

(Filename:Mikes92p3)

6.44 Forces of rod on a cart. A uniform rod with mass m_r rests on a cart (mass m_c) which is being pulled to the right. The rod is hinged at one end (with a frictionless hinge) and has no friction at the contact with the cart. The cart is rolling on wheels that are modeled as having no mass and no bearing friction (ideal massless wheels). Find:

a) The force on the rod from the cart at point B. Answer in terms of g, m_r, m_c, θ and F.

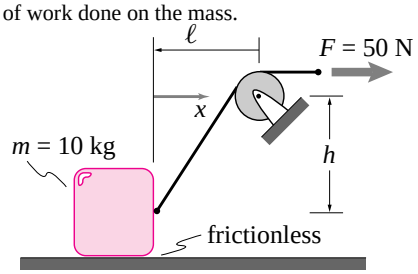
b) The force on the rod from the cart at point A.



problem 6.44:

(Filename:pfigure.s94h3p3)

6.45 At the instant shown, the mass is moving to the right at speed $v = 3\,\text{m/s}$. Find the rate of work done on the mass.



problem 6.45:

(Filename:pfig2.3.rp9)

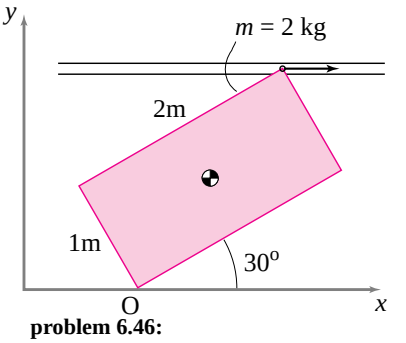
6.46 The box shown in the figure is dragged in the x-direction with a constant acceleration $\vec{\boldsymbol{a}} = 0.5\,\text{m/s}^2\hat{\boldsymbol{\imath}}$. At the instant shown, the velocity of (every point on) the box is $\vec{\boldsymbol{v}} = 0.8\,\text{m/s}\hat{\boldsymbol{\imath}}$.

a) Find the linear momentum of the box.

b) Find the rate of change of linear momentum of the box.

c) Find the angular momentum of the box about the contact point O.

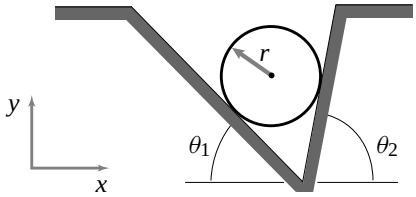
d) Find the rate of change of angular momentum of the box about the contact point O.



problem 6.46:

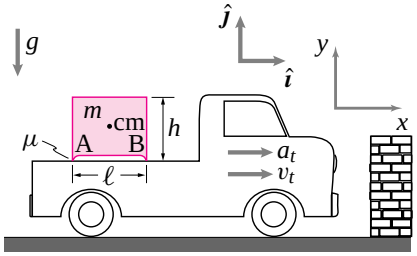
(Filename:pfigure3.mom.rp1)

6.47 (See also problem 4.15.) The groove and disk accelerate upwards, $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\jmath}}$. Neglecting gravity, what are the forces on the disk due to the groove?



problem 6.47:

(Filename:ch2.5.ba)



problem 6.48:

(Filename:pfigure.blue.22.1)

6.48 The following problems concern a box that is in the back of a pickup truck. The pickup truck is accelerating forward at an acceleration of a_t. The truck's speed is v_t. The box has sharp feet at the front and back ends so the only place it contacts the truck is at the feet. The center of mass of the box is at the geometric center of the box. The box has height h, length ℓ and depth w (into the paper.) Its mass is m. There is gravity. The friction coefficient between the truck and the box edges is μ.

In the problems below you should express your solutions in terms of the variables given in the figure, ℓ, h, μ, m, g, a_t, and v_t. If any variables do not enter the expressions comment on why they do not. In all cases you may assume that the box does not rotate (though it might be on the verge of doing so).

a) Assuming the box does not slide, what is the total force that the truck exerts on the box (i.e. the sum of the reactions at A and B)?

b) Assuming the box does not slide what are the reactions at A and B? [Note: You cannot find both of them without additional assumptions.]

c) Assuming the box does slide, what is the total force that the truck exerts on the box?

d) Assuming the box does slide, what are the reactions at A and B?

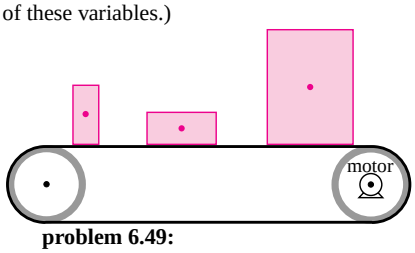
e) Assuming the box does not slide, what is the maximum acceleration of the truck for which the box will not tip over (hint: just at that critical acceleration what is the vertical reaction at B?)?

f) What is the maximum acceleration of the truck for which the block will not slide?

g) The truck hits a brick wall and stops instantly. Does the block tip over?

Assuming the block does not tip over, how far does it slide on the truck before stopping (assume the bed of the truck is sufficiently long)?

6.49 A collection of uniform boxes with various heights h and widths w and masses m sit on a horizontal conveyer belt. The acceleration $a(t)$ of the conveyer belt gets extremely large sometimes due to an erratic over-powered motor. Assume the boxes touch the belt at their left and right edges only and that the coefficient of friction there is μ. It is observed that some boxes never tip over. What is true about μ, g, w, h, and m for the boxes that always maintain contact at both the right and left bottom edges? (Write an inequality that involves some or all of these variables.)

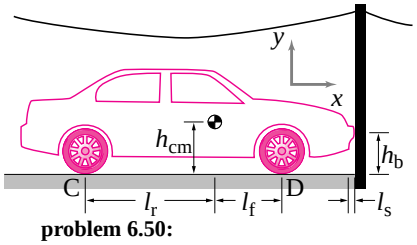


problem 6.49:

(Filename:pfigure.f93q3)

6.50 After failure of her normal brakes, a driver pulls the emergency brake of her old car. This action locks the rear wheels (friction coefficient $= \mu$) but leaves the well lubricated and light front wheels spinning freely. The car, braking inadequately as is the case for rear wheel braking, hits a stiff and slippery phone pole which compresses the car bumper. The car bumper is modeled here as a linear spring (constant $= k$, rest length $= l_0$, present length $= l_s$). The car is still traveling forward at the moment of interest. The bumper is at a height h_b above the ground. Assume that the car, excepting the bumper, is a non-rotating rigid body and that the wheels remain on the ground (that is, the bumper is compliant but the suspension is stiff).

- What is the acceleration of the car in terms of g, m, μ, l_f, l_r, k, h_b, h_{cm}, l_0, and l_s (and any other parameters if needed)?



problem 6.50:

(Filename:pfigure.s94q4p1)

6.51 Car braking: front brakes versus rear brakes versus all four brakes. There are a few puzzles in dynamics concerning the differences between front and rear braking of a car. Here is one you can deal with now. What is the peak deceleration of a car when you apply: the front brakes till they skid, the rear brakes till they skid, and all four brakes till they skid? Assume that the coefficient of friction between rubber and road is $\mu = 1$ (about right, the coefficient of friction between rubber and road varies between about .7 and 1.3) and that $g = 10\,\mathrm{m/s^2}$ (2% error). Pick the dimensions and mass of the car, but assume the center of mass height h is above the ground. The height h, should be less than half the wheel base w, the distance between the front and rear wheel. Further assume that the CM is halfway between the front and back wheels (*i.e.*, $l_f = l_r = w/2$). Assume also that the car has a stiff suspension so the car does not move up or down or tip during braking; i. e., the car does not rotate in the xy-plane. Neglect the mass of the rotating wheels in the linear and angular momentum balance equations. Treat this problem as two-dimensional problem; i. e., the car is symmetric left to right, does not turn left or right, and that the left and right wheels carry the same loads. To organize your work, here are some steps to follow.

a) Draw a FBD of the car assuming rear wheel is skidding. The FBD should show the dimensions, the gravity force, what you know *a priori* about the forces on the wheels from the ground (i.e., that the friction force $F_r = \mu N_r$, and that there is no friction at the front wheels), and the coordinate directions. Label points of interest that you will use in your momentum balance equations. (Hint: also draw a free body diagram of the rear wheel.)

b) Write down the equation of linear momentum balance.

c) Write down the equation of angular momentum balance relative to a point of your choosing. Some particularly useful points to use are: the point above the front wheel and at the height of the center of mass; the point at the height of the center of mass, behind the rear wheel that makes a 45 degree angle line down to the rear wheel ground contact point; and the point on the ground straight under the front wheel that is as deep as the wheel base is long.

d) Solve the momentum balance equations for the wheel contact forces and the deceleration of the car. If you have used any or all of the recommendations from part (c) you will have the pleasure of only solving one equation in one unknown at a time. *

e) Repeat steps (a) to (d) for front-wheel skidding. Note that the advantageous points to use for angular momentum balance are now different. Does a car

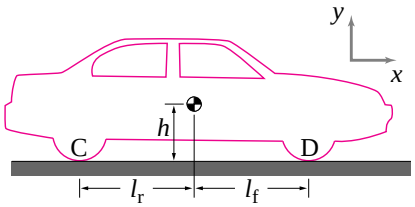
stop faster or slower or the same by skidding the front instead of the rear wheels? Would your solution to (e) be different if the center of mass of the car was at ground level(h=0)? *

f) Repeat steps (a) to (d) for all-wheel skidding. There are some shortcuts here. You determine the car deceleration without ever knowing the wheel reactions (or using angular momentum balance) if you look at the linear momentum balance equations carefully. *

g) Does the deceleration in (f) equal the sum of the deceleration in (d) and (e)? Why or why not? *

h) What peculiarity occurs in the solution for front-wheel skidding if the wheel base is twice the height of the CM above ground and $\mu = 1$? *

i) What impossibility does the solution predict if the wheel base is shorter than twice the CM height? What wrong assumption gives rise to this impossibility? What would really happen if one tried to skid a car this way? *



problem 6.51:

(Filename:pfigure.s94h3p6)

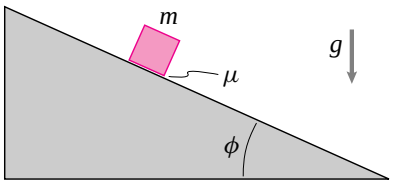
6.52 At time $t = 0$, the block of mass m is released at rest on the slope of angle ϕ. The coefficient of friction between the block and slope is μ.

a) What is the acceleration of the block for $\mu > 0$? *

b) What is the acceleration of the block for $\mu = 0$? *

c) Find the position and velocity of the block as a function of time for $\mu > 0$. *

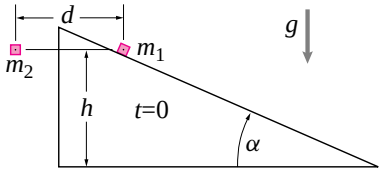
d) Find the position and velocity of the block as a function of time for $\mu = 0$. *



problem 6.52:

(Filename:Danef94s1q5)

6.53 A small block of mass m_1 is released from rest at altitude h on a frictionless slope of angle α. At the instant of release, another small block of mass m_2 is dropped vertically from rest at the same altitude. The second block does not interact with the ramp. What is the velocity of the first block relative to the second block after t seconds have passed?



problem 6.53:

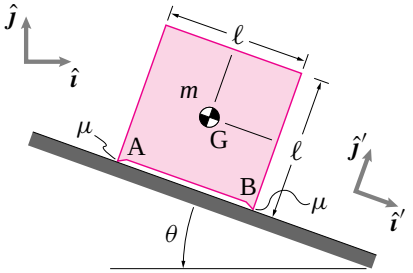
(Filename:ch8.7)

6.54 Block sliding on a ramp with friction. A square box is sliding down a ramp of angle θ with instantaneous velocity $v\hat{\boldsymbol{\imath}}'$. It is assumed to not tip over.

a) What is the force on the block from the ramp at point A? Answer in terms of any or all of $\theta, \ell, m, g, \mu, v, \hat{\boldsymbol{\imath}}'$, and $\hat{\boldsymbol{\jmath}}'$. As a check, your answer should reduce to $\frac{mg}{2}\hat{\boldsymbol{\jmath}}'$ when $\theta = \mu = 0$. *

b) In addition to solving the problem by hand, see if you can write a set of computer commands that, if θ, μ, ℓ, m, v and g were specified, would give the correct answer.

c) Assuming $\theta = 80^\circ$ and $\mu = 0.9$, can the box slide this way or would it tip over? Why? *

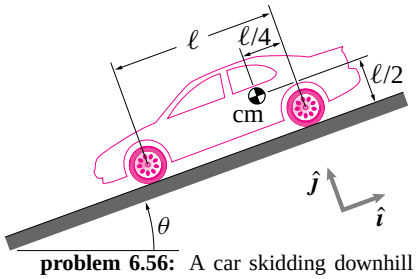


problem 6.54:

(Filename:pfigure3.f95p1p1)

6.55 A coin is given a sliding shove up a ramp with angle ϕ with the horizontal. It takes twice as long to slide down as it does to slide up. What is the coefficient of friction μ between the coin and the ramp. Answer in terms of some or all of m, g, ϕ and the initial sliding velocity v.

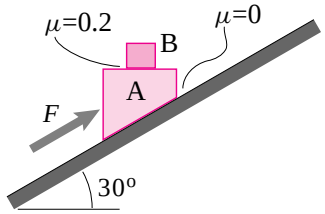
6.56 A skidding car. What is the braking acceleration of the front-wheel braked car as it slides down hill. Express your answer as a function of any or all of the following variables: the slope θ of the hill, the mass of the car m, the wheel base ℓ, and the gravitational constant g. Use $\mu = 1$. *



problem 6.56: A car skidding downhill on a slope of angle θ

(Filename:pfigure3.car)

6.57 Two blocks A and B are pushed up a frictionless inclined plane by an external force F as shown in the figure. The coefficient of friction between the two blocks is $\mu = 0.2$. The masses of the two blocks are $m_A = 5\,\text{kg}$ and $m_B = 2\,\text{kg}$. Find the magnitude of the maximum allowable force such that no relative slip occurs between the two blocks.



problem 6.57:

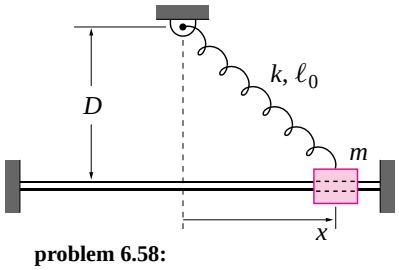
(Filename:summer95f.1)

6.58 A bead slides on a frictionless rod. The spring has constant k and rest length ℓ_0. The bead has mass m.

a) Given x and $\dot{x}$ find the acceleration of the bead (in terms of some or all of $D, \ell_0, x, \dot{x}, m, k$ and any base vectors that you define).

b) If the bead is allowed to move, as constrained by the slippery rod and the spring, find a differential equation that must be satisfied by the variable x. (Do not try to solve this somewhat ugly nonlinear equation.)

c) In the special case that $\ell_0 = 0$* find how long it takes for the block to return to its starting position after release with no initial velocity at $x = x_0$.



problem 6.58:

(Filename:p.s96.p1.2)

6.59 A bead oscillates on a straight frictionless wire. The spring obeys the equation $F = k\,(\ell - \ell_o)$, where ℓ = length of the spring and ℓ_0 is the 'rest' length. Assume

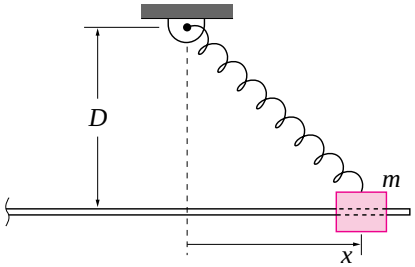
$$x(t = 0) = x_0,\ \dot{x}(t = 0) = 0.$$

a) Write a differential equation satisfied by $x(t)$.

b) What is $\dot{x}$ when $x = 0$? [hint: Don't try to solve the equation in (a)!]

c) Note the simplification in (a) if $\ell_o = 0$ (spring is then a so-called "zero–length" spring).

d) For this special case ($\ell_o = 0$) solve the equation in (a) and show the result agrees with (b) in this special case.



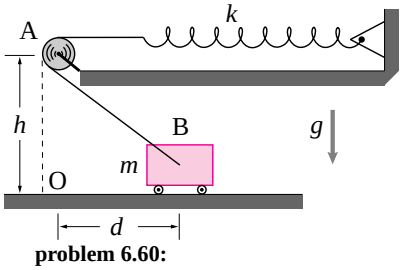
problem 6.59:

(Filename:pfigure.blue.60.1)

6.60 A cart on a springy leash. A cart B (mass m) rolls on a frictionless level floor. One end of an inextensible string is attached to the the cart. The string wraps around a pulley at point A and the other end is attached to a spring with constant k. When the cart is at point O, it is in static equilibrium. The spring and room height are such that the spring would be relaxed if the end of the cart B was in the air at the ceiling pulley, point A. The ceiling height is h. The gravitational constant is g. The cart is pulled a horizontal distance d from the center of the room (at O) and released.

a) Assuming that the cart never leaves the floor, what is the speed of the cart when it passes through the center of the room, in terms of m, h, g and d. *

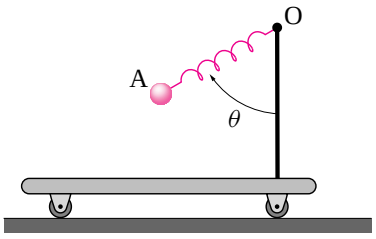
b) Does the cart undergo simple harmonic motion for small or large oscillations (specify which if either)? (Simple harmonic motion is when position is sine-wave and/or cosine-wave function of time.) *



problem 6.60:

(Filename:cartosc)

6.61 The cart moves to the right with constant acceleration a. The ball has mass m. The spring has unstretched length ℓ_0 and spring constant k. Assuming the ball is stationary with respect to the cart find the distance from O to A in terms of k, ℓ_0, and a . [Hint: find θ first.]



problem 6.61:

(Filename:Danef94s3q6)

6.62 Consider a person, modeled as a rigid body, riding an accelerating motorcycle (2-D). The motorcycle is accelerating. The person is a rigid body. The person is sitting on the seat and cannot slide fore or aft, but is free to rock in the plane of the motorcycle(as if there is a hinge connecting the motorcycle to the rider at the seat). The person's feet are off the pegs and the legs are sticking down and not touching anything. The person's arms are like cables (they are massless and only carry tension). Assume all dimensions and masses are known (you have to define them carefully with a sketch and words). Assume the forward acceleration of the motorcycle is known. You may use numbers and/or variables to describe the quantities of interest.

a) Draw a clear sketch of the problem showing needed dimension information and the coordinate system you will use.

b) Draw a Free Body Diagram of the rider.

c) Write the equations of linear and angular momentum balance for the rider.

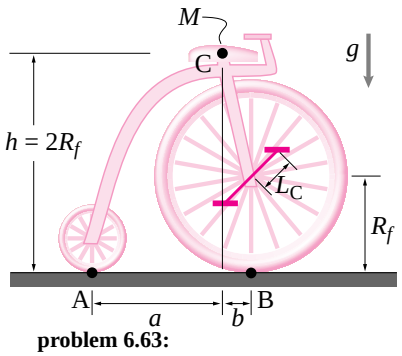
d) Find all forces on the rider from the motorcycle (i.e., at the hands and the seat).

e) What are the forces on the motorcycle from the rider?

6.63 Acceleration of a bicycle on level ground. 2-D . A very compact bicycler (modeled as a point mass M at the bicycle seat C with height h, and distance b behind the front wheel contact), rides a very light old-fashioned bicycle (all components have negligible mass) that is well maintained (all bearings have no frictional torque) and streamlined (neglect air resistance). The rider applies a force F_p to the pedal perpendicular to the pedal crank (with length L_c). No force is applied to the other pedal. The radius of the front wheel is R_f.

a) Assuming no slip, what is the forward acceleration of the bicycle? [Hint: draw a FBD of the front wheel and crank, and another FBD of the whole bicycle-rider system.] *

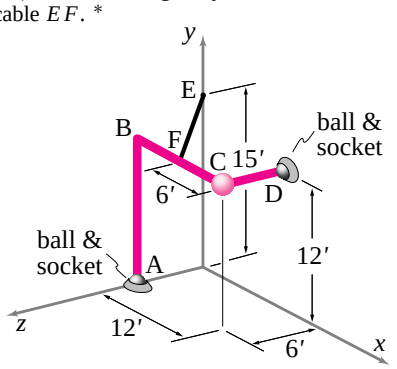
b) (Harder) Assuming the rider can push arbitrarily hard but that $\mu = 1$, what is the maximum possible forward acceleration of the bicycle. *



problem 6.63:

(Filename:f.3)

6.64 A 320 lbm mass is attached at the corner C of a light rigid piece of pipe bent as shown. The pipe is supported by ball-and-socket joints at A and D and by cable EF. The points A, D, and E are fastened to the floor and vertical sidewall of a pick-up truck which is accelerating in the z-direction. The acceleration of the truck is $\vec{\boldsymbol{a}} = 5\,\text{ft/s}^2\hat{\boldsymbol{k}}$. There is gravity. Find the tension in cable EF. *



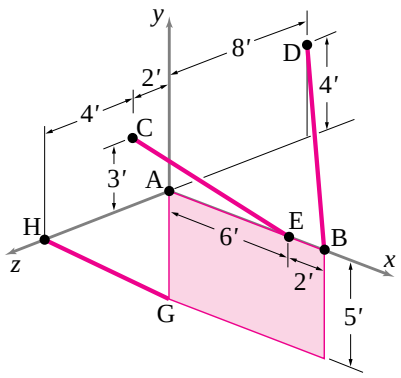
problem 6.64:

(Filename:mikef91p1)

6.65 A 5 ft by 8 ft rectangular plate of uniform density has mass $m = 10$ lbm and is supported by a ball-and-socket joint at point A and the light rods CE, BD, and GH. The entire system is attached to a truck which is moving with acceleration $\vec{\boldsymbol{a}}_T$. The plate is moving without rotation or angular acceleration relative to the truck. Thus, the center of mass acceleration of the plate is the same as the truck's. Dimensions are as shown. Points A, C, and D are fixed to the truck but the truck is not touching the plate at any other points. Find the tension in rod BD.

a) If the truck's acceleration is $\boldsymbol{a}_{cm} = (5\,\text{ft/s}^2)\boldsymbol{k}$, what is the tension or compression in rod BD? *

b) If the truck's acceleration is $\vec{\boldsymbol{a}}_{\text{cm}} = (5\,\text{ft/s}^2)\hat{\boldsymbol{j}} + (6\,\text{ft/s}^2)\hat{\boldsymbol{k}}$, what is the tension or compression in rod GH? *



problem 6.65:

(Filename:Mikesp92p1)

6.66 Hanging a shelf. A shelf with negligible mass supports a 0.5 kg mass at its center. The shelf is supported at one corner with a ball and socket joint and the other three corners with strings. At the moment of interest the shelf is in a rocket in outer space and accelerating at $10\,\mathrm{m/s^2}$ in the $\boldsymbol{k}$ direction. The shelf is in the xy plane.

a) Draw a FBD of the shelf.

b) Challenge: without doing any calculations on paper can you find one of the reaction force components or the tension in any of the cables? Give yourself a few minutes of staring to try this approach. If you can't, then come back to this question after you have done all the calculations. *

c) Write down the linear momentum balance equation (a vector equation). *

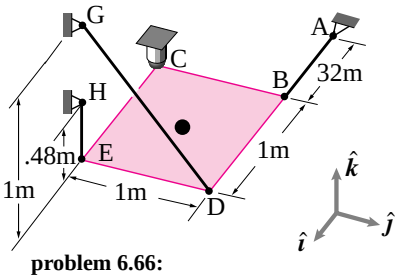
d) Write down the angular momentum balance equation using the center of mass as a reference point. *

e) By taking components, turn (b) and (c) into six scalar equations in six unknowns. *

f) Solve these equations by hand or on the computer. *

g) Instead of using a system of equations try to find a single equation which can be solved for T_{EH}. Solve it and compare to your result from before. *

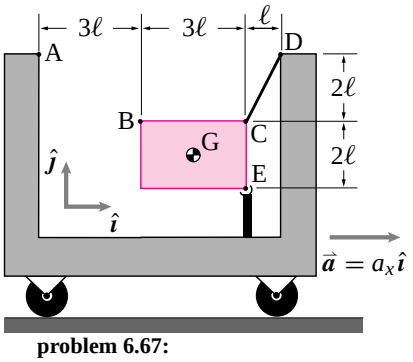
h) Challenge: For how many of the reactions can you find one equation which will tell you that particular reaction without knowing any of the other reactions? [Hint, try angular momentum balance about various axes as well as linear momentum balance in an appropriate direction. It is possible to find five of the six unknown reaction components this way.] Must these solutions agree with (d)? Do they?



problem 6.66:

(Filename:pfigure.s94h2p10)

6.67 See also problem 6.38. A uniform rectangular plate of mass m is supported by an inextensible cable CD and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart has acceleration $a_x\hat{\boldsymbol{\imath}}$. There is gravity. Find the tension in cable CD.



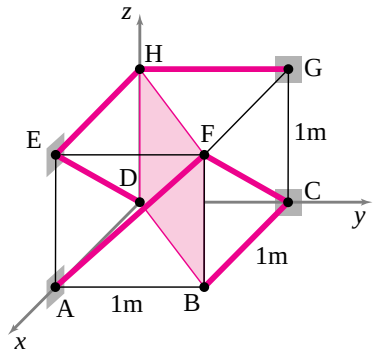
problem 6.67:

(Filename:ch3.11)

6.68 The uniform 2 kg plate DBFH is held by six massless rods (AF, CB, CF, GH, ED, and EH) which are hinged at their ends. The support points A, C, G, and E are all accelerating in the x-direction with acceleration $\boldsymbol{a} = 3\,\mathrm{m/s^2}\boldsymbol{i}$. There is no gravity.

a) What is $\{\sum \vec{\boldsymbol{F}}\}\cdot\hat{\boldsymbol{\imath}}$ for the forces acting on the plate?

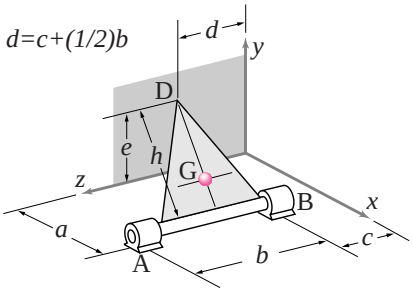
b) What is the tension in bar CB?



problem 6.68:

(Filename:pfigure.s94q3p1)

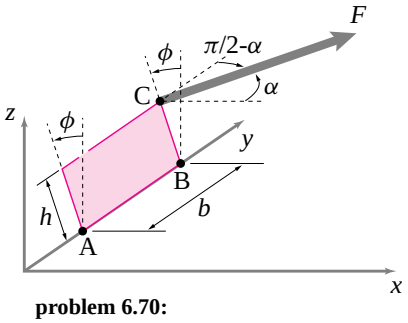
6.69 A massless triangular plate rests against a frictionless wall of a pick-up truck at point D and is rigidly attached to a massless rod supported by two ideal bearings fixed to the floor of the pick-up truck. A ball of mass m is fixed to the centroid of the plate. There is gravity. The pick-up truck skids across a road with acceleration $\vec{\boldsymbol{a}} = a_x\hat{\boldsymbol{\imath}} + a_z\hat{\boldsymbol{k}}$. What is the reaction at point D on the plate?



problem 6.69:

(Filename:ch3.1a)

6.70 Towing a bicycle. A bicycle on the level xy plane is steered straight ahead and is being towed by a rope. The bicycle and rider are modeled as a uniform plate with mass m (for the convenience of the artist). The tow force F applied at C has no z component and makes an angle α with the x axis. The rolling wheel contacts are at A and B. The bike is tipped an angle ϕ from the vertical. The towing force F is just the magnitude needed to keep the bike accelerating in a straight line without tipping any more or less than the angle ϕ. What is the acceleration of the bicycle? Answer in terms of some or all of b, h, α, ϕ, m, g and $\hat{\boldsymbol{\jmath}}$ (Note: F should *not* appear in your final answer.)



problem 6.70:

(Filename:s97f3)

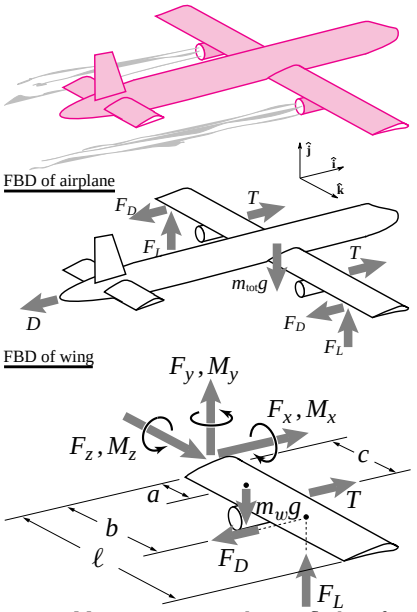
6.71 An airplane in flight. An airplane is in straight level flight but is accelerating in the forward direction. In terms of some or all of the following parameters,

- m_{tot} ≡ the total mass of the plane (including the wings),
- D= the drag force on the fuselage,
- F_D= the drag force on each wing,
- g= gravitational constant, and,
- T = the thrust of one engine.

a) What is the lift on each wing F_L? *

b) What is the acceleration of the plane $\vec{\boldsymbol{a}}_P$? *

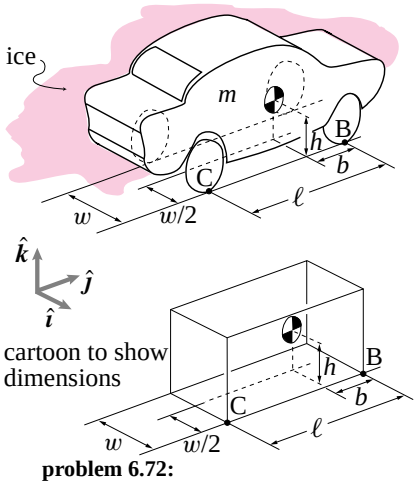
c) A free body diagram of one wing is shown. The mass of one wing is m_w. What, in terms of m_{tot}, m_w, F_L, F_D, g, a,b, c, and ℓ are the reactions at the base of the wing (where it is attached to the plane), $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{M}} = M_x\hat{\boldsymbol{\imath}} + M_y\hat{\boldsymbol{\jmath}} + M_z\hat{\boldsymbol{k}}$? *



problem 6.71: An airplane in flight, a free body diagram of the plane, and a free body diagram of one wing

(Filename:pfigure3.airplane)

6.72 A rear-wheel drive car on level ground. The two left wheels are on perfectly slippery ice. The right wheels are on dry pavement. The negligible-mass front right wheel at B is steered straight ahead and rolls without slip. The right rear wheel at C also rolls without slip and drives the car forward with velocity $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\jmath}}$ and acceleration $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\jmath}}$. Dimensions are as shown and the car has mass m . What is the sideways force from the ground on the right front wheel at B? Answer in terms of any or all of m, g, a, b, ℓ, w, and $\hat{\boldsymbol{\imath}}$. *



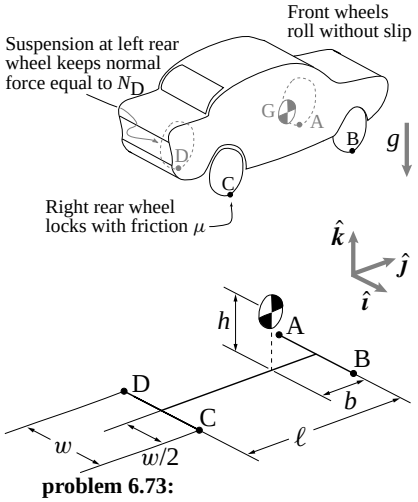
problem 6.72:

(Filename:pfigure3.f95p1p3)

6.73 A somewhat crippled car slams on the brakes. The suspension springs at A, B, and C are frozen and keep the car level and at constant height. The normal force at D is kept equal to N_D by the only working suspension spring which is on the left rear wheel at D. The only brake which is working is that of the right rear wheel at C which slides on the ground with friction coefficient μ. Wheels A, B, and D roll freely without slip. Dimensions are as shown.

a) Find the acceleration of the car in terms of some or all of m, w, ℓ, b, h, g, μ, $\hat{\boldsymbol{\jmath}}$, and N_D.

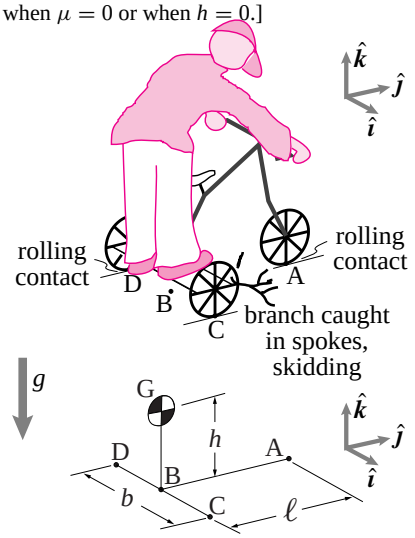
b) From the information given could you also find all of the reaction forces at all of the wheels? If so, why? If not, what can't you find and why? (No credit for correct answer. Credit depends on clear explanation.)



problem 6.73:

(Filename:s97p2.3)

6.74 Speeding tricycle gets a branch caught in the right rear wheel. A scared-stiff tricyclist riding on level ground gets a branch stuck in the right rear wheel so the wheel skids with friction coefficient μ. Assume that the center of mass of the tricycle-person system is directly above the rear axle. Assume that the left rear wheel and the front wheel have negligible mass, good bearings, and have sufficient friction that they roll in the $\hat{\boldsymbol{\jmath}}$ direction without slip, thus constraining the overall motion of the tricycle. Dimensions are shown in the lower sketch. *Find the acceleration of the tricycle* (in terms of some or all of ℓ, h, b, m, $[I^{cm}]$, μ, g, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$).
[Hint: check your answer against special cases for which you might guess the answer, such as when $\mu = 0$ or when $h = 0$.]



problem 6.74:

(Filename:p.f96.f.1)

Problems for Chapter 7

7.1 Kinematics of a particle in circular motion

skills

7.1 If a particle moves along a circle at constant rate (constant $\dot{\theta}$) following the equation

$$\vec{\boldsymbol{r}}(t) = R\cos(\dot{\theta}t)\hat{\boldsymbol{\imath}} + R\sin(\dot{\theta}t)\hat{\boldsymbol{\jmath}}$$

which of these things are true and why? If not true, explain why.

(a) $\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$

(b) $\vec{\boldsymbol{v}} =$ constant

(c) $|\vec{\boldsymbol{v}}| =$ constant

(d) $\vec{\boldsymbol{a}} = \vec{\boldsymbol{0}}$

(e) $\vec{\boldsymbol{a}} =$ constant

(f) $|\vec{\boldsymbol{a}}| =$ constant

(g) $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{a}}$

7.2 The motion of a particle is described by the following equations:

$$\begin{aligned} x(t) &= 1\,\text{m}\cdot\cos((5\,\text{rad/s})\cdot t), \\ y(t) &= 1\,\text{m}\cdot\sin((5\,\text{rad/s})\cdot t). \end{aligned}$$

a) Show that the speed of the particle is constant.

b) There are two points marked on the path of the particle: P with coordinates $(0, 1\,\text{m})$ and Q with coordinates $(1\,\text{m}, 0)$. How much time does the particle take to go from P to Q?

c) What is the acceleration of the particle at point Q?

7.3 Find a planar motion of a particle where the velocity and acceleration are always orthogonal. *

7.4 A bead goes around a circular track of radius 1 ft at a constant speed. It makes around the track in exactly 1 s.

a) Find the speed of the bead.

b) Find the magnitude of acceleration of the bead.

7.5 A 200 mm diameter gear rotates at a constant speed of 100 rpm.

a) What is the speed of a peripheral point on the gear?

b) If no point on the gear is to exceed the centripetal acceleration of $25\,\text{m/s}^2$, find the maximum allowable angular speed (in rpm) of the gear.

7.6 A particle executes circular motion in the xy-plane at a constant angular speed $\dot{\theta} = 2\,\text{rad/s}$. The radius of the circular path is 0.5 m. The particle's motion is tracked from the instant when $\theta = 0$, i.e., at $t = 0$, $\theta = 0$. Find the velocity and acceleration of the particle at

a) $t = 0.5\,\text{s}$ and

b) $t = 15\,\text{s}$.

Draw the path and show the position of the particle at each instant.

7.7 A particle undergoes constant rate circular motion in the xy-plane. At some instant t_0, its velocity is $v(t_0) = -3\,\text{m/s}\hat{\boldsymbol{\imath}} + 4\,\text{m/s}\hat{\boldsymbol{\jmath}}$ and after 5 s the velocity is $v(t_0 + 5\,\text{s}) = 5/\sqrt{2}\,\text{m/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$. If the particle has not yet completed one revolution between the two instants, find

a) the angular speed of the particle,

b) the distance traveled by the particle in 5 s, and

c) the acceleration of the particle at the two instants.

7.8 See also problem 7.9. A bead on a circular path of radius R in the xy-plane has rate of change of angular speed $\alpha = bt^2$. The bead starts from rest at $\theta = 0$.

a) What is the bead's angular position θ (measured from the positive x-axis) and angular speed ω as a function of time ?

b) What is the angular speed as function of angular position?

7.9 Reconsider the bead from problem 7.8. This time, it has rate of change of angular speed proportional to angular position rather than time, $\alpha = c\theta^{3/2}$. The bead starts from rest at $\theta = 0$. What is the angular position and speed of the bead as a function of time? [Hint: this problem has 2 answers! (one of which you can find with a quick guess.)]

7.10 Repeat problem 7.51 but assume that the moment of interest is at position B with $\theta = 60^\circ$.

7.11 Solve $\dot{\omega} = \alpha$, given $\omega(0) = \omega_0$ and α is a constant.

7.12 Solve $\ddot{\theta} = \alpha$, given $\theta(0) = \theta_0$, $\dot{\theta}(0) = \dot{\omega}_0$, and α a constant.

7.13 Given $\dot{\omega} = \frac{d\dot{\theta}}{dt} = \alpha$ (a constant), find an expression for ω as a function of θ if $\omega(\theta = 0) = w_0$.

7.14 Given that $\ddot{\theta} - \lambda^2\theta = 0$, $\theta(0) = \pi/2$, and $\dot{\theta}(0) = 0$, find the value of θ at $t = 1\,\text{s}$.

7.15 Two runners run on a circular track side-by-side at the same constant angular rate ω about the center of the track. The inside runner is in a lane of radius r_i and the outside runner is in a lane of radius r_o. What is the velocity of the outside runner relative to the inside runner in polar coordinates?

7.16 A particle oscillates on the arc of a circle with radius R according to the equation $\theta = \theta_0\cos(\lambda t)$. What are the conditions on R, θ_0, and λ so that the maximum acceleration in this motion occurs at $\theta = 0$. "Acceleration" here means the magnitude of the acceleration vector.

7.2 Dynamics of a particle in circular motion

skills

7.17 Force on a person standing on the equator. The total force acting on an object of mass m, moving with a constant angular speed ω on a circular path with radius r, is given by $F = m\omega^2 r$. Find the magnitude of the total force acting on a 150 lbm person standing on the equator. Neglect the motion of the earth around the sun and of the sun around the solar system, etc. The radius of the earth is 3963 mi. Give your solution in both pounds (lbf) and Newtons (N). *

7.18 The sum of forces acting on a mass $m = 10\,\text{lbm}$ is $\vec{\boldsymbol{F}} = 100\,\text{lbf}\hat{\boldsymbol{\imath}} - 120\,\text{lbf}\hat{\boldsymbol{\jmath}}$. The particle is going in circles at constant rate with $r = 18\,\text{in}$ and $\hat{\boldsymbol{e}}_r = \cos 30^\circ\hat{\boldsymbol{\imath}} + \sin 30^\circ\hat{\boldsymbol{\jmath}}$. Using $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, find v.

7.19 The acceleration of a particle in planar circular motion is given by $\vec{\boldsymbol{a}} = \alpha r\hat{\boldsymbol{e}}_\theta - \omega^2 r\hat{\boldsymbol{e}}_r$, where α is the angular acceleration, ω is the angular speed and r is the radius of the circular path. Using $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, find the expressions for $\sum F_x$ and $\sum F_y$ in terms of α, ω, r, and θ, given that $\hat{\boldsymbol{e}}_r = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.

7.20 Consider a particle with mass m in circular motion. Let $\ddot{\theta} = \alpha$, $\dot{\theta} = \omega$, and $\vec{\boldsymbol{r}} = -r\hat{\boldsymbol{\imath}}$. Let $\sum \vec{\boldsymbol{F}} = \sum F_r\hat{\boldsymbol{e}}_r + \sum F_\theta\hat{\boldsymbol{e}}_\theta$, where $\hat{\boldsymbol{e}}_r = -\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}}$. Using $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, express $\sum F_r$ and $\sum F_\theta$ in terms of α, ω, r, and m.

7.21 A bead of mass m on a circular path of radius R in the xy-plane has rate of change of angular speed $\alpha = ct^3$. The bead starts from rest at $\theta = 0$.

a) What is the angular momentum of the bead about the origin at $t = t_1$?

b) What is the kinetic energy of the bead at $t = t_1$.

7.22 A 200 gm particle goes in circles about a fixed center at a constant speed $v = 1.5\,\text{m/s}$. It takes 7.5 s to go around the circle once.

a) Find the angular speed of the particle.

b) Find the magnitude of acceleration of the particle.

c) Take center of the circle to be the origin of a xy-coordinate system. Find the net force on the particle when it is at $\theta = 30^o$ from the x-axis.

7.23 A race car cruises on a circular track at a constant speed of 120 mph. It goes around the track once in three minutes. Find the magnitude of the centripetal force on the car. What applies this force on the car? Does the driver have any control over this force?

7.24 A particle moves on a counter-clockwise, origin-centered circular path in the xy-plane at a constant rate. The radius of the circle is r, the mass of the particle is m, and the particle completes one revolution in time τ.

a) Neatly draw the following things:

1. The path of the particle.
2. A dot on the path when the particle is at $\theta = 0^\circ$, 90°, and 210°, where θ is measured from the x-axis (positive counter-clockwise).
3. Arrows representing $\hat{\boldsymbol{e}}_R$, $\hat{\boldsymbol{e}}_\theta$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{a}}$ at each of these points.

b) Calculate all of the quantities in part (3) above at the points defined in part (2), (represent vector quantities in terms of the cartesian base vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$). *

c) If this motion was imposed by the tension in a string, what would that tension be? *

d) Is radial tension enough to maintain this motion or is another force needed to keep the motion going (assuming no friction) ? *

e) Again, if this motion was imposed by the tension in a string, what is F_x, the x component of the force in the string, when $\theta = 210^\circ$? Ignore gravity.

7.25 The velocity and acceleration of a 1 kg particle, undergoing constant rate circular motion, are known at some instant t:

$$\vec{\boldsymbol{v}} = -10\,\text{m/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}), \quad \vec{\boldsymbol{a}} = 2\,\text{m/s}^2(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}).$$

a) Write the position of the particle at time t using $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ base vectors.

b) Find the net force on the particle at time t.

c) At some later time t^*, the net force on the particle is in the $-\hat{\boldsymbol{\jmath}}$ direction. Find the elapsed time $t - t^*$.

d) After how much time does the force on the particle reverse its direction.

7.26 A particle of mass 3 kg moves in the xy-plane so that its position is given by

$$\boldsymbol{r}(t) = 4\,\text{m}[\cos(\frac{2\pi t}{s})\hat{\boldsymbol{\imath}} + \sin(\frac{2\pi t}{s})\hat{\boldsymbol{\jmath}}].$$

a) What is the path of the particle? Show how you know what the path is.

b) What is the angular velocity of the particle? Is it constant? Show how you know if it is constant or not.

c) What is the velocity of the particle in polar coordinates?

d) What is the speed of the particle at $t = 3\,\text{s}$?

e) What net force does it exert on its surroundings at $t = 0\,\text{s}$? Assume the x and y axis are attached to a Newtonian frame.

f) What is the angular momentum of the particle at $t = 3\,\text{s}$ about point O located at the origin of the coordinate system that the particle is referenced to?

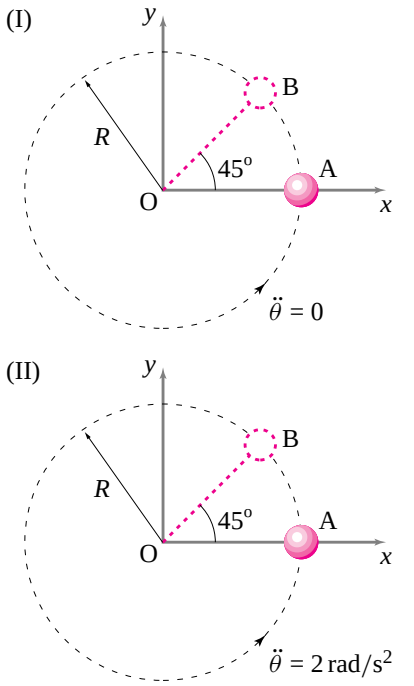
7.27 Going in circles — a comparison of constant and nonconstant rate motion. A 100 gm mass is going in circles of radius $R = 20\,\text{cm}$ at a constant rate $\dot{\theta} = 3\,\text{rad/s}$. Another identical mass is going in circles of the same radius but at a non-constant rate. The second mass is accelerating at $\ddot{\theta} = 2\,\text{rad/s}^2$ and at position A, it happens to have the same angular speed as the first mass.

a) Find and draw the accelerations of the two masses (call them I and II) at position A.

b) Find $\vec{\vec{\boldsymbol{H}}}_O$ for both masses at position A. *

c) Find $\vec{\vec{\boldsymbol{H}}}_O$ for both masses at positions A and B. Do the changes in $\vec{\vec{\boldsymbol{H}}}_O$ between the two positions reflect (qualitatively) the results obtained in (b)? *

d) If the masses are pinned to the center O by massless rigid rods, is tension in the rods enough to keep the two motions going? Explain.



problem 7.27:

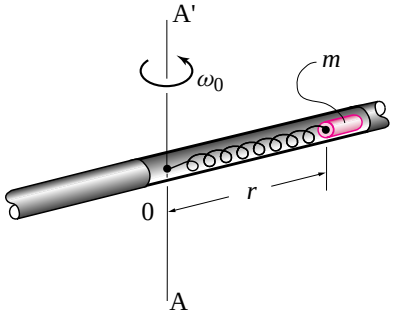
(Filename:pfigure.s94h8p2)

7.28 A small mass m is connected to one end of a spring. The other end of the spring is fixed to the center of a circular track. The radius of the track is R, the unstretched length of the spring is $l_0 (< R)$, and the spring constant is k.

a) With what speed should the mass be launched in the track so that it keeps going at a constant speed?

b) If the spring is replaced by another spring of same relaxed length but twice the stiffness, what will be the new required launch speed of the particle?

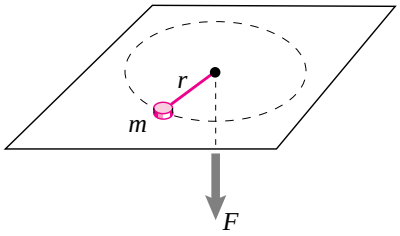
7.29 A bead of mass m is attached to a spring with constant k. The bead slides without friction in the tube shown. The tube is driven at a constant angular rate ω_0 about axis AA' by a motor (not pictured). There is no gravity. The unstretched spring length is r_0. Find the radial position r of the bead if it is stationary with respect to the rotating tube. *



problem 7.29:

(Filename:Danef94s2q6)

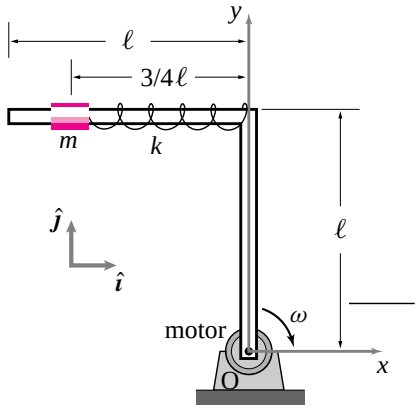
7.30 A particle of mass m is restrained by a string to move with a constant angular speed ω around a circle of radius R on a horizontal frictionless table. If the radius of the circle is reduced to r, by pulling the string with a force F through a hole in the table, what will the particle's angular velocity be? Is kinetic energy is conserved? Why or why not?



problem 7.30:

(Filename:pfigure.blue.44.1)

7.31 An 'L' shaped rigid, massless, and frictionless bar is made up of two uniform segments of length $\ell = 0.4\,\text{m}$ each. A collar of mass $m = 0.5\,\text{kg}$, attached to a spring at one end, slides frictionlessly on one of the arms of the 'L'. The spring is fixed to the elbow of the 'L' and has a spring constant $k = 6\,\text{N/m}$. The structure rotates clockwise at a constant rate $\omega = 2\,\text{rad/s}$. If the collar is steady at a distance $\frac{3}{4}\ell = 0.3\,\text{m}$ away from the elbow of the 'L', find the relaxed length of the spring, ℓ_0. * .



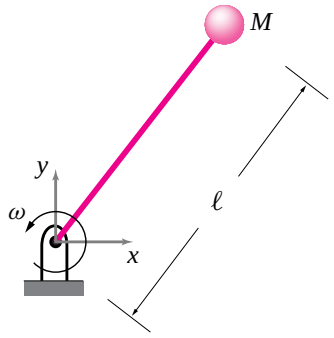
problem 7.31: Forces in constant rate circular motion.

(Filename:pfigure.Lbar)

7.32 A massless rigid rod with length ℓ attached to a ball of mass M spins at a constant angular rate ω which is maintained by a motor (not shown) at the hinge point. The rod can only withstand a tension of T_{cr} before breaking. Find the maximum angular speed of the ball so that the rod does not break assuming

a) there is no gravity, and

b) there is gravity (neglect bending stresses).



problem 7.32:

(Filename:Danef94s2q5)

7.33 A 1 m long massless string has a particle of 10 grams mass at one end and is tied to a stationary point O at the other end. The particle rotates counter-clockwise in circles on a frictionless horizontal plane. The rotation rate is 2π rev/sec. Assume an xy-coordinate system in the plane with its origin at O.

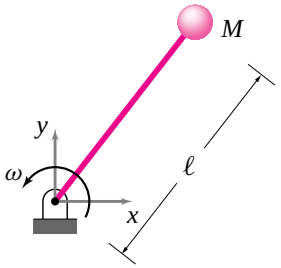
a) Make a clear sketch of the system.

b) What is the tension in the string (in Newtons)? *

c) What is the angular momentum of the mass about O? *

d) When the string makes a 45° angle with the positive x and y axis on the plane, the string is quickly and cleanly cut. What is the position of the mass 1 sec later? Make a sketch. *

7.34 A ball of mass M fixed to an inextensible rod of length ℓ and negligible mass rotates about a frictionless hinge as shown in the figure. A motor (not shown) at the hinge point accelerates the mass-rod system from rest by applying a constant torque M_O. The rod is initially lined up with the positive x-axis. The rod can only withstand a tension of T_{cr} before breaking. At what time will the rod break and after how many revolutions? Include gravity if you like.



problem 7.34:

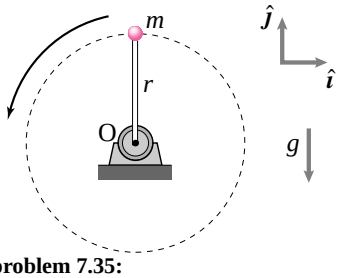
(Filename:pfigure5.mike.sum95)

7.35 A particle of mass m, tied to one end of a rod whose other end is fixed at point O to a motor, moves in a circular path in the vertical plane at a constant rate. Gravity acts in the $-\hat{\boldsymbol{\jmath}}$ direction.

a) Find the difference between the maximum and minimum tension in the rod. *

b) Find the ratio $\frac{\Delta T}{T_{max}}$ where $\Delta T = T_{max} - T_{min}$. A criterion for ignoring gravity might be if the variation in tension is less than 2% of the maximum tension; i.e., when $\frac{\Delta T}{T_{max}} < 0.02$. For a given length r of the rod, find the rotation rate ω for which this condition is met. * .

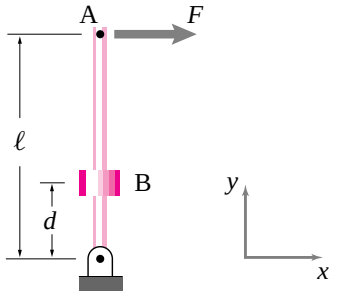
c) For $\omega = 300\,\text{rpm}$, what would be the length of the rod for the condition in part (b) to be satisfied? * .



problem 7.35:

(Filename:pfigure4.rph)

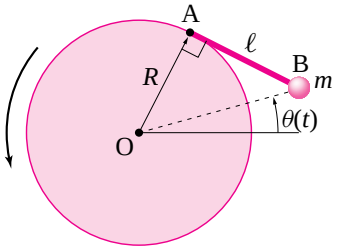
7.36 A massless rigid bar of length L is hinged at the bottom. A force F is applied at point A at the end of the bar. A mass m is glued to the bar at point B, a distance d from the hinge. There is no gravity. What is the acceleration of point A at the instant shown? Assume the angular velocity is initially zero.



problem 7.36:

(Filename:pfigure.blue.10.1)

7.37 The mass m is attached rigidly to the rotating disk by the light rod AB of length ℓ. Neglect gravity. Find M_A (the moment on the rod AB from its support point at A) in terms of $\dot{\theta}$ and $\ddot{\theta}$. What is the sign of M_A if $\dot{\theta} = 0$ and $\ddot{\theta} > 0$? What is the sign if $\ddot{\theta} = 0$ and $\dot{\theta} > 0$?



problem 7.37:

(Filename:pfigure.blue.84.2)

7.38 Pendula using energy methods. Find the equations of motion for the pendula in problem 7.94 using energy methods. * .

7.39 Tension in a simple pendulum string. A simple pendulum of length 2 m with mass 3 kg is released from rest at an initial angle of 60° from the vertically down position.

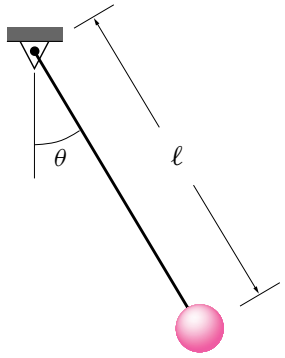
a) What is the tension in the string just after the pendulum is released?

b) What is the tension in the string when the pendulum has reached 30° from the vertical?

7.40 Simply the simple pendulum. Find the nonlinear governing differential equation for a simple pendulum

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

as many different ways as you can.



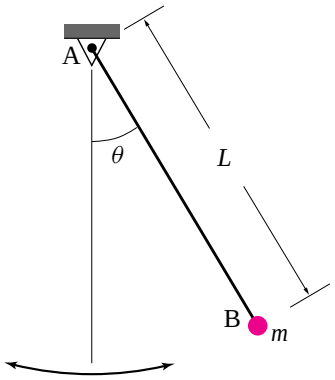
problem 7.40: A simple pendulum.

(Filename:spend)

7.41 Tension in a rope-swing rope. Model a swinging person as a point mass. The swing starts from rest at an angle $\theta = 90°$. When the rope passes through vertical the tension in the rope is higher (it is hard to hang on). A person wants to know ahead of time if she is strong enough to hold on. How hard does she have to hang on compared, say, to her own weight? You are to find the solution two ways. Use the same m, g, and L for both solutions.

a) Find $\ddot{\theta}$ as a function of g, L, θ, and m. This equation is the governing differential equation. Write it as a system of first order equations. Solve them numerically. Once you know $\dot{\theta}$ at the time the rope is vertical you can use other mechanics relations to find the tension. If you like, you can plot the tension as a function of time as the mass falls.

b) Use conservation of energy to find $\dot{\theta}$ at $\theta = 0$. Then use other mechanics relations to find the tension. *



problem 7.41:

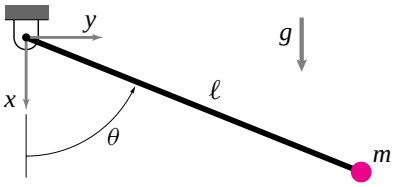
(Filename:pfigure.s94h8p5)

7.42 Pendulum. A pendulum with a negligible-mass rod and point mass m is realeased from rest at the horizontal position $\theta = \pi/2$.

a) Find the acceleration (a vector) of the mass just after it is released at $\theta = \pi/2$ in terms of ℓ, m, g and any base vectors you define clearly.

b) Find the acceleration (a vector) of the mass when the pendulum passes through the vertical at $\theta = 0$ in terms of ℓ, m, g and any base vectors you define clearly

c) Find the string tension when the pendulum passes through the vertical at $\theta = 0$ (in terms of ℓ, m and g).



problem 7.42:

(Filename:s97p2.1)

7.43 Simple pendulum, extended version. A point mass $M = 1\,\text{kg}$ hangs on a string of length $L = 1\,\text{m}$. Gravity pulls down on the mass with force Mg, where $g = 10\,\text{m/s}^2$. The pendulum lies in a vertical plane. At any time t, the angle between the pendulum and the straight-down position is $\theta(t)$. There is no air friction.

a) **Equation of motion.** Assuming that you know both θ and $\dot{\theta}$, find $\ddot{\theta}$. There are several ways to do this problem. Use any ways that please you. *

b) **Tension.** Assuming that you know θ and $\dot{\theta}$, find the tension T in the string.

c) **Reaction components.** Assuming you know θ and $\dot{\theta}$, find the x and y components of the force that the hinge support causes on the pendulum . Define your coordinate directions sensibly.

d) **Reduction to first order equations.** The equation that you found in (a) is a nonlinear second order ordinary differential equation. It can be changed to a pair of first order equations by defining a new variable $\alpha \equiv \dot{\theta}$. Write the equation from (a) as a pair of first order equations. Solving these equations is equivalent to solving the original second order equation. *

e) **Numerical solution.** Given the initial conditions $\theta(t = 0) = \pi/2$ and $\alpha(t = 0) = \dot{\theta}(t = 0) = 0$, one should be able to find what the position and speed of the pendulum is as a function of time. Using the results from (b) and (c) one can also find the reaction components. Using any computer and any method you like, find: $\theta(t)$, $\dot{\theta}(t)$ & $T(t)$. Make a single plot, or three vertically aligned plots, of these variables for one full oscillation of the pendulum.

f) **Maximum tension.** Using your numerical solutions, find the maximum value of the tension in the rod as the mass swings. *

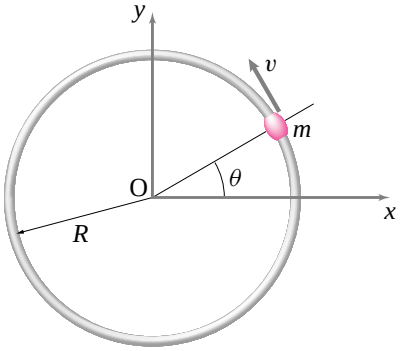
g) **Period of oscillation.** How long does it take to make one oscillation?

h) **Other observations.** Make any observations that you think are interesting about this problem. Some questions: Does the solution to (f) depend on the length of the string? Is the solution to (f) exactly 30 or just a number near 30? How does the period found in (g) compare to the period found by solving the linear equation $\ddot{\theta} + (g/l)\theta = 0$, based on the (inappropriate-to-use in this case) small angle approximation $\sin\theta = \theta$?

7.44 Bead on a hoop with friction. A bead slides on a rigid, stationary, circular wire. The coefficient of friction between the bead and the wire is μ. The bead is loose on the wire (not a tight fit but not so loose that you have to worry about rattling). Assume gravity is negligible.

a) Given v, m, R, & μ; what is $\dot{v}$? *

b) If $v(\theta = 0) = v_0$, how does v depend on θ, μ, v_0 and m? *



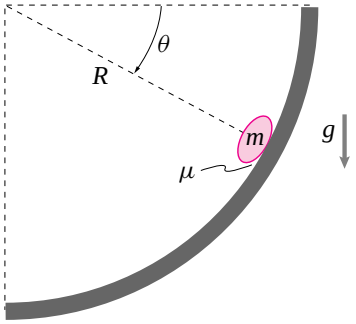
problem 7.44:

(Filename:pfigure.s94h10p1)

7.45 Particle in a chute. One of a million non-interacting rice grains is sliding in a circular chute with radius R. Its mass is m and it slides with coefficient of friction μ (Actually it slides, rolls and tumbles — μ is just the effective coefficient of friction from all of these interactions.) Gravity g acts downwards.

a) Find a differential equation that is satisfied by θ that governs the speed of the rice as it slides down the hoop. Parameters in this equation can be m, g, R and μ [*Hint: Draw FBD, write eqs of mechanics, express as ODE.*]

b) Find the particle speed at the bottom of the chute if $R = .5m$, $m = .1$ grams, $g = 10\,\mathrm{m/s^2}$, and $\mu = .2$ as well as the initial values of $\theta_0 = 0$ and its initial downward speed is $v_0 = 10\,\mathrm{m/s}$. [Hint: you are probably best off seeking a numerical solution.]



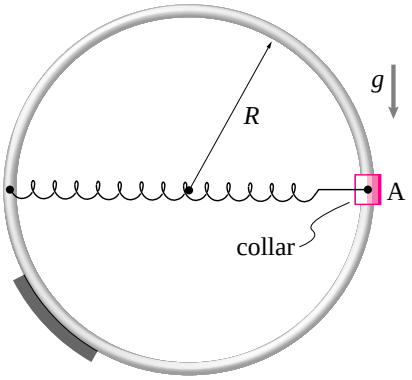
problem 7.45:

(Filename:p.f96.p3.1)

7.46 Due to a push which happened in the past, the collar with mass m is sliding up at speed v_0 on the circular ring when it passes through the point A. The ring is frictionless. A spring of constant k and unstretched length R is also pulling on the collar.

a) What is the acceleration of the collar at A. Solve in terms of R, v_0, m, k, g and any base vectors you define.

b) What is the force on the collar from the ring when it passes point A? Solve in terms of R, v_0, m, k, g and any base vectors you define.



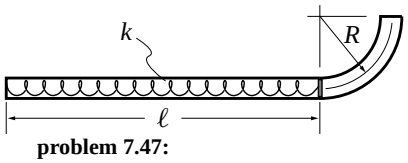
problem 7.46:

(Filename:s93q4Ruina)

7.47 A toy used to shoot pellets is made out of a thin tube which has a spring of spring constant k on one end. The spring is placed in a straight section of length ℓ; it is unstretched when its length is ℓ. The straight part is attached to a (quarter) circular tube of radius R, which points up in the air.

a) A pellet of mass m is placed in the device and the spring is pulled to the left by an amount $\Delta\ell$. Ignoring friction along the travel path, what is the pellet's velocity $\vec{v}$ as it leaves the tube? *

b) What force acts on the pellet just prior to its departure from the tube? What about just after? *



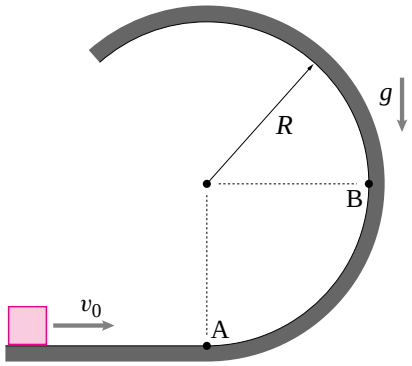
problem 7.47:

(Filename:pfigure.blue.toygun)

7.48 A block with mass m is moving to the right at speed v_0 when it reaches a circular frictionless portion of the ramp.

a) What is the speed of the block when it reaches point B? Solve in terms of R, v_0, m and g.

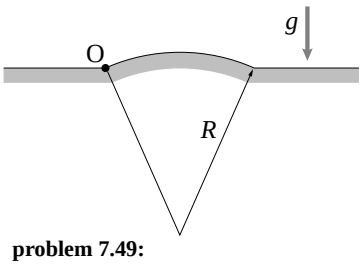
b) What is the force on the block from the ramp just after it gets onto the ramp at point A? Solve in terms of R, v_0, m and g. Remember, force is a vector.



problem 7.48:

(Filename:s93q4Sachse)

7.49 A car moves with speed v along the surface of the hill shown which can be approximated as a circle of radius R. The car starts at a point on the hill at point O. Compute the magnitude of the speed v such that the car just leaves the ground at the top of the hill.



problem 7.49:

(Filename:pfigure.blue.65.1)

7.3 Kinematics of a rigid body in planar circular motion

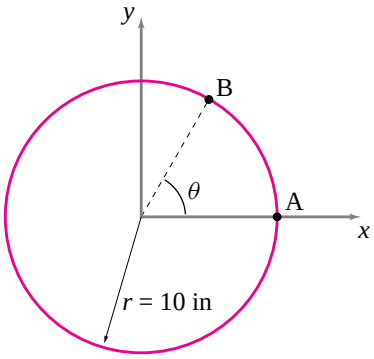
skills

7.50 Find $\vec{v} = \vec{\omega} \times \vec{r}$, if $\vec{\omega} = 1.5\,\mathrm{rad/s}\hat{k}$ and $\vec{r} = 2\,\mathrm{m}\hat{\imath} - 3\,\mathrm{m}\hat{\jmath}$.

7.51 A particle goes around a circle shown in the figure, at the moment of interest, with angular velocity $\vec{\omega} = 5\,\mathrm{rad/s}\hat{k}$ and angular acceleration $\vec{\alpha} = 2\,\mathrm{rad/s^2}\hat{k}$. Find, draw, and label the tangential and normal acceleration

a) of the particle at the location A. *

b) of the particle at the location B.



problem 7.51:

(Filename:efig1.2.18)

7.52 A motor turns a uniform disc of radius R counter-clockwise about its mass center at a constant rate ω. The disc lies in the xy-plane and its angular displacement θ is measured (positive counter-clockwise) from the x-axis. What is the angular displacement $\theta(t)$ of the disc if it starts at $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \omega$? What are the velocity and acceleration of a point P at position $\vec{r} = x\hat{\imath} + y\hat{\jmath}$?

7.53 A disc is rotating at 15 rpm. How many seconds does it take to rotate by 180 degrees? What is the angular speed of the disc in rad/s?

7.54 Two discs $\mathcal{A}$ and $\mathcal{B}$ rotate at constant speeds about their centers. Disc $\mathcal{A}$ rotates at 100 rpm and disc $\mathcal{B}$ rotates at 10 rad/s. Which is rotating faster?

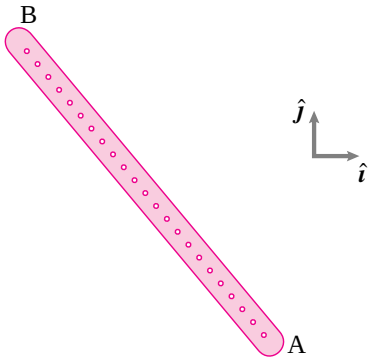
7.55 Find the angular velocities of the second, minute, and hour hands of a clock.

7.56 A motor turns a uniform disc of radius R counter-clockwise about its mass center at a constant rate ω. The disc lies in the xy-plane and its angular displacement θ is measured (positive counter-clockwise) from the x-axis. What are the velocity and acceleration of a point P at position $\vec{r}_P = c\hat{\imath} + d\hat{\jmath}$ relative to the velocity and acceleration of a point Q at position $\vec{r}_Q = 0.5(-d\hat{\imath} + c\hat{\jmath})$ on the disk? ($c^2 + d^2 < R^2$.)

7.57 A 0.4 m long rod AB has many holes along its length such that it can be pegged at any of the various locations. It rotates counter-clockwise at a constant angular speed about a peg whose location is not known. At some instant t, the velocity of end B is $\vec{v}_B = -3\,\text{m/s}\hat{\jmath}$. After $\frac{\pi}{20}$ s, the velocity of end B is $\vec{v}_B = -3\,\text{m/s}\hat{\imath}$. If the rod has not completed one revolution during this period,

a) find the angular velocity of the rod, and

b) find the location of the peg along the length of the rod.



problem 7.57:

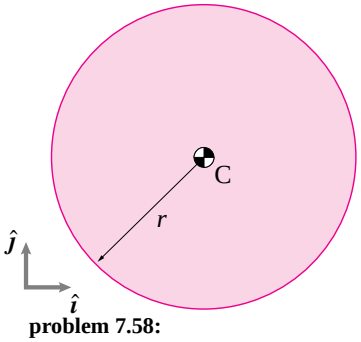
(Filename:pfigure4.3.rp9)

7.58 A circular disc of radius $r = 250\,\text{mm}$ rotates in the xy-plane about a point which is at a distance $d = 2r$ away from the center of the disk. At the instant of interest, the linear speed of the center C is 0.60 m/s and the magnitude of its centripetal acceleration is $0.72\,\text{m/s}^2$.

a) Find the rotational speed of the disk.

b) Is the given information enough to locate the center of rotation of the disk?

c) If the acceleration of the center has no component in the $\hat{\jmath}$ direction at the moment of interest, can you locate the center of rotation? If yes, is the point you locate unique? If not, what other information is required to make the point unique?



problem 7.58:

(Filename:pfigure4.3.rp8)

7.59 A disc $\mathcal{C}$ spins at a constant rate of two revolutions per second counter-clockwise about its geometric center, G, which is fixed. A point P is marked on the disk at a radius of one meter. At the moment of interest, point P is on the x-axis of an xy-coordinate system centered at point G.

a) Draw a neat diagram showing the disk, the particle, and the coordinate axes.

b) What is the angular velocity of the disk, $\vec{\omega}_{\mathcal{C}}$?

c) What is the angular acceleration of the disk, $\vec{\alpha}_{\mathcal{C}}$?

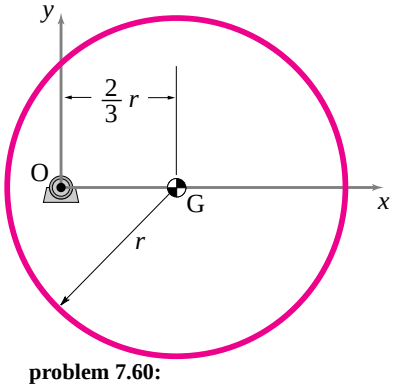
d) What is the velocity $\vec{v}_P$ of point P?

e) What is the acceleration $\vec{a}_P$ of point P?

7.60 A uniform disc of radius $r = 200\,\text{mm}$ is mounted eccentrically on a motor shaft at point O. The motor rotates the disc at a constant angular speed. At the instant shown, the velocity of the center of mass is $\vec{v}_G = -1.5\,\text{m/s}\hat{\jmath}$.

a) Find the angular velocity of the disc.

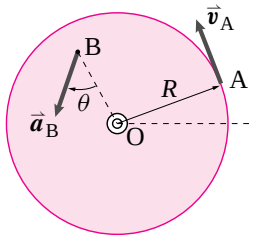
b) Find the point with the highest linear speed on the disc. What is its velocity?



problem 7.60:

(Filename:pfigure4.3.rp4)

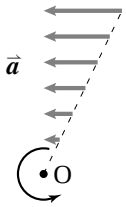
7.61 The circular disc of radius $R = 100\,\text{mm}$ rotates about its center O. At a given instant, point A on the disk has a velocity $v_A = 0.8\,\text{m/s}$ in the direction shown. At the same instant, the tangent of the angle θ made by the total acceleration vector of any point B with its radial line to O is 0.6. Compute the angular acceleration α of the disc.



problem 7.61:

(Filename:pfigure.blue.43.2)

7.62 Show that, for non-constant rate circular motion, the acceleration of all points in a given radial line are parallel.



problem 7.62:

(Filename:pfigure.DH.5.1a)

7.63 A motor turns a uniform disc of radius R about its mass center at a variable angular rate ω with rate of change $\dot{\omega}$, counter-clockwise. The disc lies in the xy-plane and its angular displacement θ is measured from the x-axis, positive counter-clockwise. What are the velocity and acceleration of a point P at position $\vec{r}_P = c\hat{\imath} + d\hat{\jmath}$ relative to the velocity and acceleration of a point Q at position $\vec{r}_Q = 0.5(-d\hat{\imath} + y\hat{\jmath})$ on the disk? ($c^2 + d^2 < R^2$.)

7.64 Bit-stream kinematics of a CD. A Compact Disk (CD) has bits of data etched on concentric circular tracks. The data from a track is read by a beam of light from a head that is positioned under the track. The angular speed of the disk remains constant as long as the head is positioned over a particular track. As the head moves to the next track, the angular speed of the disk changes, so that the *linear speed* at any track is always the same. The data stream comes out at a constant rate 4.32×10^6 bits/second. When the head is positioned on the outermost track, for which $r = 56\,\text{mm}$, the disk rotates at 200 rpm.

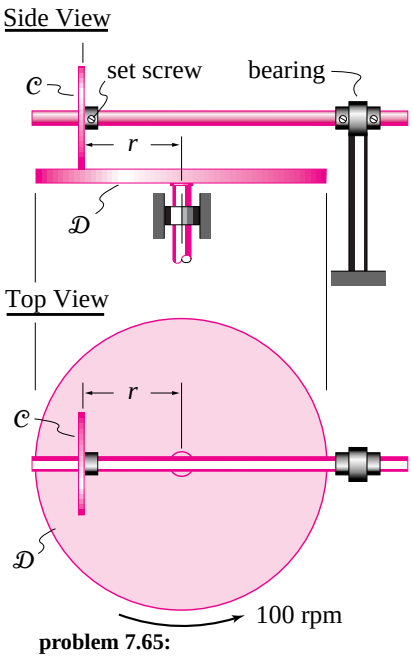
a) What is the number of bits of data on the outermost track.

b) find the angular speed of the disk when the head is on the innermost track ($r = 22\,\text{mm}$), and

c) find the numbers of bits on the innermost track.

7.65 A horizontal disk $\mathcal{D}$ of diameter $d = 500\,\text{mm}$ is driven at a constant speed of 100 rpm. A small disk $\mathcal{C}$ can be positioned anywhere between $r = 10\,\text{mm}$ and $r = 240\,\text{mm}$ on disk $\mathcal{D}$ by sliding it along the overhead shaft and then fixing it at the desired position with a set screw (see the figure). Disk

$\mathcal{C}$ rolls without slip on disk $\mathcal{D}$. The overhead shaft rotates with disk $\mathcal{C}$ and, therefore, its rotational speed can be varied by varying the position of disk $\mathcal{C}$. This gear system is called *brush gearing*. Find the maximum and minimum rotational speeds of the overhead shaft. *



problem 7.65:

(Filename:pfigure4.rpc)

7.66 Two points A and B are on the same machine part that is hinged at an as yet unkown location C. Assume you are given that points at positions $\vec{r}_A$ and $\vec{r}_B$ are supposed to move in given directions, indicated by unit vectors as $\hat{\boldsymbol{\lambda}}_A$ and $\hat{\boldsymbol{\lambda}}_B$. For each of the parts below, illustrate your results with two numerical examples (in consistent units): i) $\vec{r}_A = 1\hat{\imath}$, $\vec{r}_B = 1\hat{\jmath}$, $\hat{\boldsymbol{\lambda}}_A = 1\hat{\jmath}$, and $\hat{\boldsymbol{\lambda}}_B = -1\hat{\imath}$ (thus $\vec{r}_C = \vec{0}$), and ii) a more complex example of your choosing.

a) Describe in detail what equations must be satisfied by the point $\vec{r}_C$.

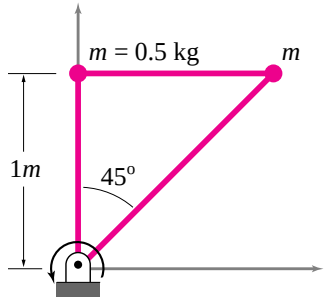
b) Write a computer program that takes as input the 4 pairs of numbers $[\vec{r}_A]$, $[\vec{r}_B]$, $[\hat{\boldsymbol{\lambda}}_A]$ and $[\hat{\boldsymbol{\lambda}}_B]$ and gives as output the pair of numbers $[\hat{\boldsymbol{\lambda}}_C]$.

c) Find a formula of the form $\vec{r}_C = \ldots$ that explicitly gives the position vector for point C in terms of the 4 given vectors.

7.4 Dynamics of a rigid body in planar circular motion

skills

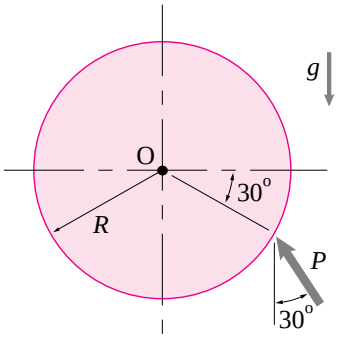
7.67 The structure shown in the figure consists of two point masses connected by three rigid, massless rods such that the whole structure behaves like a rigid body. The structure rotates counterclockwise at a constant rate of 60 rpm. At the instant shown, find the force in each rod.



problem 7.67:

(Filename:pfig4.1.rp10)

7.68 The hinged disk of mass m (uniformly distributed) is acted upon by a force P shown in the figure. Determine the *initial* angular acceleration and the reaction forces at the pin O.



problem 7.68:

(Filename:pfigure.blue.41.3)

7.69 A thin uniform circular disc of mass M and radius R rotates in the xy plane about its center of mass point O. Driven by a motor, it has rate of change of angular speed proportional to angular position, $\alpha = d\theta^{5/2}$. The disc starts from rest at $\theta = 0$.

a) What is the rate of change of angular momentum about the origin at $\theta = \frac{\pi}{3}$ rad?

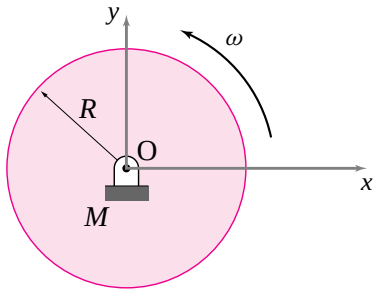
b) What is the torque of the motor at $\theta = \frac{\pi}{3}$ rad?

c) What is the total kinetic energy of the disk at $\theta = \frac{\pi}{3}$ rad?

7.70 A uniform circular disc rotates at constant angular speed ω about the origin, which is also the center of the disc. It's radius is R. It's total mass is M.

a) What is the total force and moment required to hold it in place (use the origin as the reference point of angular momentum and torque).

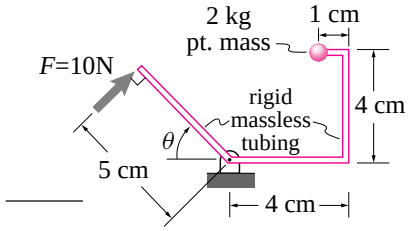
b) What is the total kinetic energy of the disk?



problem 7.70:

(Filename:pfigure.blue.87a.1)

7.71 Neglecting gravity, calculate $\alpha = \dot{\omega} = \ddot{\theta}$ at the instant shown for the system in the figure.



problem 7.71:

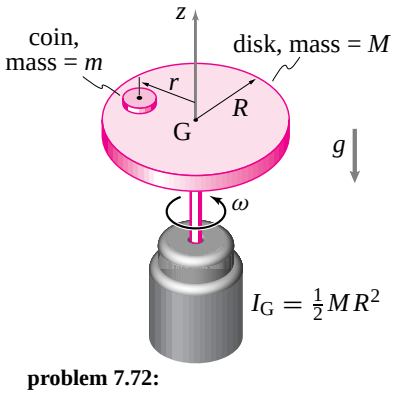
(Filename:pfigure.blue.20.1)

7.72 A disk of mass M and radius R is attached to an electric motor as shown. A coin of mass m rests on the disk, with the center of the coin a distance r from the center of the disk. Assume that $m \ll M$, and that the coefficient of friction between the coin and the disk is μ. The motor delivers a constant power P to the disk. The disk starts from rest when the motor is turned on at $t = 0$.

a) What is the angular velocity of the disk as a function of time?

b) What is its angular acceleration?

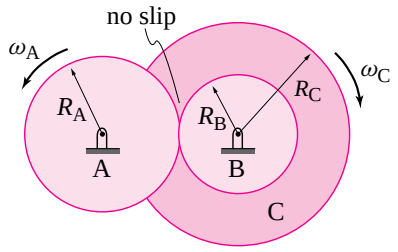
c) At what time does the coin begin to slip off the disk? (It will suffice here to give the equation for t that must be solved.)



problem 7.72:

(Filename:pfigure.blue.45.1)

7.73 2-D constant rate gear train. The angular velocity of the input shaft (driven by a motor not shown) is a constant, $\omega_{\text{input}} = \omega_A$. What is the angular velocity $\omega_{\text{output}} = \omega_C$ of the output shaft and the speed of a point on the outer edge of disc C, in terms of R_A, R_B, R_C, and ω_A?



problem 7.73: A set of gears turning at constant rate.

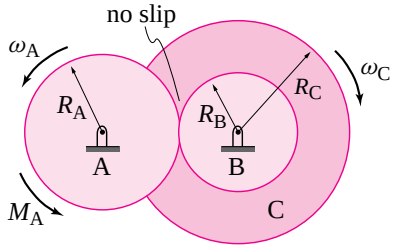
(Filename:ch4.3)

7.74 2-D constant speed gear train. Gear A is connected to a motor(not shown) and gear B, which is welded to gear C, is connected to a taffy-pulling mechanism. Assume you know the torque $M_{\text{input}} = M_A$ and angular velocity $\omega_{\text{input}} = \omega_A$ of the input shaft. Assume the bearings and contacts are frictionless.

a) What is the input power?

b) What is the output power?

c) What is the output torque $M_{\text{output}} = M_C$, the torque that gear C applies to its surroundings in the clockwise direction?



problem 7.74:

(Filename:pfigure.s94h6p5)

7.75 Accelerating rack and pinion. The two gears shown are welded together and spin on a frictionless bearing. The inner gear has radius 0.5 m and negligible mass. The outer disk has 1 m radius and a uniformly distributed mass of 0.2 kg. They are loaded as shown with the force $F = 20$ N on the massless rack which is held in place by massless frictionless rollers. At the time of interest the angular velocity is $\omega = 2$ rad/s (though ω is not constant). The point P is on the disk a distance 1 m from the center. At the time of interest, point P is on the positive y axis.

a) What is the speed of point P?

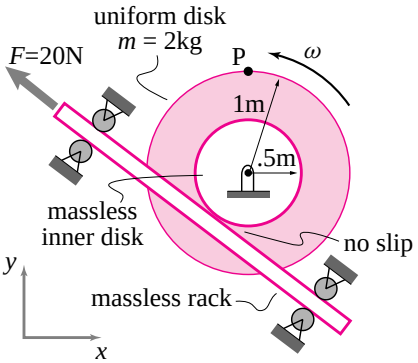
b) What is the velocity of point P?

c) What is the angular acceleration α of the gear?

d) What is the acceleration of point P?

e) What is the magnitude of the acceleration of point P?

f) What is the rate of increase of the speed of point P?



problem 7.75: Accelerating rack and pinion

(Filename:pg45.2)

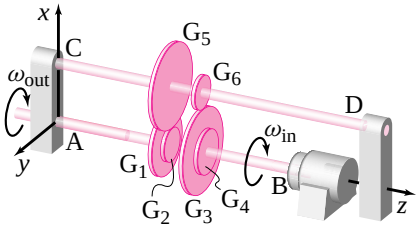
7.76 A 2-D constant speed gear train. Shaft B is rigidly connected to gears G_4 and G_3. G_3 meshes with gear G_6. Gears G_6 and G_5 are both rigidly attached to shaft CD. Gear G_5 meshes with G_2 which is welded to shaft A. Shaft A and shaft B spin independently. The input torque $M_{\text{input}} = 500$ N·m and the spin rate $\omega_{\text{input}} = 150$ rev/min.. Assume the bearings and contacts are frictionless.

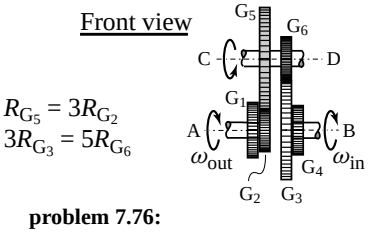
a) What is the input power? *

b) What is the output power? *

c) What is the angular velocity ω_{output} of the output shaft? *

d) What is the output torque M_{output}? *





problem 7.76:

(Filename:pfigure.s94h6p6)

7.77 Two gears rotating at constant rate. See also problems 4.16 and **??**. At the input to a gear box a 100 lbf force is applied to gear A. At the output, the machinery (not shown) applies a force of F_B to the output gear. Gear A rotates at constant angular rate $\omega = 2$ rad/s, clockwise.

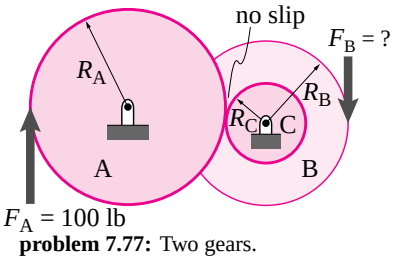
a) What is the angular speed of the right gear?

b) What is the velocity of point P?

c) What is F_B?

d) If the gear bearings had friction, would F_B have to be larger or smaller in order to achieve the same constant velocity?

e) If instead of applying a 100 lbf to the left gear it is driven by a motor (not shown) at constant angular speed ω, what is the angular speed of the right gear?



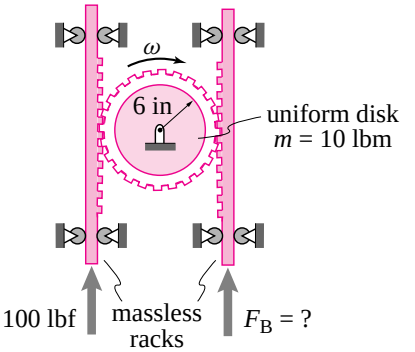
problem 7.77: Two gears.

(Filename:pg131.3)

7.78 Two racks connected by a gear. A 100 lbf force is applied to one rack. At the output the machinery (not shown) applies a force F_B to the other rack.

a) Assume the gear is spinning at constant rate and is frictionless. What is F_B?

b) If the gear bearing had friction, would that increase or decrease F_B to achieve the same constant rate?



problem 7.78: Two racks connected by a gear.

(Filename:ch4.4)

7.79 Constant rate rack and pinion. The two gears shown are welded together and spin on a frictionless bearing. The inner gear has radius 0.5 m and negligible mass. The outer gear has 1 m radius and a uniformly distributed mass of 0.2 kg. A motor (not shown) rotates the disks at constant rate $\omega = 2$ rad/s. The gears drive the massless rack which is held in place by massless frictionless rollers as shown. The gears and the rack have teeth that are not shown in the figure. The point P is on the outer gear a distance 1.0 m from the center. At the time of interest, point P is on the positive y axis.

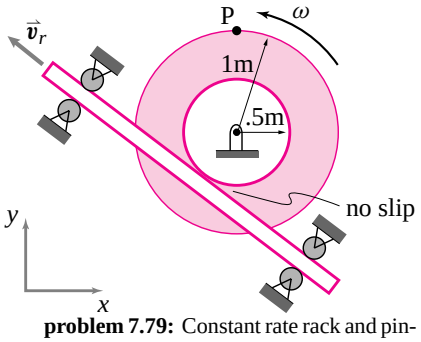
a) What is the speed of point P? *

b) What is the velocity of point P? *

c) What is the acceleration of point P? *

d) What is the velocity of the rack $\vec{v}_r$? *

e) What is the force on the rack due to its contact with the inner gear? *



problem 7.79: Constant rate rack and pinion

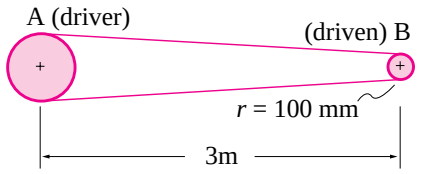
(Filename:ch4.2)

7.80 *Belt drives* are used to transmit power between parallel shafts. Two parallel shafts, 3 m apart, are connected by a belt passing over the pulleys A and B fixed to the two shafts. The driver pulley A rotates at a constant 200 rpm. The speed ratio between the pulleys A and B is 1:2.5. The input torque is 350 N·m. Assume no loss of power between the two shafts.

a) Find the input power. *

b) Find the rotational speed of the driven pulley B * .

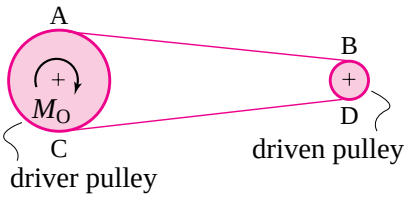
c) Find the output torque at B. *



problem 7.80:

(Filename:pfigure4.rpd)

7.81 In the belt drive system shown, assume that the driver pulley rotates at a constant angular speed ω. If the motor applies a constant torque M_O on the driver pulley, show that the tensions in the two parts, AB and CD, of the belt must be different. Which part has a greater tension? Does your conclusion about unequal tension depend on whether the pulley is massless or not? Assume any dimensions you need.



problem 7.81:

(Filename:pfigure4.rpe)

7.82 A belt drive is required to transmit 15 kW power from a 750 mm diameter pulley rotating at a constant 300 rpm to a 500 mm diameter pulley. The centers of the pulleys are located 2.5 m apart. The coefficient of friction between the belt and pulleys is $\mu = 0.2$.

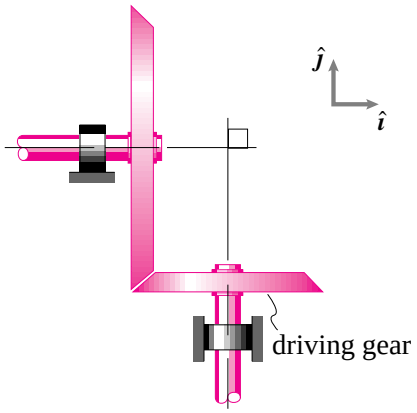
a) (See problem 7.102.) Draw a neat diagram of the pulleys and the belt-drive system and find the *angle of lap*, the contact angle θ, of the belt on the driver pulley.

b) Find the rotational speed of the driven pulley.

c) (See the figure in problem 7.102.) The power transmitted by the belt is given by *power = net tension × belt speed*, i.e., $P = (T_1 - T_2)v$, where v is the linear speed of the belt. Find the maximum tension in the belt. [Hint: $\frac{T_1}{T_2} = e^{\mu\theta}$ (see problem 7.102).]

d) The belt in use has a 15 mm×5 mm rectangular cross-section. Find the maximum tensile stress in the belt.

7.83 A bevel-type gear system, shown in the figure, is used to transmit power between two shafts that are perpendicular to each other. The driving gear has a mean radius of 50 mm and rotates at a constant speed $\omega = 150$ rpm. The mean radius of the driven gear is 80 mm and the driven shaft is expected to deliver a torque of $M_{out} = 25$ N·m. Assuming no power loss, find the input torque supplied by the driving shaft



problem 7.83:

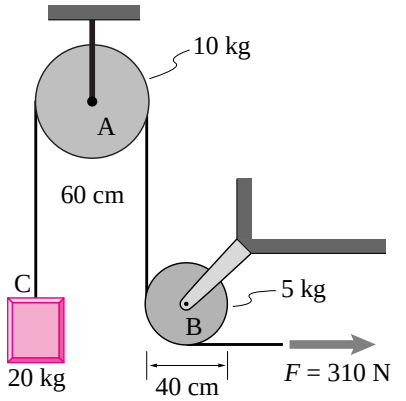
(Filename:pfigure4.rpa)

7.84 Disk pulleys. Two uniform disks A and B of *non-negligible* masses 10 kg and 5 kg respectively, are used as pulleys to hoist a block of mass 20 kg as shown in the figure. The block is pulled up by applying a force $F = 310$ N at one end of the string. Assume the string to be massless but 'frictionful' enough to not slide on the pulleys. Use $g = 10\,\text{m/s}^2$.

a) Find the angular acceleration of pulley B. *

b) Find the acceleration of block C. *

c) Find the tension in the part of the string between the block and the overhead pulley. *



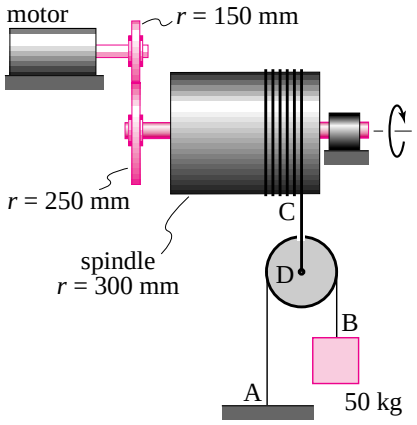
problem 7.84:

(Filename:pfigure.s94h8p4)

7.85 A spindle and pulley arrangement is used to hoist a 50 kg mass as shown in the figure. Assume that the pulley is to be of negligible mass. When the motor is running at a constant 100 rpm,

a) Find the velocity of the mass at B.

b) Find the tension in strings AB and CD.



problem 7.85:

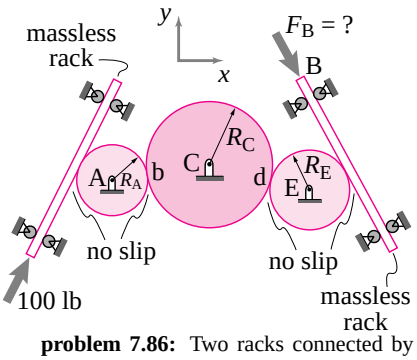
(Filename:pfigure4.rpg)

7.86 Two racks connected by three constant rate gears. See also problem 4.19. A 100 lbf force is applied to one rack. At the output, the machinery (not shown) applies a force of F_B to the other rack.

a) Assume the gear-train is spinning at constant rate and is frictionless. What is F_B? *

b) If the gear bearings had friction would that increase or decrease F_B to achieve the same constant rate?

c) If instead of applying a 100 lbf to the left rack it is driven by a motor (not shown) at constant speed v, what is the speed of the right rack? *



problem 7.86: Two racks connected by three gears.

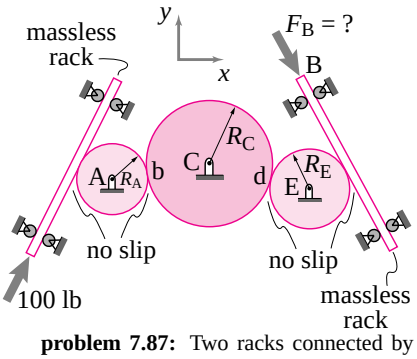
(Filename:ch4.5)

7.87 Two racks connected by three accelerating gears. A 100 lbf force is applied to one rack. At the output, the machinery (not shown) applies a force of F_B to the other rack.

a) Assume the gear-train is spinning at constant rate and is frictionless, what is F_B?

b) If the gear bearings had friction would that increase or decrease F_B to achieve the same constant rate?

c) If the angular velocity of the gear is increasing at rate α does this increase or decrease F_B at the given ω.



problem 7.87: Two racks connected by three gears.

(Filename:pg131.2)

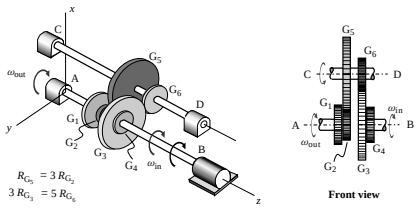
7.88 3-D accelerating gear train. This is really a 2-D problem; each gear turns in a different parallel plane. Shaft B is rigidly connected to gears G_4 and G_5. G_5 meshes with gear G_6. Gears G_6 and G_5 are both rigidly attached to shaft AD. Gear G_5 meshes with G_2 which is welded to shaft A. Shaft A and shaft B spin independently. Assume you know the torque M_{input}, angular velocity ω_{input} and the angular acceleration α_{input} of the input shaft. Assume the bearings and contacts are frictionless.

a) What is the input power?

b) What is the output power?

c) What is the angular velocity ω_{output} of the output shaft?

d) What is the output torque M_{output}?

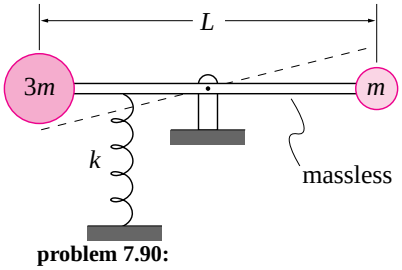


problem 7.88: A 3-D set of gears turning at constant rate.

(Filename:3dgeartrain)

7.89 A uniform disk of mass M and radius R rotates about a hinge O in the xy-plane. A point mass m is fixed to the disk at a distance $R/2$ from the hinge. A motor at the hinge drives the disk/point mass assembly with constant angular acceleration α. What torque at the hinge does the motor supply to the system?

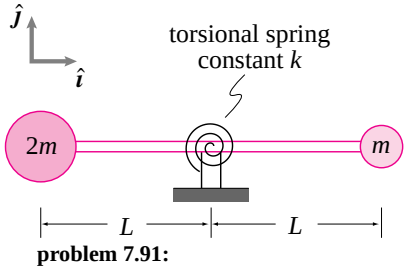
7.90 The asymmetric dumbbell shown in the figure is pivoted in the center and also attached to a spring at one quarter of its length from the bigger mass. When the bar is horizontal, the compression in the spring is y_s. At the instant of interest, the bar is at an angle θ from the horizontal; θ is small enough so that $y \approx \frac{L}{2}\theta$. If, at this position, the velocity of mass 'm' is $v\hat{\boldsymbol{j}}$ and that of mass $3m$ is $-v\hat{\boldsymbol{j}}$, evaluate the power term $(\sum \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}})$ in the energy balance equation.



problem 7.90:

(Filename:pfig2.3.rp4)

7.91 The dumbbell shown in the figure has a torsional spring with spring constant k (torsional stiffness units are $\frac{\text{kN}\cdot\text{m}}{\text{rad}}$). The dumbbell oscillates about the horizontal position with small amplitude θ. At an instant when the angular velocity of the bar is $\dot{\theta}\hat{\boldsymbol{k}}$, the velocity of the left mass is $-L\dot{\theta}\,\hat{\boldsymbol{j}}$ and that of the right mass is $L\dot{\theta}\,\hat{\boldsymbol{j}}$. Find the expression for the power P of the spring on the dumbbell at the instant of interest.



problem 7.91:

(Filename:pfig2.3.rp10)

7.92 A physical pendulum. A swinging stick is sometimes called a 'physical' pendulum. Take the 'body', the system of interest, to be the whole stick.

a) Draw a free body diagram of the system.

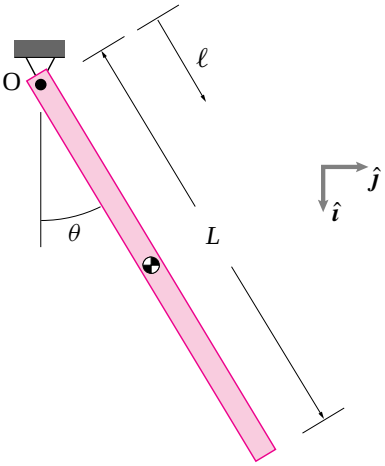
b) Write the equation of angular momentum balance for this system about point O.

c) Evaluate the left-hand-side as explicitly as possible in terms of the forces showing on your Free Body Diagram.

d) Evaluate the right hand side as completely as possible. You may use the following facts:

$$\begin{aligned}\vec{\boldsymbol{v}} &= l\dot{\theta}\cos\theta\,\hat{\boldsymbol{j}} + -l\dot{\theta}\sin\theta\hat{\boldsymbol{i}}\\ \vec{\boldsymbol{a}} &= -l\dot{\theta}^2\left[\cos\theta\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}}\right]\\ &\quad + l\ddot{\theta}\left[\cos\theta\,\hat{\boldsymbol{j}} - \sin\theta\hat{\boldsymbol{i}}\right]\end{aligned}$$

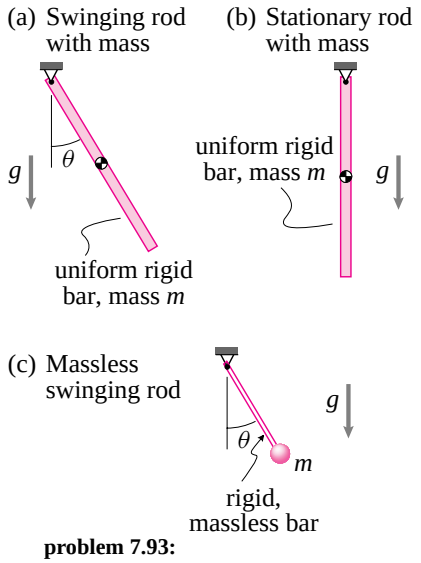
where ℓ is the distance along the pendulum from the top, θ is the angle by which the pendulum is displaced counter-clockwise from the vertically down position, $\hat{\boldsymbol{i}}$ is vertically down, and $\hat{\boldsymbol{j}}$ is to the right. You will have to set up and evaluate an integral.



problem 7.92:

(Filename:ppend)

7.93 Which of (a), (b), and (c) are two force members?



problem 7.93:

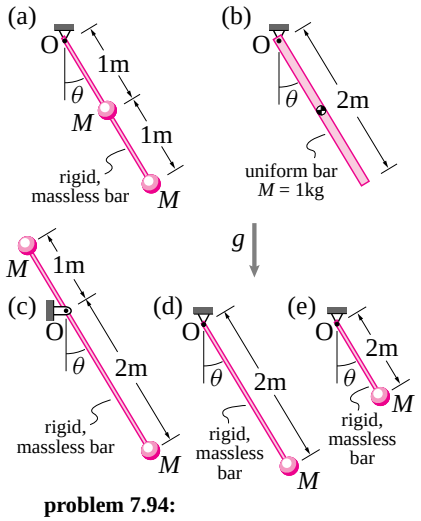
(Filename:pfigure2.two.force)

7.94 For the pendula in the figure :

a) Without doing any calculations, try to figure out the relative durations of the periods of oscillation for the five pendula (i.e. the order, slowest to fastest) Assume small angles of oscillation.

b) Calculate the period of small oscillations. [Hint: use balance of angular momentum about the point 0]. *

c) Rank the relative duration of oscillations and compare to your intuitive solution in part (a), and explain in words why things work the way they do.



problem 7.94:

(Filename:pg144.1n)

7.95 A massless 10 meter long bar is supported by a frictionless hinge at one end and has a 3.759 kg point mass at the other end. It is released at $t = 0$ from a tip angle of $\phi = .02$ radians measured from vertically upright position (hinge at the bottom). Use $g = 10\text{m}\,\text{s}^{-2}$.

a) Using a small angle approximation and the solution to the resulting linear differential equation, find the angle of tip at $t = 1$s and $t = 7$s. Use a calculator, not a numerical integrator.

b) Using numerical integration of the nonlinear differential equation for an inverted pendulum find ϕ at $t = 1$s and $t = 7$s.

c) Make a plot of the angle versus time for your numerical solution. Include on the same plot the angle versus time from the approximate linear solution from part (a).

d) Comment on the similarities and differences in your plots.

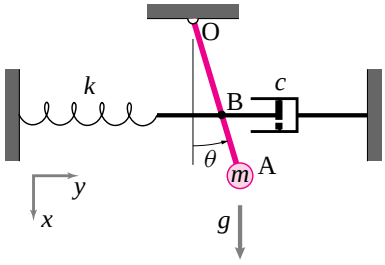
7.96 A spring-mass-damper system is depicted in the figure. The horizontal damping force applied at B is given by $F_D = -c\dot{y}_B$
The dimensions and parameters are as follows:

$$\begin{aligned} r_{B/0} &= 2\,ft \\ r_{A/0} &= \ell = 3\,ft \\ k &= 2\,\text{lbf/ ft} \\ c &= 0.3\,\text{lbf}\cdot\text{s/ ft} \end{aligned}$$

For small θ, assume that $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$.

a) Determine the natural circular frequency of small oscillations about equilibrium for the pendulum shown. The static equilibrium position is $\theta = 0$ (pendulum hanging vertically), so the spring is at its rest point in this position. Idealize the pendulum as a point mass attached to a rigid massless rod of length 1, so $I^O = m\ell^2$. Also use the "small angle approximation" where appropriate.

b) Sketch a graph of θ as a function of $t\,(t \geq 0)$ if the pendulum is released from rest at position $\theta = 0.2$ rad when $t = 0$. Your graph should show the correct qualitative behavior, but calculations are not necessary.



problem 7.96:

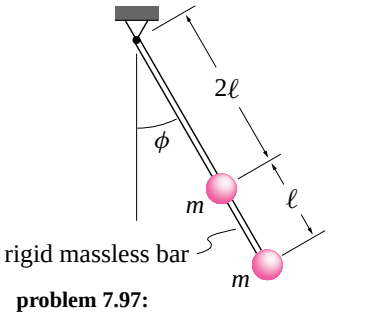
(Filename:pfigure.blue.157.2)

7.97 A rigid massless rod has two equal masses m_B and m_C ($m_B = m_C = m$) attached to it at distances $2L$ and 3ℓ, respectively, measured along the rod from a frictionless hinge located at a point A. The rod swings freely from the hinge. There is gravity. Let ϕ denote the angle of the rod measured from the vertical. Assume that ϕ and $\dot{\phi}$ are known at the moment of interest.

a) What is $\ddot{\phi}$? Solve in terms of m, ℓ, g, ϕ and $\dot{\phi}$.

b) What is the force of the hinge on the rod? Solve in terms of m, ℓ, g, ϕ, $\dot{\phi}$, $\ddot{\phi}$ and any unit vectors you may need to define.

c) Would you get the same answers if you put a mass $2m$ at 2.5ℓ? Why or why not?



problem 7.97:

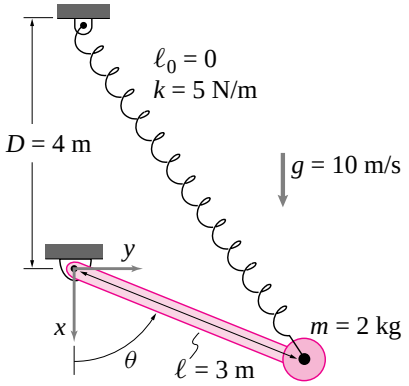
(Filename:pfigure.twomasspend)

7.98 See also problem 4.3. A zero length spring (relaxed length $\ell_0 = 0$) with stiffness $k = 5$ N/m supports the pendulum shown.

a) Find $\ddot{\theta}$ assuming $\dot{\theta} = 2$ rad/s, $\theta = \pi/2$. *

b) Find $\ddot{\theta}$ as a function of $\dot{\theta}$ and θ (and k, ℓ, m, and g.) *

[Hint: use vectors (otherwise it's hard)]
[Hint: For the special case, $kD = mg$, the solution simplifies greatly.]



problem 7.98:

(Filename:pfigure.blue.80.2)

7.99 Robotics problem: Simplest balancing of an inverted pendulum. You are holding a stick upside down, one end is in your hand, the other end sticking up. To simplify things, think of the stick as massless but with a point mass at the upper end. Also, imagine that it is only a two-dimensional problem (either you can ignore one direction of falling for simplicity or imagine wire guides that keep the stick from moving in and out of the plane of the paper on which you draw the problem).

You note that if you model your holding the stick as just having a stationary hinge then you

get $\ddot{\phi} = \frac{g}{l}\sin\phi$. Assuming small angles, this hinge leads to exponentially growing solutions. Upside-down sticks fall over. How can you prevent this falling?

One way to do keep the stick from falling over is to firmly grab it with your hand, and if the stick tips, apply a torque in order to right it. This corrective torque is (roughly) how your ankles keep you balanced when you stand upright. Your task in this assignment is to design a robot that keeps an inverted pendulum balanced by applying appropriate torque.

Your model is: Inverted pendulum, length ℓ, point mass m, and a hinge at the bottom with a motor that can apply a torque T_m. The stick might be tipped an angle ϕ from the vertical. A *horizontal* disturbing force $F(t)$ is applied to the mass (representing wind, annoying friends, etc).

a) Draw a picture and a FBD

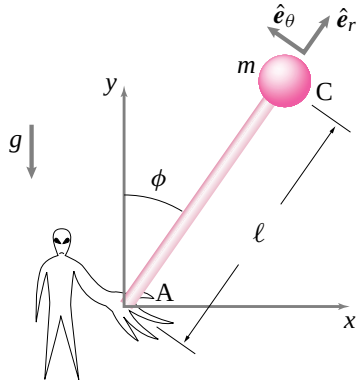
b) Write the equation for angular momentum balance about the hinge point. *

c) Imagine that your robot can sense the angle of tip ϕ and its rate of change $\dot{\phi}$ and can apply a torque in response to that sensing. That is you can make T_m any function of ϕ and $\dot{\phi}$ that you want. Can you find a function that will make the pendulum stay upright? Make a guess (you will test it below).

d) Test your guess the following way: plug it into the equation of motion from part (b), linearize the equation, assume the disturbing force is zero, and see if the solution of the differential equation has exponentially growing (i.e. unstable) solutions. Go back to (c) if it does and find a control strategy that works.

e) Pick numbers and model your system on a computer using the full non-linear equations. Use initial conditions both close to and far from the upright position and plot ϕ versus time.

f) If you are ambitious, pick a non-zero forcing function $F(t)$ (say a sine wave of some frequency and amplitude) and see how that affects the stability of the solution in your simulations.



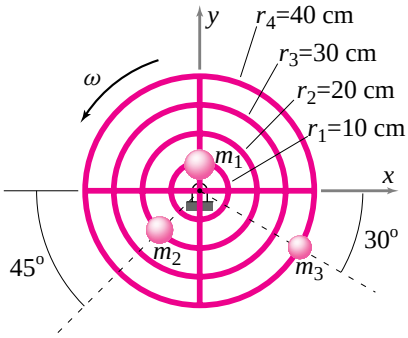
problem 7.99:

(Filename:pfigure.s94h9p1)

7.100 Balancing a system of rotating particles. A wire frame structure is made of four concentric loops of massless and rigid wires, connected to each other by four rigid wires presently coincident with the x and y axes. Three masses, $m_1 = 200$ grams, $m_2 = 150$ grams and $m_3 = 100$ grams, are glued to the structure as shown in the figure. The structure rotates counter-clockwise at a constant rate $\dot{\theta} = 5$ rad/s. There is no gravity.

a) Find the net force exerted by the structure on the support at the instant shown. *

b) You are to put a mass m at an appropriate location on the third loop so that the net force on the support is zero. Find the appropriate mass and the location on the loop.



problem 7.100:

(Filename:pfigure.s94h6p2)

7.101 A rope of length ℓ and total mass m is held fixed at one end and whirled around in circular motion at a constant rate ω in the horizontal plane. Ignore gravity.

a) Find the tension in the rope as a function of r, the radial distance from the center of rotation to any desired location on the rope. *

b) Where does the maximum tension occur in the rope? * .

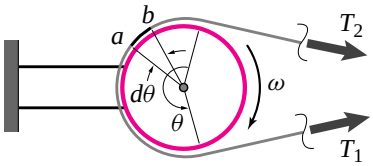
c) At what distance from the center of rotation does the tension drop to half its maximum value? * .

7.102 Assume that the pulley shown in figure(a) rotates at a constant speed ω. Let the angle of contact between the belt and pulley surface be θ. Assume that the belt is massless and that the condition of impending slip exists between the pulley and the belt. The free body diagram of an infinitesimal section ab of the belt is shown in figure(b).

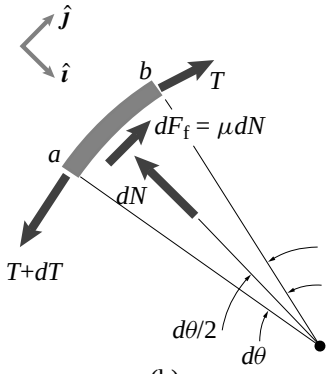
a) Write the equations of linear momentum balance for section ab of the belt in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions.

b) Eliminate the normal force N from the two equations in part (a) and get a differential equation for the tension T in terms of the coefficient of friction μ and The contact angle θ.

c) Show that the solution to the equation in part (b) satisfies $\frac{T_1}{T_2} = e^{\mu\theta}$, where T_1 and T_2 are the tensions in the lower and the upper segments of the belt, respectively.



(a)



(b)

problem 7.102:

(Filename:pfigure4.rpf)

7.5 Polar moment of inertia: I_{zz}^{cm} and I_{zz}^{O}

Skills

7.103 A point mass $m = 0.5\,\text{kg}$ is located at $x = 0.3\,\text{m}$ and $y = 0.4\,\text{m}$ in the xy-plane. Find the moment of inertia of the mass about the z-axis. *

7.104 A small ball of mass 0.2 kg is attached to a 1 m long inextensible string. The ball is to execute circular motion in the xy-plane with the string fully extended.

a) What is value of I_{zz} of the ball about the center of rotation? *

b) How much must you shorten the string to reduce the moment of inertia of the ball by half? *

7.105 Two identical point masses are attached to the two ends of a rigid massless bar of length ℓ (one mass at each end). Locate a point along the length of the bar about which the polar moment of inertia of the system is 20% more than that calculated about the mid point of the bar. *

7.106 A dumbbell consists of a rigid massless bar of length ℓ and two identical point masses m and m, one at each end of the bar.

a) About which point on the dumbbell is its polar moment of inertia I_{zz} a minimum and what is this minimum value? *

b) About which point on the dumbbell is its polar moment of inertia I_{zz} a maximum and what is this maximum value? *

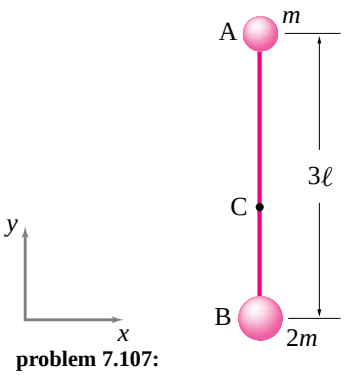
7.107 Think first, calculate later. A light rigid rod AB of length 3ℓ has a point mass m at end A and a point mass $2m$ at end B. Point C is the center of mass of the system. First, answer the following questions without any calculations and then do calculations to verify your guesses.

a) About which point A, B, or C, is the polar moment of inertia I_{zz} of the system a minimum? *

b) About which point is I_{zz} a maximum? *

c) What is the ratio of I_{zz}^{A} and I_{zz}^{B}? *

d) Is the radius of gyration of the system greater, smaller, or equal to the length of the rod? *



problem 7.107:

(Filename:pfigure4.4.rp1)

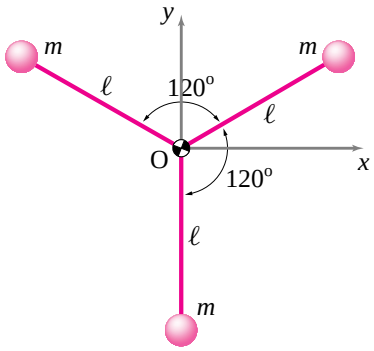
7.108 Do you understand the perpendicular axis theorem? Three identical particles of mass m are connected to three identical massless rods of length ℓ and welded together at point O as shown in the figure.

a) Guess (no calculations) which of the three moment of inertia terms I_{xx}^{O}, I_{yy}^{O}, I_{zz}^{O} is the smallest and which is the biggest. *

b) Calculate the three moments of inertia to check your guess. *

c) If the orientation of the system is changed, so that one mass is along the x-axis, will your answer to part (a) change?

d) Find the radius of gyration of the system for the polar moment of inertia. *



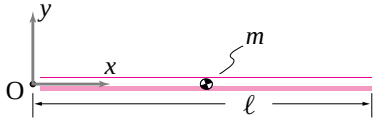
problem 7.108:

(Filename:pfigure4.4.rp3)

7.109 Show that the polar moment of inertia I_{zz}^{O} of the uniform bar of length ℓ and mass m, shown in the figure, is $\frac{1}{3}m\ell^2$, in two different ways:

a) by using the basic definition of polar moment of inertia $I_{zz}^{O} = \int r^2\,dm$, and

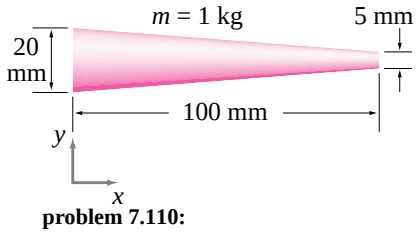
b) by computing I_{zz}^{cm} first and then using the parallel axis theorem.



problem 7.109:

(Filename:pfigure4.4.rp5)

7.110 Locate the center of mass of the tapered rod shown in the figure and compute the polar moment of inertia I_{zz}^{cm}. [Hint: use the variable thickness of the rod to define a variable mass density per unit length.]



problem 7.110:

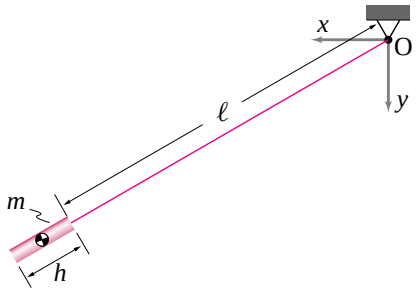
(Filename:pfigure4.4.rp6)

7.111 A short rod of mass m and length h hangs from an inextensible string of length ℓ.

a) Find the moment of inertia I_{zz}^{O} of the rod.

b) Find the moment of inertia of the rod I_{zz}^{O} by considering it as a point mass located at its center of mass.

c) Find the percent error in I_{zz}^{O} in treating the bar as a point mass by comparing the expressions in parts (a) and (b). Plot the percent error versus h/ℓ. For what values of h/ℓ is the percentage error less than 5%?



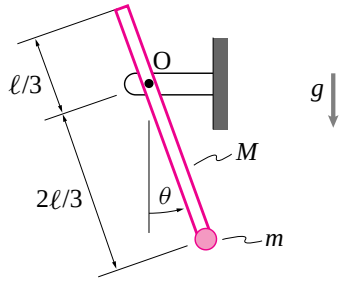
problem 7.111:

(Filename:pfigure4.4.rp11)

7.112 A small particle of mass m is attached to the end of a thin rod of mass M (uniformly distributed), which is pinned at hinge O, as depicted in the figure.

a) Obtain the equation of motion governing the rotation θ of the rod.

b) What is the natural frequency of the system for small oscillations θ?



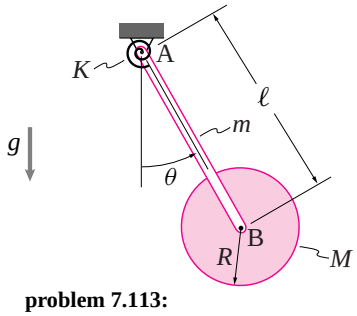
problem 7.112:

(Filename:pfigure.blue.151.3)

7.113 A thin rod of mass m and length ℓ is hinged with a torsional spring of stiffness K at A, and is connected to a thin disk of mass M and radius R at B. The spring is uncoiled when $\theta = 0$. Determine the natural frequency ω_n of the system for small oscillations θ, assuming that the disk is:

a) welded to the rod, and *

b) pinned frictionlessly to the rod. *



problem 7.113:

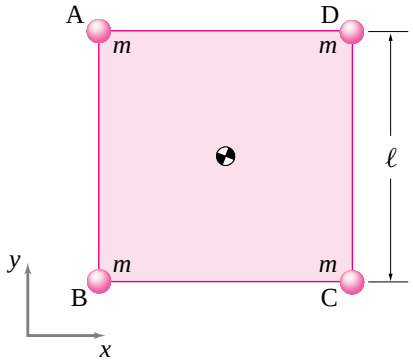
(Filename:pfigure.blue.151.4)

7.114 Do you understand the parallel axis theorem? A massless square plate $ABCD$ has four identical point masses located at its corners.

a) Find the polar moment of inertia I_{zz}^{cm}. *

b) Find a point P on the plate about which the system's moment of inertia I_{zz} is maximum? *

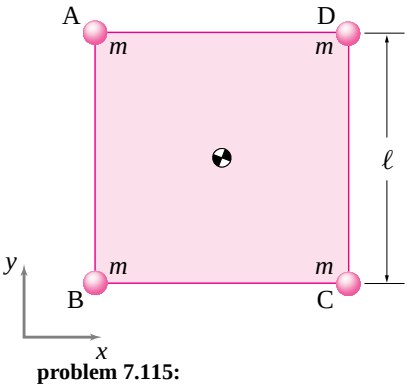
c) Find the radius of gyration of the system. *



problem 7.114:

(Filename:pfigure4.4.rp2)

7.115 Perpendicular axis theorem and symmetry. For the massless square plate with four point masses on the corners, the polar moment of inertia $I_{zz}^{cm} = 0.6\,\text{kg}\cdot\text{m}^2$. Find I_{xx}^{cm} of the system. *



problem 7.115:

(Filename:pfigure4.4.rp4)

7.116 A uniform square plate (2 m on edge) has a corner cut out. The total mass of the remaining plate is 3 kg. It spins about the origin at a constant rate of one revolution every π s.

a) What is the moment of inertia of the plate about point O?

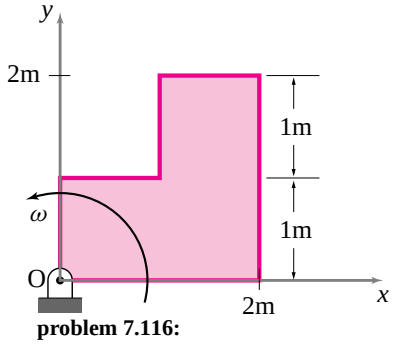
b) Where is the center of mass of the plate at the instant shown?

c) What are the velocity and acceleration of the center of mass at the instant shown?

d) What is the angular momentum of the plate about the point O at the instant shown?

e) What are the total force and moment required to maintain this motion when the plate is in the configuration shown?

f) What is the total kinetic energy of the plate?



problem 7.116:

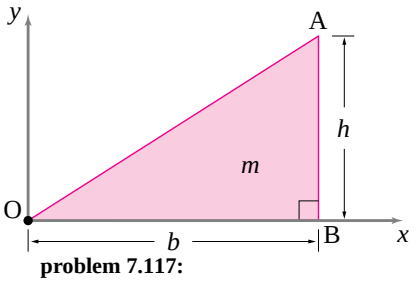
(Filename:pfigure.blue.87b.1)

7.117 A uniform thin triangular plate of mass m, height h , and base b lies in the xy-plane.

a) Set up the integral to find the polar moment of inertia I_{zz}^{O} of the plate. *

b) Show that $I_{zz}^{O} = \dfrac{m}{6}(h^2 + 3b^2)$ by evaluating the integral in part (a).

c) Locate the center of mass of the plate and calculate I_{zz}^{cm}.



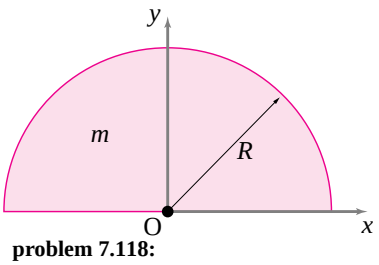
problem 7.117:

(Filename:pfigure4.4.rp9)

7.118 A uniform thin plate of mass m is cast in the shape of a semi-circular disk of radius R as shown in the figure.

a) Find the location of the center of mass of the plate

b) Find the polar moment of inertia of the plate, I_{zz}^{cm}. [Hint: It may be easier to set up and evaluate the integral for I_{zz}^{O} and then use the parallel axis theorem to calculate I_{zz}^{cm}.]



problem 7.118:

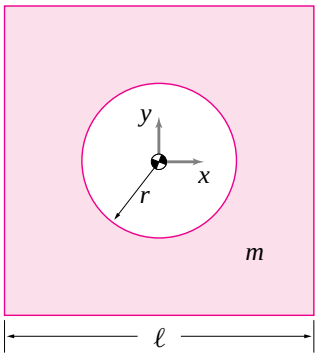
(Filename:pfigure4.4.rp10)

7.119 A uniform square plate of side $\ell = 250\,\text{mm}$ has a circular cut-out of radius $r = 50\,\text{mm}$. The mass of the plate is $m = \dfrac{1}{2}\,\text{kg}$.

a) Find the polar moment of inertia of the plate.

b) Plot I_{zz}^{cm} versus r/ℓ.

c) Find the limiting values of I_{zz}^{cm} for $r = 0$ and $r = \ell$.



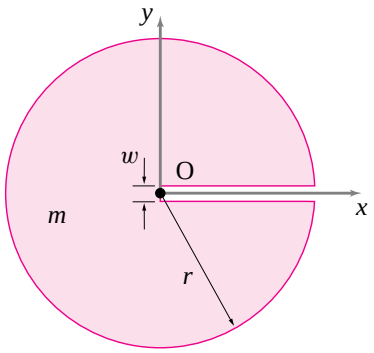
problem 7.119:

(Filename:pfigure4.4.rp7)

7.120 A uniform thin circular disk of radius $r = 100\,\text{mm}$ and mass $m = 2\,\text{kg}$ has a rectangular slot of width $w = 10\,\text{mm}$ cut into it as shown in the figure.

a) Find the polar moment of inertia I_{zz}^{O} of the disk.

b) Locate the center of mass of the disk and calculate I_{zz}^{cm}.

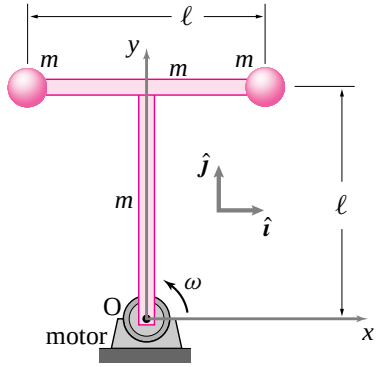


problem 7.120:

(Filename:pfigure4.4.rp8)

7.6 Using I_{zz}^{cm} and I_{zz}^{O} in mechanics equations

7.121 Motor turns a dumbbell. Two uniform bars of length ℓ and mass m are welded at right angles. At the ends of the horizontal bar are two more masses m. The bottom end of the vertical rod is attached to a hinge at O where a motor keeps the structure rotating at constant rate ω (counter-clockwise). What is the net force and moment that the motor and hinge cause on the structure at the instant shown?



problem 7.121:

(Filename:pfigure.s94h5p5)

7.122 An object consists of a massless bar with two attached masses m_1 and m_2. The object is hinged at O.

a) What is the moment of inertia of the object about point O (I_{zz}^{O})?

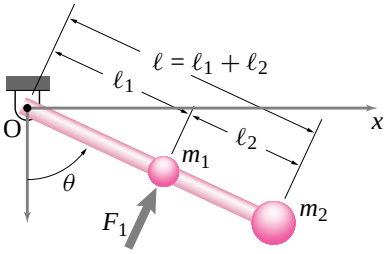
b) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is $\vec{\boldsymbol{H}}_{\mathrm{O}}$, the angular momentum about point O?

c) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$, the rate of change of angular momentum about point O?

d) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is T, the total kinetic energy?

e) Assume that you don't know θ, $\dot{\theta}$ or $\ddot{\theta}$ but you do know that F_1 is applied to the rod, perpendicular to the rod at m_1. What is $\ddot{\theta}$? (Neglect gravity.)

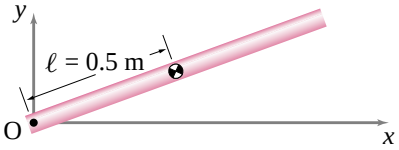
f) If F_1 were applied to m_2 instead of m_1, would $\ddot{\theta}$ be bigger or smaller?



problem 7.122:

(Filename:pfigure.blue.82.1)

7.123 A uniform rigid rod rotates at constant speed in the xy-plane about a peg at point O. The center of mass of the rod may not exceed a specified acceleration $a_{max} = 0.5\,\mathrm{m/s^2}$. Find the maximum angular velocity of the rod.



problem 7.123:

(Filename:pfigure4.3.rp3)

7.124 A uniform one meter bar is hung from a hinge that is at the end. It is allowed to swing freely. $g = 10\,\mathrm{m/s^2}$.

a) What is the period of small oscillations for this pendulum?

b) Suppose the rod is hung 0.4 m from one end. What is the period of small oscillations for this pendulum? Can you explain why it is longer or shorter than when it is is hung by its end?

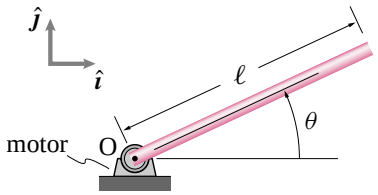
7.125 A motor turns a bar. A uniform bar of length ℓ and mass m is turned by a motor whose shaft is attached to the end of the bar at O. The angle that the bar makes (measured counter-clockwise) from the positive x axis is $\theta = 2\pi\ t^2/\mathrm{s^2}$. Neglect gravity.

a) Draw a free body diagram of the bar.

b) Find the force acting on the bar from the motor and hinge at $t = 1$ s.

c) Find the torque applied to the bar from the motor at $t = 1$ s.

d) What is the power produced by the motor at $t = 1$ s?



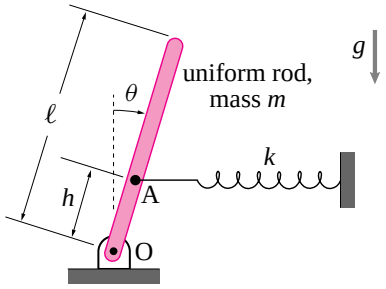
problem 7.125:

(Filename:pfigure.f93f1)

7.126 The rod shown is uniform with total mass m and length ℓ. The rod is pinned at point 0. A linear spring with stiffness k is attached at the point A at height h above 0 and along the rod as shown. When $\theta = 0$, the spring is unstretched. Assume that θ is small for both parts of this problem.

a) Find the natural frequency of vibration (in radians per second) in terms of m, g, h, ℓ and k.

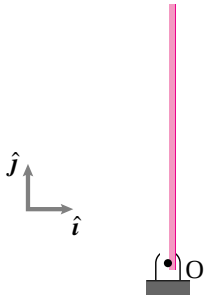
b) If you have done the calculation above correctly there is a value of h for which the natural frequency is zero. Call this value of h, h_{crit}. What is the behavior of the system when $h < h_{crit}$? (Desired is a phrase pointing out any qualitative change in the type of motion with some justification.)



problem 7.126:

(Filename:pfigure.blue.153.2)

7.127 A uniform stick of length ℓ and mass m is a hair away from vertically up position when it is released with no angular velocity (a 'hair' is a technical word that means 'very small amount, zero for some purposes'). It falls to the right. What is the force on the stick at point O when the stick is horizontal. Solve in terms of ℓ, m, g, $\hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{\jmath}}$. Carefully define any coordinates, base vectors, or angles that you use.



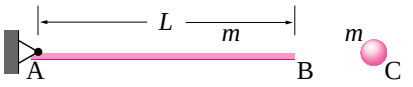
problem 7.127:

(Filename:pfigure.s94q8p1)

7.128 Acceleration of a trap door. A uniform bar AB of mass m and a ball of the same mass are released from rest from the same horizontal position. The bar is hinged at end A. There is gravity.

a) Which point on the rod has the same acceleration as the ball, immediately after release. *

b) What is the reaction force on the bar at end A just after release? *



problem 7.128:

(Filename:pfigure.s94h8p3)

7.129 A pegged compound pendulum. See also problem 7.130. A uniform bar of mass m and length ℓ hangs from a peg at point C and swings in the vertical plane about an axis passing through the peg. The distance d from the center of mass of the rod to the peg can be changed by putting the peg at some other point along the length of the rod.

a) Find the angular momentum of the rod as about point C.

b) Find the rate of change of angular momentum of the rod about C.

c) How does the period of the pendulum vary with d? Show the variation by plotting the period against $\frac{d}{\ell}$. [Hint, you must first find the equations of motion, linearize for small θ, and then solve.] *

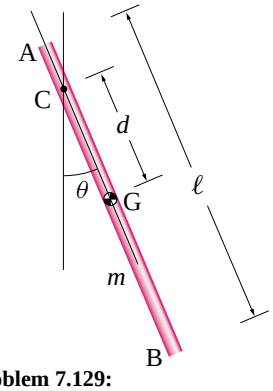
d) Find the total energy of the rod (using the height of point C as a datum for potential energy).

e) Find $\ddot{\theta}$ when $\theta = \pi/6$.

f) Find the reaction force on the rod at C, as a function of m, d, ℓ, θ, and $\dot{\theta}$.

g) For the given rod, what should be the value of d (in terms of ℓ) in order to have the fastest pendulum? *

h) **Test of Schuler's pendulum.** The pendulum with the value of d obtained in (g) is called the Schuler's pendulum. It is not only the fastest pendulum but also the "most accurate pendulum". The claim is that even if d changes slightly over time due to wear at the support point, the period of the pendulum does not change much. Verify this claim by calculating the percent error in the time period of a pendulum of length $\ell = 1$ m under the following three conditions: (i) initial $d = 0.15$ m and after some wear $d = 0.16$ m, (ii) initial $d = 0.29$ m and after some wear $d = 0.30$ m, and (iii) initial $d = 0.45$ m and after some wear $d = 0.46$ m. Which pendulum shows the least error in its time period? Do you see any connection between this result and the plot obtained in (c)?

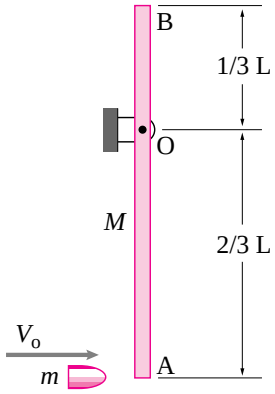


problem 7.129:

(Filename:pfigure.s94h8p6)

7.130 Given $\ddot{\theta}$, $\dot{\theta}$, and θ, what is the total kinetic energy of the pegged compound pendulum in problem 7.129?

7.131 A slender uniform bar AB of mass M is hinged at point O, so it can rotate around O without friction. Initially the bar is at rest in the vertical position as shown. A bullet of mass m and horizontal velocity V_o strikes the end A of the bar and sticks to it (an inelastic collision). Calculate the angular velocity of the system — the bar with its embedded bullet, immediately after the impact.



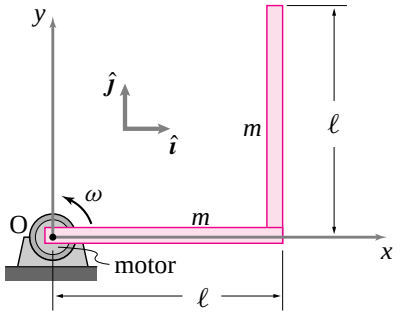
problem 7.131:

(Filename:pfigure.blue.78.1)

7.132 Motor turns a bent bar. Two uniform bars of length ℓ and uniform mass m are welded at right angles. One end is attached to a hinge at O where a motor keeps the structure rotating at a constant rate ω (counterclockwise). What is the net force and moment that the motor and hinge cause on the structure at the instant shown.

a) neglecting gravity *

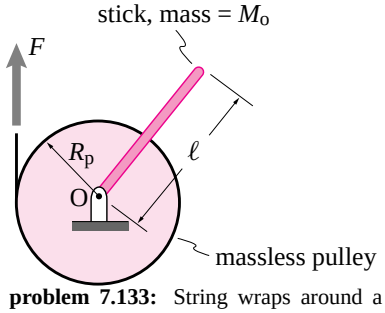
b) including gravity. *



problem 7.132: A bent bar is rotated by a motor.

(Filename:pg85.3)

7.133 2-D problem, no gravity. A uniform stick with length ℓ and mass M_o is welded to a pulley hinged at the center O. The pulley has negligible mass and radius R_p. A string is wrapped many times around the pulley. At time $t = 0$, the pulley, stick, and string are at rest and a force F is suddenly applied to the string. How long does it take for the pulley to make one full revolution? *



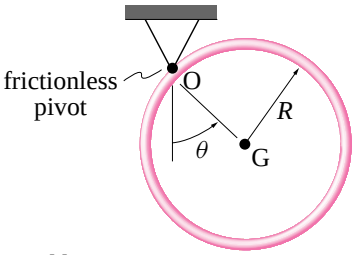
problem 7.133: String wraps around a pulley with a stick glued to it.

(Filename:p2.3)

7.134 A thin hoop of radius R and mass M is hung from a point on its edge and swings in its plane. Assuming it swings near to the position where its center of mass G is below the hinge:

a) What is the period of its swinging oscillations?

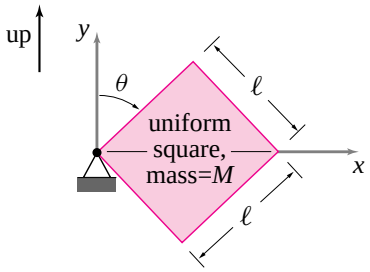
b) If, instead, the hoop was set to swinging in and out of the plane would the period of oscillations be greater or less?



problem 7.134:

(Filename:pfigure.blue.145.1)

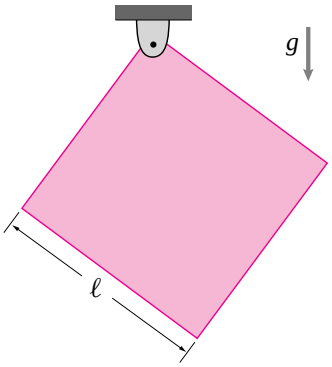
7.135 The uniform square shown is released from rest at $t = 0$. What is $\alpha = \dot{\omega} = \ddot{\theta}$ immediately after release?



problem 7.135:

(Filename:pfigure.blue.42.2)

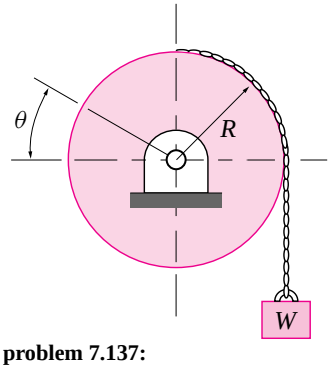
7.136 A square plate with side ℓ and mass m is hinged at one corner in a gravitational field g. Find the period of small oscillation.



problem 7.136:

(Filename:p.f96.p3.2)

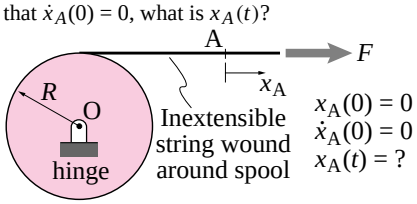
7.137 A wheel of radius R and moment of inertia I about the axis of rotation has a rope wound around it. The rope supports a weight W. Write the equation of conservation of energy for this system, and differentiate to find the equation of motion in terms of acceleration. Check the solution obtained by drawing separate free-body diagrams for the wheel and for the weight, writing the equations of motion for each body, and solving the equations simultaneously. Assume that the mass of the rope is negligible, and that there is no energy loss during the motion.



problem 7.137:

(Filename:pfigure.blue.136.1)

7.138 A disk with radius R has a string wrapped around it which is pulled with a force F. The disk is free to rotate about the axis through O normal to the page. The moment of inertia of the disk about O is I_o. A point A is marked on the string. Given that $x_A(0) = 0$ and that $\dot{x}_A(0) = 0$, what is $x_A(t)$?

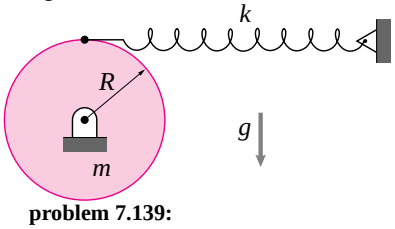


Spool has mass moment of inertia, I_O

problem 7.138:

(Filename:pfigure.blue.42.1)

7.139 Oscillating disk. A uniform disk with mass m and radius R pivots around a frictionless hinge at its center. It is attached to a massless spring which is horizontal and relaxed when the attachment point is directly above the center of the disk. Assume small rotations and the consequent geometrical simplifications. Assume the spring can carry compression. What is the period of oscillation of the disk if it is disturbed from its equilibrium configuration? [You may use the fact that, for the disk shown, $\dot{\vec{H}}_O = \frac{1}{2}mR^2\ddot{\theta}\hat{\boldsymbol{k}}$, where θ is the angle of rotation of the disk.] *



problem 7.139:

(Filename:pg147.2)

7.140 This problem concerns a narrow rigid hoop. For reference, here are dimensions and values you should use in this problem: mass of hoop $m_{\text{hoop}} = 1\,\text{kg}$ radius of hoop $R_{\text{hoop}} = 3\,\text{m}$, and gravitational acceleration $g = 10\,\text{m/s}^2$.

a) The hoop is hung from a point on its edge and swings in its plane. Assuming its swings near to the position where its center of mass is below the hinge.

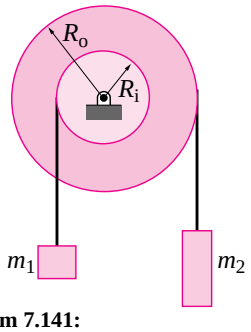
b) What is the period of its swinging oscillations?

c) If, instead, the hoop was set to swinging in and out of the plane would the period of oscillations be greater or less?

7.141 The compound pulley system shown in the figure consists of two pulleys rigidly connected to each other. The radii of the two pulleys are: $R_i = 0.2$ m and $R_o = 0.4$ m. The combined moment of intertia of the two pulleys about the axis of rotation is 2.7 kg·m^2. The two masses, $m_1 = 40$ kg and $m_2 = 100$ kg, are released from rest in the configuration shown. Just after release,

a) find the angular acceleration of the pulleys, and *

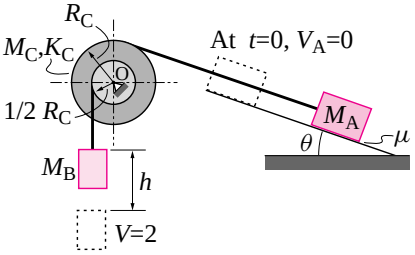
b) find the tension in each string. *



problem 7.141:

(Filename:summer95p2.1)

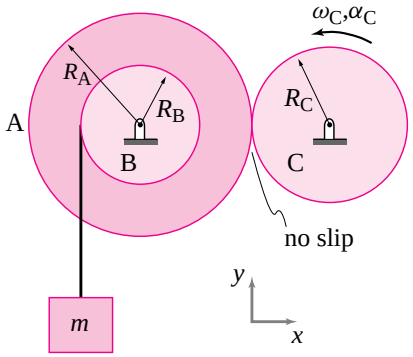
7.142 Consider a system of two blocks A and B and the reel C mounted at the fixed point O, as shown in the figure. Initially the system is at rest. Calculate the velocity for the block B after it has dropped a vertical distance h. Given: h, mass of block A, M_A, coefficient of friction μ, slope angle θ, mass of the reel, M_C, moment of inertia I about point O radius of gyration of the reel K_C, outer radius of the reel R_C, inner radius of the reel $\frac{1}{2}R_C$, mass of the block B M_B. The radius of gyration K is defined by $I_{zz}^{\text{cm}} = mK^2$. (Apply the work-energy principle.) *



problem 7.142:

(Filename:pfigure.blue.135.1)

7.143 Gear A with radius $R_A = 400$ mm is rigidly connected to a drum B with radius $R_B = 200$ mm. The combined moment of inertia of the gear and the drum about the axis of rotation is $I_{zz} = 0.5\ \text{kg}\cdot\text{m}^2$. Gear A is driven by gear C which has radius $R_C = 300$ mm. As the drum rotates, a 5 kg mass m is pulled up by a string wrapped around the drum. At the instant of interest, The angular speed and angular acceleration of the driving gear are 60 rpm and 12 rpm/s, respectively. Find the acceleration of the mass m. *



problem 7.143:

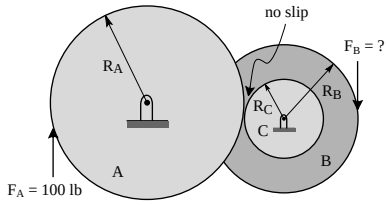
(Filename:summer95f.2)

7.144 Two gears accelerating. See also problems 7.77 and 4.16. At the input to a gear box a 100 lbf force is applied to gear A. At the output the machinery (not shown) applies a force of F_B to the output gear.

a) Assume the gear is spinning at constant rate and is frictionless, what is F_B?

b) If the gear bearing had friction would that increase or decrease F_B?

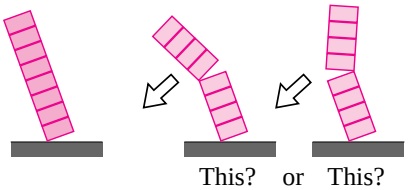
c) If the angular velocity of the gear is increasing at rate α does this increase or decrease F_B at the given ω.



problem 7.144: Two gears.

(Filename:pg131.3.aa)

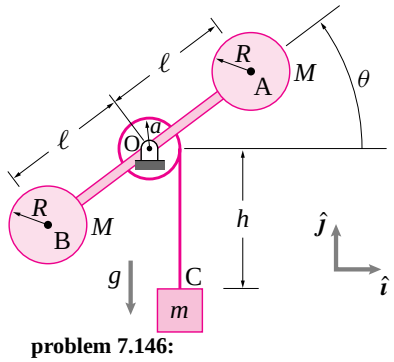
7.145 Frequently parents will build a tower of blocks for their children. Just as frequently, kids knock them down. In falling (even when they start to topple aligned), these towers invariably break in two (or more) pieces at some point along their length. Why does this breaking occur? What condition is satisfied at the point of the break? Will the stack bend towards or away from the floor after the break?



problem 7.145:

(Filename:pfigure.blue.129.1)

7.146 Massless pulley, dumbell and a hanging mass. A mass M falls vertically but is withheld by a string which is wrapped around an ideal massless pulley with radius a. The pulley is welded to a dumbell made of a massless rod welded to uniform solid spheres at A and B of radius R, each of whose center is a distance ℓ from O. At the instant in question, the dumbell makes an angle θ with the positive x axis and is spinning at the rate $\dot\theta$. Point C is a distance h down from O. In terms of some or all of m, M, a, R, ℓ, h, g, θ, $\dot\theta$, and $\hat{\boldsymbol{j}}$, find the acceleration of the mass.



problem 7.146:

(Filename:s97p3.1)

7.147 Two racks connected by a gear. A 100 lbf force is applied to one rack. At the output the machinery (not shown) applies a force F_B to the other rack.

a) Assume the gear is spinning at constant rate and is frictionless. What is F_B?

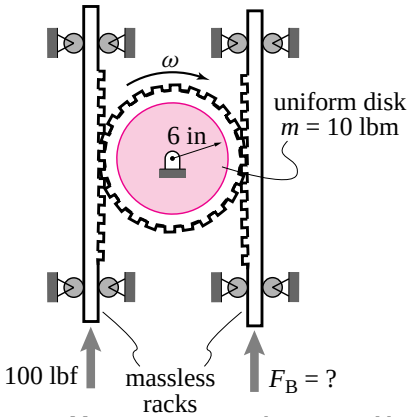
b) If the gear bearing had friction, would that increase or decrease F_B to achieve the same constant rate?

c) If the angular velocity of the gear is increasing at rate α, does this increase or decrease F_B at the given ω.

d) If the output load F_B is given then the motion of the machine can be found from the input load. Assume that the machine starts from rest with a given output load. So long as rack B moves in the opposite direction of the output force F_B the output power is positive.

(a) For what values of F_B is the output power positive?

(b) For what values of F_B is the output work maximum if the machine starts from rest and runs for a fixed amount of time?



problem 7.147: Two racks connected by a gear.

(Filename:pg131.1)

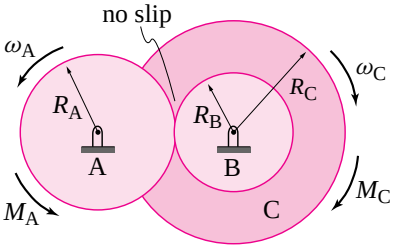
7.148 2-D accelerating gear train. See also problem 7.74. Assume you know the torque $M_{\text{input}} = M_A$ and angular velocity $\omega_{\text{input}} = \omega_A$ of the input shaft. Assume the bearings and contacts are frictionless. Assume you also know the input angular acceleration $\dot\omega$ and the moments of inertia I_A, I_B and I_C of each of the disks about their centers .

a) What is the input power?

b) What is the output power?

c) What is the angular velocity $\omega_{\text{output}} = \omega_C$ of the output shaft?

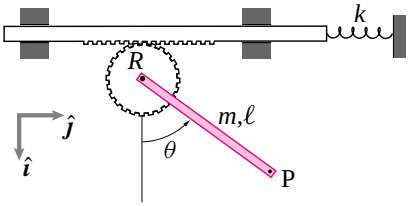
d) What is the output torque $M_{\text{output}} = M_C$?



problem 7.148: A set of gears turning at variable rate.

(Filename:pg129.1)

7.149 A stick welded to massless gear that rolls against a massless rack which slides on frictionless bearings and is constrained by a linear spring. Neglect gravity. The spring is relaxed when the angle $\theta = 0$. Assume the system is released from rest at $\theta = \theta_0$. What is the acceleration of the point P at the end of the stick when $\theta = 0$? Answer in terms of any or all of m, R, ℓ, θ_0, k, $\hat{\boldsymbol{i}}$, and $\hat{\boldsymbol{j}}$. [Hint: There are several steps of reasoning required. You might want to draw FBD(s), use angular momentum balance, set up a differential equation, solve it, plug values into this solution, and use the result to find the quantities of interest.



problem 7.149:

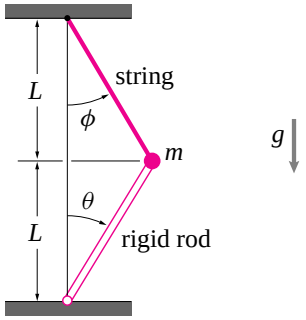
(Filename:p.f96.p2.2)

7.150 A tipped hanging sign is represented by a point mass m. The sign sits at the end of a massless, rigid rod which is hinged at its point of attachment to the ground. A taut massless elastic cord helps keep the rod vertical. The tension T in the very stretchy cord is idealized as constant during small displacements. (Note also that $\phi \equiv \theta$ during such motions). Consider all hinges to be frictionless and motions to take place in the plane of the paper.

a) Write the angular momentum of the mass about 0 when the rod has an angular velocity $\dot\theta$.

b) Find the differential equation that governs the mass's motion for small θ.

c) Describe the motion for $T > \frac{mg}{2}$, $T = \frac{mg}{2}$, and $T < \frac{mg}{2}$. Interpret the differences of these cases in physical terms.



problem 7.150:

(Filename:pfigure.blue.153.1)

Problems for Chapter 8

Advanced topics in circular motion

8.1 3-D description of circular motion

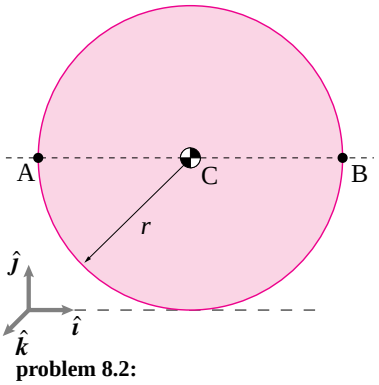
8.1 If $\vec{A}$ and $\vec{B}$ are perpendicular, what is $|\vec{A}\times(\vec{A}\times\vec{B})|$ in terms of $|\vec{A}|$ and $|\vec{B}|$? (Hint: make a neat 3-D sketch.)

8.2 A circular disk of radius $r = 100\,\text{mm}$ rotates at constant speed ω about a fixed but unknown axis. At the instant when the disk is in the xy-plane, the angular velocity of each point on the diameter AB (which is parallel to the x-axis) is $\vec{v} = -2\,\text{m/s}\hat{k}$. At the same instant, the magnitude of the acceleration of the center of mass C is $50\,\text{m/s}^2$.

a) Find the angular velocity of the disk.

b) Find the location of the axis of rotation.

c) Find the acceleration of points A and B.



problem 8.2:

(Filename:pfigure4.3.rp10)

8.3 In the expression for normal acceleration $\vec{a}_n = \vec{\omega}\times(\vec{\omega}\times\vec{r})$, the parenthesis around $\vec{\omega}\times\vec{r}$ is important because vector product is nonassociative. Showing $\vec{\omega}\times(\vec{\omega}\times\vec{r}) \neq (\vec{\omega}\times\vec{\omega})\times\vec{r}$ is almost trivial. (why?) Take $\vec{\omega}_1 = \omega_1\hat{k}$, $\vec{\omega}_2 = \omega_2\hat{i} + \omega_1\hat{k}$ and $\vec{r} = a\hat{i} - b\hat{j}$ and show that $\vec{\omega}_1\times(\vec{\omega}_2\times\vec{r}) \neq (\vec{\omega}_1\times\vec{\omega}_2)\times\vec{r}$.

8.4 In circular motion the velocity is $\vec{v} = \vec{\omega}\times\vec{r}$, where $\vec{\omega}$ is the angular velocity and $\vec{r}$ is the position vector of the point of interest relative to a point on the axis of rotation. Compute $\vec{v}$ in the three cases below, draw the circular orbit, and show the vectors $\vec{\omega}$, $\vec{r}$, and $\vec{v}$.

a) $\vec{\omega} = 2\,\text{rad/s}\hat{k}$, $\vec{r} = 3\,\text{m}\hat{i} + 4\,\text{m}\hat{j}$.

b) $\vec{\omega} = 4.33\,\text{rad/s}\hat{i} + 2.5\,\text{rad/s}\hat{j}$, $\vec{r} = 0.5\,\text{m}\hat{i} - 0.87\,\text{m}\hat{j}$.

c) $\vec{\omega} = 2\,\text{rad/s}\hat{i} + 2\,\text{rad/s}\hat{j} + 2.83\,\text{rad/s}\hat{k}$, $\vec{r} = 3.5\,\text{m}\hat{i} + 3.5\,\text{m}\hat{j} - 4.95\,\text{m}\hat{k}$.

8.5 The following questions are about the velocity and acceleration formulae for the non-constant rate circular motion about a fixed axis:

$$\vec{v} = \vec{\omega}\times\vec{r}$$
$$\vec{a} = \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

a) For planar circular motion about the z-axis, take

$$\dot{\vec{\omega}} = \alpha\hat{k}, \quad \vec{\omega} = \omega\hat{k}, \quad \text{and} \quad \vec{r} = R\hat{e}_R$$

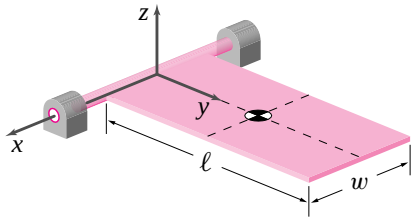
and show that the velocity and acceleration formulae reduce to the formulae from elementary physics: $\vec{v} = \omega R\hat{e}_\theta$ and $\vec{a} = -\omega^2 R\hat{e}_R + R\alpha\hat{e}_\theta$.

b) Draw a schematic picture of a particle (or a point on a rigid body) going in circles about a fixed axis in 3-D. Take two distinct points on the axis and draw vectors $\vec{r}_1$ and $\vec{r}_2$ from these two points to the particle of interest. Show, using the formulae for $\vec{v}$ and $\vec{a}$ above, that the acceleration and velocity of the particle are the same irrespective of which $\vec{r}$ is used and thus conclude that the vector $\vec{r}$ can be taken from any point on the axis of rotation.

8.6 A rectangular plate with width $w = 1\,\text{m}$ and length $\ell = 2\,\text{m}$ is rigidly attached to a shaft along one of its edges. The shaft rotates about the x-axis. The center of mass of the plate goes around at a constant linear speed $v = 2.5\,\text{m/s}$.

a) Find the angular velocity of the plate.

b) At the instant shown, find the velocity of point P.



problem 8.6:

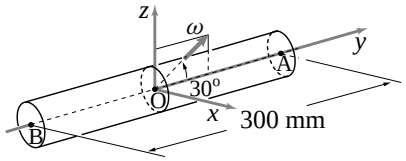
(Filename:pfigure4.3.rp1)

8.7 The solid cylinder shown has an angular velocity $\vec{\omega}$ of magnitude 40 rad/s. The vector $\vec{\omega}$ lies in the yz-plane. The origin O of the coordinate frame is at the center of the cylinder and is at rest. Find the following quantities:

a) The vector $\vec{\omega}$.

b) The velocity $\vec{v}_{A/B}$ of point A relative to point B.

c) The acceleration $\vec{a}_A$ of point A.



problem 8.7:

(Filename:pfigure.blue.37.1)

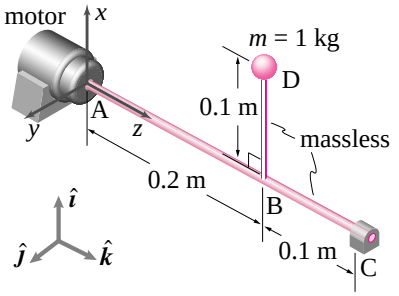
8.8 A rigid body rotates about a fixed axis and has an angular acceleration $\boldsymbol{\alpha} = 3\pi^2(\hat{i} + 1.5\,\hat{j} + 1.5\,\hat{k})\,\text{rad/s}^2$. At a certain instant, the velocity of a point A of the body has x-component and y-component equal to 3 and -4 m/s, respectively. Find the magnitude v_A of the velocity of point A.

8.9 Particle attached to a shaft, no gravity. See also problem 8.10. A shaft of negligible mass spins at a constant rate. The motor at A imposes this rate if needed. The bearing at A prevents translation of its end of the shaft in any direction but causes no torques other than that of the motor (which causes a torque in the $\hat{k}$ direction). The frictionless bearing at C holds that end of the shaft from moving in the $\hat{i}$ and $\hat{j}$ directions but allows slip in the $\hat{k}$ direction. A bar of negligible mass is welded at right angles to the shaft. A small sphere of mass 1 kg is attached to the free end of the bar. There is no gravity. The shaft is spinning counterclockwise (when viewed from the positive $\hat{k}$ direction) at 3 rad/s. In the configuration of interest the rod BD is parallel to the $\hat{i}$ direction.

a) What is the position vector of the mass in the given coordinate system at the instant of interest?

b) What is angular velocity vector of the shaft?

c) What are the velocity and acceleration of the particle at the instant of interest?



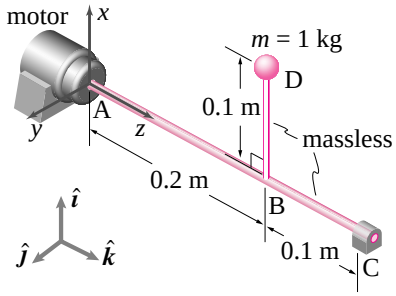
problem 8.9:

(Filename:ch4.9)

8.10 Reconsider the particle on a shaft in problem 8.9.

a) What is the velocity of a point halfway down the rod BD relative to the velocity of the mass at D?

b) What is the acceleration of a point halfway down the rod BD relative to the acceleration of the mass at D?



problem 8.10: Particle attached to a shaft, no gravity

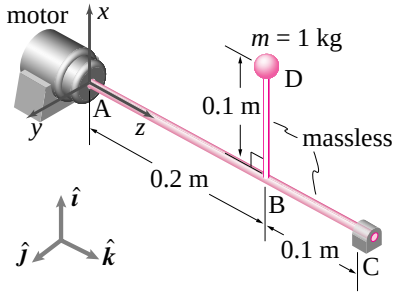
(Filename:pfigure4.particleshaft1.a)

8.11 Particle attached to a shaft, no gravity. See also problem 8.12. A shaft of negligible mass spins at variable rate. The motor at A imposes this rate if needed. The bearing at A prevents translation of its end of the shaft in any direction but causes no torques other than that of the motor (which causes a torque in the $\hat{k}$ direction). The frictionless bearing at C holds that end of the shaft from moving in the $\hat{\imath}$ and $\hat{\jmath}$ directions but allows slip in the $\hat{k}$ direction. The shaft is welded orthogonally to a bar, also of negligible mass, which is attached to a small ball of (non-negligible) mass of 1 kg. There is no gravity. At the instant of interest, the shaft is spinning counterclockwise (when viewed from the positive $\hat{k}$ direction) at $\omega = 3\,\text{rad/s}$ and $\dot{\omega} = 5\,\text{rad/s}^2$. In the configuration of interest the rod BD is parallel with the $\hat{\imath}$ direction.

a) In the coordinate system shown, what is the position vector of the particle at the instant of interest?

b) In the coordinate system shown, what are angular velocity and angular acceleration vectors of the shaft?

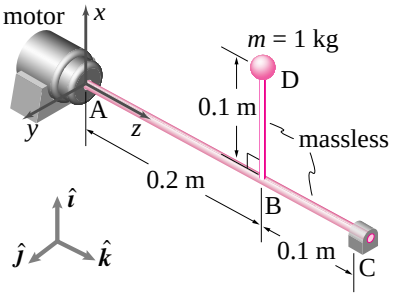
c) In the coordinate system shown, what is the velocity and acceleration of the particle at the instant of interest?



problem 8.11:

(Filename:ch5.3)

8.12 Reconsider the particle on a shaft in problem 8.11. What is the velocity and acceleration of a point halfway down the rod BD relative the velocity and acceleration of the particle?



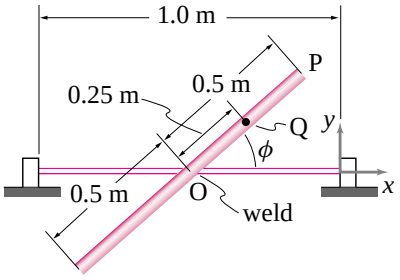
problem 8.12:

(Filename:ch5.3.copy)

8.13 Crooked rod. A one meter long uniform rod is welded at its center at an angle ϕ to a shaft as shown. The shaft is supported by bearings at its ends and spins at a constant rate ω. At the instant of interest, the crooked rod lies in the xy-plane.

a) In the coordinate system shown, what is the position vector of the point P on the rod at the instant of interest? *

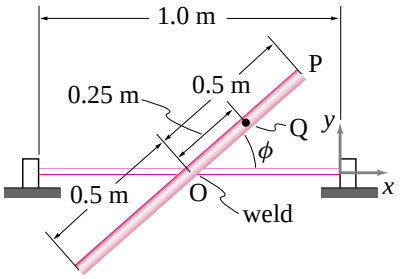
b) In the coordinate system shown, what is the velocity and acceleration of point P at the instant of interest? *



problem 8.13: A crooked rod spins on a shaft.

(Filename:ch4.7)

8.14 Crooked rod, again. Reconsider the crooked rod on a shaft in problem 8.13. What are the velocity and acceleration of point P relative to point Q?



problem 8.14: A crooked rod spins on a shaft.

(Filename:pfigure4.crooked.a)

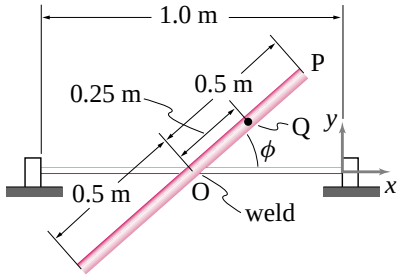
8.15 Crooked rod. A one meter long uniform rod is welded at its center at an angle ϕ to a shaft as shown. The shaft is supported by bearings at its ends and spins at a variable rate. At the instant of interest the shaft is spinning counterclockwise (when viewed from the positive x direction) at angular speed ω with angular acceleration $\dot{\omega}$. At the instant of interest, the crooked rod lies in the xy-plane.

a) In the coordinate system shown, what is the position vector of the point P on the rod?

b) In the coordinate system shown, what are angular velocity and angular acceleration vectors of the rod and shaft?

c) In the coordinate system shown, what is the velocity and acceleration of point P?

d) What is the velocity and acceleration of point P relative to point Q?



problem 8.15: A crooked rod spins on a shaft.

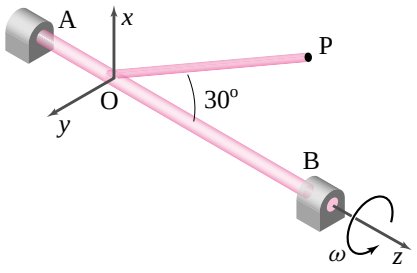
(Filename:ch5.4)

8.16 A rigid rod OP of length $L = 500\,\text{mm}$ is welded to a shaft AB at an angle $\theta = 30°$. The shaft rotates about its longitudinal axis at a constant rate $\omega = 120\,\text{rpm}$. At the instant shown, the rod OP is in the the xz-plane.

a) Neatly draw the path of point P. At the instant shown, draw the basis vectors $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$ and express them in terms of the basis vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

b) Find the radius of the circular path of point P and calculate the velocity and acceleration of the point using planar circular motion formulae $\vec{\boldsymbol{v}} = \dot{\theta}\hat{\boldsymbol{e}}_\theta$ and $\vec{\boldsymbol{a}} = -\dot{\theta}^2 R\hat{\boldsymbol{e}}_r$.

c) Find the position vector $\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{r}}_{P/O}$ of point P. Use this position vector to compute the velocity and acceleration of point P using the general formulae $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$. Show that the answers obtained here are the same as those in (b).



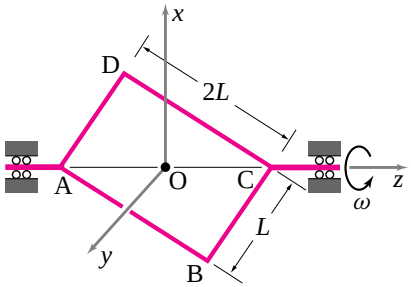
problem 8.16:

(Filename:pfigure4.2.rp1)

8.17 A uniform rectangular plate $ABCD$ of width ℓ and length 2ℓ rotates about its diagonal AC at a constant angular speed $\omega = 5$ rad/s. At the instant shown, the plate is in the xz-plane.

a) Find the velocity and acceleration of point B at the instant shown.

b) Find the velocity and acceleration of point B when the plate is in the yz-plane.



problem 8.17:

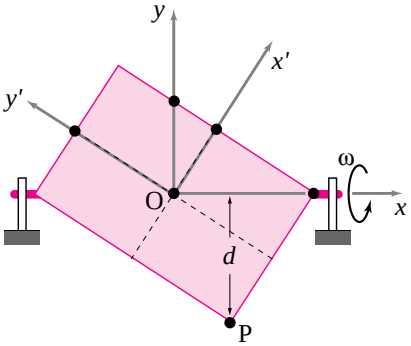
(Filename:pfigure4.2.rp2)

8.18 A rectangular plate of width $w = 100$ mm and length $\ell = 200$ mm rotates about one of its diagonals at a constant angular rate $\omega = 3$ rad/s. There is a fixed coordinate system xyz with the origin at the center of the plate and the x-axis aligned with the axis of rotation. Another set of axes $x'y'z'$, aligned with the principal axes of the plate, has the same origin but is attached to the plate and therefore, rotates with the plate. At the instant of interest, the plate is in the xy-plane and the two axes z and z' coincide. Find the velocity of the corner point P as follows:

a) Write the angular velocity $\vec{\boldsymbol{\omega}}$ of the plate in terms of its components in the two coordinate systems.

b) Find the velocity of point P in both coordinate systems using the expressions for $\vec{\boldsymbol{\omega}}$ found in part (a).

c) Find the perpendicular distance d of point P from the axis of rotation, and show that the linear speed of point P calculated from velocities found in part (b) is ωd in each case.



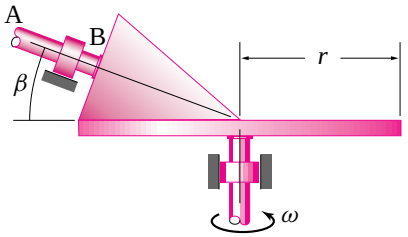
problem 8.18:

(Filename:pfigure4.3.rp11)

8.19 A cone of angle β sits on its side on a horizontal disk of radius $r = 100$ mm as shown in the figure. The cone is attached to shaft AB that turns in the bearing. The disk rotates at a constant angular speed $\omega = 120$ rpm. Assume that the cone rolls without slip on the disk. Thus, the disk drives the cone-and-shaft assembly causing it to turn in the bearing. Find the angular velocity of the cone as a function of the cone angle β. In particular, find the angular velocity of the cone for

a) $\beta = 30^\circ$, and

b) $\beta = 90^\circ$ (or 90° minus a hair if that makes you more comfortable).



problem 8.19:

(Filename:pfigure4.rpb)

8.2 Dynamics of fixed axis rotation

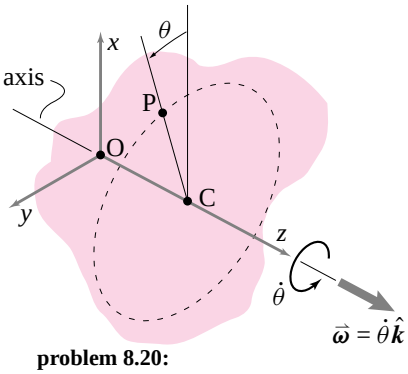
8.20 Let $P(x, y, z)$ be a point mass on an abstract massless body that rotates at a constant rate ω about the z-axis. Let $C(0, 0, z)$ be the center of the circular path that P describes during its motion.

a) Show that the rate of change of angular momentum of mass P about the origin O is given by

$$\dot{\vec{\boldsymbol{H}}}_O = -m\omega^2 Rz\hat{\boldsymbol{e}}_\theta$$

where $R = \sqrt{x^2 + y^2}$ is the radius of the circular path of point P.

b) Is there any point on the axis of rotation (z-axis) about which $\vec{\boldsymbol{H}}$ is zero. Is this point unique?



problem 8.20:

(Filename:pfigure4.2.rp5)

8.21 System of Particles going in a circle. Assume for the problems below that you have two mass points going in circles about a common axis at a fixed common angular rate.

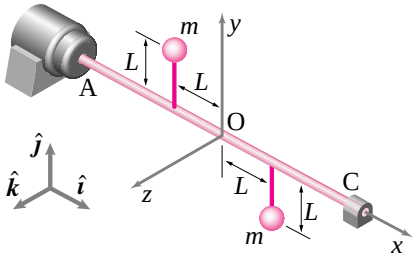
a) 2-D (particles in the plane, axis of rotation normal to the plane). Show, by calculating the sum in $\dot{\mathbf{H}}_C$ that the result is the same when using the two particles or when using a single particle at the center of mass (with mass equal to the sum of the two particles). You may do the general problem or you may pick particular appropriate locations for the masses, particular masses for the masses, a specific point C, a specific axis and a particular rotation rate.

b) 3-D (particles *not* in a common plane normal to the axis of rotation). Using particular mass locations that you choose, show that 2 particles give a *different* value for $\dot{\mathbf{H}}_C$ than a single particle at the center of mass. [Moral: In general, $\dot{\vec{\boldsymbol{H}}}_C \neq \vec{\boldsymbol{r}}_{cm/C} \times m_{tot}\vec{\boldsymbol{a}}_{cm}$.]

8.22 The rotating two-mass system shown in the figure has a constant angular velocity $\vec{\boldsymbol{\omega}} = 12\,\text{rad/s}\hat{\boldsymbol{\imath}}$. Assume that at the instant shown, the masses are in the xy-plane. Take $m = 1$ kg and $L = 1$ m.

a) Find the linear momentum $\vec{\boldsymbol{L}}$ and its rate of change $\dot{\vec{\boldsymbol{L}}}$ for this system.

b) Find the angular momentum $\vec{\boldsymbol{H}}_O$ about point O and its rate of change $\dot{\vec{\boldsymbol{H}}}_O$ for this system.



problem 8.22:

(Filename:pfigure4.2.rp4)

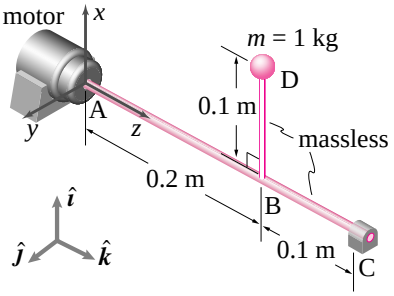
8.23 Particle attached to a shaft, no gravity. A shaft of negligible mass spins at constant rate. The motor at A imposes this rate if needed. The bearing at A prevents translation of its end of the shaft in any direction but causes no torques other than that of the motor (which causes a torque in the $\hat{\boldsymbol{k}}$ direction). The frictionless bearing at C holds that end of the shaft from moving in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions but allows slip in the $\hat{\boldsymbol{k}}$ direction. The shaft is welded orthogonally to a bar, also of negligible mass, which is attached to a small ball of(non-negligible) mass of 1 kg. There is no gravity. The shaft is spinning counterclockwise (when viewed from the positive $\hat{\boldsymbol{k}}$ direction) at 3 rad/s. In the configuration of interest the rod BD is parallel with the $\hat{\boldsymbol{\imath}}$ direction.

a) What is the tension in the rod BD? *

b) What are all the reaction forces at A and C? *

c) What is the torque that the motor at A causes on the shaft? *

d) There are two ways to do problems (b) and (c): one is by treating the shaft, bar and mass as one system and writing the momentum balance equations. The other is to use the result of (a) with action-reaction to do statics on the shaft BC. Repeat problems (b) and (c) doing it the way you did not do it the first time.



problem 8.23:

(Filename:pfigure.s94h7p1)

8.24 Particle attached to a shaft, with gravity. Consider the same situation as in problem 8.23 with the following changes: There is gravity (pointing in the $-\mathbf{i}$ direction). At the time of interest the rod BD is parallel to the $+\mathbf{j}$ direction.

a) What is the force on D from the rod BD? (Note that the rod BD is *not* a 'two-force' member. Why not?)

b) What are all the reaction forces at A and C?

c) What is the torque that the motor at A causes on the shaft?

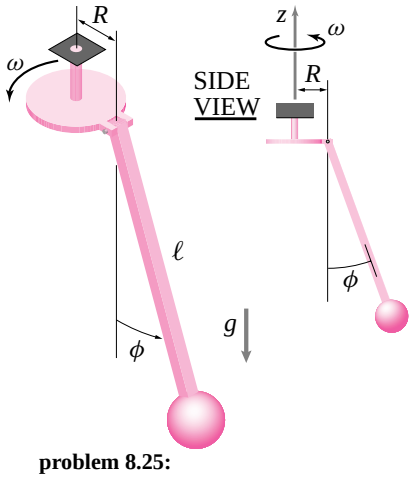
d) Do problems (b) and (c) again a different way.

8.25 A turntable rotates at the constant rate of $\omega = 2\,\text{rad/s}$ about the z axis. At the edge of the 2 m radius turntable swings a pendulum which makes an angle ϕ with the negative z direction. The pendulum length is $\ell = 10\,\text{m}$. The pendulum consists of a massless rod with a point mass $m = 2\,\text{kg}$ at the end. The gravitational constant is $g = 10\,\text{m/s}^2$ pointing in the $-z$ direction. A perspective view and a side view are shown.

a) What possible angle(s) could the pendulum make in a steady motion. In other words, find ϕ for steady circular motion of the system. It turns out that this quest leads to a transcendental equation. You are not asked to solve this equation but you should have the equation clearly defined. *

b) How many solutions are there? [hint, sketch the functions involved in your final equation.] *

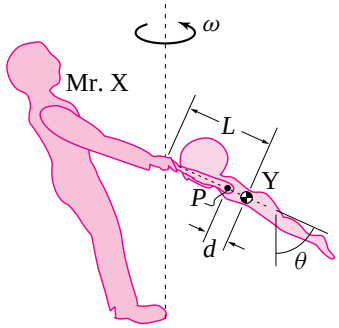
c) Sketch very approximately any equilibrium solution(s) not already sketched on the side view in the picture provided.



problem 8.25:

(Filename:pfigure.s95q6)

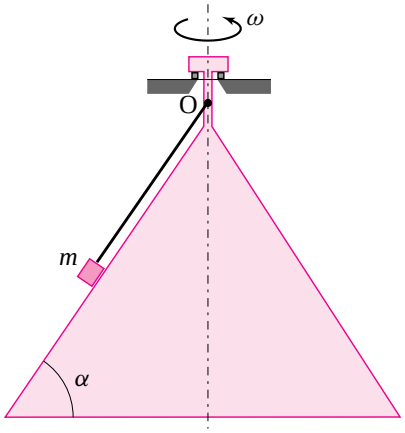
8.26 Mr. X swings his daughter Y about the vertical axis at a constant rate $\omega = 1\,\text{rad/s}$. Y is merely one year old and weighs 12 kg. Assuming Y to be a point mass located at the center of mass, estimate the force on each of her shoulder joints.



problem 8.26:

(Filename:pfigure4.2.rp12)

8.27 A block of mass $m = 2\,\text{kg}$ supported by a light inextensible string at point O sits on the surface of a rotating cone as shown in the figure. The cone rotates about its vertical axis of symmetry at constant rate ω. Find the value of ω in rpm at which the block loses contact with the cone.



problem 8.27:

(Filename:pfigure4.2.rp11)

8.28 A small rock swings in circles at constant angular rate ω. It is hung by a string with length ℓ the other end of which is at the origin of a good coordinate system. The rock has mass m and the gravitational constant is g. The string makes an angle of ϕ with the negative z axis. The mass rotates so that if you follow it with the fingers of your right hand, your right thumb points up the z-axis. At the moment of interest the mass is passing through the yz-plane.

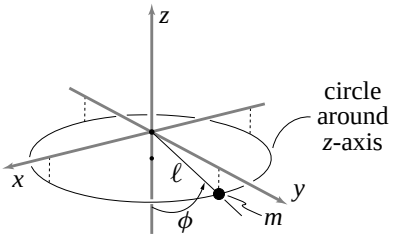
a) Draw a free body diagram of the rock.

b) How long does it take to make one revolution? How does this compare with the period of oscillation of a simple pendulum. *

c) Find the following quantities in any order that pleases you. All solutions should be in terms of m, g, ℓ, ϕ, ω and the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$.

- The tension in the string (T).
- The velocity of the rock ($\vec{\boldsymbol{v}}$).
- The acceleration of the rock ($\vec{\boldsymbol{a}}$).
- The rate of change of the angular momentum about the $+x$ axis (dH_x/dt).

*

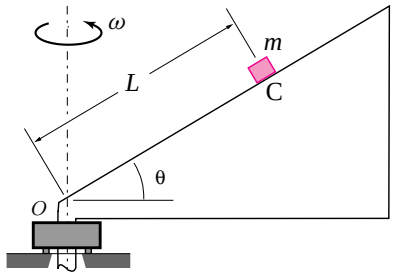


problem 8.28:

(Filename:pfigure.blue.14.2)

8.29 Consider the configuration in problem 8.28. What is the relation between the angle (measured from the vertical) at which it hangs and the speed at which it moves? Do an experiment with a string, weight and stop watch to check your calculation. *

8.30 A small block of mass $m = 5\,\text{kg}$ sits on an inclined plane which is supported by a bearing at point O as shown in the figure. At the instant of interest, the block is at a distance $L = 0.5\,\text{m}$ from where the surface of the incline intersects the axis of rotation. By experiments, the limiting rate of rotation of the turntable before the block starts sliding up the inclined plane is found to be $\omega = 125\,\text{rpm}$. Find the coefficient of friction between the block and the incline.



problem 8.30:

(Filename:pfigure4.2.rp9)

8.31 Particle sliding in circles in a parabolic bowl. As if in a James Bond adventure in a big slippery radar bowl, a particle-like human with mass m is sliding around in circles at speed v. The equation describing the bowl is $z = CR^2 = C(x^2 + y^2)$.

a) Find v in terms of any or all of R, g, and C.

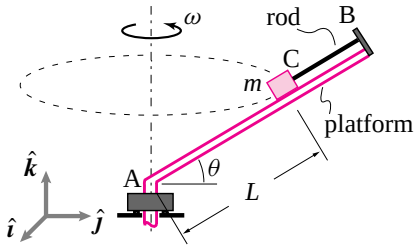
b) Now say you are given ω, C and g. Find v and R if you can. Explain any oddities.



problem 8.31:

(Filename:p.f96.p2.1)

8.32 Mass on a rotating inclined platform. A block of mass $m = 5$ kg is held on an inclined platform by a rod BC as shown in the figure. The platform rotates at a constant angular rate ω. The coefficient of friction between the block and the platform is μ. $\theta = 30^o$, $L = 1$ m, and $g = 9.81\,\text{m/s}^2$. (a) Assuming $\omega = 5$ rad/s and $\mu = 0$ find the tension in the rod. (b) If $\mu = .3$ what is the range of ω for which the block would stay on the ramp without slipping even if there were no rod? *



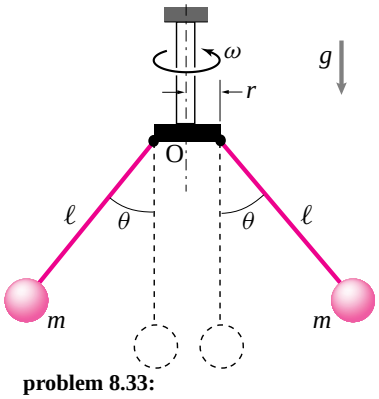
problem 8.32:

(Filename:pfigure.s94h5p6)

8.33 A circular plate of radius $r = 10$ cm rotates in the horizontal plane about the vertical axis passing through its center O. Two identical point masses hang from two points on the plate diametrically opposite to each other as shown in the figure. At a constant angular speed ω, the two masses maintain a steady state angle θ from their vertical static equilibrium.

a) Find the linear speed and the magnitude of the centripetal acceleration of either of the two masses.

b) Find the steady state ω as a function of θ. Do the limiting values of θ, $\theta \to 0$ and $\theta \to \pi/2$ give sensible values of ω?



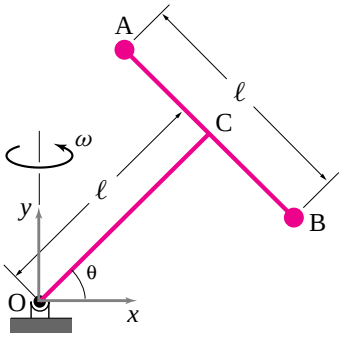
problem 8.33:

(Filename:pfigure4.2.rp6)

8.34 For the rod and mass system shown in the figure, assume that $\theta = 60^\circ$, $\omega \equiv \dot\theta = 5$ rad/s, and $\ell = 1$ m. Find the acceleration of mass B at the instant shown by

a) calculating $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ where $\vec{\boldsymbol{r}}$ is the position vector of point B with respect to point O, i.e., $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{B/O}$, and by

b) calculating $\vec{\boldsymbol{a}} = -R\dot\theta^2\hat{\boldsymbol{e}}_R$ where R is the radius of the circular path of mass B.



problem 8.34:

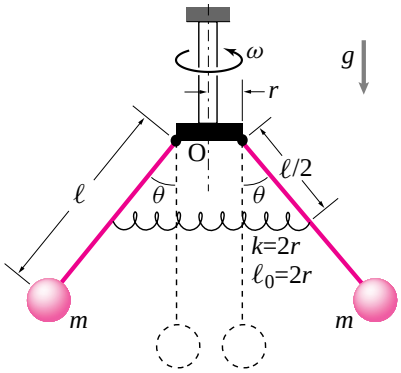
(Filename:pfigure4.2.rp7)

8.35 A spring of relaxed length ℓ_0 (same as the diameter of the plate) is attached to the mid point of the rods supporting the two hanging masses.

a) Find the steady angle θ as a function of the plate's rotation rate ω and the spring constant k. (Your answer may include other given constants.)

b) Verify that, for $k = 0$, your answer for θ reduces to that in problem 8.33.

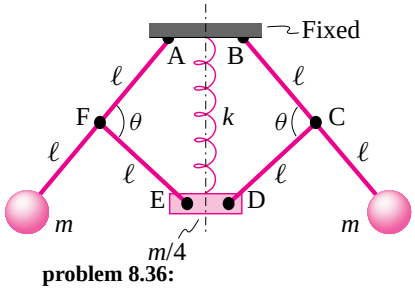
c) Assume that the entire system sits inside a cylinder of radius 20 cm and the axis of rotation is aligned with the longitudinal axis of the cylinder. The maximum operating speed of the rotating shaft is 200 rpm (i.e., $\omega_{max} =$ 200 rpm). If the two masses are not to touch the walls of the cylinder, find the required spring stiffness k.



problem 8.35:

(Filename:pfigure4.2.rp8)

8.36 Flyball governors are used to control the flow of working fluid in steam engines, diesel engines, steam turbines, etc. A spring-loaded flyball governor is shown schematically in the figure. As the governor rotates about the vertical axis, the two masses tend to move outwards. The rigid and massless links that connect the outer arms to the collar, in turn, try to lift the central collar (mass $m/4$) against the spring. The central collar can move along the vertical axis but the top of the governor remains fixed. The mass of each ball is $m = 5$ kg, and the length of each link is $l = 0.25$ m. There are frictionless hinges at points A, B, C, D, E, F where the links are connected. Find the amount the string is elongated, Δx, when the governor rotates at 100 rpm, given that the spring constant $k = 500$ N/m, and that in this steady state, $\theta = 90^o$. *



problem 8.36:

(Filename:summer95p2.2)

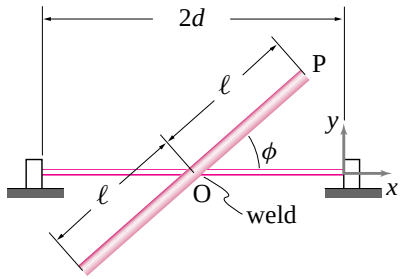
8.37 Crooked rod. A 2ℓ long uniform rod of mass m is welded at its center to a shaft at an angle ϕ as shown. The shaft is supported by bearings at its ends and spins at a constant rate ω.

a) What are $\dot{\vec{\boldsymbol{H}}}_O$ and $\dot{\vec{\boldsymbol{L}}}$?

b) Calculate $\dot{\mathbf{H}}_{/O}$ using the integral definition of angular momentum. *

c) What are the six reaction components if there is no gravity? *

d) Repeat the calculations, including gravity g. Which reactions change and by how much? *



problem 8.37:

(Filename:h6.4)

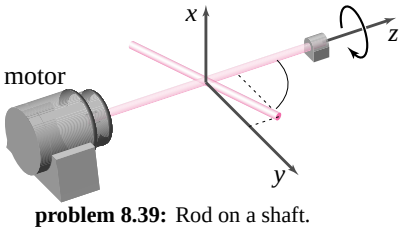
8.38 For the configuration in problem 8.37, find two positions along the rod where concentrating one-half the mass at each of these positions gives the same $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$.

8.39 Rod on a shaft. A uniform rod is welded to a shaft which is connected to frictionless bearings. The rod has mass m and length ℓ and is attached at an angle γ at its midpoint to the midpoint of the shaft. The length of the shaft is $2d$. A motor applies a torque M_{motor} along the axis of the shaft. At the instant shown in the figure, assume m, ℓ, d, ω and M_{motor} are known and the rod lies in the yz-plane.

a) Find the angular acceleration of the shaft.

b) The moment that the shaft applies to the rod.

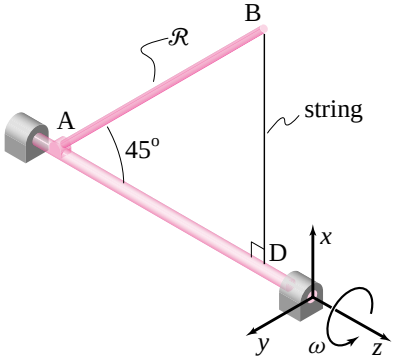
c) The reaction forces at the bearings.



problem 8.39: Rod on a shaft.

(Filename:pg79.2)

8.40 Rod spins on a shaft. A uniform rod $\mathcal{R}$ with length R and mass m spins at constant rate ω about the z axis. It is held by a hinge at A and a string at B. Neglect gravity. Find the tension in the string in terms of ω, R and m. *



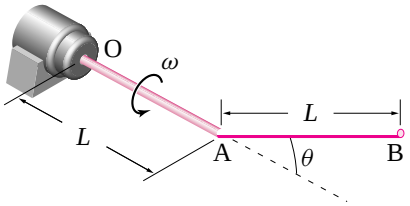
problem 8.40:

(Filename:pfigure.s95f4)

8.41 A uniform narrow shaft of length $2L$ and mass $2M$ is bent at an angle θ half way down its length. It is spun at constant rate ω.

a) What is the magnitude of the force that must be applied at O to keep the shaft spinning?

b) (harder) What is the magnitude of the moment that must be applied at O to keep the shaft spinning?



problem 8.41: Spinning rod. No gravity.

(Filename:f.4)

8.42 For the configuration in problem 8.41, find two positions along the bent portion of the shaft where concentrating one-half its mass at each of these positions gives the same $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$ and $\dot{\vec{\boldsymbol{L}}}$.

8.43 Rod attached to a shaft. A uniform 1 kg, $\sqrt{2}$ m rod is attached to a shaft which spins at a constant rate of 1 rad/s. One end of the rod is connected to the shaft by a ball-and-socket joint . The other end is held by two strings (or pin jointed rods) that are connected to a rigid cross bar. The cross bar is welded to the main shaft. Ignore gravity.

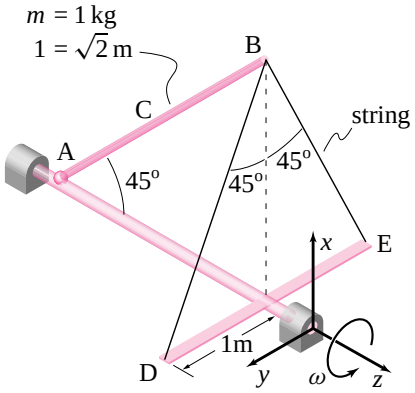
a) Draw a free body diagram of the rod AB, including a bit of the strings.

b) Find the tension in the string DB. Two approaches:

(i) Write the six equations of linear and angular momentum balance (about any point of your choosing) and solve them (perhaps using a computer) for the tension T_{DB}.

(ii) Write the equation of angular momentum balance about the axis AE. (That is, take the equation of angular momentum balance about point A and dot both sides with a vector in the direction of $\boldsymbol{r}_{E/A}$.) Use this calculation to find T_{DB} and compare to method ii.

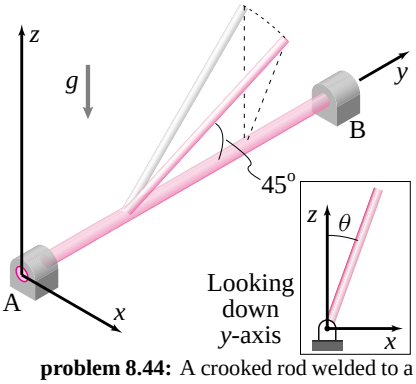
c) Find the reaction **force** (a vector) at A.



problem 8.43: Uniform rod attached to a shaft.

(Filename:pg89.1)

8.44 3D, crooked bar on shaft falls down. A massless, rigid shaft is connected to frictionless hinges at A and B. A bar with mass M_b and length ℓ is welded to the shaft at 45^o. At time $t = 0$ the assembly is tipped a small angle θ_0 from the yz-plane but has no angular velocity. It then falls under the action of gravity. What is θ at $t = t_1$ (assuming t_1 is small enough so that θ_1 is much less than 1)?

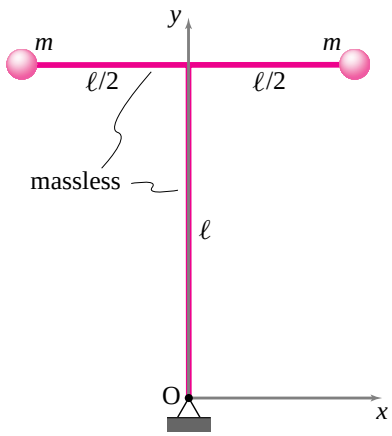


problem 8.44: A crooked rod welded to a shaft falls down.

(Filename:p2.4)

8.3 Moment of inertia matrices: $[I^{\mathrm{cm}}]$ and $[I^{\mathrm{O}}]$

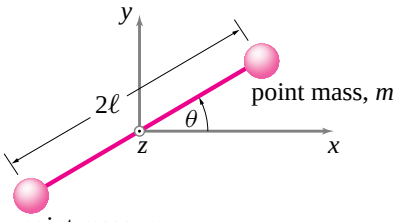
8.45 Find the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$ of the system shown in the figure, given that $m = 0.25\,\mathrm{kg}$ and $\ell = 0.5\,\mathrm{m}$. Which components in the matrix $[\boldsymbol{I}^{\mathrm{O}}]$ will change if one mass is removed and the other one is doubled?



problem 8.45:

(Filename:pfigure4.4.rp12)

8.46 A dumbbell (two masses connected by a massless rod of length ℓ) lies in the xy plane. Compute the moments and products of inertia of this body about the x, y, and z axes. What are the principal axes of this body?



problem 8.46:

(Filename:pfigure.blue.38.2)

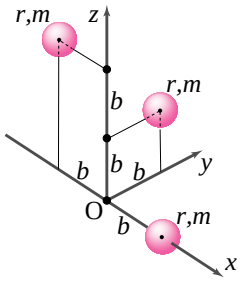
8.47 Equivalent dumbbell. It is a great fact that, for the purposes of dynamics, every rigid body can be replaced by a massless rigid structure holding six masses. The structure looks like a child's jack. It has three orthogonal bars with three pairs of point masses along the bars. The center of the jack is at the center of mass. The bars are in the directions of the eigenvectors of the moment of inertia matrix of the body (the principal directions). For flat objects one only needs four point masses. Where should one put four masses on a massless disk to make it equivalent to a uniform and 'massful' disk in problem 8.50(a). That is, what set of four masses has the same total mass, the same center of mass location, and the same moment of inertia matrix as a uniform disk?

8.48 Three identical solid spheres of mass m and radius r are situated at $(b, 0, 0)$, $(0, b, b)$, and $(-b, 0, 2b)$ relative to the origin O. Find all nine components of $[\boldsymbol{I}^{\mathrm{O}}]$ of the system of three spheres.

Hint: $[\boldsymbol{I}^{\mathrm{cm}}]$ for each sphere is

$$\begin{bmatrix} \frac{2}{5}mr^2 & 0 & 0 \\ 0 & \frac{2}{5}mr^2 & 0 \\ 0 & 0 & \frac{2}{5}mr^2 \end{bmatrix}.$$

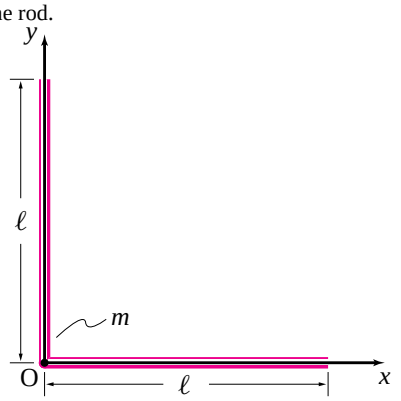
Use the parallel axis theorem.



problem 8.48:

(Filename:pfigure.blue.49.2)

8.49 Find the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$ for the 'L'-shaped rod shown in the figure. The mass of the rod is $m = 1.5\,\mathrm{kg}$ and each leg is $\ell = 0.4\,\mathrm{m}$ long. You may ignore the width of the rod.



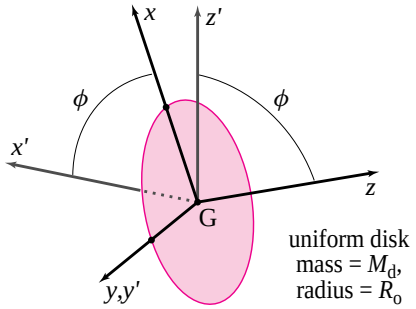
problem 8.49:

(Filename:pfigure4.4.rp13)

8.50 Moment of Inertia Matrix. This problem concerns a uniform flat disk of radius R_0 with mass $M_d = M_{disc}$. In the two parts of this problem, you are to calculate the moment of inertia matrix for the disk. In the first part, do it by direct integration and, in the second part, by using a change of basis.

a) Using a coordinate system that is lined up with the disk so that the z axis is normal to the disk and the origin of the coordinate system is at the center of mass (the center) of the disk, calculate the moment of inertia matrix for the disk. [Hint: use polar coordinates, and use $dm = (M/\pi R_O^2)\,dA$, and $dA = R\,dR\,d\theta$.]

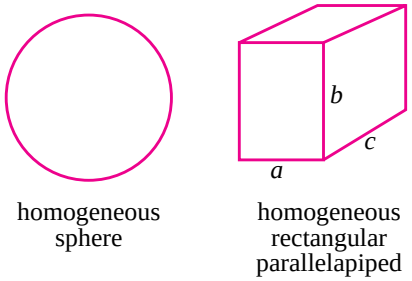
b) Use a coordinate system $x'y'z'$ aligned as follows: the origin of the coordinate system is at the center of the disk, the z' axis makes an angle ϕ with the z axis which is normal to the plane of the (measured from the z axis towards the z' axis), the disk intersects the $x'y'$ plane along the y' axis. The y axis and the y' axis are coincident. Calculate the moment of inertia matrix for the disk. Alternate method: use the solution from (a) above with a change of basis.



problem 8.50:

(Filename:f92h7p1)

8.51 For each of the shapes below, mark the approximate location of the center of mass and the principal axes about the center of mass



homogeneous sphere

homogeneous rectangular parallelapiped



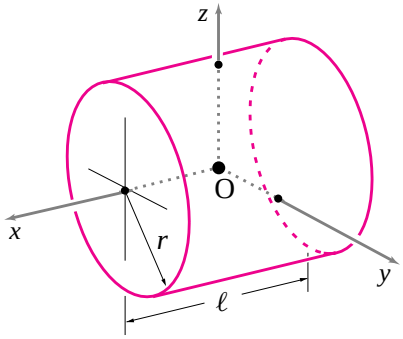
massless rod holding two equal masses

capital "J"

problem 8.51:

(Filename:pfigure.blue.52.2)

8.52 Find the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{cm}}]$ for the uniform solid cylinder of length ℓ and radius r shown in the figure.

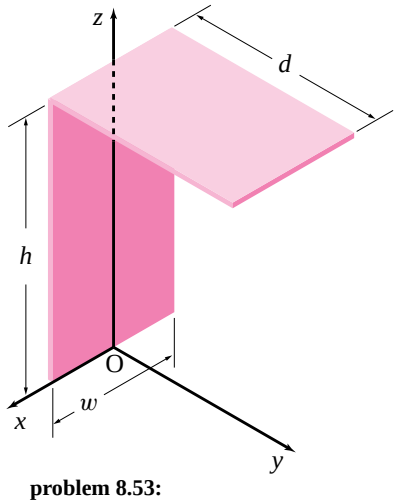


problem 8.52:

(Filename:pfigure4.4.rp16)

8.53 An 'L'-shaped structure consists of two thin uniform rectangular plates each of width $w = 200\,\mathrm{mm}$. The vertical plate has height

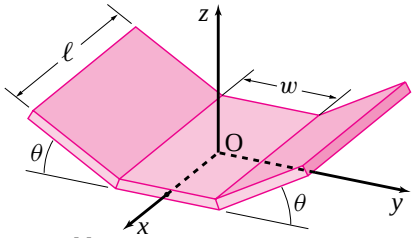
$h = 400$ mm and the horizontal plate has depth $d = 300$ mm. The plates are welded at right angles along the side measuring 200 mm. The total mass of the structure is $m = 0.7$ kg. Find the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$.



problem 8.53:

(Filename:pfigure4.4.rp14)

8.54 A solar panel consists of three rectangular plates each of mass $m = 1.2$ kg, length $\ell = 1$ and width $w = 0.5$ m The two side panels are inclined at 30° with the plane of the middle panel. Evaluate all components of the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{O}}]$ of the solar panel.



problem 8.54:

(Filename:pfigure4.4.rp15)

8.4 Mechanics using $[I^{\mathrm{cm}}]$ and $[I^{\mathrm{O}}]$

8.55 Find the angle between $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{H}} = [\boldsymbol{I}^{\mathrm{G}}]\vec{\boldsymbol{\omega}}$, if $\vec{\boldsymbol{\omega}} = 3\,\mathrm{rad/s}\hat{\boldsymbol{k}}$ and

$$[\boldsymbol{I}^{\mathrm{G}}] = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 40 \end{bmatrix} \mathrm{kg\cdot m^2}.$$

8.56 Find the projection of $\vec{\boldsymbol{H}} = [\boldsymbol{I}^{\mathrm{G}}]\vec{\boldsymbol{\omega}}$ in the direction of $\vec{\boldsymbol{\omega}}$, if $\vec{\boldsymbol{\omega}} = (2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}})$ rad/s and

$$[\boldsymbol{I}^{\mathrm{G}}] = \begin{bmatrix} 5.4 & 0 & 0 \\ 0 & 5.4 & 0 \\ 0 & 0 & 10.8 \end{bmatrix} \mathrm{lbm\,ft^2}.$$

8.57 Compute $\vec{\boldsymbol{\omega}} \times ([\boldsymbol{I}^{\mathrm{G}}]\cdot\vec{\boldsymbol{\omega}})$ for the following two cases. In each case $\vec{\boldsymbol{\omega}} = 2.5\,\mathrm{rad/s}\hat{\boldsymbol{\jmath}}$.

a) $[\boldsymbol{I}^{\mathrm{G}}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ kg m^2, and

*

b) $[\boldsymbol{I}^{\mathrm{G}}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & -4 \\ 0 & -4 & 8 \end{bmatrix}$ kg m^2. *

8.58 Find the z component of $\vec{\boldsymbol{H}} = [\boldsymbol{I}^{\mathrm{G}}]\vec{\boldsymbol{\omega}}$ for $\vec{\boldsymbol{\omega}} = 2\,\mathrm{rad/s}\hat{\boldsymbol{k}}$ and

$$[\boldsymbol{I}^{\mathrm{G}}] = \begin{bmatrix} 10.20 & -2.56 & 0 \\ -2.56 & 10.20 & 0 \\ 0 & 0 & 14.00 \end{bmatrix} \mathrm{lbm\,ft^2}.$$

8.59 For $\vec{\boldsymbol{\omega}} = 5\,\mathrm{rad/s}\hat{\boldsymbol{\lambda}}$, find $\hat{\boldsymbol{\lambda}}$ such that the angle between $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{H}} = [\boldsymbol{I}^{\mathrm{G}}]\cdot\vec{\boldsymbol{\omega}}$ is zero if

$$[\boldsymbol{I}^{\mathrm{G}}] = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 6 & 0 \\ 2 & 0 & 3 \end{bmatrix} \mathrm{kg\,m^2}.$$

8.60 Calculation of $\vec{\boldsymbol{H}}_{\mathrm{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$. You will consider a disc spinning about various axes through the center of the disc O. For the most part, you are concerned with calculating $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$. But, you should keep in mind that the purpose of the calculation might be to calculate reaction forces and moments. Every point on the disk moves in circles about the same axis at the same number of radians per second. These problems can be done at least three different ways: using the[I]'s from problem 8.50, direct integration of the definition of $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$, and using the equivalent dumbbell (with the definition of $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$). You should do them all three ways, first using the way you are most comfortable with.

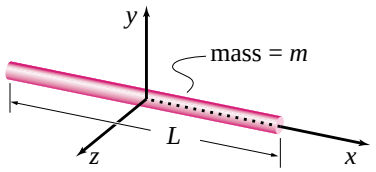
a) Assume that the configuration is as in problem 8.50(a) and that the disk is spinning about the z axis at the constant rate of $\omega\mathbf{k}$. What are $\vec{\boldsymbol{H}}_{\mathrm{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$?

b) Assume that the configuration is as in problem 8.50(b) and that the disk is spinning about the z axis at the constant rate of $\omega\mathbf{k}$. What are $\vec{\boldsymbol{H}}_{\mathrm{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$? (In this problem there are two ways to use [I]: one is in the xyz coordinate system, the other is in a coordinate system aligned nicely with the disk.)

8.61 What is the angular momentum $\vec{\boldsymbol{H}}$ for:

a) A uniform sphere rotating with angular velocity $\vec{\boldsymbol{\omega}}$ (radius R, mass m)?

b) A rod shown in the figure rotating about its center of mass with these angular velocities: $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{\omega}} = \omega(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})/\sqrt{2}$?



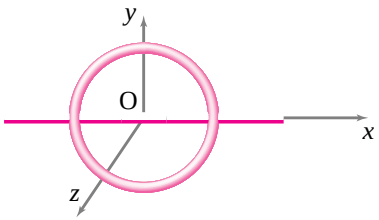
problem 8.61:

(Filename:pfigure.blue.99.1)

8.62 A hoop is welded to a long rigid massless shaft which lies on a diameter of the hoop. The shaft is parallel to the x-axis. At the moment of interest the hoop is in the xy-plane of a coordinate system which has its origin O at the center of the hoop. The hoop spins about the x-axis at 1 rad/s. There is no gravity.

a) What is $[I_G]$ of the hoop?

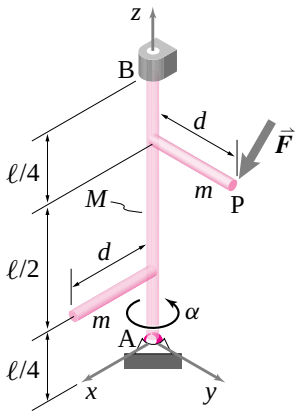
b) What is $\vec{\boldsymbol{H}}_{\mathrm{O}}$ of the hoop?



problem 8.62:

(Filename:pfigure.blue.47.2)

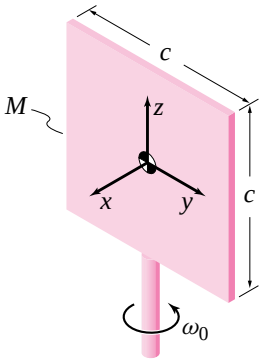
8.63 Two thin rods, each of mass m and length d, are welded perpendicularly to an axle of mass M and length ℓ, which is supported by a ball-and-socket joint at A and a journal bearing at B. A force $\vec{\boldsymbol{F}} = -F\hat{\boldsymbol{\imath}} + G\hat{\boldsymbol{k}}$ acts at the point P as shown in the figure. Determine the initial angular acceleration α and initial reactions at point B.



problem 8.63:

(Filename:pfigure.blue.89.1)

8.64 The thin square plate of mass M and side c is mounted vertically on a vertical shaft which rotates with angular speed ω_0. In the fixed x, y, z coordinate system with origin at the center of mass (with which the plate is currently aligned) what are the vector components of $\vec{\boldsymbol{\omega}}$? What are the components of $\vec{\boldsymbol{H}}_{cm}$?



problem 8.64:

(Filename:pfigure.blue.35.2)

8.65 If you take any rigid body (with weight), skewer it with an axle, hold the axle with hinges that cause no moment about the axle, and hold it so that it cannot slide along the axle, it will swing like a pendulum. That is, it will obey the equations:

$$\ddot{\theta} = -C\sin\theta, \quad \text{or} \quad (a)$$
$$\ddot{\phi} = +C\sin\phi, \quad (b)$$

where

θ is the amount of rotation about the axis measuring zero when the center of mass is directly below the z'-axis; i. e., when the center of mass is in a plane containing the z'-axis and the gravity vector,

ϕ is the amount of rotation about the axis measuring zero when the center of mass is directly above the z'-axis,

$C = M_{total}\, gd \sin\gamma / I_{z'z'}$,

M_{total} is the total mass of the body,

g is the gravitational constant,

d is the perpendicular distance from the axis to the center of mass of the object,

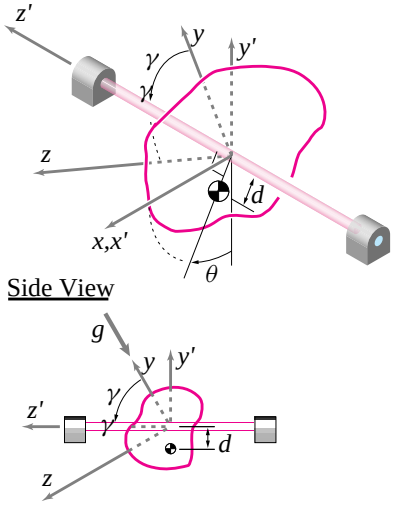
γ is the angle of tip of the axis from the vertical,

$I_{z'z'}$ is the moment of inertia of the object about the skewer axis.

a) How many examples of such pendula can you think of and what would be the values of the constants? (e.g. a simple pendulum?, a broom upside down on your hand?, a door?, a box tipping on one edge?, a bicycler stopped at a red light and stuck in her new toe clips?, a person who got *rigor mortis* and whose ankles went totally limp at the same time?,...)

b) A door with frictionless hinges is misaligned by half an inch (the top hinge is half an inch away from the vertical line above the other hinge). Estimate the period with which it will swing back and forth. Use reasonable magnitudes of any quantities you need.

c) Can you verify the formula (a simple pendulum and a swinging stick are special cases)?



problem 8.65: 3-D pendulum on a hinge.

(Filename:pfigure4.3Dpend)

8.66 A hanging rod going in circles. A uniform rod with length ℓ and mass m hangs from a well greased ball and socket joint at O. It swings in circles at constant rate $\vec{\omega} = \omega\boldsymbol{k}$ while the rod makes an angle ϕ with the vertical (so it sweeps a cone).

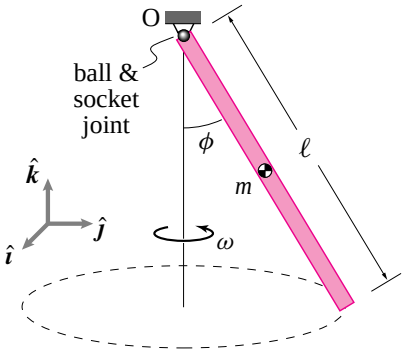
a) Calculate $\dot{\vec{H}}_O$ and $\ddot{\vec{H}}_O$.

b) Using $\ddot{\vec{H}}_O$ and angular momentum balance, find the rotation rate ω in terms of ϕ, g, ℓ, and m. How long does it take to make one revolution? *

c) If another stick rotates twice as fast while making the same angle ϕ, what must be its length compared to ℓ?

d) Is there a simple way to describe what two uniform sticks of different length and different cone angle ϕ both have in common if their rate of rotation ω is the same? (If you accurately draw two such sticks, you should see the relation.)

e) Find the reaction force at the ball and socket joint. Is this force parallel to the stick?



problem 8.66:

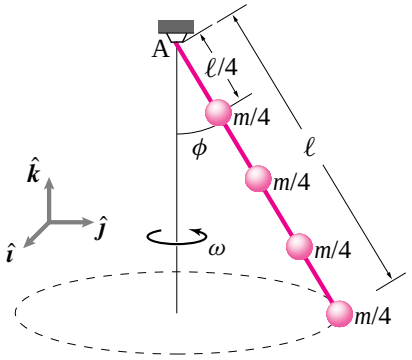
(Filename:pfigure.s94h6p3)

8.67 From discrete to continuous. n equal beads of mass m/n each are glued to a rigid massless rod at equal distances $l = L/n$ from each other. For $n = 4$, the system is shown in the figure, but you are to consider the general problem with n masses. The rod swings around the vertical axis maintaining angle ϕ.

a) Find the rate of rotation of the system as a function of g, L, n, and ϕ for constant rate circular motion. [Hint: You may need these series sums: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ and $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.] *

b) Set $n = 1$ in the expression derived in (a) and check the value with the rate of rotation of a *conical pendulum* of mass m and length ℓ.

c) Now, put $n \to \infty$ in the expression obtained in part (a). Does this rate of rotation make sense to you? [Hint: How does it compare with the rate of rotation for the uniform stick of Problem 8.66?]



problem 8.67:

(Filename:pfigure.s94h6p4)

8.68 Spinning dumbbell. Two point masses are connected by a rigid rod of negligible mass and length $2\sqrt{2}L$. The rod is welded to a shaft that is along the y axis. The shaft is spinning at a constant rate ω driven by a motor at A (Not shown). At the time of interest the dumbbell is in the yz-plane. Neglect gravity.

a) What is the moment of inertia matrix about point O for the dumbbell at the instant shown? *

b) What is the angular momentum of the dumbbell at the instant shown about point O. *

c) What is the rate of change of angular momentum of the dumbbell at the instant shown? Calculate the rate of change of angular momentum two different ways:

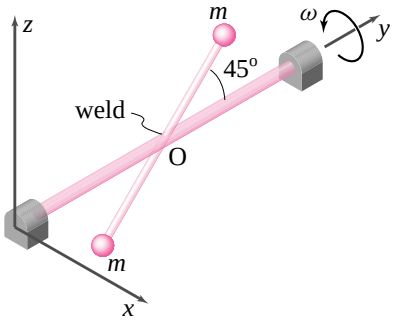
(i) by adding up the contribution of both of the masses to the rate of change of angular momentum (i.e. $\dot{\vec{H}} = \sum \vec{r}_i \times (m_i \vec{a}_i)$) and

(ii) by calculating I_{xy} and I_{yz}, calculating $\vec{H}$, and then calculating $\dot{\vec{H}}$ from $\vec{\omega} \times \vec{H}$. *

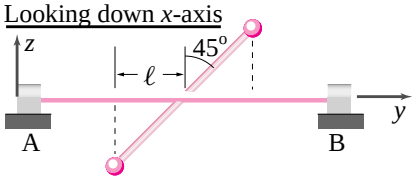
d) What is the total force and moment (at the CM of the dumbbell) required to keep this motion going? Do the calculation in the following ways:

(i) Using angular momentum balance (about any point of your choice) for the shaft dumbbell system.

(ii) By drawing free body diagrams of the masses alone and calculating the forces on the masses. Then use action and reaction and draw a free body diagram of the remaining (massless) part of the shaft-dumbbell system. Then use moment balance for this system. *



Looking down *x*-axis



problem 8.68: Crooked spinning dumbbell.

(Filename:pg84.3)

8.69 A uniform rectangular plate of mass m, height h, and width b spins about its diagonal at a constant angular speed ω, as shown in the figure. The plate is supported by frictionless bearings at O and C. The plate lies in the YZ-plane at the instant shown.

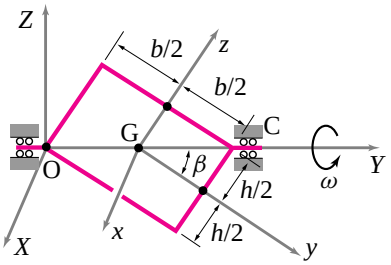
(Note: $\cos(\beta) = b/\sqrt{(h^2+b^2)}$, $\sin(\beta) = h/\sqrt{(h^2+b^2)}$)

a) Compute the angular momentum of the plate about G (with respect to the body axis $Gxyz$ shown).

b) Ignoring gravity determine the net torque (moment) that the supports impose on the body at the instant shown. (Express your solution using base vectors and components associated with the inertial XYZ frame.)

c) Calculate the reactions at O or C for the configuration shown. *

d) In the case when $h = b = 1\,\text{m}$ what is the vertical reaction at C (still neglecting gravity).



OXYZ - Fixed Frame
(base vectors, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$)

Gxyz - Moving "Body" Frame
(base vectors, $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$, $\hat{\boldsymbol{e}}_z$)

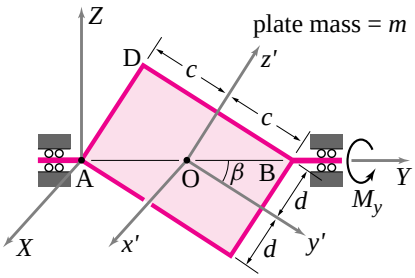
problem 8.69:

(Filename:pfigure.blue.36.1)

8.70 A uniform rectangular plate spins about a fixed axis. The constant torque M_y is applied about the Y axis. The plate is held by bearings at the corners of the plate at A and B. At the instant shown the plate and the $y'z'$ axis attached to the plate are in the YZ plane. Neglect gravity. At $t = 0$ the plate is at rest.

a) At $t = 0^+$ (i.e., *just* after the start), what is the acceleration of point D? Give your answer in terms of any or all of c, d, m, M_y, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ (the base vectors aligned with the fixed XYZ axes), and I^{cm}_{YY}
[You may use I^{cm}_{YY} in your answer. Or you could use this result from the change of coordinates formulae in linear algebra: $I^{cm}_{YY} = (\cos^2\beta) I^{cm}_{y'y'} + (\sin^2\beta) I^{cm}_{z'z'}$ to get an answer in terms of c, d, m, and M_y.]
*

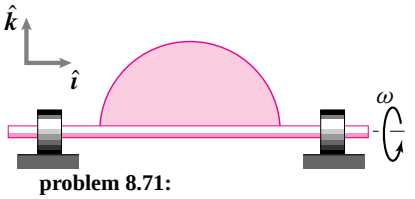
b) At $t = 0^+$, what is the reaction at B? *



problem 8.70:

(Filename:pfigure.s95q8)

8.71 A uniform semi-circular disk of radius $r = 0.2\,\text{m}$ and mass $m = 1.5\,\text{kg}$ is attached to a shaft along its straight edge. The shaft is tipped very slightly from a vertical upright position. Find the dynamic reactions on the bearings supporting the shaft when the disk is in the vertical plane (xz) and below the shaft.



problem 8.71:

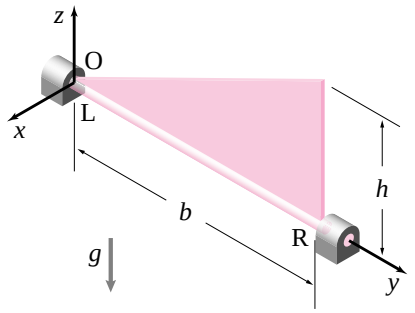
(Filename:pfigure4.rpi)

8.72 A thin triangular plate of base b, height h, and mass m is attached to a massless shaft which passes through the bearings L and R. The plate is initially vertical and then, provoked by a teensy disturbance, falls , pivoting around the shaft. The bearings are frictionless. Given,

$$\begin{aligned}
\vec{\boldsymbol{r}}_{cm} &= (0, \tfrac{2}{3}b, \tfrac{1}{3}h),\\
I^O_{xx} &= \frac{m}{2}(b^2 + h^2/3),\\
I^O_{yy} &= mh^2/6,\\
I^O_{zz} &= mb^2/2,\\
I^{cm}_{yy} &= mh^2/18,\\
I^O_{zy} &= -mbh/4,
\end{aligned}$$

a) what is the plate's angular velocity as it passes through the downward vertical position?

b) Compute the bearing reactions when the plate swings through the downward vertical position.

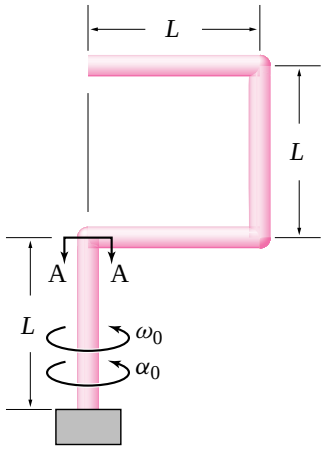


problem 8.72:

(Filename:pfigure.blue.86.1)

8.73 A welded structural tubing framework is proposed to support a heavy emergency searchlight. To select properly the tubing dimensions it is necessary to know the forces and moments at the critical section AA, when the framework is moving. The frame moves as shown.

a) Determine the magnitude and direction of the dynamic moments (bending and twisting) and forces (axial and shear) at section AA of the frame if, at the instant shown, the frame has angular velocity ω_o and angular acceleration α_o. Assume that the mass per unit length of the pipe is ρ.



problem 8.73:

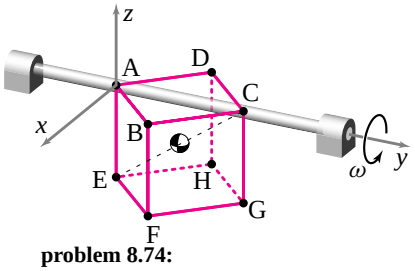
(Filename:pfigure.blue.88.1)

8.74 A uniform cube with side 0.25 m is glued to a pipe along the diagonal of one of its faces $ABCD$. The pipe rotates about the y-axis at a constant rate $\omega = 30$ rpm. At the instant of interest, the face $ABCD$ is in the xy-plane.

a) Find the velocity and acceleration of point F on the cube.

b) Find the position of the center of mass of the cube and compute its acceleration $\vec{\boldsymbol{a}}_{cm}$.

c) Suppose that the entire mass of the cube is concentrated at its center of mass. What is the angular momentum of the cube about point A?



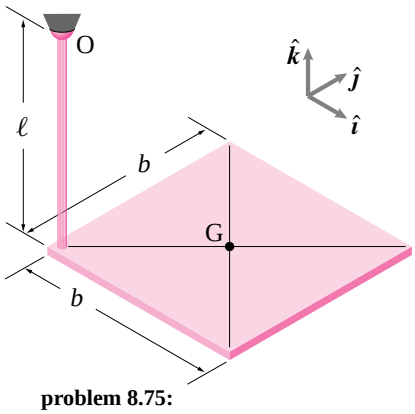
problem 8.74:

(Filename:pfigure4.2.rp3)

8.75 A uniform square plate of mass m and side length b is welded to a narrow shaft at one corner. The shaft is perpendicular to the plate and has length ℓ. The upper end of the shaft is connected to a ball and socket joint at O. The assembly is rotating about the z axis through O at the fixed rate $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ which happens to be just right to make the plate horizontal and the shaft vertical, as shown. The gravitational constant is g.

a) What is ω in terms of m, ℓ, b, and g?

b) Is the reaction at O on the rod-plate assembly in the direction of the line from G to O? (Why?)



problem 8.75:

(Filename:pfigure.s94q6p1)

8.76 A spinning box. A solid box $\mathcal{B}$ with dimensions a, a, and $2a$ (where $a = .1$ m) and mass = 1 kg. It is supported with ball and socket joints at opposite diagonal corners at A and B. It is spinning at constant rate of $(50/\pi)$ rev/sec and at the instant of interest is aligned with the fixed axis $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. Neglect gravity.

a) What is $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$?

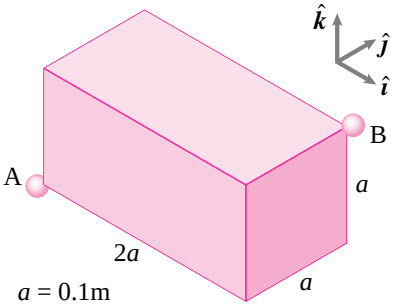
b) What is $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}}$?

c) What is $\vec{\boldsymbol{H}}_{\mathcal{B}/cm}$?

d) What is $\dot{\vec{\boldsymbol{H}}}_{\mathcal{B}/cm}$?

e) What is $\vec{\boldsymbol{r}}_{B/A} \times \vec{\boldsymbol{F}}_B$ (where $\vec{\boldsymbol{F}}_B$ is the reaction force on $\mathcal{B}$ at B)? A dimensional vector is desired.

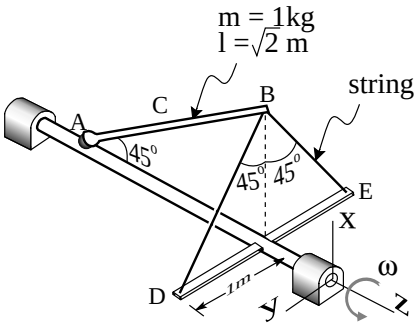
f) If the hinges were suddenly cut off when $\mathcal{B}$ was in the configuration shown, what would be $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}}$ immediately afterwards?



problem 8.76:

(Filename:pfigure.s94f1p2)

8.77 Uniform rod attached to a shaft. A uniform 1 kg, $\sqrt{2}$ m rod is attached to a shaft which spins at a constant rate of 1 rad/s. One end of the rod is connected to the shaft by a ball-and-socket joint. The other end is held by two strings (or pin jointed rods) that are connected to a rigid cross bar. The cross bar is welded to the main shaft. Ignore gravity. Find the reaction **force** at A.



problem 8.77:

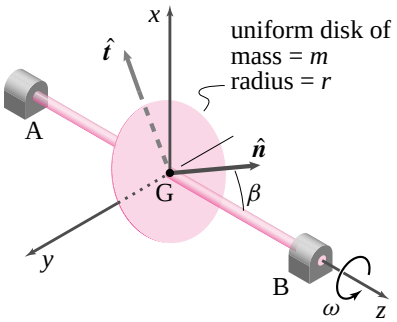
(Filename:s97f2)

8.78 A round disk spins crookedly on a shaft. A thin homogeneous disk of mass m and radius r is welded to the horizontal axle AB. The normal to the plane of the disk forms an angle β with the axle. The shaft rotates at the constant rate ω.

a) What is the x component of the angular momentum of the disk about point G the center of mass of the disk?

b) What is the x component of the rate of change of angular momentum of the disk about G?

c) At the given instant, what is the total force and moment (about G) required to keep the disk spinning at constant rate?



problem 8.78: A round disk spins crookedly on a shaft.

(Filename:pg92.2)

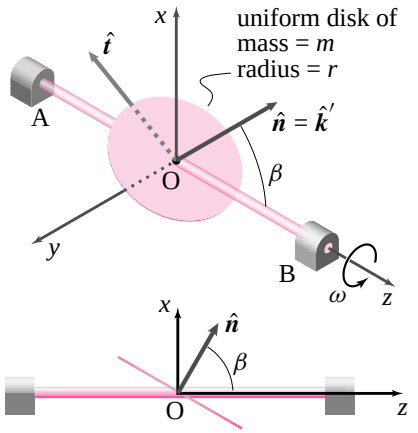
8.79 A uniform disk of mass $m = 2$ kg and radius $r = 0.2$ m is mounted rigidly on a massless axle. The normal to the disk makes an angle $\beta = 60^{\circ}$ with the axle. The axle rotates at a constant rate $\omega = 60$ rpm. At the instant shown in the figure,

a) Find the angular momentum $\vec{\boldsymbol{H}}_{\mathrm{O}}$ of the disk about point O. *

b) Draw the angular momentum vector indicating its magnitude and direction. *

c) As the disk rotates (with the axle), the angular momentum vector changes. Does it change in magnitude or direction, or both? *

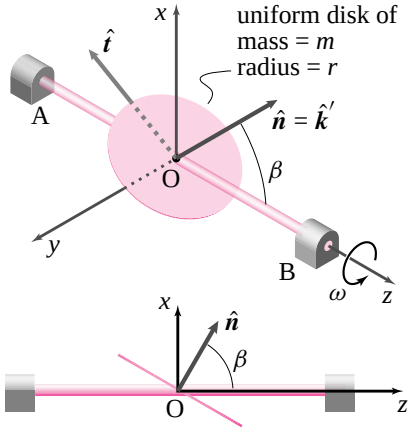
d) Find the direction of the net torque on the system at the instant shown. Note, only the *direction* is asked for. You can answer this question based on parts (b) and (c). *



problem 8.79:

(Filename:summer95p2.3)

8.80 A uniform disk spins at constant angular velocity. The uniform disk of mass m and radius r spins about the z axis at the constant rate ω. Its normal $\hat{\boldsymbol{n}}$ is inclined from the z axis by the angle β and is in the xz plane at the moment of interest. What is the total force and moment (relative to O) needed to keep this disk spinning at this rate? Answer in terms of m, r, ω, β and any base vectors you need.



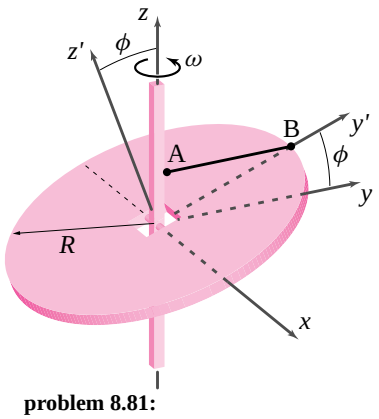
problem 8.80:

(Filename:p.f96.p3.3)

8.81 Spinning crooked disk. A rigid uniform disk of mass m and radius R is mounted to a shaft with a hinge. An apparatus not shown keeps the shaft rotating around the z axis at a constant ω. At the instant shown the hinge axis is in the x direction. A massless wire, perpendicular to the shaft, from the shaft at A to the edge of the disk at B keeps the normal to the plate (the z' direction) at an angle ϕ with the shaft.

a) What is the tension in the wire? Answer in terms of some or all of R, m, ϕ and ω.

b) What is the reaction force at the hinge at O?



problem 8.81:

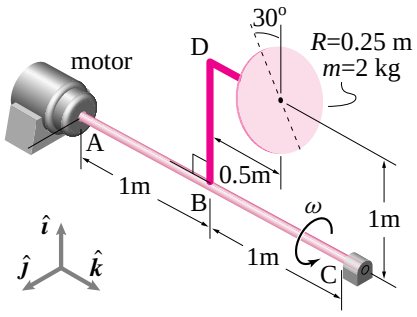
(Filename:s97p3.3)

8.82 A disoriented disk rotating with a shaft. A uniform thin circular disk with mass $m = 2\,\text{kg}$ and radius $R = 0.25\,\text{m}$ is mounted on a massless shaft as shown in the figure. The shaft spins at a constant speed $\omega = 3\,\text{rad/s}$. At the instant shown, the disk is slanted at 30° with the xy-plane. Ignore gravity.

a) Find the rate of change of angular momentum of the disk about its center of mass. *

b) Find the rate of change of angular momentum about point A. *

c) Find the total force and moment required at the supports to keep the motion going. *



problem 8.82:

(Filename:pfigure.s94h8p1)

8.5 Dynamic balance

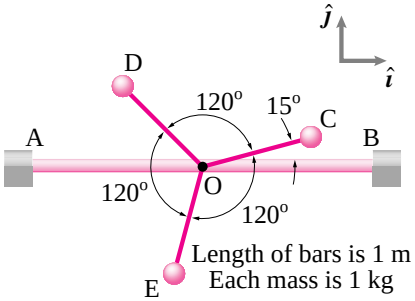
8.83 Is $\vec{\boldsymbol{\omega}} = (2\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{k}})$ rad/s an eigenvector of

$$[\boldsymbol{I}^{\text{G}}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & -4 \\ 0 & -4 & 8 \end{bmatrix} \text{ kg m}^2?$$

8.84 Find the eigenvalues and eigenvectors of

$$[\boldsymbol{I}^{\text{G}}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -3 \\ 0 & -3 & 10 \end{bmatrix} \text{ kg m}^2.$$

8.85 Three equal length (1 m) massless bars are welded to the rigid shaft AB which spins at constant rate $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{\imath}}$. Three equal masses (1 kg) are attached to the ends of the bars. At the moment of interest the masses are all in the xy plane as shown. Where should what masses be added to make the system shown dynamically balanced? [The solution is not unique.]



problem 8.85:

(Filename:pfigure.s94q7p1)

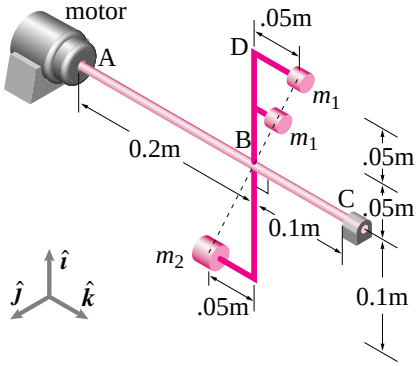
8.86 Dynamic balance of a system of particles. Three masses are mounted on a massless shaft AC with the help of massless rigid rods. Mass $m_1 = 1\,\text{kg}$ and mass $m_2 = 1.5\,\text{kg}$. The shaft rotates with a constant angular velocity $\vec{\boldsymbol{\omega}} = 5\,\text{rad/s}\hat{\boldsymbol{k}}$. At the instant shown, the three masses are in the xz-plane. Assume frictionless bearings and ignore gravity.

a) Find the net reaction force (the total of all the forces) on the shaft. Is the system statically balanced?

b) Find the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/A}$, using $\vec{\boldsymbol{H}}_{/A} = \sum_i \vec{\boldsymbol{r}}_{i/A} \times m_i \vec{\boldsymbol{a}}_i$.

c) Find the angular momentum about point A, $\vec{\boldsymbol{H}}_{/A}$. Use this intermediate result to find the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/A}$, using $\dot{\vec{\boldsymbol{H}}}_{/A} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_{/A}$ and verify the result obtained in (b).

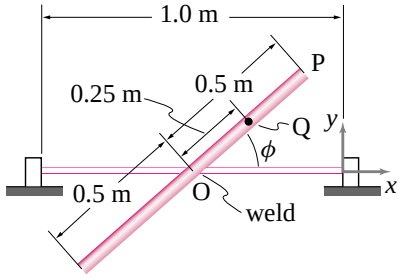
d) What is the value of the resultant unbalanced torque on the shaft? Propose a design to balance the system by adding an appropriate mass in an appropriate location.



problem 8.86:

(Filename:pfigure.s94h7p2)

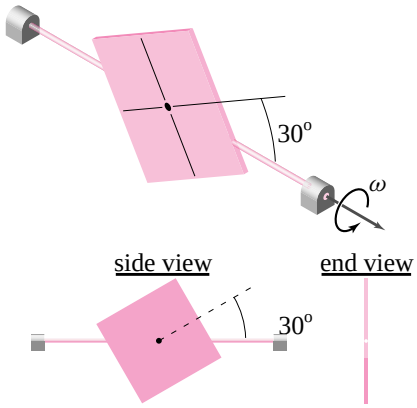
8.87 For what orientation of the rod ϕ is the system shown dynamically balanced?



problem 8.87:

(Filename:ch4.7.c)

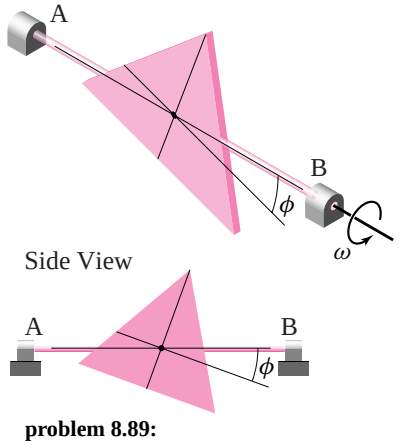
8.88 A uniform square plate with 1 m sides and 1 kg mass is mounted (as shown) with a shaft that is in the plane of the square but not parallel to the sides. What is the net force and moment required to maintain rotation at a constant rate?



problem 8.88:

(Filename:pfigure.blue.87d.1)

8.89 An equilateral uniform triangular plate with sides of length ℓ plate spins at constant rate ω about an axis through its center of mass. The axis is coplanar with the triangle. The angle ϕ is arbitrary. When $\phi = 0^\circ$, the base of the triangle is parallel to the axis of rotation. Find the net force and moment required to maintain rotation of the plate at a constant rate for any ϕ. For what angle ϕ is the plate dynamically balanced?



problem 8.89:

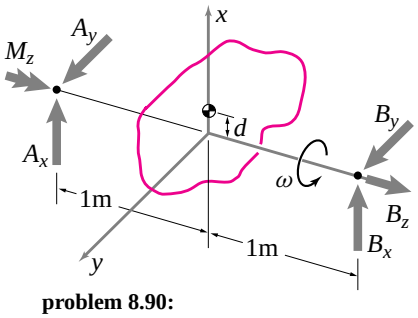
(Filename:Mikef91p3)

8.90 An abstract problem. This problem concerns a system rigidly attached to a massless rod spinning about a fixed axis at a fixed rate. That is, whether or not the system is a rigid body, it is moving as if it were a rigid body. Every point in the system is moving in circles about the same axis at the same number of radians per second. The position of a particle of the system relative to the axis is $\vec{\boldsymbol{R}}$. The position relative to some point C is $\vec{\boldsymbol{r}}_C$. In all cases, the body rotates about the axis shown. The total mass of all the objects is M_{tot}. Point O is at the center of mass of the system and is at a distance d from the rotation axis and is on the x-axis at the moment of interest. Assume there is no gravity. At the point A, the rod is free to slide in the z-direction. But, at point B, the rod is constrained from moving in the z-direction. The FBD of the system is shown in the figure. Referring to the free body diagram, there are six scalar unknown reactions in this 3 dimensional problem.

a) **The general case:** Imagine that you know $\dot{\vec{\boldsymbol{H}}}_O = \dot{H}_{0x}\hat{\boldsymbol{\imath}} + \dot{H}_{0y}\hat{\boldsymbol{\jmath}} + \dot{H}_{0z}\hat{\boldsymbol{k}}$ and also $\dot{\vec{\boldsymbol{L}}} = \dot{L}_x\hat{\boldsymbol{\imath}} + \dot{L}_y\hat{\boldsymbol{\jmath}} + \dot{L}_z\hat{\boldsymbol{k}}$. How would you find the six unknown reactions A_x, A_y, M_{Az}, B_x, B_y, B_z? [Reduce the problem to one of solving six equations in six unknowns.]

b) **The General Result:** Is M_{Az} necessarily equal to zero? Prove your result or find a counter-example. *

c) **Static Balance:**. If $d = 0$ is it true that all reaction forces are equal to zero? Prove your result or find a counter example. *



problem 8.90:

(Filename:pfigure.s94h7p3)

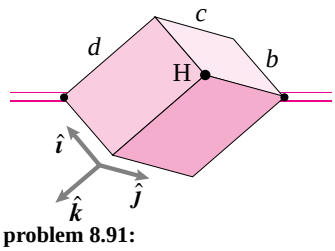
8.91 A uniform block spins at constant angular velocity. The magnitude of the angular velocity is ω. The center of mass G is on the axis of rotation. The $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ directions are parallel to the sides with lengthes b, c and d, respectively.
Answer in terms of some or all of ω, m, b, c, d, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$.

a) Find $\vec{\boldsymbol{v}}_H$.

b) Find $\vec{\boldsymbol{a}}_H$.

c) What is the *sum* of all the moments about point G of all the reaction forces at A and B?

d) Are there any values of b, c, and d for which this block is dynamically balanced for rotation about an axis through a pair of corners? Why?



problem 8.91:

(Filename:p.f96.p2.3)

8.92 Dynamic Balance. Consider each of the objects shown in turn (so to speak), each spinning about the z axis which is horizontal. In each case the center of mass is half way between the bearings (not shown). Gravity is in the $-\hat{\boldsymbol{\jmath}}$ direction. State whether it is possible to spin the object at constant rate with constant bearing reactions of $mg\hat{\boldsymbol{\jmath}}/2$ *and* no other applied torques ('yes') or might not be ('no'). Give a brief explanation of your answer (enough to distinguish an informed answer from a guess).

a) Axis along centerline of a uniform rectangular plate.

b) Axis orthogonal to uniform rectangular plate and through its centroid.

c) Axis orthogonal to uniform rectangular plate and *not* through its centroid.

d) Axis orthogonal to uniform but totally irregular shaped plate and through its centroid.

e) Axis along the diagonal of a uniform rhombus.

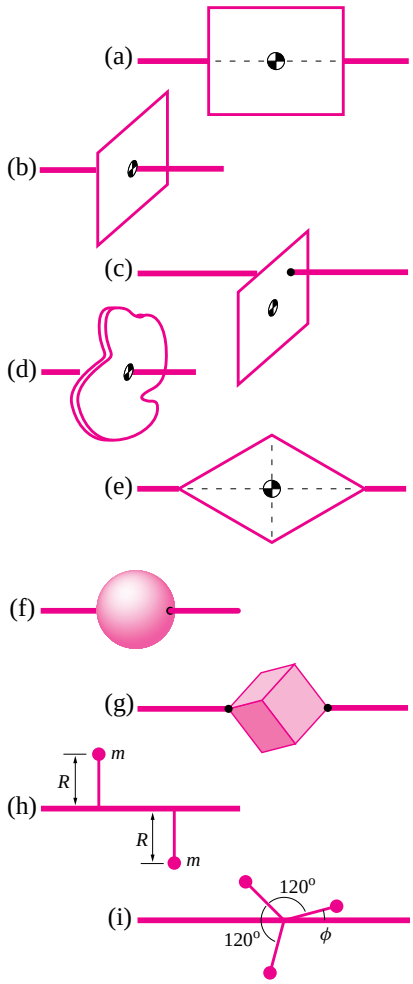
f) Axis through the center of a uniform sphere.

g) Axis through the diagonal of a uniform cube.

h) Axis with two equal masses attached as shown.

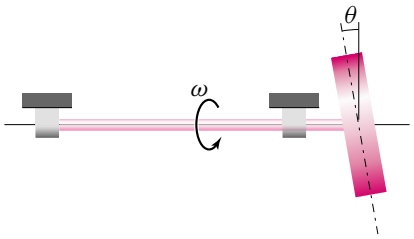
i) Axis with three equal masses attached as shown, all in a common plane with the axis, with ϕ not equal to 30°.

j) An arbitrary rigid body which has center of mass on the axis and two of its moment-of-inertia eigenvectors perpendicular to the axis. (no picture)



problem 8.92:

(Filename:s97p3.2)



problem 8.93:

(Filename:pfigure.s94h7p5)

8.93 Dynamic tire balancing. Where should you place what weights in order to dynamically balance the tire?

Assume that a car tire can be modeled as a uniform disk of radius one foot and mass 20 lbm. Assume the tire has been mounted crooked by one degree (the plane of the disk makes an 89° angle with the car axle). This crooked wheel is *statically* balanced since its center of mass is on the axis of rotation. But it is dynamically unbalanced since it wobbles and will wobble the car. Car shops put little weights on unbalanced wheels to balance them. They figure out where to put the weights by using a little machine that measures the wheel wobble. But you don't need such a machine for this problem because you have been told where all the mass is located.

There are many approaches to solving this problem and there are also many correct solutions. You should find any correct solution. [one approach: guess a good location to put two masses and then figure out how big they have to be to make the tire dynamically balanced.]

Problems for Chapter 9

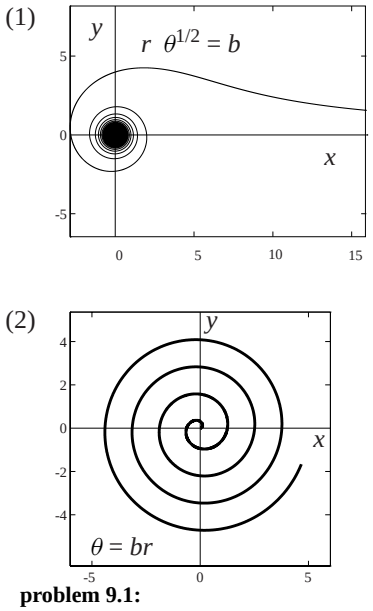
General planar motion of particles and rigid bodies

9.1 Kinematics of planar rigid-body motion

9.1 A particle moves along the two paths (1) and (2) as shown.

a) In each case, determine the velocity of the particle in terms of b, θ, and $\dot{\theta}$. *

b) Find the x and y coordinates of the path as functions of b and r or b and θ. *



problem 9.1:

(Filename:polarplot0)

9.2 For the particle path (1) in problem 9.1, find the acceleration of the particle in terms of b, θ, and $\dot{\theta}$.

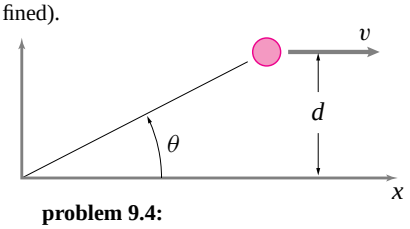
9.3 For the particle path (2) in in problem 9.1, find the acceleration of the particle in terms of b, θ, and $\dot{\theta}$.

9.4 A body moves with constant velocity V in a straight line parallel to and at a distance d from the x-axis.

a) Calculate $\dot{\theta}$ in terms of V, d, and θ.

b) Calculate the $\hat{\boldsymbol{e}}_\theta$ component of acceleration.

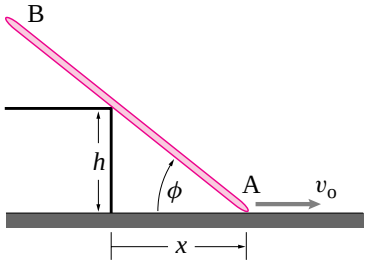
Carefully define, using a sketch and/or words, any variables, coordinate systems, and reference frames you use. Express your answers using any convenient coordinate system (just make sure its orientation has been clearly defined).



problem 9.4:

(Filename:pfigure.blue.62.1)

9.5 The slender rod AB rests against the step of height h, while end "A" is moved along the ground at a constant velocity v_o. Find $\dot{\phi}$ and $\ddot{\phi}$ in terms of x, h, and v_o. Is $\dot{\phi}$ positive or negative? Is $\ddot{\phi}$ positive or negative?



problem 9.5:

(Filename:pfigure.blue.98.1)

9.6 A ten foot ladder is leaning between a floor and a wall. The top of the ladder is sliding down the wall at one foot per second. (The foot is simultaneously sliding out on the floor). When the ladder makes a 45 degree angle with the vertical what is the speed of the midpoint of the ladder?

9.7 A stone in a wheel. A round wheel rolls to the right. At time $t = 0$ it picks up a stone the road. The stone is stuck in the edge of the wheel. You want to know the direction of the rock's motion just before and after it next hits the ground. Here are some candidate answers:

- When the stone approaches the ground its motion is tangent to the ground.
- The stone approaches the ground at angle x (you name it).
- When the stone approaches the ground its motion is perpendicular to the ground.
- The stone approaches the ground at various angles depending on the following conditions(...you list the conditions.)

Although you could address this question analytically, you are to try to get a clear answer by looking at computer generated plots. In particular, you are to plot the pebble's path for a small interval of time near when the stone next touches the ground. You should pick the parameters that make your case for an answer the strongest. You may make more than one plot.

Here are some steps to follow:

a) Assuming the wheel has radius R_w and the pebble is a distance R_p from the center (not necessarily equal to R_w). The pebble is directly below the center of the wheel at time $t = 0$. The wheel spins at constant clockwise rate ω. The x-axis is on the ground and $x(t = 0) = 0$. The wheel rolls without slipping. Using a clear well labeled drawing (use a compass and ruler or a computer drawing program), show that

$$x(t) = \omega t R_w - R_p \sin(\omega t)$$

$$y(t) = R_w - R_p \cos(\omega t)$$

b) Using this relation, write a program to make a plot of the path of the pebble as the wheel makes a little more than one revolution. Also show the outline of the wheel and the pebble itself at some intermediate time of interest. [Use any software and computer that pleases you.]

c) Change whatever you need to change to make a good plot of the pebble's path for a small amount of time as the pebble approaches and leaves the road. Also show the wheel and the pebble at some time in this interval.

d) In this configuration the pebble moves a *very* small distance in a small time so your axes need to be scaled. But make sure your x- and y- axes have the same scale so that the path of the pebble and the outline of the wheel will not be distorted.

e) How does your computer output buttress your claim that the pebble approaches and leaves the ground at the angles you claim?

f) Think of something about the pebble in the wheel that was not explicitly asked in this problem and explain it using the computer, and/or hand calculation and/or a drawing.

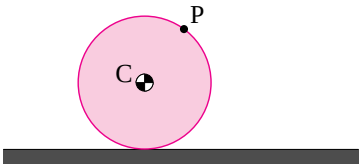
9.8 A uniform disk of radius r rolls at a constant rate without slip. A small ball of mass m is attached to the outside edge of the disk. What is the force required to hold the disk in place when the mass is above the center of the disk?

9.9 Rolling at constant rate. See also problem 9.39. A round disk rolls on the ground at constant rate. It rolls $1\frac{1}{4}$ revolutions over the time of interest.

a) **Particle paths.** Accurately plot the paths of three points: the center of the disk C, a point on the outer edge that is initially on the ground, and a point that is initially half way between the former two points. [Hint: Write a parametric equation for the position of the points. First find a relation between ω and v_C. Then note that the position of a point is the position of the center plus the position of the point relative to the center.] Draw the paths on the computer, make sure x and y scales are the same (In MATLAB, use the `axis('equal')` command for the same scaling).

b) **Velocity of points.** Find the velocity of the points at a few instants in the motion: after $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 revolution. Draw the velocity vector (by hand) on your plot. Draw the direction accurately and draw the lengths of the vectors in proportion to their magnitude. You can find the velocity by differentiating the position vector or by using relative motion formulas appropriately. Draw the disk at its position after one quarter revolution. Note that the velocity of the points is perpendicular to the line connecting the points to the ground contact.

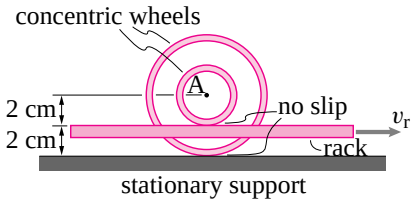
c) **Acceleration of points.** Do the same as above but for acceleration. Note that the acceleration of the points is parallel to the line connecting the points to the center of the disk.



problem 9.9:

(Filename:pfigure.s94h11p2)

9.10 The concentric wheels are welded to each other and roll without slip on the rack and stationary support. The rack moves to the right at $v_r = 1\,\text{m/s}$. What is the velocity of point A in the middle of the wheels shown?



problem 9.10:

(Filename:pfigure.blue.118.1)

9.11 Questions (a) - (e) refer to the cylinders in the configuration shown figure. Question (f) is closely related. Answer the questions in terms of the given quantities (and any other quantities you define if needed).

a) What is the speed (magnitude of velocity) of point c?

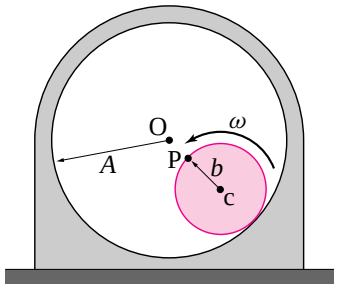
b) What is the speed of point P?

c) What is the magnitude of the acceleration of point c?

d) What is the magnitude of the acceleration of point P?

e) What is the radius of curvature of the path of the particle P at the point of interest?

f) In the special case of $A = 2b$ what is the curve which particle P traces (for all time)? Sketch the path.



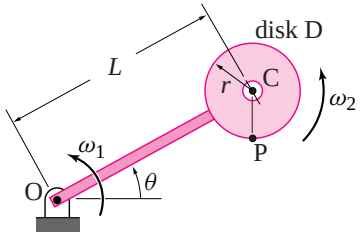
problem 9.11: A little cylinder (with outer radius b and center at point c) rolls without slipping inside a bigger fixed cylinder (with inner radius A and center at point O). The *absolute* angular velocity of the little cylinder ω is *constant*. P is attached to the outside edge of the little cylinder. At the instant of interest, P is on the line between O and c.

(Filename:pfigure.blue.50.1)

9.12 See also problems 10.31, 10.35, and 11.28. Arm OC rotates with constant rate ω_1. Disc D of radius r rotates about point C at constant rate ω_2 measured with respect to the arm OC. What are the absolute velocity and acceleration of point P on the disc, $\vec{\boldsymbol{v}}_P$ and $\vec{\boldsymbol{a}}_P$? (To do this problem will require defining a moving frame of reference. More than one choice is possible.) *

a) Pick a suitable moving frame and do the problem.

b) Pick another suitable moving frame and redo the problem. Make sure the answers are the same.



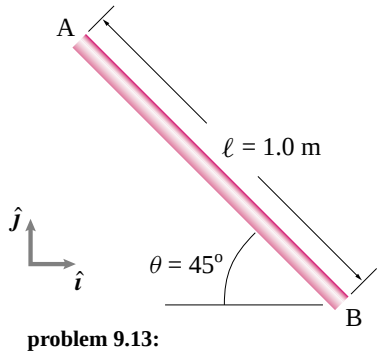
problem 9.12:

(Filename:Mikes93p1)

9.13 A uniform rigid rod AB of length $\ell = 1\,\text{m}$ rotates at a constant angular speed ω about an unknown fixed point. At the instant shown, the velocities of the two ends of the rod are $\vec{\boldsymbol{v}}_A = -1\,\text{m/s}\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{v}}_B = 1\,\text{m/s}\hat{\boldsymbol{\jmath}}$.

a) Find the angular velocity of the rod.

b) Find the center of rotation of the rod.



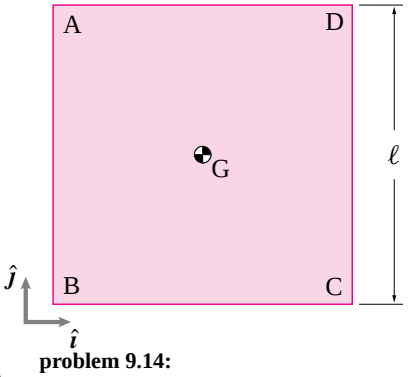
problem 9.13:

(Filename:pfigure4.3.rp6)

9.14 A square plate $ABCD$ rotates at a constant angular speed about an unknown point in its plane. At the instant shown, the velocities of the two corner points A and D are $\vec{\boldsymbol{v}}_A = -2\,\text{ft/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$ and $\vec{\boldsymbol{v}}_D = -(2\,\text{ft/s})\hat{\boldsymbol{\imath}}$, respectively.

a) Find the center of rotation of the plate.

b) Find the acceleration of the center of mass of the plate.



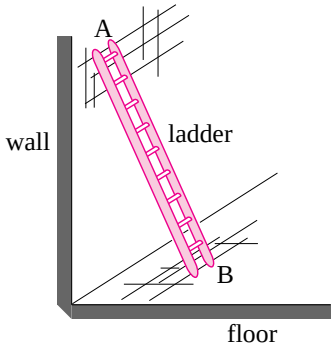
problem 9.14:

(Filename:pfigure4.3.rp7)

9.15 Consider the motion of a rigid ladder which can slide on a wall and on the floor as shown in the figure. The point A on the ladder moves parallel to the wall. The point B moves parallel to the floor. Yet, at a given instant, both have velocities that are consistent with the ladder rotating about some special point, the center of rotation (COR). Define appropriate dimensions for the problem.

a) Find the COR for the ladder when it is at some given position (and moving, of course). Hint, if a point is A is 'going in circles' about another point C, that other point C must be in the direction perpendicular to the motion of A.

b) As the ladder moves, the COR changes with time. What is the set of points on the plane that are the COR's for the ladder as it falls from straight up to lying on the floor?



problem 9.15:

(Filename:pfigure.blue.96.1)

9.16 A car driver on a very boring highway is carefully monitoring her speed. Over a one hour period, the car travels on a curve with constant radius of curvature, $\rho = 25\,\text{mi}$, and its speed increases uniformly from 50 mph to 60 mph. What is the acceleration of the of the car half-way through this one hour period, in path coordinates?

9.17 Find expressions for $\hat{\boldsymbol{e}}_t$, a_t, a_n, $\hat{\boldsymbol{e}}_n$, and the radius of curvature ρ, at any position (or time) on the given particle paths for

a) problem 5.81, *

b) problem 5.83,

c) problem 10.5,

d) problem 5.85,

e) problem 5.84, and

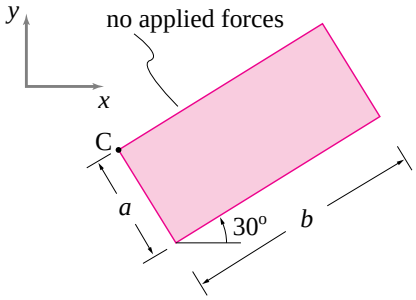
f) problem 5.82.

9.18 A particle travels at non-constant speed on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$. Carefully sketch the ellipse for particular values of a and b. For various positions of the particle on the path, sketch the position vector $\vec{\boldsymbol{r}}(t)$; the polar coordinate basis vectors $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$; and the path coordinate basis vectors $\hat{\boldsymbol{e}}_n$ and $\hat{\boldsymbol{e}}_t$. At what points on the path are $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_n$ parallel(or $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_t$ parallel)?

9.2 Unconstrained Dynamics of 2-D rigid-body planar motion

9.19 A uniform disc of mass m and radius r rotates with angular velocity $\omega\hat{\boldsymbol{k}}$. Its center of mass translates with velocity $\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$ in the xy-plane. What is the total kinetic energy of the disk?

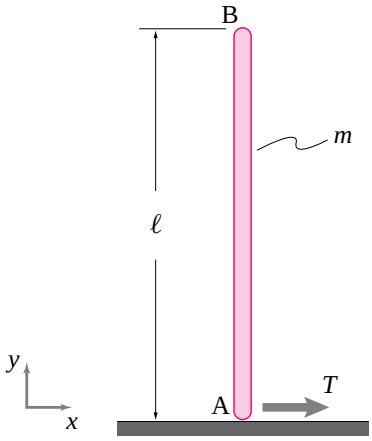
9.20 The uniform rectangle of width $a = 1\,\text{m}$, length $b = 2\,\text{m}$, and mass $m = 1\,\text{kg}$ in the figure is sliding on the $x - y$ plane with no friction. At the moment in question, point C is at $x_C = 3\,\text{m}$ and $y_C = 2\,\text{m}$. The linear momentum is $\vec{\boldsymbol{L}} = 4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$ (kg·m/s) and the angular momentum about the center of mass is $\vec{\boldsymbol{H}}_{cm} = 5\hat{\boldsymbol{k}}(\text{kg·m/s}^2)$. Find the acceleration of any point on the body that you choose. (Mark it.) [Hint: You have been given some useless information.]



problem 9.20:

(Filename:pfigure.blue.101.1)

9.21 The vertical pole AB of mass m and length ℓ is initially, *at rest* on a frictionless surface. A tension T is suddenly applied at A. What is $\ddot{x}_{cm}$? What is $\ddot{\theta}_{AB}$? What is $\ddot{x}_B$? Gravity may be ignored.



problem 9.21:

(Filename:pfigure.blue.95.1)

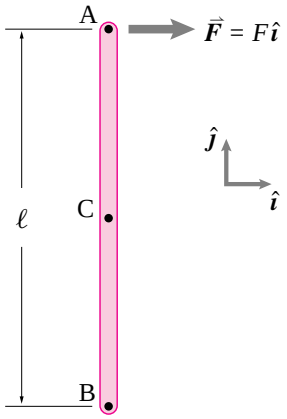
9.22 Force on a stick in space. 2-D . No gravity. A uniform thin stick with length ℓ and mass m is, at the instant of interest, parallel to the y axis and has no velocity and no angular velocity. The force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$ with $F > 0$ is suddenly applied at point A. The questions below concern the instant after the force $\vec{\boldsymbol{F}}$ is applied.

a) What is the acceleration of point C, the center of mass? *

b) What is the angular acceleration of the stick? *

c) What is the acceleration of the point A? *

d) (relatively harder) What additional force would have to be applied to point B to make point B's acceleration zero? *



problem 9.22:

(Filename:pfigure.s94h10p2)

9.23 A uniform thin rod of length ℓ and mass m stands vertically, with one end resting on a frictionless surface and the other held by someone's hand. The rod is released from rest, displaced slightly from the vertical. No forces are applied during the release. There is gravity.

a) Find the path of the center of mass.

b) Find the force of the floor on the end of the rod just before the rod is horizontal.

9.24 A uniform disk, with mass center labeled as point G, is sitting motionless on the frictionless xy plane. A massless peg is attached to a point on its perimeter. This disk has radius of 1 m and mass of 10 kg. A constant force of $F = 1000\,\text{N}\hat{\boldsymbol{\imath}}$ is applied to the peg for .0001 s (one ten-thousandth of a second).

a) What is the velocity of the center of mass of the center of the disk after the force is applied?

b) Assuming that the idealizations named in the problem statement are exact is your answer to (a) exact or approximate?

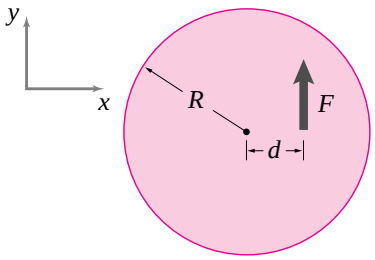
c) What is the angular velocity of the disk after the force is applied?

d) Assuming that the idealizations named in the problem statement are exact is your answer to (c) exact or approximate?

9.25 A uniform thin flat disc is floating in space. It has radius R and mass m. A force F is applied to it a distance d from the center in the y direction. Treat this problem as two-dimensional.

a) What is the acceleration of the center of the disc? *

b) What is the angular acceleration of the disk? *



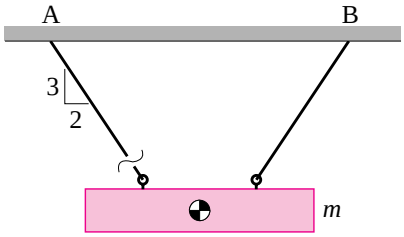
problem 9.25:

(Filename:pfigure.blue.102.1)

9.26 A uniform rectangular metal beam of mass m hangs symmetrically by two strings as shown in the figure.

a) Draw a free body diagram of the beam and evaluate $\sum \vec{F}$.

b) Repeat (a) immediately after the left string is cut.



problem 9.26:

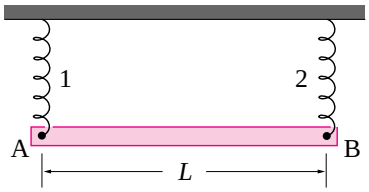
(Filename:pfig2.2.rp8)

9.27 A uniform slender bar AB of mass m is suspended from two springs (each of spring constant K) as shown. If spring 2 breaks, determine at that instant

a) the angular acceleration of the bar,

b) the acceleration of point A, and

c) the acceleration of point B.



problem 9.27:

(Filename:pfigure.blue.50.2)

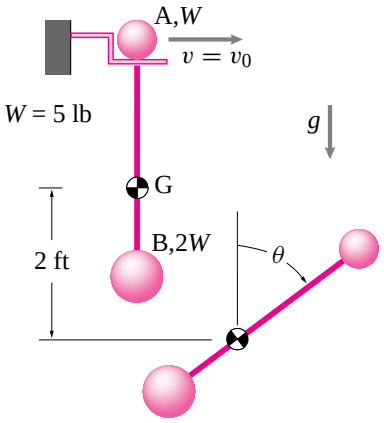
9.28 Two small spheres A and B are connected by a rigid rod of length $\ell = 1.0$ ft and negligible mass. The assembly is hung from a hook, as shown. Sphere A is struck, suddenly breaking its contact with the hook and giving it a horizontal velocity $v_0 = 3.0$ ft/s which sends the assembly into free fall. Determine the angular momentum of the assembly about its mass center at point G immediately after A is hit. After the center of mass has fallen two feet, determine:

a) the angle θ through which the rod has rotated,

b) the velocity of sphere A,

c) the total kinetic energy of the assembly of spheres A and B and the rod, and

d) the acceleration of sphere A.



problem 9.28:

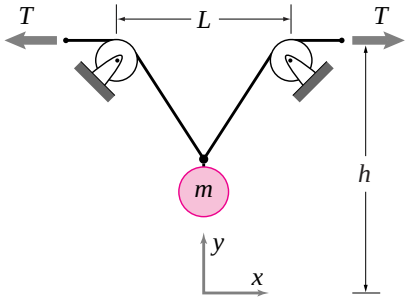
(Filename:pfigure.blue.105.1)

The next several problems concern Work, power and energy

9.29 Verify that the expressions for work done by a force F, $W = F\Delta S$, and by a moment M, $W = M\Delta\theta$, are dimensionally correct if ΔS and $\Delta\theta$ are linear and angular displacements respectively.

9.30 Calculate the energy stored in a spring using the expression $E_P = \frac{1}{2}k\delta^2$ if the spring is compressed by 100 mm and the spring constant is 100 N/m.

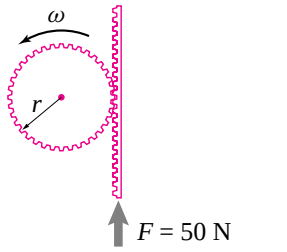
9.31 A point mass 'm' is pulled straight up by two strings. The two strings pull the mass symmetrically about the vertical axis with constant and equal force t. At an instant in time t, the position and the velocity of the mass are $y(t)\,\hat{\boldsymbol{j}}$ and $\dot{y}(t)\,\hat{\boldsymbol{j}}$, respectively. Draw a free body diagram of the mass at the instant of interest and evaluate the left hand side of the energy balance equation ($P = \dot{E}_K$).



problem 9.31:

(Filename:pfig2.3.rp2)

9.32 In a rack and pinion system, the rack is acted upon by a constant force $F = 50$ N and has speed $v = 2$ m/s in the direction of the force. Find the input power to the system.



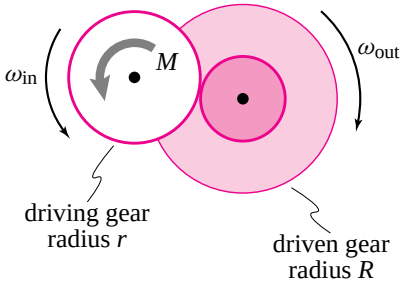
problem 9.32:

(Filename:pfig2.3.rp6)

9.33 The driving gear in a compound gear train rotates at constant speed ω_0. The driving torque is M_{in}. If the driven gear rotates at a constant speed ω_{out}, find:

a) the input power to the system, and

b) the output torque of the system assuming there is no power loss in the system; i.e., power in = power out.



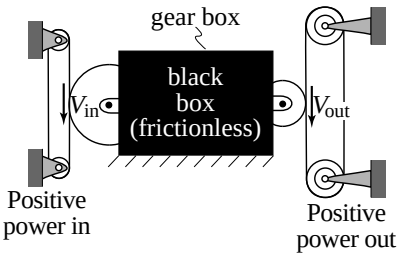
problem 9.33:

(Filename:pfig2.3.rp7)

9.34 An elaborate frictionless gear box has an input and output roller with $V_{in} = const$. Assuming that $V_{out} = 7V_{in}$ and the force between the left belt and roller is $F_{in} = 3$ lb:

a) What is F_{out} (draw a picture defining the signs of F_{in} and F_{out})?. *

b) Is F_{out} greater or less than the F_{in}? (Assume $F_{in} > 0$.) Why? *



problem 9.34:

(Filename:pfigure.blue.138.1)

9.3 Special topics in planar kinematics

9.4 Mechanics of contacting bodies: rolling and sliding

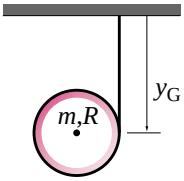
9.35 A uniform disc of mass m and radius r rolls without slip at constant rate. What is the total kinetic energy of the disk?

9.36 A non-uniform disc of mass m and radius r rolls without slip at constant rate. The center of mass is located at a distance $\frac{r}{2}$ from the center of the disc. What is the total kinetic energy of the disc when the center of mass is directly above the center of the disc?

9.37 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R. G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

a) Find y_G at a later time t in terms of any or all of m, R, g, and t.

b) Does G move sideways as the hoop falls and unrolls?



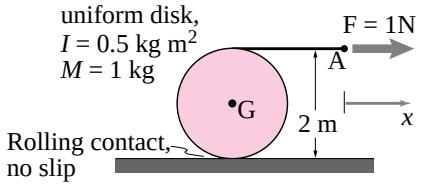
problem 9.37:

(Filename:p.s96.p3.2)

9.38 A uniform disk with radius R and mass m has a string wrapped around it. The string is pulled with a force F. The disk rolls without slipping.

a) What is the angular acceleration of the disk, $\vec{\boldsymbol{\alpha}}_{Disk} = -\ddot{\theta}\hat{\boldsymbol{k}}$? Make any reasonable assumptions you need that are consistent with the figure information and the laws of mechanics. State your assumptions. *

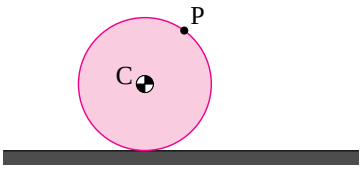
b) Find the acceleration of the point A in the figure. *



problem 9.38:

(Filename:pfigure.blue.43.1)

9.39 If a pebble is stuck to the edge of the wheel in problem 9.9, what is the maximum speed of the pebble during the motion? When is the force on the pebble from the wheel maximum? Draw a good FBD including the force due to gravity.



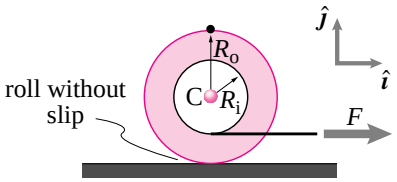
problem 9.39:

(Filename:pfigure.s94h11p2.a)

9.40 Spool Rolling without Slip and Pulled by a Cord. See also problem 9.42. The lightweight spool is nearly empty but a lead ball with mass m has been placed at its center. A force F is applied in the horizontal direction to the cord wound around the wheel. Dimensions are as marked. Coordinate directions are as marked.

a) What is the acceleration of the center of the spool? *

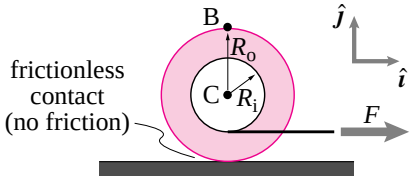
b) What is the horizontal force of the ground on the spool? *



problem 9.40:

(Filename:pfigure.s94h11p5)

9.41 A film spool is placed on a very slippery table. Assume that the film and reel (together) have mass distributed the same as a uniform disk of radius R_i. What, in terms of R_i, R_o, m, g, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and F are the accelerations of points C and B at the instant shown (the start of motion)?



problem 9.41:

(Filename:p.f96.f.4)

9.42 Again, Spool Rolling without Slip and Pulled by a Cord. Reconsider the spool from problem 9.40. This time, a force F is applied vertically to the cord wound around the wheel. In this case, what is the acceleration of the center of the spool? Is it possible to pull the cord at some angle between horizontal and vertical so that the angular acceleration of the spool or the acceleration of the center of mass is zero? If so, find the angle in terms of R_i, R_0, m, and F.

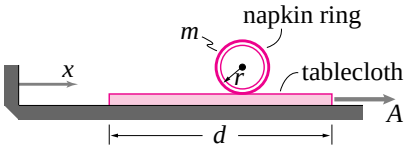
9.43 A napkin ring lies on a thick velvet tablecloth. The thin ring (of mass m, radius r) rolls without slip as a mischievous child pulls the tablecloth (mass M) out with acceleration A. The ring starts at the right end ($x = d$). You can make a reasonable physical model of this situation with an empty soda can and a piece of paper on a flat table.

a) What is the ring's acceleration as the tablecloth is being withdrawn?

b) How far has the tablecloth moved to the right from its starting point $x = 0$ when the ring rolls off its left-hand end?

c) Clearly describe the subsequent motion of the ring. Which way does it end up rolling at what speed?

d) Would your answer to the previous question be different if the ring slipped on the cloth as the cloth was being pulled out?



problem 9.43:

(Filename:pfigure.blue.51.1)

9.44 A block of mass M is supported by two rollers (uniform cylinders) each of mass m and radius r. They roll without slip on the block and the ground. A force F is applied in the horizontal direction to the right, as shown in the figure. Given F, m, r, and M, find:

a) the acceleration of the block,

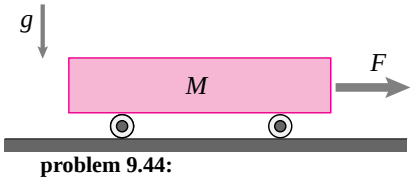
b) the acceleration of the center of mass of this block/roller system,

c) the reaction at the wheel bases,

d) the force of the right wheel on the block,

e) the acceleration of the wheel centers, and

f) the angular acceleration of the wheels.



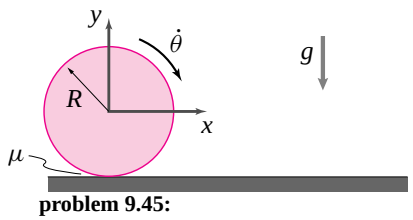
problem 9.44:

(Filename:pfigure.blue.108.2)

9.45 Dropped spinning disk. 2-D . A uniform disk of radius R and mass m is gently dropped onto a surface and doesn't bounce. When it is released it is spinning clockwise at the rate $\dot{\theta}_0$. The disk skids for a while and then is eventually rolling.

a) What is the speed of the center of the disk when the disk is eventually rolling (answer in terms of g, μ, R, $\dot{\theta}_0$, and m)? *

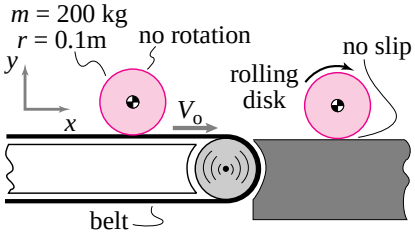
b) In the transition from slipping to rolling, energy is lost to friction. How does the amount lost depend on the coefficient of friction (and other parameters)? How does this loss make or not make sense in the limit as $\mu \to 0$ and the dissipation rate $\to$ zero? *



problem 9.45:

(Filename:pfigure.s95q10)

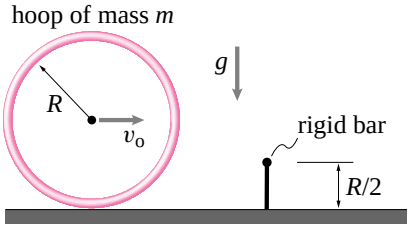
9.46 Disk on a conveyor belt. A uniform metal cylinder with mass of 200 kg is carried on a conveyer belt which moves at $V_0 = 3$ m/s. The disk is not rotating when on the belt. The disk is delivered to a flat hard platform where it slides for a while and ends up rolling. How fast is it moving (i.e. what is the speed of the center of mass) when it eventually rolls? *



problem 9.46:

(Filename:pfigure.s94h11p3)

9.47 A rigid hoop with radius R and mass m is rolling without slip so that its center has translational speed v_o. It then hits a narrow bar with height $R/2$. When the hoop hits the bar suddenly it sticks and doesn't slide. It does hinge freely about the bar, however. The gravitational constant is g. How big is v_o if the hoop just barely rolls over the bar? *



problem 9.47:

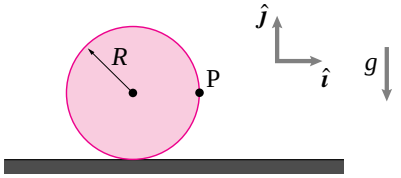
(Filename:pfigure.blue.87.5)

9.48 2-D rolling of an unbalanced wheel. A wheel, modeled as massless, has a point mass (mass = m) at its perimeter. The wheel is released from rest at the position shown. The wheel makes contact with coefficient of friction μ.

a) What is the acceleration of the point P just after the wheel is released if $\mu = 0$?

b) What is the acceleration of the point P just after the wheel is released if $\mu = 2$?

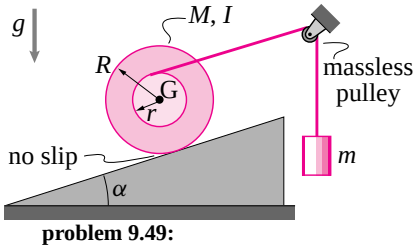
c) Assuming the wheel rolls without slip (no-slip requires, by the way, that the friction be high: $\mu = \infty$) what is the velocity of the point P just before it touches the ground?



problem 9.48:

(Filename:pfigure.s94f1p1)

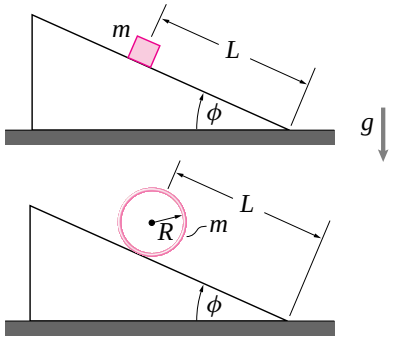
9.49 Spool and mass. A reel of mass M and moment of inertia $I_{zz}^{cm} = I$ rolls without slipping upwards on an incline with slope-angle α. It is pulled up by a string attached to mass m as shown. Find the acceleration of point G in terms of some or all of M, m, I, R, r, α, g and any base vectors you clearly define.



problem 9.49:

(Filename:s97f4)

9.50 Two objects are released on two identical ramps. One is a sliding block (no friction), the other a rolling hoop (no slip). Both have the same mass, m, are in the same gravity field and have the same distance to travel. It takes the sliding mass 1 s to reach the bottom of the ramp. How long does it take the hoop? [Useful formula: "$s = \frac{1}{2}at^2$"]



problem 9.50:

(Filename:pfigure.blue.51.2)

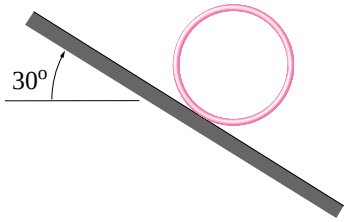
9.51 The hoop is rolled down an incline that is 30° from horizontal. It does not slip. It does not fall over sideways. It is let go from rest at t = 0.

a) At t = 0^+ what is the acceleration of the hoop center of mass?

b) At t = 0^+ what is the acceleration of the point on the hoop that is on the incline?

c) At t = 0^+ what is the acceleration of the point on the hoop that is furthest from the incline?

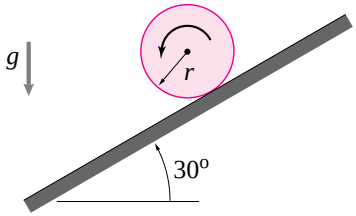
d) After the hoop has descended 2 vertical meters (and traveled an appropriate distance down the incline) what is the acceleration of the point on the hoop that is (at that instant) furthest from the incline?



problem 9.51:

(Filename:pfigure.blue.47.1)

9.52 A uniform cylinder of mass m and radius r rolls down an incline *without slip*, as shown below. Determine: (a) the angular acceleration α of the disk; (b) the minimum value of the coefficient of (static) friction μ that will insure no slip.



problem 9.52:

(Filename:pfigure.blue.49.3)

9.53 Race of rollers. A uniform disk with mass M_0 and radius R_0 is allowed to roll down the frictionless but quite slip-resistant ($\mu = 1$) 30° ramp shown. It is raced against four other objects (A, B, C and D), one at a time. Who wins the races, or are there ties? First try to construct any plausible reasoning. Good answers will be based, at least in part, on careful use of equations of mechanics. *

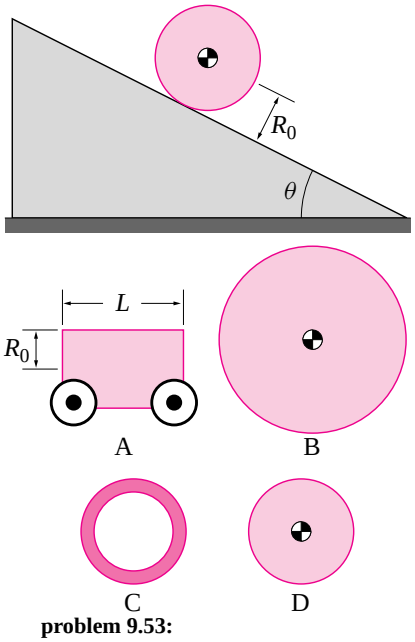
a) Block A has the same mass and has center of mass a distance R_0 from the ramp. It rolls on massless wheels with frictionless bearings.

b) Uniform disk B has the same mass ($M_B = M_0$) but twice the radius ($R_B = 2R_0$).

c) Hollow pipe C has the same mass ($M_C = M_0$) and the same radius ($R_C = R_0$).

d) Uniform disk D has the same radius ($R_D = R_0$) but twice the mass ($M_D = 2M_0$).

Can you find a round object which will roll as fast as the block slides? How about a massless cylinder with a point mass in its center? Can you find an object which will go slower than the slowest or faster than the fastest of these objects? What would they be and why? (This problem is harder.)



problem 9.53:

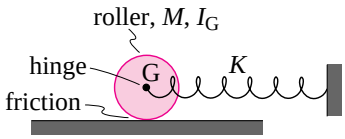
(Filename:pfigure.s94h11p4)

9.54 A roller of mass M and polar moment of inertia about the center of mass I_G is connected to a spring of stiffness K by a frictionless hinge as shown in the figure. Consider two kinds of friction between the roller and the surface it moves on:

1. Perfect slipping (no friction), and
2. Perfect rolling (infinite friction).

a) What is the period of oscillation in the first case?

b) What is the period of oscillation in the second case?



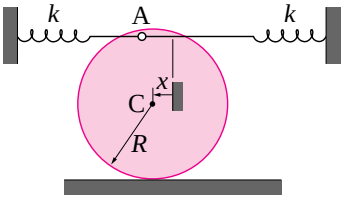
problem 9.54:

(Filename:pfigure.blue.141.2)

9.55 A uniform cylinder of mass m and radius R rolls back and forth without slipping through small amplitudes (i.e., the springs attached at point A on the rim act linearly and the vertical change in the height of point A is negligible). The springs, which act both in compression and tension, are unextended when A is directly over C.

a) Determine the differential equation of motion for the cylinder's center.

b) Calculate the natural frequency of the system for small oscillations.



problem 9.55:

(Filename:pfigure.blue.152.1)

9.56 A disk rolls in a cylinder. For all of the problems below, the disk rolls without slip and s back and forth due to gravity.

a) **Sketch.** Draw a neat sketch of the disk in the cylinder. The sketch should show all variables, coordinates and dimension used in the problem.

b) **FBD.** Draw a free body diagram of the disk.

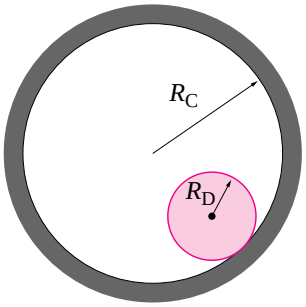
c) **Momentum balance.** Write the equations of linear and angular momentum balance for the disk. Use the point on the cylinder which touches the disk for the angular momentum balance equation. Leave as unknown in these equations variables which you do not know.

d) **Kinematics.** The disk rolling in the cylinder is a *one*-degree-of-freedom system. That is, the values of only *one* coordinate and its derivatives are enough to determine the positions, velocities and accelerations of all points. The angle that the line from the center of the cylinder to the center of the disk makes from the vertical can be used as such a variable. Find all of the velocities and accelerations needed in the momentum balance equation in terms of this variable and it's derivative. [Hint: you'll need to think about the rolling contact in order to do this part.]

e) **Equation of motion.** Write the angular momentum balance equation as a single second order differential equation.

f) **Simple pendulum?** Does this equation reduce to the equation for a pendulum with a point mass and length equal to the radius of the cylinder, when the disk radius gets arbitrarily small? Why, or why not?

g) **How many?** How many parts can one simple question be divided into?

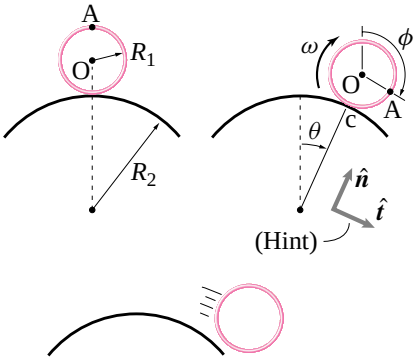


problem 9.56: A disk rolls without slip inside a bigger cylinder.

(Filename:h12.3)

9.57 A uniform hoop of radius R_1 and mass m rolls from rest down a semi-circular track of radius R_2 as shown. Assume that no slipping occurs. At what angle θ does the hoop leave the track and what is its angular velocity ω and the linear velocity $\vec{v}$ of its center of mass at that instant? If the hoop slides down the track without friction, so that it does not rotate, will it leave at a smaller or larger angle θ than if it rolls without slip (as above)? Give a qualitative argument to justify your answer.

HINT: Here is a geometric relationship between angle ϕ hoop turns through and angle θ subtended by its center when no slipping occurs: $\phi = [(R_1 + R_2)/R_2]\theta$. (You may or may not need to use this hint.)

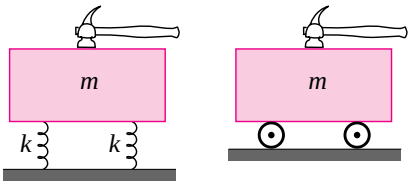


problem 9.57:

(Filename:pfigure.blue.46.1)

9.5 Collisions

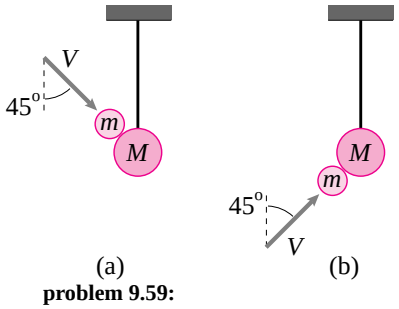
9.58 The two blocks shown in the figure are identical except that one rests on two springs while the other one sits on two massless wheels. Draw free body diagrams of each mass as each is struck by a hammer. Here we are interested in the free body diagrams only during collision. Therefore, ignore all forces that are much smaller than the *impulsive* forces. State in words why the forces you choose to show should not be ignored during the collision.



problem 9.58:

(Filename:pfig2.1.rp9)

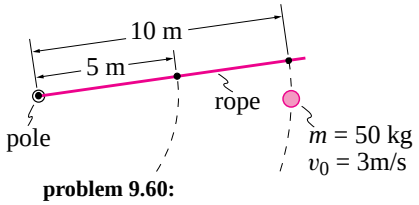
9.59 These problems concerns two colliding masses. In the first case In (a) the smaller mass hits the hanging mass from above at an angle 45° with the vertical. In (b) second case the smaller mass hits the hanging mass from below at the same angle. Assuming perfectly elastic impact between the balls, find the velocity of the hanging mass just after the collision. [Note, these problems are *not* well posed and can only be solved if you make additional modeling assumptions.]



problem 9.59:

(Filename:pfig2.2.rp7)

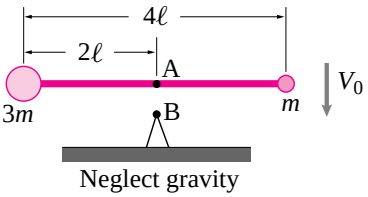
9.60 A narrow pole is in the middle of a pond with a 10 m rope tied to it. A frictionless ice skater of mass 50 kg and speed 3 m/s grabs the rope. The rope slowly wraps around the pole. What is the speed of the skater when the rope is 5 m long? (A tricky question.)



problem 9.60:

(Filename:pfigure.blue.49.1)

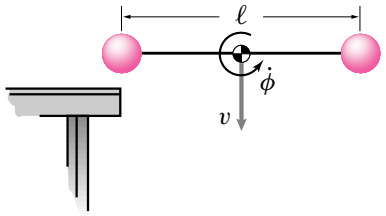
9.61 The masses m and $3m$ are joined by a light-weight bar of length 4ℓ. If point A in the center of the bar strikes fixed point B vertically with velocity V_0, and is not permitted to rebound, find $\dot{\theta}$ of the system immediately after impact.



problem 9.61: Neglect gravity!

(Filename:pfigure.blue.81.1)

9.62 Two equal masses each of mass m are joined by a massless rigid rod of length ℓ. The assembly strikes the edge of a table as shown in the figure, when the center of mass is moving downward with a linear velocity v and the system is rotating with angular velocity $\dot{\phi}$ in the counter-clockwise sense. The impact is 'elastic'. Find the immediate subsequent motion of the system, assuming that no energy is lost during the impact and that there is no gravity. Show that there is an interchange of translational and rotational kinetic energy.



problem 9.62:

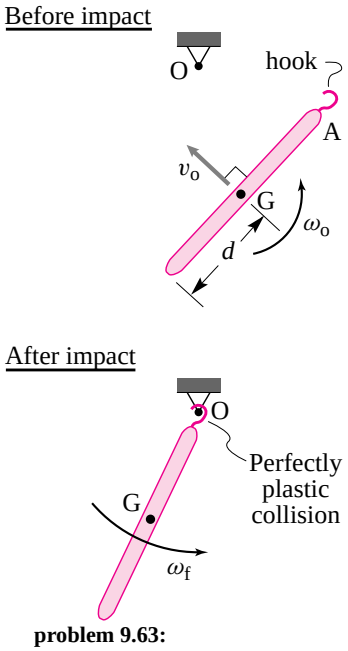
(Filename:pfigure.blue.137.2)

9.63 In the *absence of gravity*, a thin rod of mass m and length ℓ is initially tumbling with constant angular speed ω_o, in the counter-clockwise direction, while its mass center has constant speed v_o, directed as shown below. The end A then makes a perfectly plastic collision with a rigid peg O (via a hook). The velocity of the mass center happens to be perpendicular to the rod just before impact.

a) What is the angular speed ω_f immediately after impact?

b) What is the angular speed 10 seconds after impact? Why?

c) What is the loss in energy in the plastic collision?



problem 9.63:

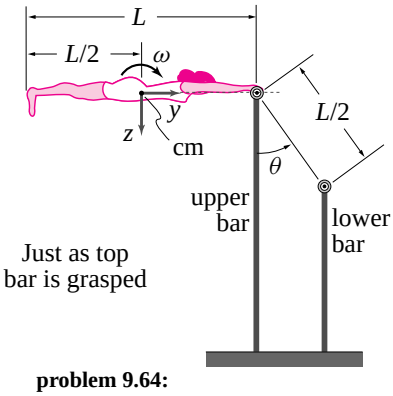
(Filename:pfigure.blue.78.2)

9.64 A gymnast of mass m and extended height L is performing on the uneven parallel bars. About the x, y, z axes which pass through her center of mass, her radii of gyration are k_x, k_y, and k_z, respectively. Just before she grasps the top bar, her fully extended body is horizontal and rotating with angular rate ω; her center of mass is then stationary. Neglect any friction between the bar and her hands and assume that she remains rigid throughout the entire stunt.

a) What is the gymnast's rotation rate just *after* she grasps the bar? State clearly any approximations/assumptions that you make.

b) Calculate the linear speed with which her hips (CM) strike the lower bar. State all assumptions/approximations.

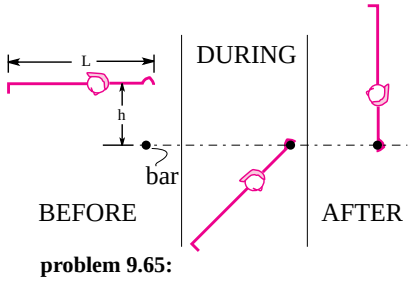
c) Describe (in words, no equations please) her motion immediately after her hips strike the lower bar if she releases her hands just prior to this impact.



problem 9.64:

(Filename:pfigure.blue.130.2)

9.65 An acrobat modelled as a rigid body. An acrobat is modeled as a uniform rigid mass m of length l. The acrobat falls without rotation in the position shown from height h where she was stationary. She then grabs a bar with a firm but slippery grip. What is h so that after the subsequent motion the acrobat ends up in a stationary handstand? [Hints: Note what quantities are preserved in what parts of the motion.] *



problem 9.65:

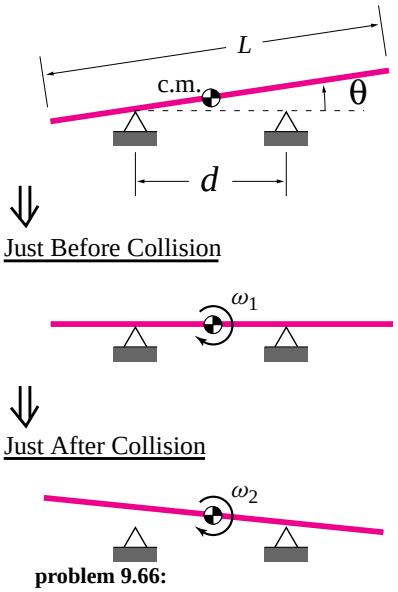
(Filename:pfigure.s94h10p4)

9.66 A crude see-saw is built with two supports separated by distance d about which a rigid plank (mass m, length L) can pivot smoothly. The plank is placed symmetrically, so that its

center of mass is midway between the supports when the plan is at rest.

a) While the left end is resting on the left support, the right end of the plank is lifted to an angle θ and released. At what angular velocity ω_1 will the plank strike the right hand support?

b) Following the impact, the left end of the plank can pivot purely about the right end if d/L is properly chosen and the right end does not bounce. Find ω_2 under these circumstances.



problem 9.66:

(Filename:pfigure.blue.81.2)

9.67 Baseball bat. In order to convey the ideas without making the calculation to complicated, some of the simplifying assumptions here are highly approximate. Assume that a bat is a uniform rigid stick with length L and mass m_s. The motion of the bat is a pivoting about one end held firmly in place with hands that rotate but do not move. The swinging of the bat occurs by the application of a constant torque M_s at the hands over an angle of $\theta = \pi/2$ until the point of impact with the ball. The ball has mass m_b and arrives perpendicular to the bat at an absolute speed v_b at a point a distance ℓ from the hands. The collision between the bat and the ball is completely elastic.

a) To maximize the speed v_{hit} of the hit ball *How heavy should a baseball bat be? Where should the ball hit the bat?* Here are some hints for one way to approach the problem.

- Find the angular velocity of the bat just before collision by drawing a FBD of the bat etc.
- Find the total energy of the ball and bat system just before the collision.
- Draw a FBD of the ball and of the bat during the collision (with this model there is an impulse at the hands on the bat). Call the magnitude of the impulse of the ball on the bat (and visa versa) $\int F dt$.
- Use various momentum equations to find the angular velocity of the bat and velocity of the ball just after the collision in terms of $\int F dt$ and other quantities above. Use these to find the energy of the system just after collion.
- Solve for the value of $\int F dt$ that conserves energy. As a check you should see if this also predicts that the relative separation speed of the ball and bat (at the impact point) is the same as the relative approach speed (it should be).
- You now know can calculte v_{hit} in terms of m_b, m_s, M_s, L, ℓ, and v_b.
- Find the maximum of the above expression by varying m_s and ℓ. Pick numbers for the fixed quantities if you like.

b) Can you explain in words what is wrong with a bat that is too light or too heavy?

c) Which aspects of the model above do you think lead to the biggest errors in predicting what a real ball player should pick for a bat and place on the bat to hit the ball?

d) Describe as clearly as possible a different model of a baseball swing that you think would give a more accurate prediction. (You need not do the calculation).

Problems for Chapter 10

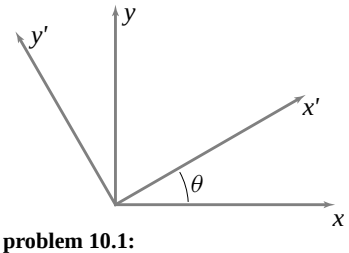
Kinematics using time-varying base vectors Problems for all three sections of this chapter are combined.

10.1 Polar coordinates and path coordinates

10.2 Rotating reference frames

10.3 General expressions for velocity and acceleration

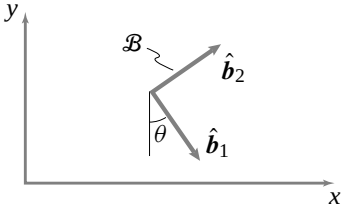
10.1 Express the basis vectors $(\hat{\boldsymbol{\imath}}', \hat{\boldsymbol{\jmath}}')$ associated with axes x' and y' in terms of the standard basis $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$ for $\theta = 30^{\circ}$.



problem 10.1:

(Filename:efig1.2.23)

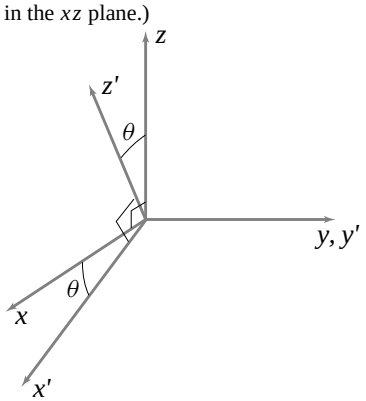
10.2 Body frames are frames of reference attached to a body in motion. The orientation of a coordinate system attached to a body frame $\mathcal{B}$ is shown in the figure at some moment of interest. For $\theta = 60^{\circ}$, express the basis vectors $(\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2)$ in terms of the standard basis $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})$.



problem 10.2:

(Filename:efig1.2.24)

10.3 Find the components of (a) $\vec{\boldsymbol{v}} = 3\,\mathrm{m/s}\hat{\boldsymbol{\imath}} + 2\,\mathrm{m/s}\hat{\boldsymbol{\jmath}} + 4\,\mathrm{m/s}\hat{\boldsymbol{k}}$ in the rotated basis $(\hat{\boldsymbol{\imath}}', \hat{\boldsymbol{\jmath}}', \hat{\boldsymbol{k}}')$ and (b) $\vec{\boldsymbol{a}} = -0.5\,\mathrm{m/s^2}\hat{\boldsymbol{\jmath}}' + 3.0\,\mathrm{m/s^2}\hat{\boldsymbol{k}}'$ in the standard basis $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$. ($y$ and y' are in the same direction and x' and z' are in the xz plane.)



problem 10.3:

(Filename:efig1.2.25)

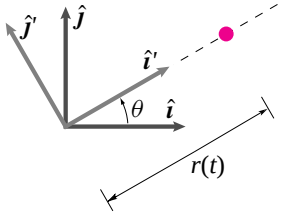
10.4 For **Q** given in problem 12.22 and $\vec{\boldsymbol{M}} = 25\,\mathrm{N{\cdot}m}\hat{\boldsymbol{E}}_1 - 32\,\mathrm{N{\cdot}m}\hat{\boldsymbol{E}}_2$, find the $\hat{\boldsymbol{e}}_2$ component of $\vec{\boldsymbol{M}}$ without calculating $[\mathbf{Q}]\{\vec{\boldsymbol{M}}\}$.

10.5 A particle travels in a straight line in the xy-plane parallel to the x-axis at a distance $y = \ell$ in the positive x direction. The position of the particle is denoted by $\vec{\boldsymbol{r}}(t)$. The angle of $\vec{\boldsymbol{r}}$ measured positive counter-clockwise from the x axis is decreasing at a constant rate with magnitude ω. If the particle starts on the y axis at $\theta = \pi/2$, what is $\vec{\boldsymbol{r}}(t)$ in cartesian coordinates?

10.6 Given that $\vec{\boldsymbol{r}}(t) = ct^2\hat{\boldsymbol{\imath}}'$ and that $\theta(t) = d\sin(\lambda t)$ find $\vec{\boldsymbol{v}}(t)$

(a) in terms of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$,

(b) in terms of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$.



problem 10.6:

(Filename:pfig3.1.DH1)

10.7 A bug walks on a straight line engraved on a rotating turntable (the bug's path in the room is not a straight line). The line passes through the center of the turntable. The bug's speed on this line is 1 in/s (the bug's absolute speed is *not* 1 in/s). The turntable rotates at a constant rate of 2 revolutions every π s in a positive sense about the z-axis. It's surface is always in the xy-plane. At time $t = 0$, the engraved line is aligned with the x-axis, the bug is at the origin and headed towards the positive x direction.

a) What is the x component of the bug's velocity at $t = \pi$ s?

b) What is the y component of the bug's velocity at $t = \pi$ seconds?

c) What is the y component of the bug's acceleration at $t = \pi$?

d) What is the radius of curvature of the bug's path at $t = 0$?

10.8 A bug walks on a turntable. In polar coordinates, the position of the bug is given by $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_r$, where the origin of this coordinate system is at the center of the turntable. The $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta)$ coordinate system is attached to the turntable and, hence, rotates with the turntable. The kinematical quantities describing the bugs motion are $\dot{R} = -\dot{R}_0$, $\ddot{R} = 0$, $\dot{\theta} = \omega_0$, and $\ddot{\theta} = 0$. A fixed coordinate system Oxy has origin O at the center of the turntable. As the bug walks through the center of the turntable:

a) What is its speed?

b) What is its acceleration?

c) What is the radius of the osculating circle (i.e., what is the radius of curvature of the bug's path?).

10.9 Actual path of bug trying to walk a straight line. A straight line is inscribed on a horizontal turntable. The line goes through the center. Let ϕ be angle of rotation of the turntable which spins at constant rate $\dot{\phi}_0$. A bug starts on the outside edge of the turntable of radius R and walks towards the center, passes through it, and continues to the opposite edge of the turntable. The bug walks at a constant speed v_A, as measured by how far her feet move per step, on the line inscribed on the table. Ignore gravity.

a) **Picture.** Make an accurate drawing of the bug's path as seen in the room (which is not rotating with the turntable). In order to make this plot, you will need to assume values of v_A and $\dot{\phi}_0$ and initial values of R and ϕ. You will need to write a parametric equation for the path in terms of variables that you can plot (probably x and y coordinates). You will also need to pick a range of times. Your plot should include the instant at which the bug walks through the origin. Make sure your $x-$ and $y-$ axes are drawn to the same scale. A computer plot would be nice.

b) Calculate the radius of curvature of the bug's path as it goes through the origin.

c) Accurately draw (say, on the computer) the osculating circle when the bug is at the origin on the picture you drew for (a) above.

d) **Force.** What is the force on the bugs feet from the turntable when she starts her trip? Draw this force as an arrow on your picture of the bug's path.

e) **Force.** What is the force on the bugs feet when she is in the middle of the turntable? Draw this force as an arrow on your picture of the bug's path.

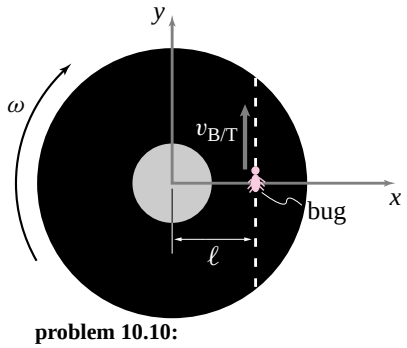
10.10 A small bug is crawling on a straight line scratched on an old record. The scratch, at its closest, is a distance $\ell = 6\,\text{cm}$ from the center of the turntable. The turntable is turning clockwise at a constant angular rate $\omega = 2\,\text{rad/s}$. The bug is walking, relative to the turntable, at a constant rate $v_{B/T} = 12\,\text{cm/s}$, straight along the scratch in the y-direction. At the instant of interest, everything is aligned as shown in figure. The bug has a mass $m_B = 1$ gram.

a) What is the bug's velocity?

b) What is the bug's acceleration?

c) What is the sum of all forces acting on the bug?

d) Sketch the path of the bug in the neighborhood of its location at the time of interest (indicate the direction the bug is moving on this path).



problem 10.10:

(Filename:pfigure.blue.55.2)

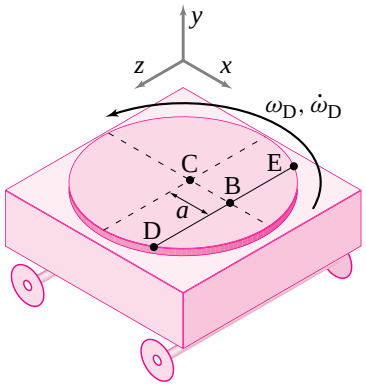
10.11 A bug of mass $m = 1\,\text{kg}$ walks on a straight line marked through the center of a turntable on the back of a pick-up truck, exactly as in problem 10.17. Evaluate the absolute velocity and acceleration of the bug in the following situations (in cartesian coordinates):

KinematicalQuantities

	x_T	$\dot{x}_T$	$\ddot{x}_T$	r_b	$\dot{r}_b$	$\ddot{r}_b$	θ	ω	$\dot{\omega}$
(a)	4 m	3 m/s	0 m/s^2	1 m	0 m/s	0 m/s^2	37°	0 rad/s	0 rad/s^2
(b)	”	”	2 m/s^2	”	”	”	”	”	”
(c)	”	”	0 m/s^2	”	”	”	$\pi/2$ rad	1 rad/s	”
(d)	”	”	”	”	”	1 m/s^2	0 rad	0 rad/s	”
(e)	”	”	”	0 m	1 m/s	0 m/s^2	”	1 rad/s	”
(f)	”	”	”	1 m	0 m/s	”	$\pi/2$ rad	0 rad/s	3 rad/s^2
(g)	”	”	2 m/s^2	”	5 m/s	−2 m/s^2	0 rad	”	0 rad/s^2
(h)	”	”	”	”	0 m/s	0 m/s^2	$\pi/2$ rad	”	2 rad/s^2
(i)	”	”	4 m/s^2	”	”	”	0 rad	2 rad/s	0 rad/s^2

*

10.12 A turntable oscillates with displacement $\vec{x}_C(t) = A\sin(st)\hat{\imath}$. The disc of the turntable rotates with angular speed and acceleration ω_D and $\dot{\omega}_D$. A small bug walks along line DE with velocity v_b relative to the turntable. At the instant shown, the turntable is at its maximum amplitude $x = A$, the line DE is currently aligned with the z-axis, and the bug is passing through point B on line DE. Point B is a distance a from the center of the turntable, point C. Find the absolute acceleration of the bug, $\vec{a}_b$. *



problem 10.12:

(Filename:Mikes93p2)

10.13 See also problem 10.15. A small .1 kg toy train engine is going *clockwise* at a constant rate (relative to the track) of 2 m/s on a circular track of radius 1 m. The track itself is on a turntable $\mathcal{B}$ that is rotating *counter- clockwise* at a constant rate of 1 rad/s. The dimensions are as shown. At the instant of interest the train is pointing due south ($-\mathbf{j}$) and is *at* the center of the turntable.

a) What is the velocity of the train relative to the turntable $\vec{v}_{/\mathcal{B}}$?

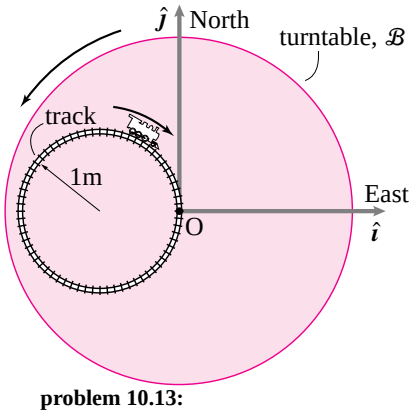
b) What is the absolute velocity of the train $\vec{v}$?

c) What is the acceleration of the train relative to the turntable $\vec{a}_{/\mathcal{B}}$?

d) What is the absolute acceleration of the train $\vec{a}$?

e) What is the total force acting on the train?

f) Sketch the path of the train for one revolution of the turntable (surprise)?

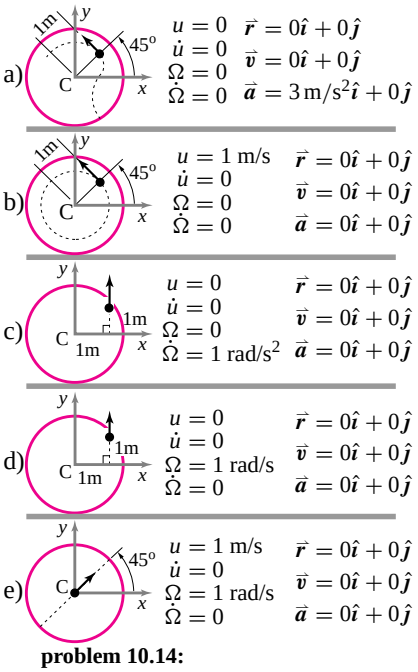


problem 10.13:

(Filename:pfigure.f93f4)

10.14 A giant bug walks on a horizontal disk. An $x-y-z$ frame is attached to the disk. The disk is rotating about the z axis (out of the paper) and simultaneously translating with respect to an inertial frame $X-Y-Z$ plane. In each of the cases shown in the figure, determine the total force acting on the bug. In each case, the dotted line is scratched on the disk and is the path the bug follows walking in the direction of the arrow. The location of the bug is marked with a dot. At the instants shown, the $x-y-z$ coordinate system shown is aligned with the inertial $X-Y-Z$ frame (which is not shown in the pictures). In this problem:

$\vec{r}$ = the position of the center of the disk,
$\vec{v}$ = the velocity of the center of the disk,
$\vec{a}$ = the acceleration of the center of the disk,
Ω the angular velocity of the disk (i.e., $\vec{\mathbf{\Omega}} = \Omega\boldsymbol{k}$),
$\dot{\Omega}$ = the angular acceleration of the disk,
u = the speed of the bug traversing the dotted line (arc length on the disk per unit time),
$\dot{u} = du/dt$, and
$m = 1\,\text{kg}$ = mass of the bug.

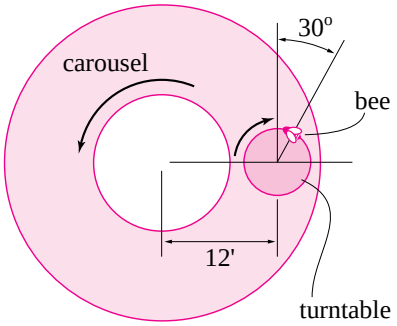


problem 10.14:

(Filename:pfigure.blue.117.1)

10.15 Repeat questions (a)-(f) for the toy train in problem 10.13 going *counter-clockwise* at constant rate (relative to the track) of 2 m/s on the circular track.

10.16 A honeybee, sensing that it can get a cheap thrill, alights on a phonograph turntable that is being carried by a carnival goer who is riding on a carousel. The situation is sketched below. The carousel has angular velocity of 5 rpm, which is increasing (accelerating) at 10 rev/min^2; the phonograph rotates at a constant 33 1/3 rpm. The honeybee is at the outer edge of the phonograph record in the position shown in the figure; the radius of the record is 7 inches. Calculate the magnitude of the acceleration of the honeybee.



problem 10.16:

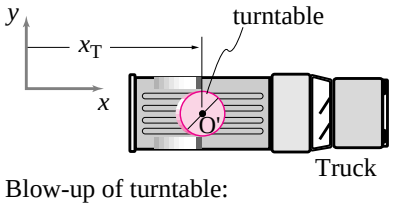
(Filename:pfigure.blue.67.2)

10.17 See also problem 10.11. Consider a turntable on the back of a pick-up truck. A bug walks on a line on the turntable. The line may or may not be drawn through the center of the turntable. The truck may or may not be going at constant rate. The turntable may or may not be spinning, and, if it spins, it may or may not go at a constant rate. The bug may be anywhere on the line and may or may not be walking at a constant rate.

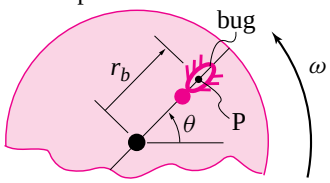
a) **Draw a picture of the situation.** Clearly define all variables you are going to use. What is the moving frame, and what is the reference point on that frame?

b) **One term at a time.** For each term in the five term acceleration formula, find a situation where all but one term is zero. Use the turntable as the moving frame, the bug as the particle of interest. *

c) **Two terms at a time.** (harder) How many situations can you find where a pair of terms is not zero but all other terms are zero? (Don't try to do all 10 cases unless you really think this infamous formula is fun. Try at least one or two.) *



Blow-up of turntable:



problem 10.17: Picking apart the five term acceleration formula.

(Filename:pfigure.blue.75.1)

10.18 Slider crank kinematics (No FBD required!). 2-D . Assume $R, \ell, \theta, \dot{\theta}, \ddot{\theta}$ are given. The crank mechanism parts move on the xy plane with the x direction being along the piston. Vectors should be expressed in terms of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ components.

a) What is the angular velocity of the crank OA? *

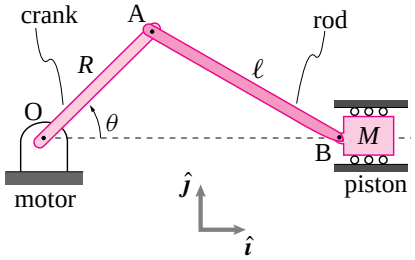
b) What is the angular acceleration of the crank OA? *

c) What is the velocity of point A? *

d) What is the acceleration of point A? *

e) What is the angular velocity of the connecting rod AB? [Geometry fact: $\vec{r}_{AB} = \sqrt{\ell^2 - R^2\sin^2\theta}\,\hat{\imath} - R\sin\theta\,\hat{\jmath}$] *

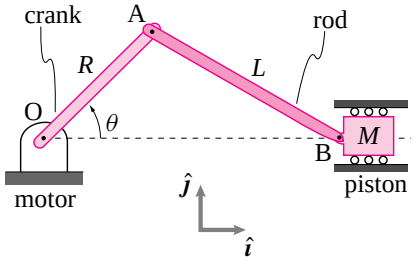
f) For what values of θ is the angular velocity of the connecting rod AB equal to zero (assume $\dot{\theta} \neq 0$)? (you need not answer part (e) correctly to answer this question correctly.) *



problem 10.18:

(Filename:pfigure.s95q12)

10.19 Slider-Crank. See also problem 11.34. Consider a slider-crank mechanism. Given θ, $\dot{\theta}$, $\ddot{\theta}$, L, and R, can you find the velocity and acceleration of B? There are many ways to do this problem.



problem 10.19:

(Filename:f.5)

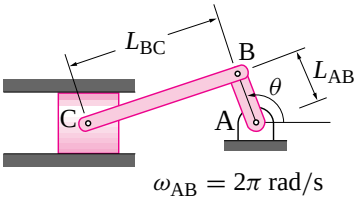
10.20 The crank AB with length $L_{AB} = 1$ inch in the crank mechanism shown rotates at a constant rate $\dot{\theta}_{AB} = \omega_{AB} = 2\pi$ rad/s counter-clockwise. The initial angle of rotation is $\theta = 0$ at $t = 0$. The connecting rod BC has a length of $L_{BC} = 2$ in.

a) What is the velocity of point B at the end of the crank when $\theta = \pi/2$ rad?

b) What is the velocity of point C at the end of the connecting rod when $\theta = \pi/2$ rad?

c) What is the angular velocity of the connecting rod BC when $\theta = \pi/2$ rad?

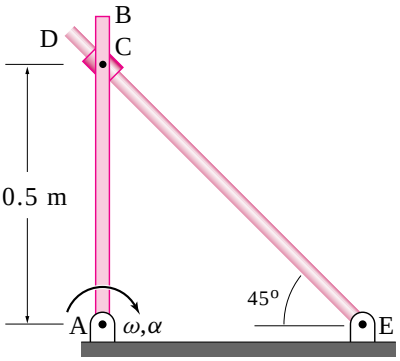
d) What is the angular velocity of the connecting rod BC as a function of time?



problem 10.20:

(Filename:pfigure.blue.116.1)

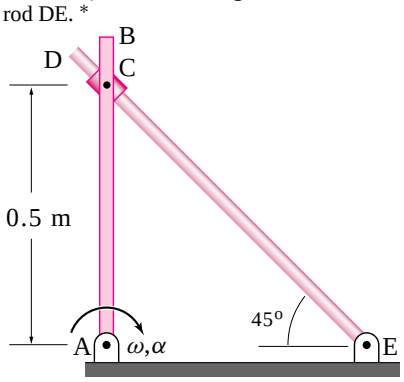
10.21 See also problem 10.22. The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3$ rad/s and angular acceleration $\alpha = 2$ rad/s^2. Find the angular velocity of rod DE. *



problem 10.21:

(Filename:summer95f.4)

10.22 Reconsider problem 10.21. The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3$ rad/s and angular acceleration $\alpha = 2\,\text{rad/s}^2$. Find the angular acceleration of rod DE. *



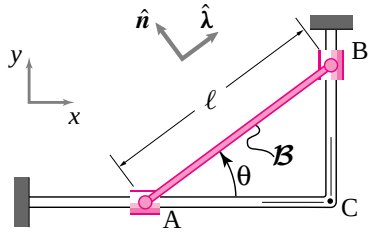
problem 10.22:

(Filename:summer95f.4.a)

10.23 Collar A is constrained to slide on a horizontal rod to the right at constant speed v_A. It is connected by a pin joint to one end of a rigid bar AB with length ℓ which makes an angle θ with the horizontal at the instant of interest. A second collar B connected by a pin joint to the other end of the rigid bar AB slides on a vertical rod.

a) Find the velocity of point B. Answer in terms of v_A, θ, ℓ, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$.

b) Find the angular speed $\dot{\theta}$ of the rod? Answer in terms of v_A, θ, ℓ, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$.



problem 10.23:

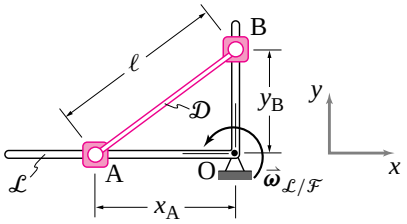
(Filename:s93q5)

10.24 A bar of length $\ell = 5$ ft, body $\mathcal{D}$, connects sliders A and B on an L-shaped frame, body $\mathcal{L}$, which itself is rotating at constant speed about an axle perpendicular to the plane of the figure through the point O and relative to a fixed frame $\mathcal{F}$, $\vec{\boldsymbol{\omega}}_{\mathcal{L}/\mathcal{F}} = 0.5\,\text{rad/s}\boldsymbol{k}$. At the instant shown, body $\mathcal{L}$ is aligned with the $x-y$ axes, slider A is $x_A = 4$ ft from point O, slider B is $y_B = 3$ ft from O. The speed and acceleration of slider B relative to the frame $\mathcal{L}$ are $\vec{\boldsymbol{v}}_{B/\mathcal{L}} = -2\,\text{ft/s}\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{a}}_{B/\mathcal{L}} = -2\,\text{ft/s}^2\hat{\boldsymbol{\jmath}}$, respectively.

Determine:

a) The absolute velocity of the slider A, $\vec{\boldsymbol{v}}_{A/\mathcal{F}}$, and

b) The absolute angular velocity of bar AB, body $\mathcal{D}$, $\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{F}}$.

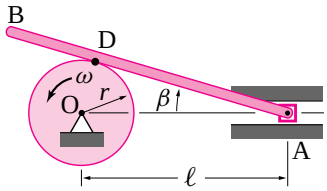


problem 10.24:

(Filename:pfigure.blue.115.1)

10.25 See also problem 11.35. The link AB is supported by a wheel at D and its end A is constrained so that it only has horizontal velocity. No slipping occurs between the wheel and the link. The wheel has an angular velocity ω and radius r. The distance $OA = \ell$.

Given: ω, r, and ℓ. ($\beta = \sin^{-1}\frac{r}{\ell}$). Determine the angular velocity of the link AB and velocity of the point A.



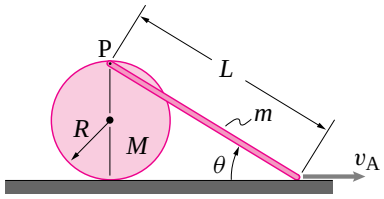
problem 10.25:

(Filename:pfigure.blue.121.1)

10.26 See also problem 11.36. A solid cylinder of radius R and mass M rolls without slip along the ground. A thin rod of mass m and length L is attached by a frictionless pin P to the cylinder's rim and its right end is dragged at a constant speed $\vec{\boldsymbol{v}}_A$ along the (frictionless) horizontal ground.

a) For the position shown (where P is directly above the contact point), find P's velocity $\vec{\boldsymbol{v}}_P$ and the rod's angular velocity $\boldsymbol{\omega}$.

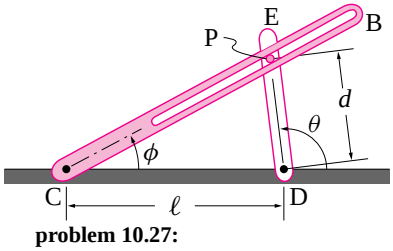
b) Find P's acceleration $\vec{\boldsymbol{a}}_P$ and the angular acceleration $\vec{\boldsymbol{\alpha}}$ of the rod.



problem 10.26:

(Filename:pfigure.blue.116.2)

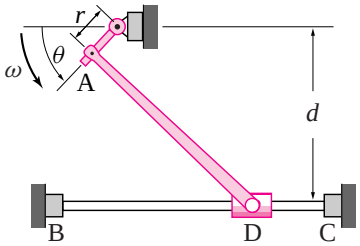
10.27 The slotted link CB is driven in an oscillatory motion by the link ED which rotates about D with constant angular velocity $\dot{\theta} = \omega_D$. The pin P is attached to ED at fixed radius d and engages the slot on CB as shown. Find the angular velocity and acceleration $\dot{\theta}$ and $\ddot{\theta}$ of CB when $\theta = \pi/2$.



problem 10.27:

(Filename:pfigure.blue.65.3)

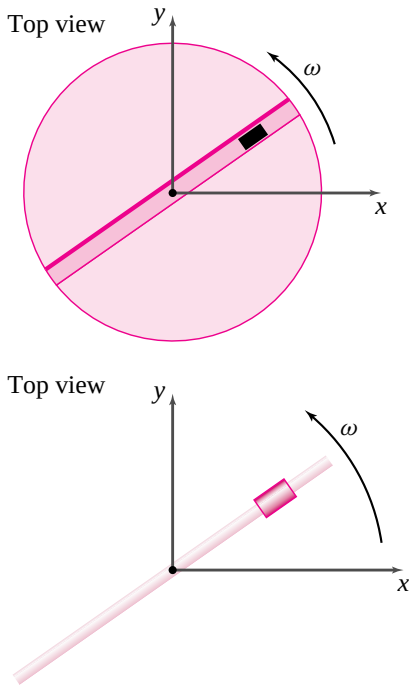
10.28 The rod of radius $r = 50$ mm shown has a constant angular velocity of $\omega = 30$ rad/s counterclockwise. Knowing that rod AD is 250 mm long and distance $d = 150$ mm, determine the acceleration of collar D when $\theta = 90°$.



problem 10.28:

(Filename:pfigure.blue.120.1)

10.29 Picking apart the polar coordinate formula for velocity. See also problems 10.30 and 11.52. This problem concerns a small mass m that sits in a slot in a turntable. Alternatively you can think of a small bead that slides on a rod. The mass always stays in the slot (or on the rod). Assume the mass is a little bug that can walk as it pleases on the rod (or in the slot) and you control how the turntable/rod rotates. Name two situations in which one of the terms is zero but the other is not in the two term polar coordinate formula for velocity, $\dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta$. You should thus gain some insight into the meaning of each of the two terms in that formula. *

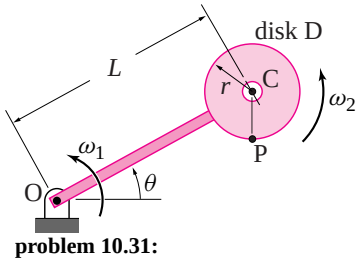


problem 10.29:

(Filename:pfigure.s94h9p4b)

10.30 Picking apart the polar coordinate formula for acceleration. Reconsider the configurations in problem 10.29. This time, name four situations in which all of the terms, but one, in the four term polar coordinate formula for acceleration, $\vec{\boldsymbol{a}} = (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta$, are zero. Each situation should pick out a different term. You should thus gain some insight into the meaning of each of the four terms in that formula. *

10.31 For the configuration in problem 9.12 what is the absolute angular velocity of the disk, D? *



problem 10.31:

(Filename:doublerot1)

10.32 A car driver on a very boring highway is carefully monitoring her speed. Over a one hour period, the car travels on a curve with constant radius of curvature, $\rho = 25$ mi, and its speed increases uniformly from 50 mph to 60 mph. What is the acceleration of the of the car half-way through this one hour period, in path coordinates?

10.33 Find expressions for $\hat{\boldsymbol{e}}_t$, a_t, a_n, $\hat{\boldsymbol{e}}_n$, and the radius of curvature ρ, at any position (or time) on the given particle paths for

a) problem 5.81, *

b) problem 5.83,

c) problem 10.5,

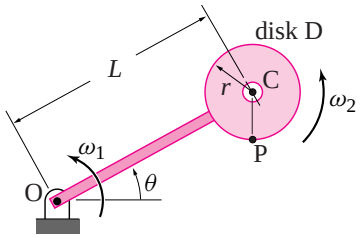
d) problem 5.85,

e) problem 5.84, and

f) problem 5.82.

10.34 A particle travels at non-constant speed on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$. Carefully sketch the ellipse for particular values of a and b. For various positions of the particle on the path, sketch the position vector $\vec{\boldsymbol{r}}(t)$; the polar coordinate basis vectors $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$; and the path coordinate basis vectors $\hat{\boldsymbol{e}}_n$ and $\hat{\boldsymbol{e}}_t$. At what points on the path are $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_n$ parallel(or $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_t$ parallel)?

10.35 For the configuration in problem 9.12, taking ω_2 to be the angular velocity of disk $\mathcal{D}$ relative to the rod, what is the absolute angular acceleration of the disk, D? What is the absolute angular acceleration of the disk if ω_1 and ω_2 are not constant?



problem 10.35:

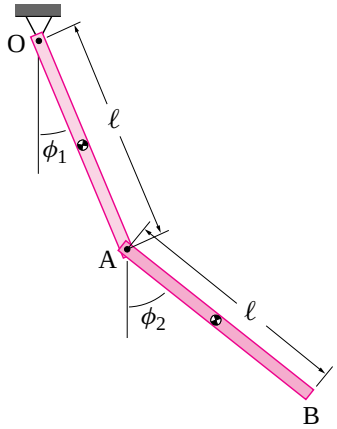
(Filename:doublerot.ang.accel)

10.36 Double pendulum. See also problem 10.37. The double pendulum shown is made up of two uniform bars, each of length ℓ and mass m. At the instant shown, ϕ_1, $\dot{\phi}_1$, ϕ_2, and $\dot{\phi}_2$ are known. For the instant shown, answer the following questions in terms of ℓ, ϕ_1, $\dot{\phi}_1$, ϕ_2, and $\dot{\phi}_2$.

a) What is the absolute velocity of point A?

b) What is the velocity of point B relative to point A?

c) What is the absolute velocity of point B?



problem 10.36:

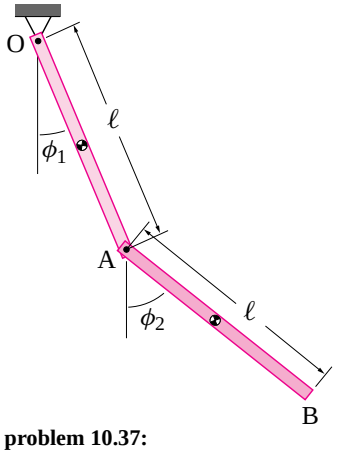
(Filename:ch7.8)

10.37 Double pendulum, Again. For the double pendulum in problem 10.36, $\ddot{\phi}_1$ and $\ddot{\phi}_2$ are also known at the instant shown. For the instant shown, answer the following questions in terms of ℓ, ϕ_1, $\dot{\phi}_1$, $\ddot{\phi}_1$, ϕ_2, $\dot{\phi}_2$, and $\ddot{\phi}_2$.

a) What is the absolute acceleration of point A?

b) What is the acceleration of point B relative to point A?

c) What is the absolute acceleration of point B?



problem 10.37:

(Filename:ch7.8.a)

Problems for Chapter 11

Mechanics of planar mechanisms

11.1 Dynamics of particles in the context of 2D mechanisms

11.1 Several of the problems below concern a narrow rigid hoop and/or a small bead. For reference here are dimensions and values you should use in working the problems: mass of bead $m_{\text{bead}} = 2$ grams, mass of hoop $m_{\text{hoop}} = 1$ kg, radius of hoop $R_{\text{hoop}} = 3$ m, gravitational acceleration $g = 10\,\text{m/s}^2$ (or zero). A bead slides on a frictionless circular hoop . The center of the circular hoop is the origin O of a fixed (Newtonian) coordinate system $Oxyz$. The hoop is on the xy plane. Gravity is in the negative z direction. The hoop is kept from moving by little angels who arrange to let the bead slide by unimpeded. At $t = 0$, the speed of the bead is 4 m/s, it is traveling counterclockwise (looking down the z-axis), and it is on the x-axis. There are no external forces applied.

a) At $t = 0$ what is the bead's kinetic energy ?

b) At $t = 0$ what is the bead's linear momentum ?

c) At $t = 0$ what is the bead's angular momentum about the origin?

d) At $t = 0$ what is the bead's acceleration?

e) At $t = 0$ what is the radius of the osculating circle of the bead's path ρ?

f) At $t = 0$ what is the force of the hoop on the bead?

g) At $t = 0$ what is the net force of the angel's hands on the hoop?

h) At $t = 27s$ what is the x-component of the bead's linear momentum?

Just when the bead above has traveled around the wire exactly 12 times the angels suddenly (but gently) let go of the hoop which is now free to slide on the frictionless xy plane. Call this time $t = 0$. At $t = 0^+$ we can assume the angels have fully let go.

i) At $t = 0^+$ what is the linear momentum of the bead and hoop system?

j) At $t = 0^+$ what is the rate of change of the linear momentum of the bead and hoop system?

k) At $t = 0^+$ What is the value of the acceleration of the bead? (Hint:$\mathbf{a}_{\text{bead}} = \mathbf{a}_{\text{hoop center}} + \mathbf{a}_{\text{bead/hoop center}}$)

11.2 A warehouse operator wants to move a crate of weight $W = 100$ lb from a 6 ft high platform to ground level by means of a roller conveyer as shown. The rollers on the inclined plane are well lubricated and thus assumed frictionless; the rollers on the horizontal conveyer are frictional, thus providing an effective friction coefficient μ. Assume the rollers are massless.

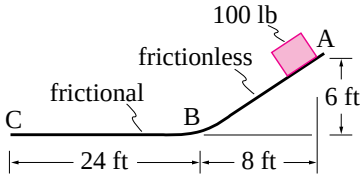
a) What are the kinetic and potential energies(pick a suitable datum) of the crate at the elevated platform, point A?

b) What are the kinetic and potential energies of the cart at the end of the inclined plane just before it moves onto the horizontal conveyor, point B.

c) Using conservation of energy, calculate the maximum speed attained by the crate at point B, assuming it starts moving from rest at point A.

d) If the crate slides a distance x, say, on the horizontal conveyor, what is the energy lost to sliding in terms of μ, W, and x?

e) Calculate the value of the coefficient of friction, μ, such that the crate comes to rest at point C.



problem 11.2:

(Filename:pfigure.blue.30.1)

11.3 On a wintry evening, a student of mass m starts down the steepest street in town, *Steep Street*, which is of height h and slope θ. At the top of the slope she starts to slide (coefficient of dynamic friction μ).

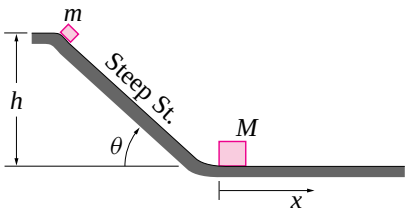
a) What is her initial kinetic and potential energy?

b) What is the energy lost to friction as she slides down the hill?

c) What is her velocity on reaching the bottom of the hill? Ignore air resistance and all cross streets (i.e. assume the hill is of constant slope).

d) If, upon reaching the bottom of the street, she collides with another student of mass M and they embrace, what is their instantaneous mutual velocity just after the embrace?

e) Assuming that friction still acts on the level flats on the bottom, how much time will it take before they come to rest?



problem 11.3:

(Filename:pfigure.blue.25.2)

11.4 Reconsider the system of blocks in problem 3.30, this time with equal mass, $m_1 = m_2 = m$. Also, now, both blocks are frictional and sitting on a frictional surface. Assume that both blocks are sliding to the right with the top block moving faster. The coefficient of friction between the two blocks is μ_1. The coefficient of friction between the lower block and the floor is μ_2. It is known that $\mu_1 > 2\mu_2$.

a) Draw free body diagrams of the blocks together and separately.

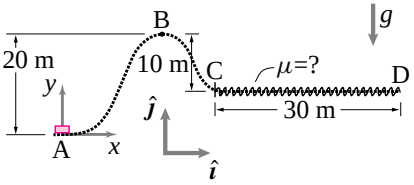
b) Write the equations of linear momentum balance for each block.

c) Find the acceleration of each block for the case of frictionless blocks. For the frictional blocks, find the acceleration of each block. What happens if $\mu_1 >> \mu_2$?

11.5 An initially motionless roller-coaster car of mass 20 kg is given a horizontal impulse $\int \vec{F}\,dt = P\boldsymbol{i}$ at position A, causing it to move along the track, as shown below.

a) Assuming that the track is perfectly frictionless from point A to C and that the car never leaves the track, determine the magnitude of the impulse I so that the car "just makes it" over the hill at B.

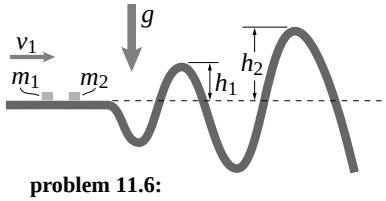
b) In the ensuing motion, assume that the horizontal track $C - D$ is frictional and determine the coefficient of friction required to bring the car to a stop a D.



problem 11.5:

(Filename:pfigure.blue.21.2)

11.6 Masses $m_1 = 1\,\text{kg}$ and $m_2\,\text{kg}$ move on the frictionless varied terrain shown. Initially, m_1 has speed $v_1 = 12\,\text{m/s}$ and m_2 is at rest. The two masses collide on the flat section. The coefficient of restitution in the collision is $e = 0.5$. Find the speed of m_2 at the top of the first hill of elevation $h_1 = 1\,\text{m}$. Does m_2 make it over the second hill of elevation $h_2 = 4\,\text{m}$?



problem 11.6:

(Filename:Danef94s2q7)

11.7 The two differential equations:

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ 2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0 \end{aligned}$$

have the general solution

$$\begin{aligned} R &= \sqrt{d^2 + (v(t-t_0))^2} \\ \theta &= \theta_0 + \tan^{-1}(v(t-t_0)/d) \end{aligned}$$

where θ_0, d, t_0, and v are arbitrary constants. This solution could be checked by plugging back into the differential equations — you need not do this (tediousl) substitution. The solution describes a curve in the plane. That is, if for a range of values of t, the values of R and θ were calculated and then plotted using polar coordinates, a curve would be drawn. *What can you say about the shape of this curve?*

[Hints:

a) Actually make a plot using some random values of the constants and see what the plot looks like.

b) Write the equation $\mathbf{F} = m\mathbf{a}$ for a particle in polar coordinates and think of a force that would be relevant to this problem.

c) The answer is something simple.]

11.8 A scotch yoke, a device for converting rotary motion into linear motion, is used to make a horizontal platform go up and down. The motion of the platform is $y = A\sin(\omega t)$ with constant ω. A dead bug with mass m is standing on the platform. Her feet have no glue on them. There is gravity.

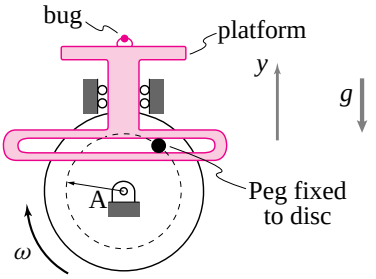
a) Draw a free body diagram of the bug.

b) What is the acceleration of the platform and, hence, the bug?

c) Write the equation of linear momentum balance for the bug.

d) What condition must be met so that the bug does not bounce the platform?

e) What is the maximum spin rate ω that can be used if the bug is not to bounce off the platform?



problem 11.8:

(Filename:pfigure.blue.34.1)

11.9 A new kind of gun. See also problems 11.10 and 11.11. Assume $\omega = \omega_0$ is a constant for the rod in the figure. Assume the mass is free to slide. At $t = 0$, the rod is aligned with the x-axis and the bead is one foot from the origin and has no radial velocity ($dR/dt = 0$).

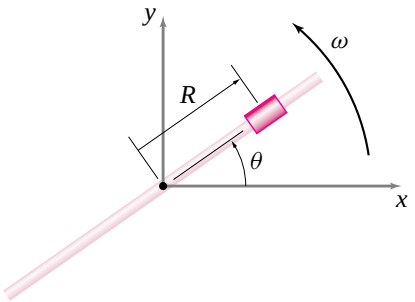
a) Find a differential equation for $R(t)$. *

b) Turn this equation into a differential equation for $R(\theta)$. *

c) How far will the bead have moved after one revolution of the rod? How far after two? *

d) What is the speed of the bead after one revolution of the rod (use $\omega_0 = 2\pi\,\text{rad/s} = 1\,\text{rev/sec}$)? *

e) How much kinetic energy does the bead have after one revolution and where did it come from? *



problem 11.9:

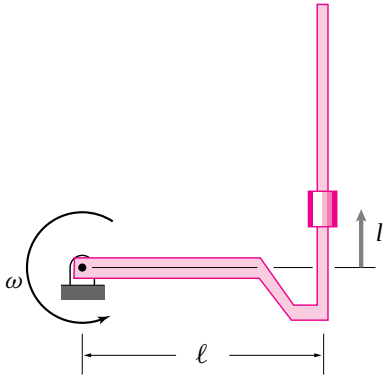
(Filename:pfigure.s94h9p5)

11.10 The new gun gets old and rusty. (See also problem 11.11.) Reconsider the bead on a rod in problem 11.9. This time, friction cannot be neglected. The friction coefficient is μ. At the instant of interest the bead with mass m has radius R_0 with rate of change $\dot{R}_0$. The angle θ is zero and ω is a constant. Neglect gravity.

a) What is $\ddot{R}$ at this instant? Give your answer in terms of any or all of R_0, $\dot{R}_0$, ω, m, μ, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_R$, $\hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{\jmath}}$. *

b) After a very long time it is observed that the angle ϕ between the path of the bead and the rod/trough is nearly constant. What, in terms of μ, is this ϕ? *

11.11 (See also problem 11.10.) **A newer kind of gun.** As an attempt to make an improvement on the 'new gun' demonstrated in problem 11.9, a person adds a length ℓ_0 to the shaft on which the bead slides. Assume there is no friction between the bead (mass m) and the wire. Assume the bead starts at $l = 0$ with $\dot{l} = v_0$. The rigid rod, on which the bead slides, rotates at a constant rate $\omega = \omega_0$. Find $l(t)$ in terms of ℓ_0, m, v_0, and ω_0.



problem 11.11:

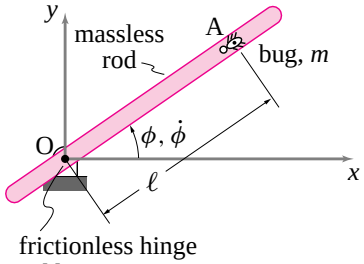
(Filename:pfigure.s94f1p5)

11.12 A bug of mass m walks straight-forwards with speed v_A and rate of change of speed $\dot{v}_A$ on a straight light (assumed to be massless) stick. The stick is hinged at the origin so that it is always horizontal but is free to rotate about the z axis. Assume that the distance the bug is from the origin , ℓ, the angle the stick makes with the x axis, ϕ, and its rate of change, $\dot{\phi}$, are known at the instant of interest. Ignore gravity. Answer the following questions in terms of ϕ, $\dot{\phi}$, ℓ, m, v_A, and $\dot{v}_A$.

a) What is $\ddot{\phi}$?

b) What is the force exerted by the rod on the support?

c) What is the acceleration of the bug?



problem 11.12:

(Filename:pfigure.blue.55.1)

11.13 Slippery bead on straight rotating stick. A long stick with mass m_s rotates at a constant rate and a bead, modeled as a point mass, slides on the stick. The stick rotates at constant rate $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. There is no gravity. The bead has mass m_b. Initially the mass is at a distance R_0 from the hinge point on the stick and has no radial velocity ($\dot{R} = 0$). The initial angle of the stick is θ_0, measured counterclockwise from the positive x axis.

a) What is the torque, as a function of the net angle the stick has rotated, $\theta - \theta_0$, required in order to keep the stick rotating at a constant rate ?

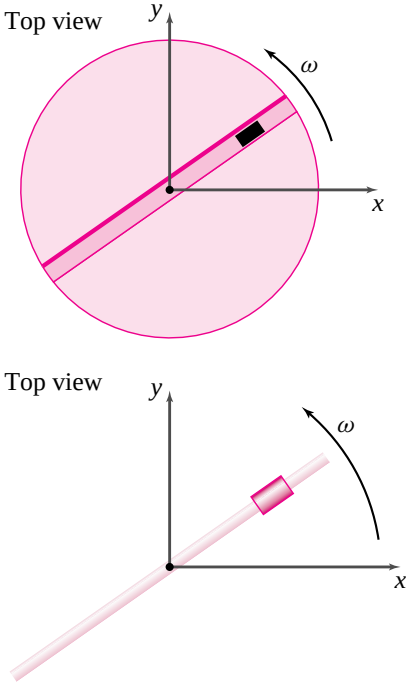
b) What is the path of the bead in the xy-plane? (Draw an accurate picture showing about one half of one revolution.)

c) How long should the stick be if the bead is to fly in the negative x-direction when it gets to the end of the stick?

d) **Add friction.** How does the speed of the bead over one revolution depend on μ, the coefficient of friction between the bead and the wire? Make a plot of $|\vec{\boldsymbol{v}}_{\text{one revolution}}|$ versus μ. [Note, you have been working with $\mu = 0$ in the problems above.]

e) **A finite rigid body.** What would change if instead of a point mass you modeled the bead as a finite rigid body (2-D), with center of mass on the stick, but contact at various places on the stick? (Do the frictionless case only.)

11.14 Mass on a lightly greased slotted turntable or spinning uniform rod. Assume that the rod/turntable in the figure is massless and also free to rotate. Assume that at $t = 0$, the angular velocity of the rod/turntable is 1 rad/s, that the radius of the bead is one meter, and that the radial velocity of the bead, dR/dt, is zero. The bead is free to slide on the rod. Where is the bead at $t = 5$ sec?

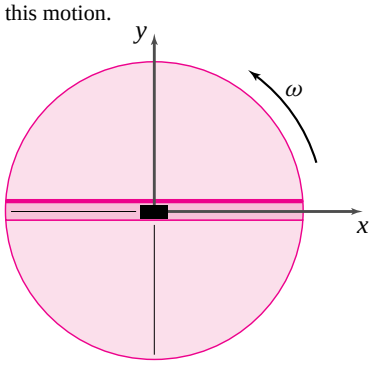


problem 11.14:

(Filename:pfigure.s94h9p4a)

11.15 A bead slides in a frictionless slot in a turntable. The turntable spins a constant rate ω. The slot is straight and goes through the center of the turntable. If the bead is at radius R_o with $\dot{R} = 0$ at $t = 0$, what are the components of the acceleration vector in the directions normal and tangential to the path of the bead after one revolution? Neglect gravity.

11.16 A bead of mass $m = 1$ gm bead is constrained to slide in a straight frictionless slot in a disk which is spinning counterclockwise at constant rate $\omega = 3$ rad/s per second. At time $t = 0$, the slot is parallel to the x-axis and the bead is in the center of the disk moving out (in the plus x direction) at a rate $\dot{R}_o = .5$ m/s. After a net rotation θ of one and one eighth (1.125) revolutions, what is the force $\vec{\boldsymbol{F}}$ of the disk on the bead? Express this answer in terms of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$. Make the unreasonable assumption that the slot is long enough to contain the bead for this motion.



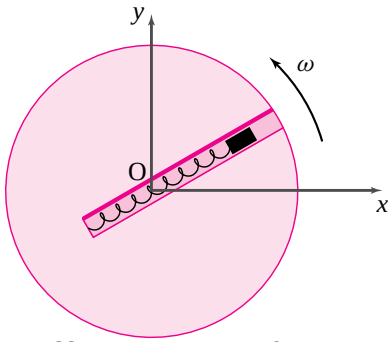
problem 11.16:

(Filename:pfigure.f93q11)

11.17 Bead on springy leash in a slot on a turntable. The bead in the figure is held by a spring that is relaxed when the bead is at the origin. The constant of the spring is k. The turntable speed is controlled by a strong stiff motor.

a) Assume $\omega = 0$ for all time. What are possible motions of the bead?

b) Assume $\omega = \omega_0$ is a constant. What are possible motions of the bead? Notice there are two cases depending on the value of ω_0. What is going on here?



problem 11.17: Mass in a slot.

(Filename:pg57.1a)

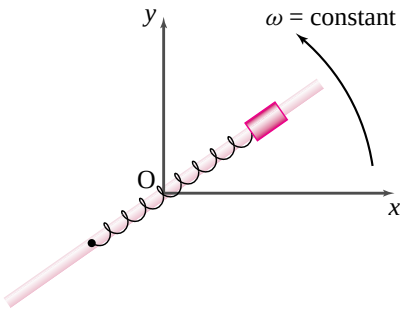
11.18 A small bead with mass m slides without friction on a rigid rod which rotates about the z axis with constant ω (maintained by a stiff motor not shown in the figure). The bead is also attached to a spring with constant K, the other end of which is attached to the rod. The spring is relaxed when the bead is at the center position. Assume the bead is released pulled to a distance d from the center of the rod and then released with an initial $\dot{R} = 0$. If needed, you may assume that $K > m\omega^2$.

a) Derive the equation of motion for the position of the bead $R(t)$.

b) How is the motion affected by large versus small values of K?

c) What is the magnitude of the force of the rod on the bead as the bead passes through the center position? Neglect gravity. Answer in terms of m, ω, d, K.

d) Write an expression for the Coriolis acceleration. Give an example of a situation in which this acceleration is important and explain why it arises.



problem 11.18:

(Filename:pfigure.s94q9p1)

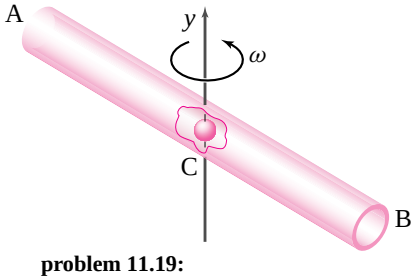
11.19 A small ball of mass $m_b = 500$ grams may slide in a slender tube of length $l = 1.2$ m. and of mass $m_t = 1.5$ kg. The tube rotates freely about a vertical axis passing through its center C. (Hint: Treat the tube as a slender rod.)

a) If the angular velocity of the tube is $\omega = 8$ rad/s as the ball passes through C with speed relative to the tube, v_0, calculate the angular velocity of the tube just before the ball leaves the tube. *

b) Calculate the angular velocity of the tube just after the ball leaves the tube. *

c) If the speed of the ball as it passes C is $v = 1.8$ m/s, determine the transverse and radial components of velocity of the ball as it leaves the tube. *

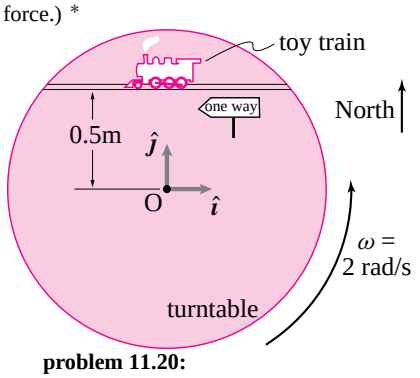
d) After the ball leaves the tube, what constant torque must be applied to the tube (about its axis of rotation) to bring it to rest in 10 s? *



problem 11.19:

(Filename:pfigure.blue.91.1)

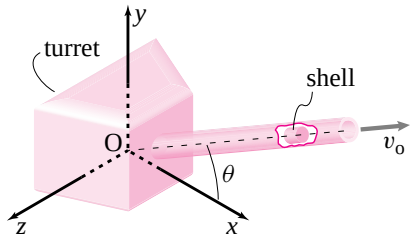
11.20 Toy train car on a turn-around. The 0.1 kg toy train's speedometer reads a constant $1m/s$ when, heading west, it passes due north of point O. The train is on a level straight track which is mounted on a spinning turntable whose center is 'pinned' to the ground. The turntable spins at the constant rate of 2 rad/s. What is the force of the turntable on the train? (Don't worry about the z-component of the force.) *



problem 11.20:

(Filename:pfigure.s94h12p1)

11.21 A shell moves down a barrel. At the instant of interest the barrel is being raised at a constant angular speed ω_1. Simultaneously the gun turret (the structure which holds the barrel) rotates about the vertical axis at a constant rate ω_2. At the instant of interest, the barrel is tipped up an angle θ and the mass m is moving at speed v along the frictionless barrel. Neglecting gravity, find the force of the barrel on the shell.



problem 11.21: A shell moves along a canon barrel.

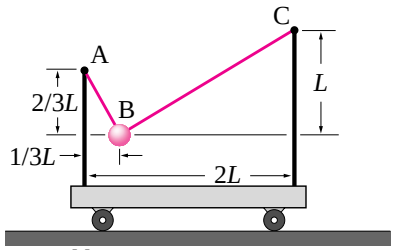
(Filename:pg122.2)

11.22 Due to forces not shown, the cart moves to the right with constant acceleration a_x. The ball B has mass m_B. At time $t = 0$, the string AB is cut. Find

a) the tension in string BC before cutting, *

b) the absolute acceleration of the mass at the instant of cutting, *

c) the tension in string BC at the instant of cutting. *



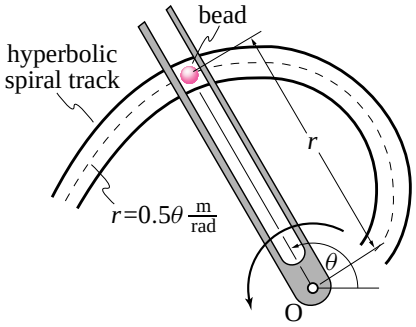
problem 11.22:

(Filename:Danef94s1q6)

11.23 The forked arm mechanism pushes the bead of mass 1 kg along a frictionless hyperbolic spiral track given by $r = 0.5\theta(\text{m/rad})$. The arm rotates about its pivot point at O with constant angular acceleration $\ddot{\theta} = 1\,\text{rad/s}^2$ driven by a motor (not shown). The arm starts from rest at $\theta = 0^o$.

a) Determine the radial and transverse components of the acceleration of the bead after 2 s have elapsed from the start of its motion.

b) Determine the magnitude of the net force on the mass at the same instant in time.



problem 11.23:

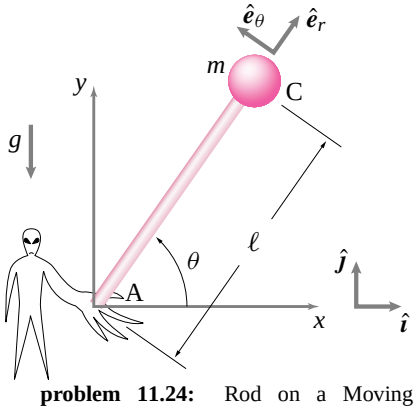
(Filename:pfigure.blue.53.1)

11.24 A rod is on the palm of your hand at point A. Its length is ℓ. Its mass m is assumed to be concentrated at its end at C. Assume that you know θ and $\dot{\theta}$ at the instant of interest. Also assume that your hand is accelerating both vertically and horizontally with $\vec{a}_{\text{hand}} = a_{hx}\hat{\imath} + a_{hy}\hat{\jmath}$. Coordinates and directions are as marked in the figure.

a) Draw a Free Body Diagram of the rod.

b) Assume that the hand is stationary. Solve for $\ddot{\theta}$ in terms of g, ℓ, m, θ, and $\dot{\theta}$.

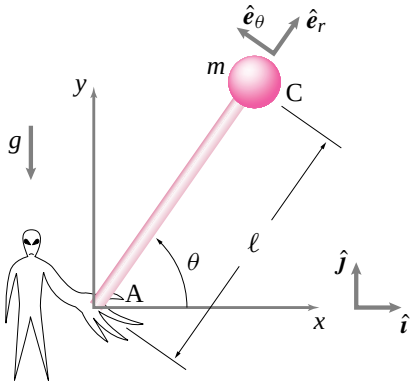
c) If the hand is not stationary but $\ddot{\theta}$ has been determined somehow, find the vertical force of the hand on the rod in terms of $\ddot{\theta}$, $\dot{\theta}$, θ, g, ℓ, m, a_{hx}, and a_{hy}.



problem 11.24: Rod on a Moving Support(2-D).

(Filename:p3.1)

11.25 Balancing a broom. Assume the hand is accelerating to the right with acceleration $\boldsymbol{a} = a\hat{\imath}$. What is the force of the hand on the broom in terms of m, ℓ, θ, $\dot{\theta}$, a, $\hat{\imath}$, $\hat{\jmath}$, and g? *(You may not have any $\hat{\boldsymbol{e}}_R$ or $\hat{\boldsymbol{e}}_\theta$ in your answer.)*



problem 11.25:

(Filename:p.f96.f.2)

11.2 Dynamics of rigid bodies in one-degree-of-freedom mechanisms

11.26 A conservative vibratory system has the following equation of conservation of energy.

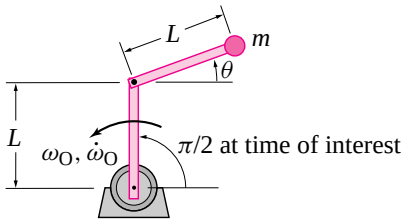
$$m(L\dot{\theta})^2 - mgL(1-\cos\theta) + K(a\theta)^2 = Constant$$

a) Obtain the differential equation of motion of this system by differentiating this energy equation with respect to t.

b) Determine the circular frequency of small oscillations of the system in part (a). HINT: (Let $\sin\theta \cong \theta \cong$ and $\cos\theta \cong 1 - \theta^2/2$).

11.27 A motor at O turns at rate ω_o whose rate of change is $\dot{\omega}_o$. At the end of a stick connected to this motor is a frictionless hinge attached to a second massless stick. Both sticks have length L. At the end of the second stick is a mass

m. For the configuration shown, what is $\ddot{\theta}$? Answer in terms of ω_o, $\dot{\omega}_o$, L, m, θ, and $\dot{\theta}$. Ignore gravity.



problem 11.27:

(Filename:pfigure.blue.119.1)

11.28 (See also problems 10.31 and 10.35.) In problem 9.12, find

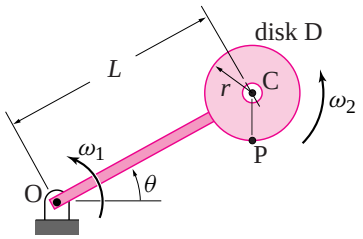
a) the angular momentum about point O, *

b) the rate of change of angular momentum of the disk about point O, *

c) the angular momentum about point C, *

d) the rate of change of angular momentum of the disk about point C. *

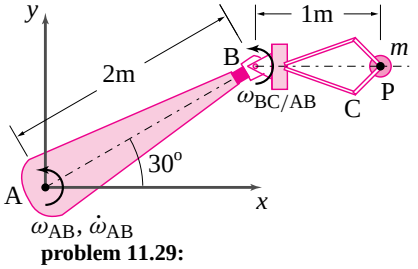
Assume the rod is massless and the disk has mass m.



problem 11.28:

(Filename:doublerot.ang.mom)

11.29 Robot arm, 2-D . The robot arm AB is rotating about point A with $\omega_{AB} = 5\,\text{rad/s}$ and $\dot{\omega}_{AB} = 2\,\text{rad/s}^2$. Meanwhile the forearm BC is rotating at a constant angular speed *with respect to AB* of $\omega_{BC/AB} = 3\,\text{rad/s}$. Gravity cannot be neglected. At the instant shown, find the *net* force acting on the object P which has mass $m = 1\,\text{kg}$. *



problem 11.29:

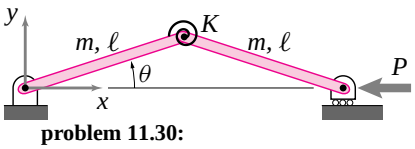
(Filename:pfigure.s94h12p5)

11.30 A crude model for a column, shown in the figure, consists of two identical rods of mass m and length ℓ with hinge connections, a linear torsional spring of stiffness K attached to the center hinge (the spring is relaxed when $\theta = 0$), and a load P applied at the top end.

a) Obtain the exact nonlinear equation of motion.

b) Obtain the squared natural frequency for small motions θ.

c) Check the stability of the straight equilibrium state $\theta = 0$ for all $P \geq 0$ via minimum potential energy. How do these results compare with those from part(b)?



problem 11.30:

(Filename:pfigure.blue.159.2)

11.31 A thin uniform rod of mass m rests against a frictionless wall and on a frictionless floor. There is gravity.

a) Draw a free body diagram of the rod.

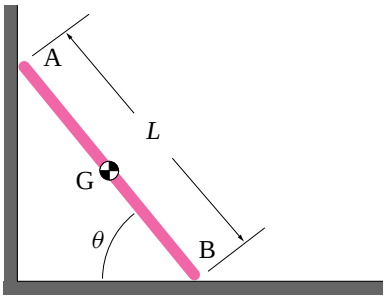
b) The rod is released from rest at $\theta = \theta_0 \neq 0$. Write the equation of motion of the rod. *

c) Using the equation of motion, find the initial angular acceleration, $\dot{\boldsymbol{\omega}}_{AB}$, and the acceleration of the center of mass, $\vec{\boldsymbol{a}}_G$, of the rod. *

d) Find the reactions on the rod at points A and B. *

e) Find the acceleration of point B. *

f) When $\theta = \frac{\theta_0}{2}$, find $\vec{\boldsymbol{\omega}}_{AB}$ and the acceleration of point A. *



problem 11.31:

(Filename:ch7.1)

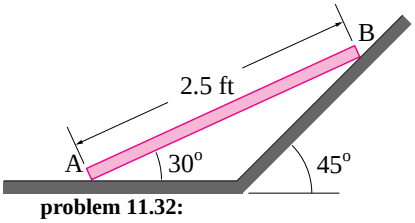
11.32 Bar leans on a crooked wall. A uniform 3 lbm bar leans on a wall and floor and is let go from rest. Gravity pulls it down.

a) Draw a Free Body Diagram of the bar.

b) Kinematics: find the velocity and acceleration of point B in terms of the velocity and acceleration of point A

c) Using equations of motion, find the acceleration of point A.

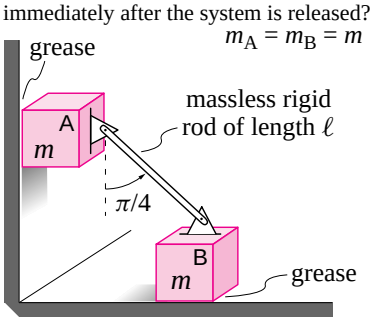
d) Do the reaction forces at A and B add up to the weight of the bar? Why or why not? (You do not need to solve for the reaction forces in order to answer this part.)



problem 11.32:

(Filename:pfigure.s94h10p3)

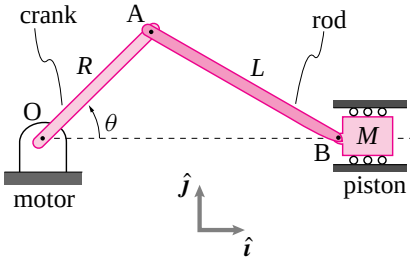
11.33 Two blocks, each with mass m, slide without friction on the wall and floor shown. They are attached with a rigid massless rod of length ℓ that is pinned at both ends. The system is released from rest when the rod makes an angle of 45°. What is the acceleration of the block B immediately after the system is released?



problem 11.33:

(Filename:pfigure.blue.98.2)

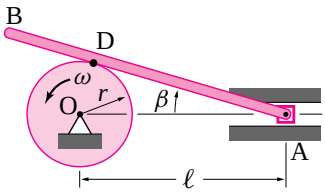
11.34 Slider Crank. 2-D . No gravity. Refer to the figure in problem 10.19. What is the tension in the massless rod AB (with length L) when the slider crank is in the position with $\theta = 0$ (piston is at maximum extent)? Assume the crankshaft has constant angular velocity ω, that the connecting rod AB is massless, that the cylinder walls are frictionless, that there is no gas pressure in the cylinder, that the piston has mass M and the crank has radius R.



problem 11.34:

(Filename:slidercrank.tension)

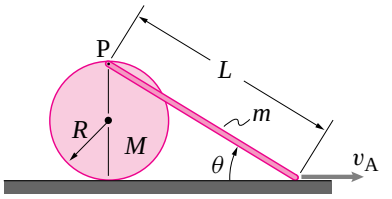
11.35 In problem 10.25, find the force on the wheel at point D due to the rod.



problem 11.35:

(Filename:roddisc.force)

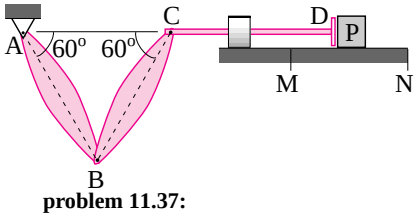
11.36 In problem 10.26, find the force on the rod at point P.



problem 11.36:

(Filename:rodcyl.force)

11.37 Slider-Crank mechanism. The slider crank mechanism shown is used to push a 2 lbm block P. Arm AB and BC are each 0.5 ft long. Given that arm AB rotates counterclockwise at a constant 2 revolutions per second, what is the force on P at the instant shown? *



problem 11.37:

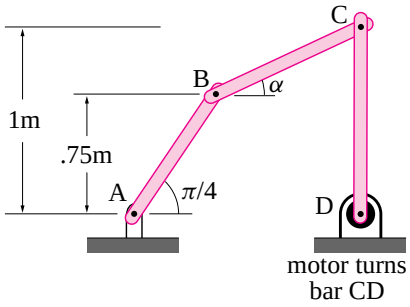
(Filename:pfigure.s94h11p1)

11.38 Consider the *4-bar linkage* AB, BC, CD and the ground DA. The bars are connected by well lubricated hinges at A, B, and C. At D, bar CD is driven by the motor which spins at constant rate. Do not ignore gravity.

a) Draw a Free Body Diagram of the linkage ABCD (with 'cuts' at A and D but not at B and C.)

b) Draw Free Body Diagrams of each of the bars AB, BC and CD. Use the law of action and reaction in these diagrams. If you can, do this problem a few times, each with a different vector notation.

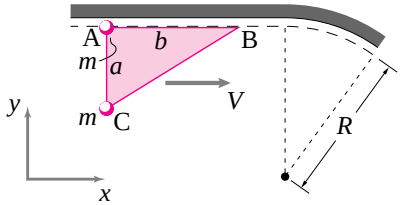
c) Repeat problem (b) but this time assume that the mass of bar BC is negligible. [Hint, this implies that the laws of statics apply to bar BC, which has connections at the *two* points B and C.]



problem 11.38: Free Body Diagram Exercise

(Filename:pfigure.s94h2p7)

11.39 The large masses m at C and A were supported by the light triangular plate ABC, whose corners follow the guide. B enters the curved guide with velocity V. Neglecting gravity, find the vertical reactions (in the y direction) at A and B. (Hint: for the rigid planar body ABC, find $\ddot{y}_A$ and $\ddot{y}_B$ in terms of a_y^{cm} and $\ddot{\theta}$, assuming $\dot{\theta} = 0$ initially.)



problem 11.39:

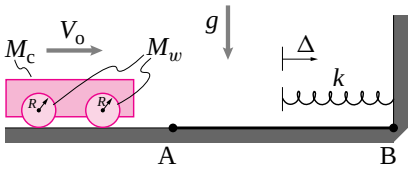
(Filename:pfigure.blue.85.2)

11.40 An idealized model for a car comprises a rigid chassis of mass M_c and four identical rigid disks (wheels) of mass M_w and radius R, as shown in the figure. Initially in motion with speed v_0, the car is momentarily brought to rest by compressing an initially uncompressed spring of stiffness k. Assume no frictional losses.

a) Assuming no wheel slip, determine the compression Δ of the spring required to stop the car. *

b) While the car is in contact with the spring but still moving forward, in which direction is the *tangential* force on any one of the wheels (due to contact with the ground) pointing? Why? (Illustrate with a sketch). *

c) Repeat part (a) assuming now that the ground is perfectly frictionless from points A to B shown in the figure. *



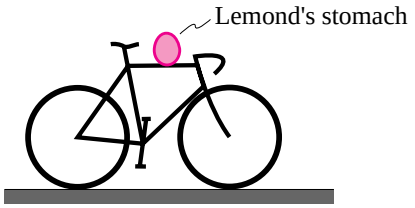
problem 11.40:

(Filename:pfigure.blue.135.2)

11.41 While walking down a flat stretch of road you see a dejected Greg Lemond riding by. His riding "tuck" is so good that you realize that you can totally neglect air resistance when you think about him and his bike. Further, you can regard his and the bike's combined mass (all 70 kg)) as concentrated at a point in his stomach somewhere. Greg's left foot has just fallen off the pedal so he is only pedaling with his right foot, which at the moment in question is at its lowest point in the motion. You note that, relative to the ground the right foot is only going 3/4 as fast as the bike (since it is going backwards relative to the bike), though you can't make out all the radii of his frictionless gears and rigid round wheels. Greg, ever in touch with his body, tells you he is pushing back on the pedal with a force of 70 N. You would like to know Greg's acceleration.

a) In your first misconceived experiment you set up a 70 kg bicycle in your laboratory and balance it with strings that cause no fore or aft forces. You tie a string to the vertically down right pedal and pull back with a force of 70 N. What acceleration do you measure?

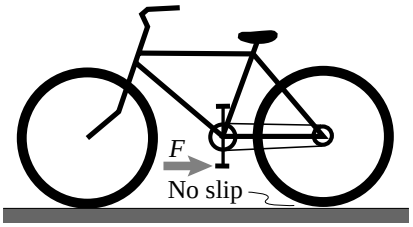
b) What is Greg's actual acceleration? [Hint: Greg's massless leg is pushing forward on his body with a force of 70 N]



problem 11.41:

(Filename:pfigure.blue.34.2)

11.42 Which way does the bike accelerate? A bicycle with all frictionless bearings is standing still on level ground. A horizontal force F is applied on one of the pedals as shown. There is no slip between the wheels and the ground. The bicycle is gently balanced from falling over sideways. It is heavy enough so that both wheels stay on the ground. Does the bicycle accelerate forward, backward, or not at all? Make any reasonable assumptions about the dimensions and mass. Justify your answer as clearly as you can, clearly enough to convince a person similar to yourself but who has not seen the experiment performed.



problem 11.42:

(Filename:pfigure.s94h14p2)

11.3 Dynamics of multi-degree-of-freedom mechanisms

11.43 Particle on a springy leash. A particle with mass m slides on a rigid horizontal frictionless plane. It is held by a string which is in turn connected to a linear elastic spring with constant k. The string length is such that the spring is relaxed when the mass is on top of the hole in the plane. The position of the particle is $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$. For each of the statements below, state the circumstances in which the statement is true (assuming the particle stays on the plane). Justify your answer with convincing explanation and/or calculation.

a) The force of the plane on the particle is $mg\hat{\boldsymbol{k}}$.

b) $\ddot{x} + \frac{k}{m}x = 0$

c) $\ddot{y} + \frac{k}{m}y = 0$

d) $\ddot{r} + \frac{k}{m}r = 0$, where $r = |\vec{\boldsymbol{r}}|$

e) $r =$ constant

f) $\dot{\theta} =$constant

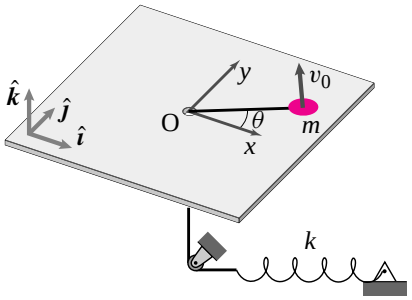
g) $r^2\dot{\theta} =$ constant.

h) $m(\dot{x}^2 + \dot{y}^2) + kr^2 =$ constant

i) The trajectory is a straight line segment.

j) The trajectory is a circle.

k) The trajectory is not a closed curve.



problem 11.43: Particle on a springy leash.

(Filename:pg69.1)

11.44 "Yo-yo" mechanism of satelite de-spinning. A sattelite, modeled as a uniform disk, is "de-spun" by the following mechanism. Before launch two equal length long strings are attached to the sattelite at diametrically opposite points and then wound around the sattelite with the same sense of rotation. At the end of the strings are placed 2 equal masses. At the start of de-spinning the two masses are released from their position wrapped against the satelite. Find the motion of the satelite as a function of time for the two cases: (a) the strings are wrapped in the same direction as the initial spin, and (b) the strings are wrapped in the opposite direction as the initial spin.

11.45 A particle with mass m is held by two long springs each with stiffness k so that the springs are relaxed when the mass is at the origin. Assume the motion is planar. Assume that the particle displacement is much smaller than the lengths of the springs.

a) Write the equations of motion in cartesian components. *

b) Write the equations of motion in polar coordinates. *

c) Express the conservation of angular momentum in cartesian coordinates. *

d) Express the conservation of angular momentum in polar coordinates. *

e) Show that (a) implies (c) and (b) implies (d) even if you didn't note them *a-priori*. *

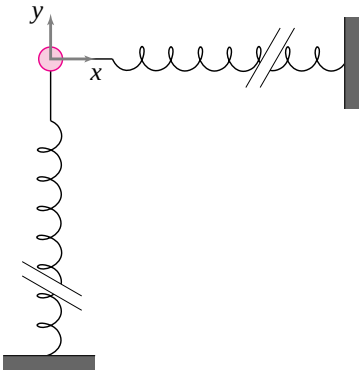
f) Express the conservation of energy in cartesian coordinates. *

g) Express the conservation of energy in polar coordinates. *

h) Show that (a) implies (f) and (b) implies (g) even if you didn't note them *a-priori*. *

i) Find the general motion by solving the equations in (a). Describe all possible paths of the mass. *

j) Can the mass move back and forth on a line which is not the x or y axis? *



problem 11.45:

(Filename:pfigure.blue.73.1)

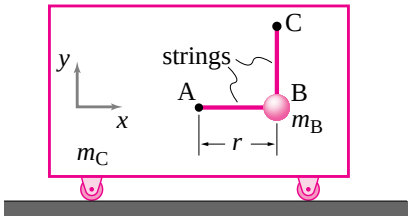
11.46 Cart and pendulum A mass $m_B = 6\,\text{kg}$ hangs by two strings from a cart with mass $m_C = 12\,\text{kg}$. Before string BC is cut string AB is horizontal. The length of string AB is $r = 1\,\text{m}$. At time $t = 0$ all masses are stationary and the string BC is cleanly and quietly cut. After some unknown time t_{vert} the string AB is vertical.

a) What is the net displacement of the cart x_C at $t = t_{\text{vert}}$? *

b) What is the velocity of the cart v_C at $t = t_{\text{vert}}$? *

c) What is the tension in the string at $t = t_{\text{vert}}$? *

d) (**Optional**) What is $t = t_{\text{vert}}$? [You will either have to leave your answer in the form of an integral you cannot evaluate analytically or you will have to get part of your solution from a computer.]



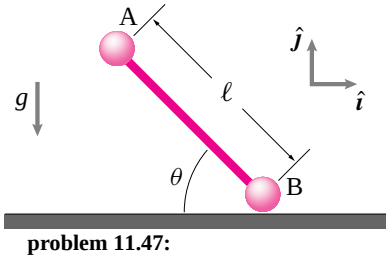
problem 11.46:

(Filename:pfigure.s94h12p3)

11.47 A dumbell slides on a floor. Two point masses m at A and B are connected by a massless rigid rod with length ℓ. Mass B slides on a frictionless floor so that it only moves horizontally. Assume this dumbell is released from rest in the configuration shown. *[Hint: What is the acceleration of A relative to B?]*

a) Find the acceleration of point B just after the dumbell is released.

b) Find the velocity of point A just before it hits the floor



problem 11.47:

(Filename:s97f5)

11.48 After a winning goal one second before the clock ran out a psychologically stunned hockey player (modeled as a uniform rod) stands nearly vertical, stationary and rigid. The players perfectly slippery skates start to pop out from under her as she falls. Her height is ℓ, her mass m, her tip from the vertical θ, and the gravitational constant is g.

a) What is the path of her center of mass as she falls? (Show clearly with equations, sketches or words.)

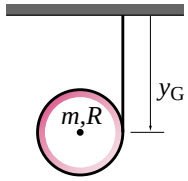
b) What is her angular velocity just before she hits the ice, a millisecond before she sticks our her hands and brakes her fall (first assume her skates remain in contact the whole time and then check the assumption)?

c) Find a differential equation that only involves θ, its time derivatives, m, g, and ℓ. (This equation could be solved to find θ as a function of time. It is a nonlinear equation and you are not being asked to solve it numerically or otherwise.)

11.49 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R. G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

a) Find y_G at a later time t in terms of any or all of m, R, g, and t.

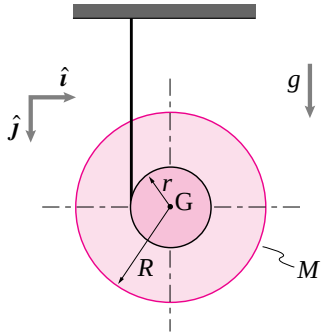
b) Does G move sideways as the hoop falls and unrolls?



problem 11.49:

(Filename:p.s96.p3.2)

11.50 A model for a yo-yo comprises a thin disk of mass M and radius R and a light drum of radius r, rigidly attached to the disk, around which a light inextensible cable is wound. Assuming that the cable unravels without slipping on the drum, determine the acceleration a_G of the center of mass.



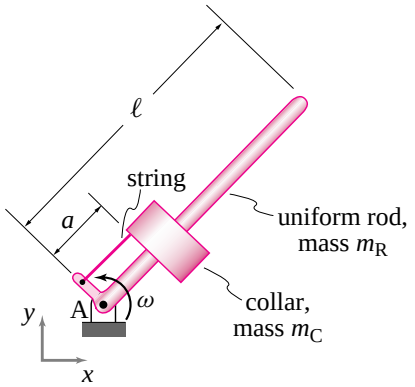
problem 11.50:

(Filename:pfigure.blue.41.2)

11.51 A uniform rod with mass m_R pivots without friction about point A in the xy-plane. A collar with mass m_C slides without friction on the rod after the string connecting it to point A is cut. There is no gravity. Before the string is cut, the rod has angular velocity ω_1.

a) What is the speed of the collar after it flies off the end of the rod? Use the following values for the constants and initial conditions: $m_R = 1\,\text{kg}$, $m_C = 3\,\text{kg}$, $a = 1\,\text{m}$, $\ell = 3\,\text{m}$, and $\omega_1 = 1\,\text{rad/s}$

b) Consider the special case $m_R = 0$. Sketch (approximately) the path of the motion of the collar from the time the string is cut until some time after it leaves the end of the rod.



problem 11.51:

(Filename:pfigure.blue.54.1)

11.52 (See also problem 11.53.) Assume the rod in the figure for problem 10.29 has polar moment of inertia $Iozz$. Assume it is free to rotate. The bead is free to slide on the rod. Assume that at $t = 0$ the angular velocity of the rod is 1 rad/s, that the radius of the bead is one meter and that the radial velocity of the bead, dR/dt, is zero.

a) Draw separate free body diagrams of the bead and rod.

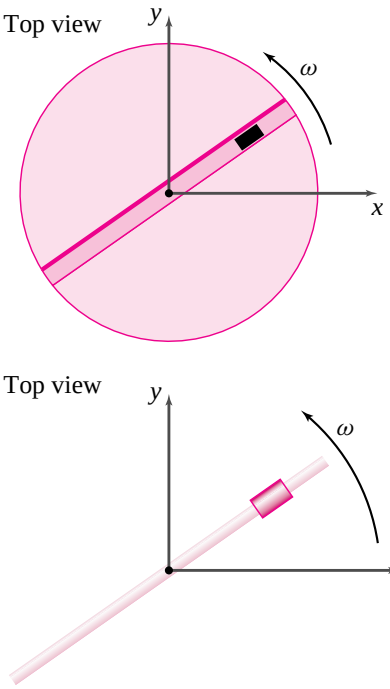
b) Write equations of motion for the system. *

c) Use the equations of motion to show that angular momentum is conserved. *

d) Find one equation of motion for the system using: (1) the equations of motion for the bead and rod and (2) conservation of angular momentum. *

e) Write an expression for conservation of energy. Let the initial total energy of the system be, say, E_0. *

f) As t goes to infinity does the bead's distance go to infinity? Its speed? The angular velocity of the turntable? The net angle of twist of the turntable? *



problem 11.52: Coupled motion of bead and rod and turntable.

(Filename:pfigure.s94h9p4b.a)

11.53 Numerically simulate the coupled system in problem 11.52. Use the simulation to show that the net angle of the turntable or rod is finite.

a) Write the equations of motion for the system from part (b) in problem 11.52 as a set of first order differential equations. *

b) Numerically integrate the equations of motion. (Below, we show a solution using MATLAB.)

```
%newgunscript.m   quiz 13 soln  - A Ruina  4/27/95
z0     = [0,3,1,0];  % [Theta, omega, R, v]
tfinal = 10000  ;
tol    = 1e-7   ;

[t,z]  = ode45('newgunfun', 0, tfinal, z0, tol);
thetamax=max(z(:,1));

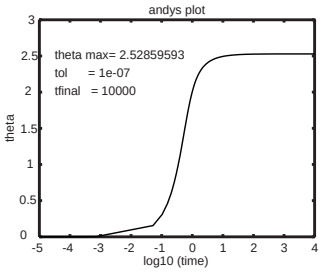
plot(log10(t),z(:,1));
title('andys plot'); xlabel('log10 (time)');
ylabel('theta');
axis([-5, log10(tfinal), 0 3])

text(-4.5,2.5 ,['theta max= ' ,num2str(thetamax,9)] )
text(-4.5,2.25,['tol      = ' ,num2str(tol,9)]      )
text(-4.5,2,   ['tfinal   = ' ,num2str(tfinal,9)]   )
```

```
function  zdot=newgunfun(t,z)
m=.25;    I=2;

theta = z(1);
omega = z(2);
R     = z(3);
v     = z(4);

zdot  = [ omega,
          -2*m*omega*R*v/(I + m*R^2),
          v,
          omega^2* R                 ];
```



problem 11.53:

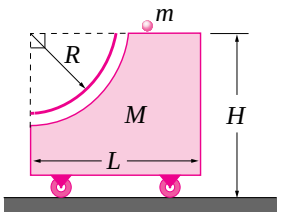
(Filename:q13soln)

11.54 A primitive gun rides on a cart (mass M) and carries a cannon ball (of mass m) on a platform at a height H above the ground. The cannon ball is dropped through a frictionless tube shaped like a quarter circle of radius R.

a) If the system starts from rest, compute the horizontal speed (relative to the ground) that the cannon ball has as it leaves the bottom of the tube. Also find the cart's speed at the same instant.

b) Compute the velocity of the cannon ball relative to the cart.

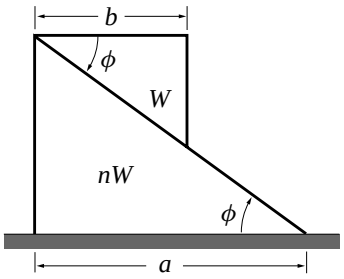
c) If two balls are dropped simultaneously through the tube, what speed does the cart have when the balls reach the bottom? Is the same final speed also achieved if one ball is allowed to depart the system entirely before the second ball is released? Why/why not?



problem 11.54:

(Filename:pfigure.blue.18.1)

11.55 Two frictionless prisms of similar right triangular sections are placed on a frictionless horizontal plane. The top prism weighs W and the lower one nW. The prisms are held in the initial position shown and then released, so that the upper prism slides down along the lower one until it just touches the horizontal plane. The center of mass of a triangle is located at one-third of its height from the base. Compute the **velocities** of the two prisms at the moment just before the upper one reaches the bottom. *



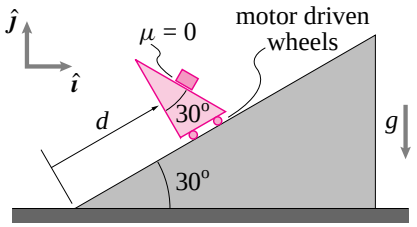
problem 11.55:

(Filename:pfigure.blue.28.2a)

11.56 Mass slides on an accelerating cart. 2D. A cart is driven by a powerful motor to move along the 30° sloped ramp according to the formula: $d = d_0 + v_o t + a_0 t^2/2$ where d_0, v_0, and a_0 are given constants. The cart is held from tipping over. The cart itself has a 30° sloped upper surface on which rests a mass (given mass m). The surface on which the mass rests is frictionless. Initially the mass is at rest with regard to the cart.

a) What is the force of the cart on the mass? [in terms of $g, d_0, v_0, t, a_0, m, g, \hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{\jmath}}$.] *

b) For what values of d_0, v_0, t and a_0 is the acceleration of the mass exactly vertical (i.e., in the $\hat{\boldsymbol{\jmath}}$ direction)? *



problem 11.56:

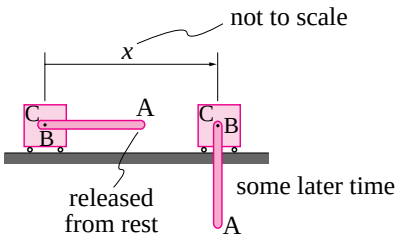
(Filename:pfigure.s95q11)

11.57 A thin rod AB of mass $W_{AB} = 10$ lbm and length $L_{AB} = 2$ ft is pinned to a cart C of mass $W_C = 10$ lbm, the latter of which is free to move along a frictionless horizontal surface, as shown in the figure. The system is released from rest with the rod in the horizontal position.

a) Determine the angular speed of the rod as it passes through the vertical position (at some later time). *

b) Determine the displacement x of the cart at the same instant. *

c) After the rod passes through vertical, it is momentarily horizontal but on the left side of the cart. How far has the cart moved when this configuration is reached?



problem 11.57:

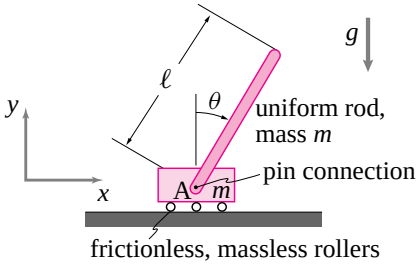
(Filename:pfigure.blue.115.3)

11.58 As shown in the figure, a block of mass m rolls without friction on a rigid surface and is at position x (measured from a fixed point). Attached to the block is a uniform rod of length ℓ which pivots about one end which is at the center of mass of the block. The rod and block have equal mass. The rod makes an angle θ with the vertical. Use the numbers below for the values of the constants and variables at the time of interest:

$$\begin{aligned} \ell &= 1\,\text{m} \\ m &= 2\,\text{kg} \\ \theta &= \pi/2 \\ d\theta/dt &= 1\,\text{rad/s} \\ d^2\theta/dt^2 &= 2\,\text{rad/s}^2 \\ x &= 1\,\text{m} \\ dx/dt &= 2\,\text{m/s} \\ d^2x/dt^2 &= 3\,\text{m/s}^2 \end{aligned}$$

a) What is the kinetic energy of the system?

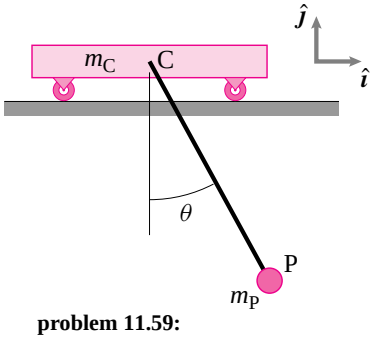
b) What is the linear momentum of the system (momentum is a vector)?



problem 11.58:

(Filename:pfigure.blue.14.1)

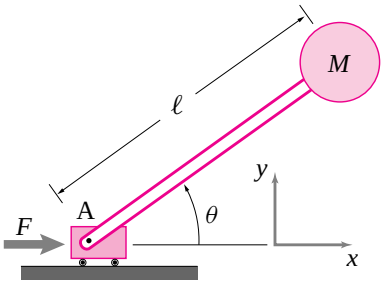
11.59 A pendulum of length ℓ hangs from a cart. The pendulum is massless except for a point mass of mass m_p at the end. The cart rolls without friction and has mass m_c. The cart is initially stationary and the pendulum is released from rest at an angle θ. What is the acceleration of the cart just after the mass is released? [Hints: $\mathbf{a}_P = \vec{\boldsymbol{a}}_C + \vec{\boldsymbol{a}}_{P/C}$. The answer is $\vec{\boldsymbol{a}}_C = (g/3)\mathbf{i}$ in the special case when $m_p = m_c$ and $\theta = \pi/4$.]



problem 11.59:

(Filename:pfigure.f93q12)

11.60 Due to the application of some unknown force F, the base of the pendulum A is accelerating with $\vec{\boldsymbol{a}}_A = a_A\boldsymbol{i}$. There is a frictionless hinge at A. The angle of the pendulum θ with the x axis and its rate of change $\dot{\theta}$ are assumed to be known. The length of the massless pendulum rod is ℓ. The mass of the pendulum bob M. There is no gravity. What is $\ddot{\theta}$? (Answer in terms of a_A, ℓ, M, θ and $\dot{\theta}$.)

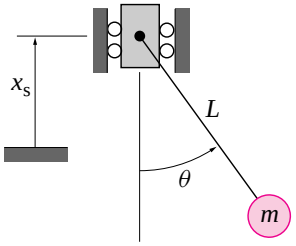


problem 11.60:

(Filename:pfigure.blue.115.2)

11.61 Pumping a Swing Can a swing be pumped by moving the support point up and down?

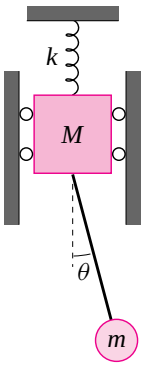
For simplicity, neglect gravity and consider the problem of swinging a rock in circles on a string. Let the rock be mass m attached to a string of fixed length ℓ. Can you speed it up by moving your hand up and down? How? Can you make a quantitative prediction? Let x_S be a function of time that you can specify to try to make the mass swing progressively faster.



problem 11.61:

(Filename:pfigure.blue.59.1)

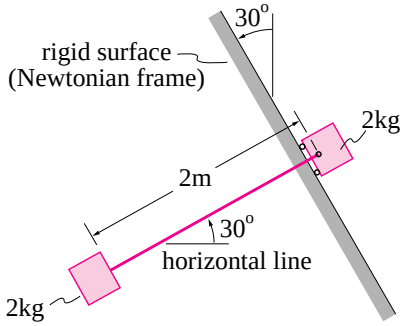
11.62 Using free body diagrams and appropriate momentum balance equations, find differential equations that govern the angle θ and the vertical deflection y of the system shown. Be clear about your datum for y. Your equations should be in terms of θ, y and their time-derivatives, as well as M, m, ℓ, and g.



problem 11.62:

(Filename:pfig2.1.rp11)

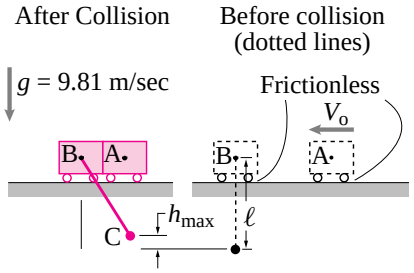
11.63 The two blocks shown are released from rest at $t = 0$. There is no friction and the cable is initially taut. (a) What is the tension in the cable immediately after release? (Use any reasonable value for the gravitational constant). (b) What is the tension after 5 s?



problem 11.63:

(Filename:pfigure.blue.18.2)

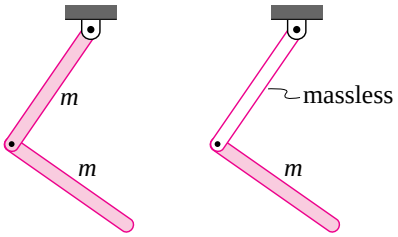
11.64 Carts A and B are free to move along a frictionless horizontal surface, and bob C is connected to cart B by a massless, inextensible cord of length ℓ, as shown in the figure. Cart A moves to the left at a constant speed $v_0 = 1$ m/s and makes a perfectly plastic collision ($e = 0$) with cart B which, together with bob C, is stationary prior to impact. Find the maximum vertical position of bob C, h_{max}, after impact. The masses of the carts and pendulum bob are $m_A = m_B = m_C = 10$ kg.



problem 11.64:

(Filename:pfigure.blue.137.3)

11.65 See also problem 3.29. A double pendulum is made of two uniform rigid rods, each of length ℓ. The first rod is massless. Find equations of motion for the second rod. Define any variables you use in your solution.



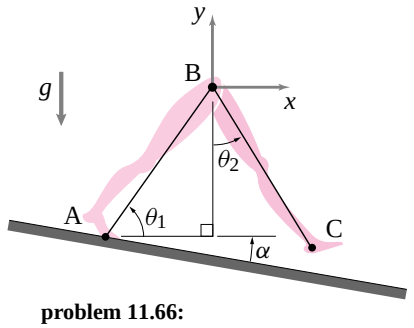
problem 11.65:

(Filename:pfig2.1.rp10)

11.66 A model of walking involves two straight legs. During the part of the motion when one foot is on the ground, the system looks like the picture in the figure, confined to motion in the $x - y$ plane. Write two equations from which one could find $\ddot{\theta}_1$, and $\ddot{\theta}_2$ given $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$ and all mass and length quantities.

Hint:

$\sum \vec{M}_{/A}$	=	$\dot{\vec{H}}_{/A}$	whole system
$\sum \vec{M}_{/B}$	=	$\dot{\vec{H}}_{/B}$	for bar BC



problem 11.66:

(Filename:pfigure.blue.100.1)

11.4 Advanced problems in 2D motion

11.67 A pendulum is hanging from a moving support in the xy-plane. The support moves in a known way given by $\vec{r}(t) = r(t)\hat{\imath}$. For the following cases, find a differential equation whose solution would determine $\theta(t)$, measured clockwise from vertical; then find an expression for $d^2\theta/dt^2$ in terms of θ and $r(t)$:

a) with no gravity and assuming the pendulum rod is massless with a point mass of mass m at the end,

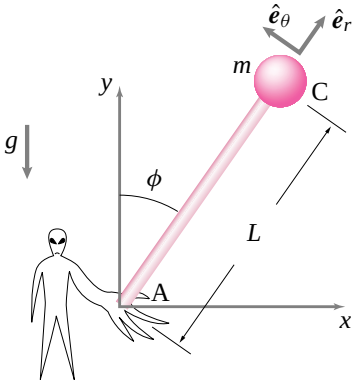
b) as above but with gravity,

c) assuming the pendulum is a uniform rod of mass m and length ℓ.

11.68 Robotics problem: balancing a broom stick by sideways motion. See also problem 7.99 and problem 8. Try to balance a broom stick by moving your hand horizontally. Model your hand contact with the broom as a hinge. You can model the broom as a uniform stick or as a point mass at the end of a stick — your choice.

a) **Equation of motion.** Given the acceleration of your hand (horizontal only), the current tip, and the rate of tip of the broom, find the angular acceleration of the broom. *

b) **Control?** Can you find a hand acceleration in terms of the tip and the tip rate that will make the broom balance upright? *



problem 11.68:

(Filename:pfigure.s94h10p5)

11.69 Balancing the broom again: vertical shaking works too. You can balance a broom by holding it at the bottom and applying appropriate torques, as in problem 7.99 or by moving your hand back and forth in an appropriate manner, as in problem 11.68. In this problem, you will to try to balance the broom differently. The lesson to be learned here is more subtle, and you should probably just wonder at it rather than try to understand it in detail.

In the previous balancing schemes, you used knowledge of the state of the broom (θ and $\dot{\theta}$) to determine what corrective action to apply. Now balance the broom by moving it in a way that ignores what the broom is doing. In the language of robotics, what you have been doing before is 'closed-loop feedback' control. The new strategy, which is simultaneously more simple minded and more subtle, is an 'open loop' control. Imagine that your hand connection to the broom is a hinge.

a) **Picture and model.** Assume your hand oscillates sinusoidally up and down with some frequency and some amplitude. The broom is instantaneously at some angle from vertical. Draw a picture which defines all the variables you will use. Use any mass distribution that you like.

b) **FBD.** Draw a FBD of the broom.

c) **Momentum balance.** Write the equation of angular momentum balance about the point instantaneously coinciding with the hinge.

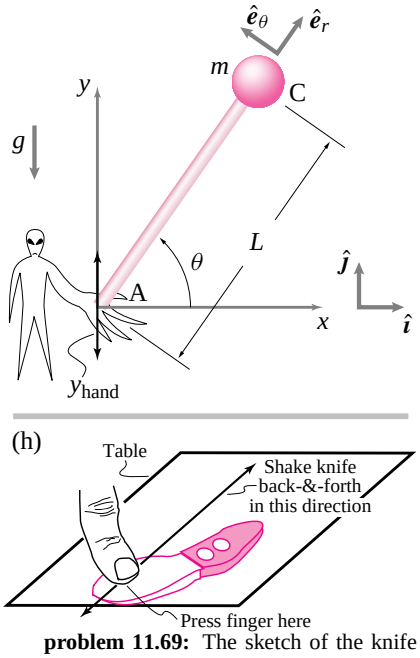
d) **Kinematics.** Use any geometry and kinematics that you need to evaluate the terms in the angular momentum balance equation in terms of the tip angle and its time derivatives and other known quantities (take the vertical motion of your hand as 'given'). [Hint: There are many approaches to this problem.]

e) **Equation of motion.** Using the angular momentum balance equation, write a governing differential equation for the tip angle. *

f) **Simulation.** Taking the hand motion as given, simulate on the computer the system you have found.

g) **Stability?** Can you find an amplitude and frequency of shaking so that the broom stays upright if started from a near upright position? You probably *cannot* find linear equations to solve that will give you a control strategy. So, this problem might best be solved by guessing on the computer. Successful strategies require the hand acceleration to be quite a bit bigger than g, the gravitational constant. The stability obtained is like the stability of an undamped uninverted pendulum — oscillations persist. You improve the stability a little by including a little friction in the hinge.

h) **Dinner table experiment for nerdy eaters.** If you put a table knife on a table and put your finger down on the tip of the blade, you can see that this experiment might work. Rapidly shake your hand back and forth, keeping the knife from sliding out from your finger but with the knife sliding rapidly on the table. Note that the knife aligns with the direction of shaking (use scratch-resistant surface). The knife is different from the broom in some important ways: there is no gravity trying to 'unalign' it and there is friction between the knife and table that is much more significant than the broom interaction with the air. Nonetheless, the experiment should convince you of the plausibility of the balancing mechanism. Because of the large accelerations required, you *cannot* do this experiment with a broom.

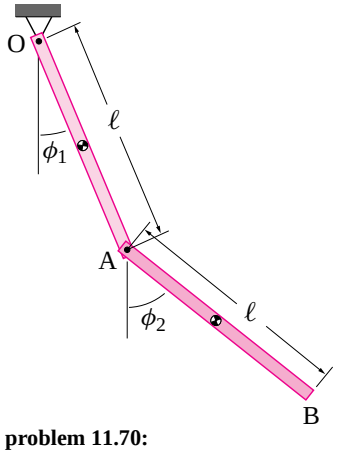


problem 11.69: The sketch of the knife on the table goes with part (h).

(Filename:pfigure.vert.shake)

11.5 Dynamics of rigid bodies in multi-degree-of-freedom 2D mechanisms

11.70 Double pendulum. The double pendulum shown is made up of two uniform bars, each of length ℓ and mass m. The pendulum is released from rest at $\phi_1 = 0$ and $\phi_2 = \pi/2$. Just after release what are the values of $\ddot{\phi}_1$ and $\ddot{\phi}_2$? Answer in terms of other quantities. *



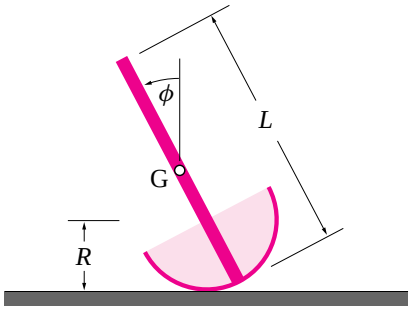
problem 11.70:

(Filename:pfigure.s94h12p2)

11.71 A rocker. A standing dummy is modeled as having massless rigid circular feet of radius R rigidly attached to their uniform rigid body of length L and mass m. The feet do not slip on the floor.

a) Given the tip angle ϕ, the tip rate $\dot{\phi}$ and the values of the various parameters (m, R, L, g) find $\ddot{\phi}$. [You may assume ϕ and $\dot{\phi}$ are small.] *

b) Using the result of (a) or any other clear reasoning find the conditions on the parameters (m, R, L, g) that make vertical passive dynamic standing stable. [Stable means that if the person is slightly perturbed from vertically up that their resulting motion will be such that they remain nearly vertically up for all future time.] answer below is questionable, says someone *



problem 11.71:

(Filename:pfigure.s94q11p1)

11.6 Advanced dynamics of planar motion

11.72 Consider a rigid spoked wheel with no rim. Assume that when it rolls a spoke hits the ground and doesn't bounce. The body just swings around the contact point until the next spoke hits the ground. The uniform spokes have length R. Assume that the mass of the wheel is m, and that the polar moment of inertia about its center is I (use $I = mR^2/2$ if you want to get a better sense of the solution). Assume that just before collision number n, the angular velocity of the wheel is ω_{n-}, the kinetic energy is T_{n-}, the potential energy (you must clearly define your datum) is U_{n-}. Just after collision n the angular velocity of the wheel is ω_{n+}. The Kinetic Energy is T_{N+}, the potential energy (you must clearly define your datum) is U_{n+}. The wheel has k spokes (pick $k = 4$ if you have trouble with abstraction). This problem is not easy. It can be answered at a variety of levels. The deeper you get into it the more you will learn.

a) What is the relation between ω_{n-} and T_{n-}?

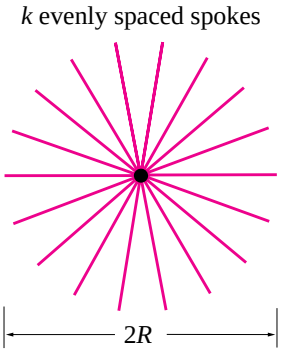
b) What is the relation between ω_{n-} and ω_{n+}?

c) Assume 'rolling' on level ground. What is the relation between ω_{n+} and ω_{n+1}?

d) Assume rolling down hill at slope θ. What is the relation between ω_{n+} and ω_{n+1}?

e) Can it be true that $\omega_{n+} = \omega_{(n+1)+}$? About how fast is the wheel going in this situation?

f) As the number of spokes m goes to infinity, in what senses does this wheel become like an ordinary wheel?



problem 11.72:

(Filename:pfigure.blue.109.1)

Problems for Chapter 12

General motion in 3-D

12.1 Orientation in 3-D

12.1 Find a rotation from a picture.

12.2 Velocity and acceleration of points on a rigid body: $\vec{\omega}$ and $\vec{\alpha}$

12.2 A particle moves on a helix. Say you know that a particle moves according to the equation

$$\boldsymbol{r} = R\cos\theta\,\hat{\boldsymbol{\imath}} + R\sin\theta\,\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

where R is a constant, $\theta = c_1 t$, and $z = c_2 t$.

a) Find $\vec{\boldsymbol{v}}$ at general time t.

b) Find $\vec{\boldsymbol{a}}$ at general time t.

c) Find $\hat{\boldsymbol{e}}_t$ at general time t.

d) Find $\hat{\boldsymbol{e}}_n$ at general time t.

e) Write $\vec{\boldsymbol{a}}$ in terms of $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$.

f) What is the radius of the osculating circle? Check your answer to see if it makes sense in the special case when $c_2 = 0$.

12.3 A curious chef tosses a potato in the air. Being a part-time dynamicist, she has just the right instruments in her kitchen and measures the absolute angular velocity of the potato to be $\vec{\boldsymbol{\omega}} = 1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}$ (rad/s) and the absolute center of mass velocity to be $\vec{\boldsymbol{v}}_G = 3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}$ (in/s), at a particular instant in time. At the same instant in time, she notes the position of an eye on the potato relative to its center of mass, point G, is $\vec{\boldsymbol{r}}_{E/G} = 1\,\text{in}\hat{\boldsymbol{\imath}}$. (The high quality potato came from the market with the center of mass already marked inside it.) $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are basis vectors in the kitchen frame.

a) What is the velocity of the potato eye relative to G, $\vec{\boldsymbol{v}}_{E/G}$?

b) What is the absolute velocity of the potato eye, $\vec{\boldsymbol{v}}_E$?

12.3 Instantaneous dynamics and "inverse dynamics" in 3-D

12.4 Under what circumstances is the linear momentum of a system conserved (that is, does not change with time)?

12.5 For a continuous system, where at a given instant in time velocity depends on position, $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})$, (a) how is linear momentum defined? (b) How is rate of change of linear momentum defined?

12.6 For what points C, with positions $\vec{\boldsymbol{r}}_C$, are the balance of angular momentum equations $\sum \vec{\boldsymbol{M}}_C = \dot{\vec{\boldsymbol{H}}}_C$, correct:

a) the origin,

b) the center of mass, and

c) any point anywhere?

12.7 Bead on a stationary 3-D wire. A bead with mass m slides on a frictionless wire that is fixed in space. The wire is twisted and curved in complicated ways. There is no gravity. The only force on the bead comes from the wire. The initial speed of the bead is v_0. Justify your answers to the following questions with text and/or equations. Add any extra assumptions if you feel they are required.

a) Is it true that $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ (where $\vec{\boldsymbol{F}}$ is the total force on the bead and $\boldsymbol{a}$ its acceleration?)?

b) Is the momentum of the bead, $m\vec{\boldsymbol{v}}$, constant?

c) Is the kinetic energy of the bead, $\frac{1}{2}mv^2$, constant?

d) What is the speed of the bead at $t = 3$ s?

e) Is the acceleration always such that $\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t = 0$ (where $\hat{\boldsymbol{e}}_t$ is the unit tangent to the wire)?

f) Is the acceleration always such that $\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_n = 0$ (where $\hat{\boldsymbol{e}}_n$ is the unit normal to the wire)?

g) Using any notation you like to describe the wire shape, write an expression for the force $\vec{\boldsymbol{F}}$ on the bead from the wire.

h) For a circular wire, what is the direction of the force from the bead on the wire?

i) For a helical wire, (say, $\vec{\boldsymbol{r}} = r_0\cos\theta\hat{\boldsymbol{\imath}} + r_0\sin\theta\,\hat{\boldsymbol{\jmath}} + C\theta\hat{\boldsymbol{k}}$), what is the direction of the force from the bead on the wire? *

12.8 A 500 kg roller coaster car travels along a track defined by the conical spiral $r = z$, $\theta = -2z$. If $\dot{\theta} = 2$ rad/s is constant, determine the force exerted on the car by the track when $z = 5$ m. Assume that the gravitational force acts in the negative z direction. Remember to express the force in vector form.

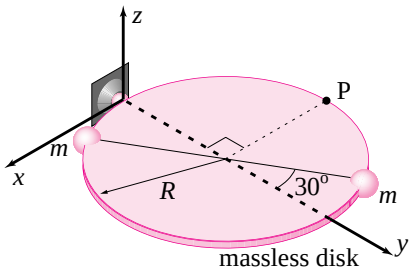
12.9 When do you use which term in $\dot{\vec{\boldsymbol{H}}}_O$ formula for a rigid body? To do the angular momentum balance about an arbitrary point O we write $\sum \vec{\boldsymbol{M}}_O = \dot{\vec{\boldsymbol{H}}}_O$ where the most general formula for $\dot{\vec{\boldsymbol{H}}}_O$ for one rigid body is $\dot{\vec{\boldsymbol{H}}}_O = \vec{\boldsymbol{r}}_{cm/o}\times m\vec{\boldsymbol{a}}_{cm} + [\mathbf{I}^{cm}]\cdot\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{\omega}}\times([\mathbf{I}^{cm}]\cdot\vec{\boldsymbol{\omega}})$. Give examples of motions of a rigid body (you may also use a point mass) in which

a) only one term at a time in the above formula for $\dot{\vec{\boldsymbol{H}}}_O$ survives (three examples) and

b) only two terms at a time in the formula survive (three examples).

12.10 (See also problems 12.33, 12.35, and 12.38.) Find the reaction supplied by the environment to the system of bodies $\mathcal{ABD}$ so that each body spins at a constant rate in problem 12.27. *

12.11 Disk suspended at corner by a ball and socket joint. See also problem 12.13. A disk with negligible mass has two point masses glued to it. It is stationary and horizontal when it is allowed to fall, but it is held back by a ball-and-socket joint at one point on its perimeter. Just after the disk is released, what is the acceleration of the point P? Is it up or down? *



problem 12.11:

(Filename:pfigure.s94h13p2)

12.12 A car driving in circles counterclockwise at constant speed so that its left rear tire has speed v_o (the middle of the tire, that is, or the location of the ground contact point.) The radius of the circle that the tire travels on is R_o. The radius of the tire is r_o. Right next to the tire is the car fender (*i.e.*, the fender overlaps the outer face of the tire. In fact the fender is rubbing on the tire just a little, but not quite enough to disturb any insects that might be in the neighborhood).

a) What is the angular velocity of the car? *

b) What is the angular velocity of the tire relative to the car? *

c) What is the angular velocity of the tire? *

d) What is the total torque applied to the car (not including the tires as part of the car, the rotating engine is also massless) relative to the car center of mass? *

e) What is the total torque on the car relative to the center of the circle around which it is traveling? *

f) What is the angular momentum of the tire relative to its center of mass (use any sensible model of the tire, say a rigid disk.)? *

g) What is the rate of change of angular momentum of the tire (relative to its center of mass)? *

h) What is the total torque of all the forces applied to the tire from the road and the car as calculated at the center of the tire? *

i) A bug is climbing straight up on the car fender named at speed v_g. At the instant of interest she is right next to the center of the wheel. What is her velocity and acceleration? What is the total force acting on her? *

j) Another bug is climbing on a straight line marked on the tire at constant speed v_b. He is in the middle of the tire at the instant of interest. What is his absolute velocity and acceleration? What is the total force acting on him? (Both bugs are at the same place at the instant of interest and walking on lines that are, at the moment, parallel.)

k) Redo the problem, questions (a) -(j), but this time the car speed is increasing with acceleration a_o. Questions (a) - (j) of this problem are a special case of question (k). You should check that your answers from (k) reduce to the answers in questions (a) - (j). Or, if you are really cocky, you should skip the first questions and only do (k). Then, display your answers for the special case $a_o = 0$.

12.13 The moment of inertia matrix. Reconsider the system of problem 12.11 — a disk with negligible mass that has two point masses glued to it.

a) Find the moment of inertia matrix $[\mathbf{I}^O]$ of the system.

b) Find the angular acceleration of the system when released from rest in the configuration shown. You may use your answer from problem 12.11.

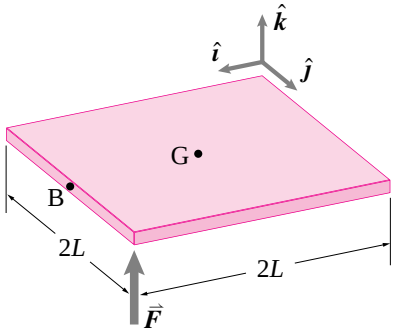
12.14 A square plate in space. A uniform square plate with mass m and sides with length $2L$ is floating stationary in space. A force of magnitude F is suddenly applied to it at point A. Using the coordinates shown the moment of inertia matrix for this plate is:

$$[I_{/cm}] = \frac{mL^2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a) What is the acceleration of the center of the plate $\vec{\boldsymbol{a}}_G$? *

b) What is the angular acceleration of the plate $\vec{\boldsymbol{\omega}}$ *

c) What is the acceleration of point B $\vec{\boldsymbol{a}}_B$? *



problem 12.14:

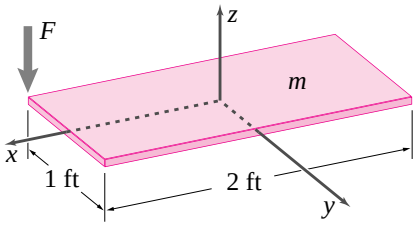
(Filename:pfigure.f93f5)

12.15 Force on a rectangular plate in space. A uniform rectangular plate of mass $m = 10\,\text{lbm}$ is floating in space. A force $F = 5\,\text{lbf}$ is suddenly applied at a corner that is perpendicular to the plate.

a) Find the acceleration of the center of mass of the plate. * $\vec{\boldsymbol{a}} = -16\,\text{ft/s}^2\hat{\boldsymbol{k}}$

b) Find the angular acceleration of the plate. *

c) Some of the plate accelerates in the direction of the force and some accelerates back towards the tail of the force vector. What part of the plate accelerates in the opposite direction of the applied force?



problem 12.15:

(Filename:pfigure.s94h13p3)

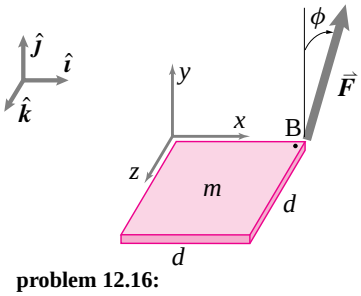
12.16 Force on a square plate. A constant force with magnitude F is applied, in the direction shown in the yz plane, to the corner B of a uniform square plate with side length d and mass m. Before the force is applied the plate is at rest in space with no other forces applied to it. Consider the time immediately after the time of the force application.

a) What, in terms of m, F, d, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$, and ϕ is the acceleration of point B?

b) For what angle ϕ is the acceleration of point B in the direction of the force?

c) Find another direction for F so that the force at B and the acceleration at B are parallel (not necessarily in the $\hat{\boldsymbol{\jmath}}$-$\hat{\boldsymbol{k}}$ plane). [Physical reasoning will probably work more easily than mathematical reasoning.]

d) Find yet another direction for F so that the force at B and the acceleration at B are parallel (not necessarily in the $\hat{\boldsymbol{\jmath}}$-$\hat{\boldsymbol{k}}$ plane).



problem 12.16:

(Filename:p.s96.p3.3)

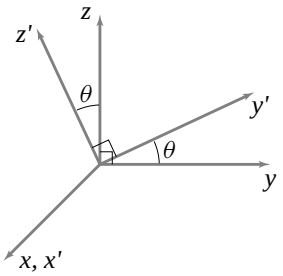
12.17 (See also problems 12.32, 12.34, and 12.37.) Find the reaction supplied by the environment to the system of bodies $\mathcal{ABD}$ so that each body spins at a constant rate in problem 12.31.

12.4 Time evolving dynamics of an unconstrained rigid body in 3-dimensions

12.18 A rigid body flies through the air and tumbles. Find the equations of motion and solve them for your choice of parameters and initial conditions.

12.5 Kinematics of motion relative to a moving and rotating reference frame in 3-D

12.19 Find the rotation matrix $[\mathbf{R}]$ for $\theta = 30^o$ such that $\{\hat{\boldsymbol{e}}'\} = [\mathbf{R}]\{\hat{\boldsymbol{e}}\}$, where $\hat{\boldsymbol{e}} = \{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$. Using the rotation matrix, find the components of $\vec{\boldsymbol{v}} = 2\,\text{m/s}\hat{\boldsymbol{\imath}} - 3\,\text{m/s}\hat{\boldsymbol{\jmath}} + 1\,\text{m/s}\hat{\boldsymbol{k}}$ in the rotated coordinate system.



problem 12.19:

(Filename:efig1.2.28)

12.20 A circular plate rotates at a constant angular velocity $\vec{\omega} = 5\,\text{rad/s}\hat{k}$. A set of coordinate axes $x'y'z'$ is glued to the plate at its center and thus rotates with $\vec{\omega}$. Find the rate of change basis vectors $\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$ using the $\dot{\vec{Q}}$ formula.

12.21 A rigid body $\mathcal{B}$ rotates in space with angular velocity $\vec{\omega} = (2\hat{\imath} + 3\hat{\jmath} - \hat{k})$ rad/s. A set of local coordinate axes $x'y'z'$ is fixed to the body at its center of mass. The position vector $\vec{r}_P$ of a point P is given in local coordinates: $\vec{r}_P = (5\hat{\imath}' + \hat{\jmath}')$ m. Find the velocity ($\dot{\vec{r}}_P$) of point P using the $\dot{\vec{Q}}$ formula.

12.22 The rotation matrix between two coordinate systems with basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ is $\mathbf{Q}$ such that $\{\hat{e}\} = [\mathbf{Q}]\{\hat{E}\}$. Find the components of $\vec{a} = (-2\hat{e}_1 + 3\hat{e}_3)$ m/s^2 in the $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ basis if $\dot{\vec{Q}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

12.23 Let $\mathcal{B}$ and $\mathcal{F}$ be two moving rigid bodies. Show that $\overset{\mathcal{B}}{\dot{\vec{\omega}}}_{\mathcal{B}/\mathcal{F}} = \overset{\mathcal{F}}{\dot{\vec{\omega}}}_{\mathcal{B}/\mathcal{F}}$.

12.24 George is driving his car on the highway on a rainy day. His windshield wipers are on. He notices that a drop of water on the wiper blade is moving along the blade at approximately 3 in/s. Although the wipers rotate at variable angular speed, he approximates that when the wiper is parallel to his nose, its angular speed is $\pi/2$ rad/s. In George's coordinate system (attached to his frame of reference and moving with him), the wiper is rotating in the positive $\hat{\jmath}$ direction and the water-drop is moving in the positive $\hat{k}$ direction at the moment of interest. Find the velocity of the water-drop in George's frame of reference.

12.25 A person sits on a bar stool which spins at angular speed Ω. Simultaneously, (s)he hoists a beer glass of mass m up at angular speed ω relative to the bar stool with a straight, rigid arm of length L. θ measures the the angle of her/his arm with respect to the XY plane. XYZ is an inertial coordinate system; YZ lies in the plane of the paper. If you use moving axes to do this problem, specify clearly his/her orientation and motion.

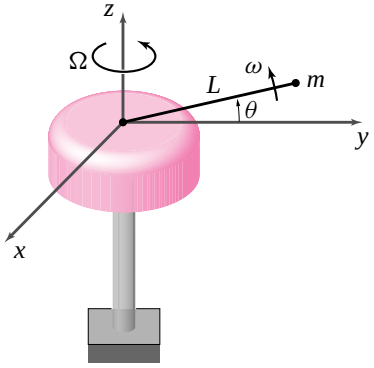
a) What is the angular velocity of the person's arm?

b) What is the angular acceleration of the person's arm?

c) What is the linear velocity of the beer glass relative to inertial space?

d) What is the linear acceleration of the beer glass relative to inertial space?

e) Describe how you would point an absolutely full glass such that the beer will not spill out.



problem 12.25:

(Filename:pfigure.blue.118.2)

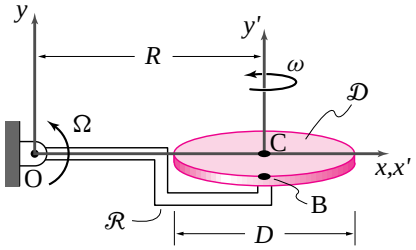
12.26 Kinematics of a disk spinning on a spinning rod. The rigid rod $\mathcal{R}$ rotates at constant rate Ω about the z axis. Attached to this rod, a distance R from the origin, is the center C of a disk $\mathcal{C}$ with diameter D. The disk is rotating at constant rate ω about an axis y' that is attached to, and rotates with, the rod. A point B is on the outer edge of the disk. At the moment of interest, the system is in the configuration shown with $\vec{r}_{CB} = (D/2)\hat{k}$.

a) What is the absolute angular velocity of the disk, $\vec{\omega}_{\mathcal{D}}$? *

b) What is the absolute angular acceleration of the disk $\vec{\alpha}_{\mathcal{D}}$? *

c) What is the absolute velocity of point B, $\vec{v}_B$? *

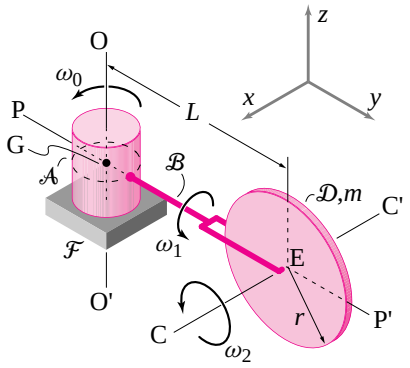
d) What is the absolute acceleration of point B, $\vec{a}_B$? *



problem 12.26:

(Filename:pfigure.s95f3a)

12.27 See also problems 12.33, 12.35, and 12.38. Body $\mathcal{A}$ rotates with respect to a Newtonian frame $\mathcal{F}$ at a constant angular rate ω_0 about axis OO'. Body $\mathcal{B}$, a 'fork', rotates with respect to body $\mathcal{A}$ at a constant angular rate ω_1 about axis PP'. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass. Body $\mathcal{D}$, a uniform disc of mass m, rotates with respect to body $\mathcal{B}$, at a constant angular rate ω_2 about axis CC'. Each of the bodies is turned at constant rate by motors(not shown.) At the instant shown, find the absolute angular velocity of body $\mathcal{D}$, $\vec{\omega}_{\mathcal{D}/\mathcal{F}}$. *



problem 12.27:

(Filename:Danef94s1q2)

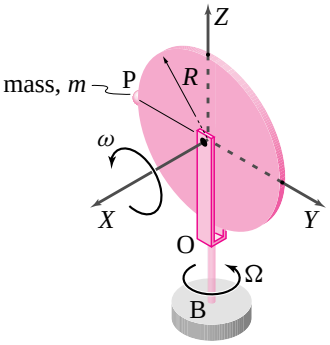
12.28 A particle P of mass m is attached to the edge of the disk of radius R, as shown in the figure. The disk spins about its center line with constant angular speed ω rad/s relative to the shaft OB. The shaft OB rotates with angular speed Ω rad/s. For the instant when the disk lies in the YZ plane, and the particle is on the Y axis, as shown:

a) Find the particle's velocity relative to XYZ.

b) Calculate the particle's acceleration relative to XYZ.

c) Calculate the force $\vec{F}$ acting on the particle.

If you choose to use another coordinate system, define it explicitly.



xyz: fixed frame

problem 12.28:

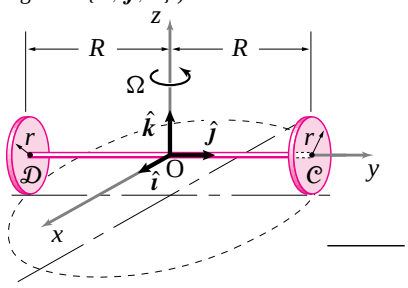
(Filename:pfigure.blue.123.1)

12.29 See also problem 12.53. Two identical thin disks, bodies $\mathcal{C}$ and $\mathcal{D}$, of radius r are connected (perpendicularly) to opposite ends of a thin axle, as depicted below. The axle rotates about the fixed Z axis at a constant angular speed Ω, and the disks roll without slipping. Determine:

(a) The total angular velocity of each disk, $\vec{\omega}_{\mathcal{C}}$ and $\vec{\omega}_{\mathcal{D}}$, and

(b) The total angular acceleration of each disk, $\boldsymbol{\alpha}_{\mathcal{C}}$ and $\boldsymbol{\alpha}_{\mathcal{D}}$.

(Express your answers relative to the rotating basis $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$.)



problem 12.29:

(Filename:pfigure.blue.119.2)

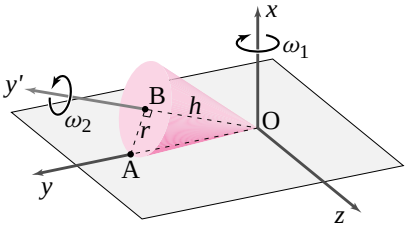
12.30 Kinematics of a rolling cone. See also problem 8.19. A solid cone with circular-base radius r and height h moves as follows. Point O is fixed in space. The cone surface always contacts the plane but may roll and/or slide. The axis of the cone OB rotates about the vertical axis with constant angular velocity $-\omega_1\hat{\boldsymbol{\imath}}$. The cone spins about this moving axis $\hat{\boldsymbol{\jmath}}'$ with constant angular rate ω_2.

a) At the instant shown what is the velocity of point B? (in terms of some or all of $\omega_1, \omega_2, r, h, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$) *

b) At the instant shown what is the angular acceleration of the cone? (in terms of some or all of $\omega_1, \omega_2, r, h, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$) *

c) If the disk is rolls with no slip, ~~find ω_1~~ in terms of ω_2, r and h. *

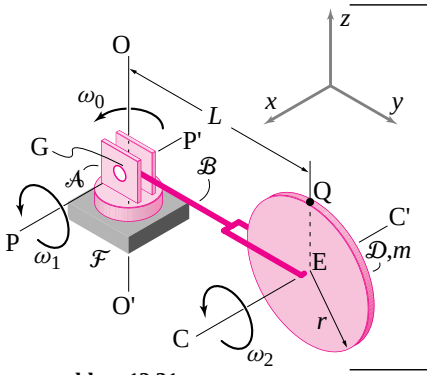
d) If the disk rolls with no slip and point B has constant speed v find the magnitude of the angular acceleration of the cone in terms of v, r and h. *



problem 12.30: Cone rolling on a plane

(Filename:pfigure.s95q14)

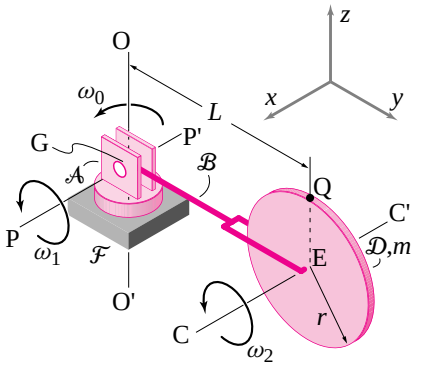
12.31 See also problems 12.32, 12.34, 12.37, and 12.17. Body $\mathcal{A}$ rotates with respect to a Newtonian frame $\mathcal{F}$ at a constant angular rate ω_0 about axis OO'. Body $\mathcal{B}$, a 'fork', rotates with respect to body $\mathcal{A}$ at a constant angular rate ω_1 about axis PP'. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass. Body $\mathcal{D}$, a uniform disc of mass m, rotates with respect to body $\mathcal{B}$, at a constant angular rate ω_2 about axis CC'. Each of the bodies is turned at constant rate by motors(not shown.) At the instant shown, find the absolute velocity and acceleration of point Q on the disk.



problem 12.31:

(Filename:Danef94s3q2)

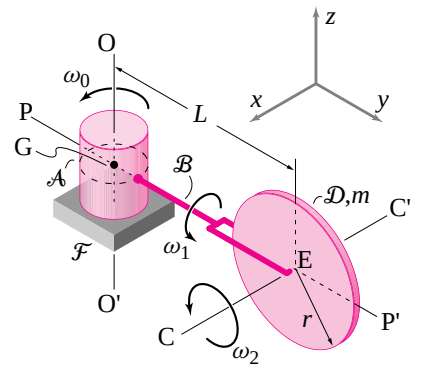
12.32 (See also problems 12.34, 12.37, and 12.17.) For the system in problem 12.31, find the absolute angular acceleration of frame $\mathcal{D}$, $\overset{\mathcal{F}}{\boldsymbol{\alpha}}_{\mathcal{D}}$, at the instant shown.



problem 12.32:

(Filename:bikefork1.ang.accel)

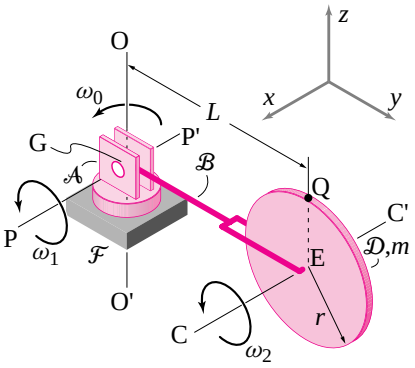
12.33 (See also problems 12.35, 12.38, and 12.10.) For the system in problem 12.27, find the absolute angular acceleration of frame $\mathcal{D}$, $\overset{\mathcal{F}}{\boldsymbol{\alpha}}_{\mathcal{D}}$, at the instant shown. *



problem 12.33:

(Filename:bikefork.ang.accel)

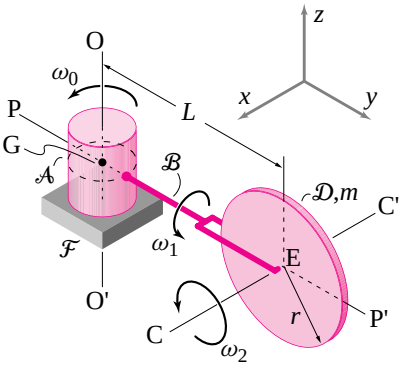
12.34 (See also problems 12.32, 12.37, and 12.17.) For the configuration in problem 12.31, find the velocity and acceleration of point Q on the disk using the alternative method.



problem 12.34:

(Filename:bikefork1.alt)

12.35 (See also problems 12.33, 12.38, and 12.10.) For the configuration in problem 12.27, find the velocity and acceleration of point the point on the edge of the disk directly above its center point E using the alternative method.



problem 12.35:

(Filename:bikefork.alt)

12.36 To simulate the flight conditions of a space vehicle, engineers have developed the centrifuge, shown diagrammatically in the figure. A main truss arm of length $\ell = 15\,\text{m}$, rotates about the AA axis. The pilot sits in the capsule which may rotate about axis CC. The seat for the pilot may rotate inside the capsule about an axis perpendicular to the page and going through the point B. These rotations are controlled by a computer that is set to simulate certain maneuvers corresponding to the entry and exit from the earth's atmosphere, malfunctions of the control system, etc. When a pilot sits in the capsule, his head, particularly his ears, has the position shown in the enlarged figure, a distance $r = 1\,\text{m}$ from point B.

The main arm is rotating at $\Omega = 10\,\text{rpm}$ and accelerating at $\dot{\Omega} = 5\,\text{rev/min}^2$, the capsule is rotating at a constant speed about CC at $\omega_c = 10\,\text{rpm}$, and the seat rotates at a speed of $\omega_s = 5\,\text{rpm}$ inside the capsule.

a) Determine the acceleration, in terms of how many g's, that the pilot's head is subjected to by choosing a moving coordinates system xyz centered at B and fixed to the capsule.

b) Repeat the calculations of (A) by letting xyz be fixed to the seat. *

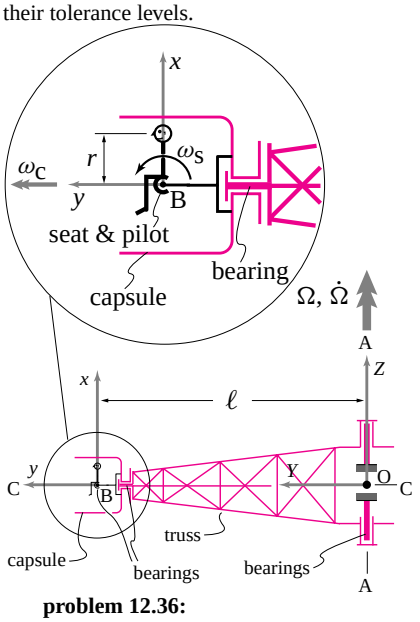
c) Choose another moving coordinates system to solve the same problem.

d) A pilot has different tolerances for acceleration components. They are roughly:

(a) In a vertical direction, i.e., toe to head – 5 g's. (Here the pilot experiences blackout or may pass out completely.)

(b) Front to rear – 1.5 g's. (Here vision becomes greatly distorted.)

Arrange a test program on the centrifuge so that each acceleration is reached separately while the other accelerations are kept below one- half their tolerance levels.



problem 12.36:

(Filename:pfigure.blue.128.1)

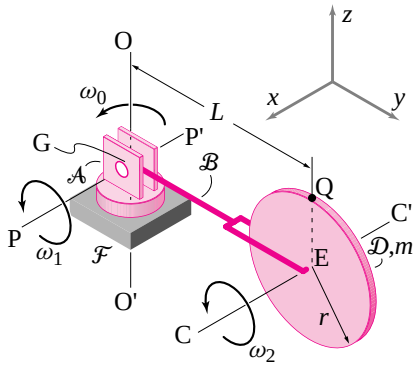
12.6 Mechanisms in 3-D

12.37 (See also problems 12.32, 12.34, and 12.17.) Reconsider the system in problem 12.31. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass and body $\mathcal{D}$ is a uniform disc of mass m. Find the angular momentum and rate of change of angular momentum of the system of bodies $\mathcal{A}\mathcal{B}\mathcal{D}$ about:

a) point G

b) point E.

Make reasonable assumptions about moments of inertia.



problem 12.37:

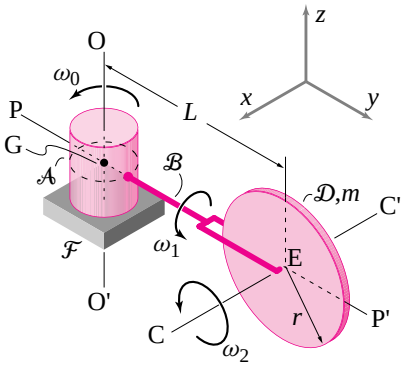
(Filename:bikefork1.ang.mom)

12.38 (See also problems 12.33, 12.35, and 12.10.) Reconsider the system in problem 12.27. Bodies $\mathcal{A}$ and $\mathcal{B}$ are of negligible mass and body $\mathcal{D}$ is a uniform disc of mass m. Find the angular momentum and rate of change of angular momentum of the system of bodies $\mathcal{A}\mathcal{B}\mathcal{D}$ about:

a) point G *

b) point E. *

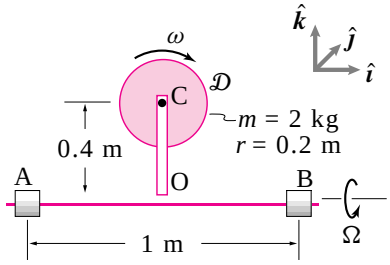
Make reasonable assumptions about moments of inertia.



problem 12.38:

(Filename:bikefork.ang.mom)

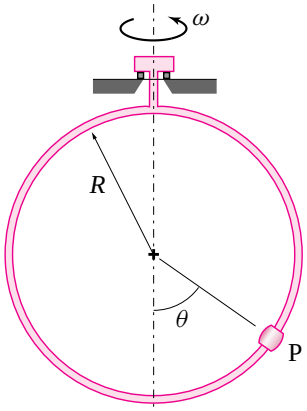
12.39 Reconsider problem 12.48. A thin uniform disk $\mathcal{D}$ of mass $m = 2\,\text{kg}$ and radius $r = 0.2\,\text{m}$ is mounted on a massless shaft AB as shown in the figure. Rod CO is welded to the shaft and has negligible mass. The shaft rotates at a constant angular speed $\Omega = 120\,\text{rpm}$ about the positive x-axis. The disk spins about its own axis at a constant rate $\omega = 60\,\text{rpm}$ with respect to the shaft (or bar CO). Find the dynamic reactions on the bearings at A and B at the instant shown. *



problem 12.39:

(Filename:summer95f.5.a)

12.40 A bead P rides on a frictionless thin circular hoop of radius R. The hoop rotates about its diameter at a constant angular speed ω in the vertical plane. Find, in terms of θ, $\dot{\theta}$ and ω and base vectors you define (a) the velocity of the bead (a vector) and (b) the acceleration of the bead (a vector).



problem 12.40:

(Filename:pfigure4.2.rp10)

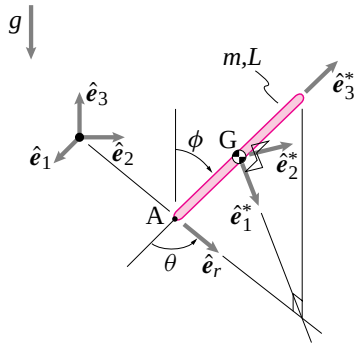
12.41 A thin rod of mass m and length L moves while making contact with the frictionless surface (the $\hat{\boldsymbol{e}}_1 - \hat{\boldsymbol{e}}_2$ plane) at its tip A, as shown in the figure. θ is assumed to be constant and the rod always remains in the $\hat{\boldsymbol{e}}_3, \hat{\boldsymbol{e}}_r$ plane.

a) What is the kinematical relation between the velocity of the center of mass, $\vec{\boldsymbol{v}}_G$, and the angular velocity of the rod, $\vec{\boldsymbol{\omega}}$?

b) Obtain the equations of motion of the rod in terms of the speed of point A and the angle ϕ.

The unit vectors $\hat{\boldsymbol{e}}_1^*$ and $\hat{\boldsymbol{e}}_3^*$ lie in the $(\hat{\boldsymbol{e}}_3 - \boldsymbol{e}_r)$ plane and $\hat{\boldsymbol{e}}_2^*$ is normal to the $(\hat{\boldsymbol{e}}_3 - \hat{\boldsymbol{e}}_r)$ plane. The starred basis vectors are written in terms of the the standard basis as follows:

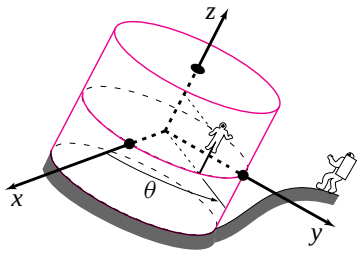
$$\begin{aligned}
\hat{\boldsymbol{e}}_r &= \cos\theta\hat{\boldsymbol{e}}_1 + \sin\theta\hat{\boldsymbol{e}}_2, \\
\hat{\boldsymbol{e}}_1^* &= \sin\phi\hat{\boldsymbol{e}}_3 + \cos\phi\hat{\boldsymbol{e}}_r, \\
\hat{\boldsymbol{e}}_3^* &= \cos\phi\hat{\boldsymbol{e}}_3 + \sin\phi\hat{\boldsymbol{e}}_r, \qquad \text{and} \\
\hat{\boldsymbol{e}}_2^* &= -\sin\theta\hat{\boldsymbol{e}}_1 + \cos\theta\hat{\boldsymbol{e}}_1.
\end{aligned}$$



problem 12.41:

(Filename:pfigure.blue.125.2)

12.42 A cosmonaut who is otherwise free to move in 3-dimensional space finds himself stuck to the surface of a giant stationary cylinder with slippery magnets. There is no friction and (net) gravity is negligible. θ and z are cylindrical coordinates that are aligned with the cylinder and mark the position of the cosmonaut relative to the cylinder. The cylinder has radius a. A nearby astronaut notes that at time $t = 0$: $\theta = \theta_0$, $\dot\theta = \dot\theta_0$, $z = z_0$, and $\dot z = \dot z_0$. What does she predict for the location of the cosmonaut (the values of θ and z) at an arbitrary later time t? You can assume that both the cylinder and the astronaut were stationary in the same Newtonian (inertial) frame.



problem 12.42:

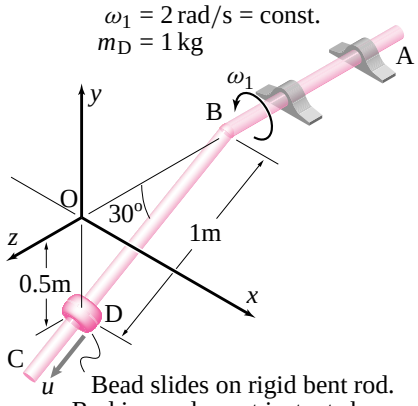
(Filename:pfigure.blue.68.1)

12.43 You are an astronaut in free space and your jet pack is no longer operating. You wish to reorient your body in order to face the central government buildings so as to salute your national leaders before reentering the atmosphere. Discuss whether this maneuver can be done in terms of momentum, energy, or other dynamical concepts. (You are free to move your arms and legs, but you are not allowed to throw anything.) [For aliens and those less patriotic, the same problem arises in the case of a trampoline jumper who decides to change direction in mid-air. Is it possible?]

12.44 A collar slides on the rigid rod ABC that has a 30° bend at B as shown in the figure. The collar is currently at point D on the rod. The rod ABC is held by fixed hinges that are aligned with the z-axis, and rotates at the constant rate $\omega_1 = 2$ rad/s. The collar has mass $m_D = 1$ kg.

a) Assume a mechanism not shown constrains the collar to slide out along the rod at the constant rate $u = 1$ m/s (u is the distance along the rod traveled per unit time). What is the net force that acts on the collar when it is 1 m from B?

b) Assume that the contact between the collar and the rod is frictionless and that no forces act on the collar besides those due to the rod (e.g. neglect gravity). At the instant shown in the picture, assume that $u = 1$ m/s (because of some initial conditions not discussed here) but is not necessarily constant in time. What is the acceleration of the collar?

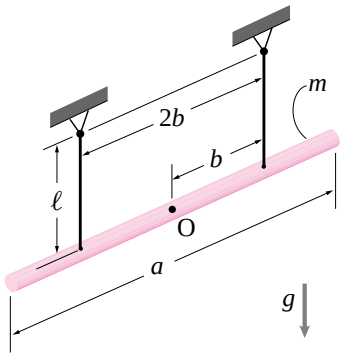


Bead slides on rigid bent rod.
Rod in yz-plane at instant shown.

problem 12.44:

(Filename:pfigure.blue.58.1)

12.45 A bifilar pendulum is formed of a rod of length a and mass m, supported by two filaments of length ℓ and separation $2b$. What is the frequency of small 'torsional' oscillations about a vertical axis though the rod's mid-point 0?



problem 12.45:

(Filename:pfigure.blue.157.1)

12.46 A trifilar pendulum is formed of a thin horizontal disk of radius a and mass m, hung by three wires a distance ℓ below the ceiling. The wire attachments are equally spaced on the perimeter of the disk. The attachment points on the ceiling form an equilateral triangle.

a) If the support wires are vertical what is the period of oscillation?

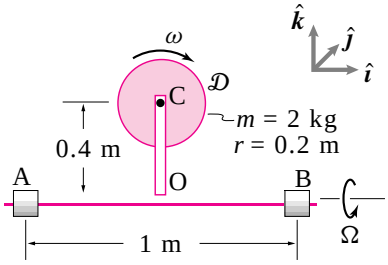
b) How big should the triangle on the ceiling be to minimize the period of oscillation?

c) To maximize the period of oscillation?

12.7 Circles on circles and other special 3-D motions

12.47 Sometimes thrown footballs (and frisbees too!) wobble and sometimes they spin smoothly. Describe as best you can why these motions differ.

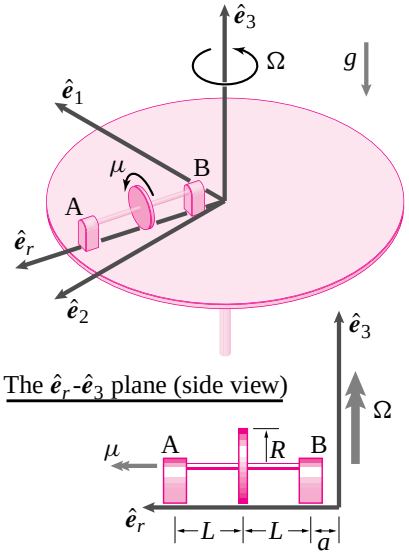
12.48 See also problem 12.39. A thin uniform disk $\mathcal{D}$ of mass $m = 2$ kg and radius $r = 0.2$ m is mounted on a massless shaft AB as shown in the figure. Rod CO is welded to the shaft and has negligible mass. The shaft rotates at a constant angular speed $\Omega = 120$ rpm about the positive x-axis. The disk spins about its own axis at a constant rate $\omega = 60$ rpm with respect to the shaft (or bar CO). Find the absolute angular velocity and acceleration of the disk. *



problem 12.48:

(Filename:summer95f.5)

12.49 A thin circular disk of mass m and radius R is mounted on a thin light axle, coincident with its axis of symmetry. The axle is supported by bearings A and B attached to a turntable, as shown in the figure. The turntable spins at a constant speed Ω about $\hat{\boldsymbol{e}}_3$, while the disk spins about its axle at a constant speed μ relative to the turntable (via motors not shown). The radial unit vector $\boldsymbol{e}_r = \cos(\Omega t)\hat{\boldsymbol{e}}_1 + \sin(\Omega t)\hat{\boldsymbol{e}}_2$ is fixed to the turntable and aligned with $\hat{\boldsymbol{e}}_1$ at time $t = 0$. Compute the vertical components of the reaction forces at A and B. Will the reaction forces change with time?



problem 12.49:

(Filename:pfigure.blue.127.2)

12.50 Steady precession of a spinning disk. A uniform disk of mass $m = 2$ kg and radius $R = 0.25$ m is rigidly mounted on a massless axle AB which is supported by frictionless bearings at A and B. The axle, in turn, is mounted on a massless semi-circular gimbal as shown in the figure. The gimbal rotates about the vertical x-axis, which passes through the center of the disk D, at a constant rate $\Omega = 2$ rad/s. The

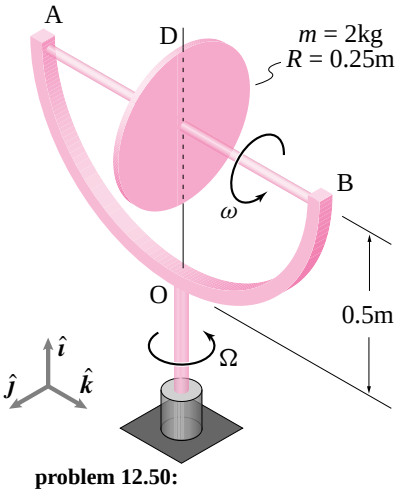
axle AB is spinning at a constant rate $\omega = 120$ rpm with respect to the gimbal. At the instant shown, the axle is aligned with the z-axis and the disk D is in the xy-plane. Ignore gravity. At the instant of interest:

a) What is the angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{D}}$ of the disk? *

b) What is the angular acceleration of the disk? *

c) What is the net torque required to keep the motion going? *

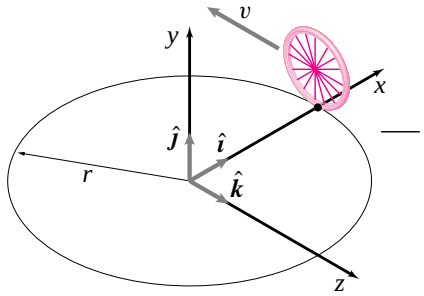
d) What are the reaction forces at the bearings A and B? *



problem 12.50:

(Filename:pfigure.s94h13p4)

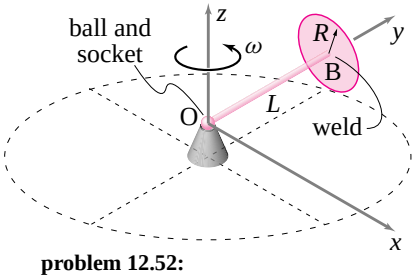
12.51 A cyclist rides a flat, circular track of radius $r = 150$ ft at a constant tangential speed of $v = 22$ ft/s on 27 in diameter wheels. Assuming that the bicycle is vertical, determine the total angular velocity and angular acceleration of the wheel of the bike in the sketch (the rest of the bicycle is not shown). Use the coordinates given in the sketch.



problem 12.51:

(Filename:pfigure.blue.90.2)

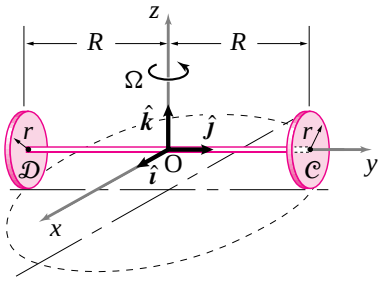
12.52 Rolling disk presses down. A rigid disk (idealized as flat) with radius R and mass m is rigidly attached (welded) to a rigid massless stick with length L which is connected by a ball-and-socket joint to the support at point O. The wheel rolls on a horizontal surface. Point O is at the height of the center of the wheel. The stick revolves around O at a rate of ω. Say gravity is negligible and, even though the wheel is rolling, you may assume there are no frictional forces acting between the wheel and the horizontal surface. What is the vertical reaction on the bottom of the wheel? *



problem 12.52:

(Filename:s92f1p7)

12.53 In problem 12.29, find the angular momentum and rate of change of angular momentum of the two disks about point O. Assume that the mass of each disk is m.



problem 12.53:

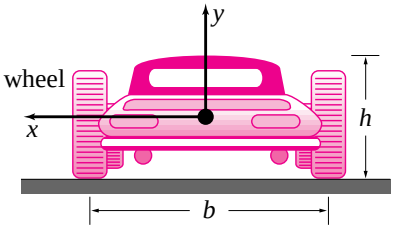
(Filename:twodisks.ang.mom)

12.54 A vehicle of mass m, width b, and height h rolls without slip along a road which runs into the page(along the z-axis.) The inertial properties about the vehicle center of mass are $I_{xx}^{cm}, I_{yy}^{cm}, I_{zz}^{cm}$, and $I_{xy}^{cm} = I_{xz}^{cm} = I_{yz}^{cm} = 0$. Make additional assumptions as needed to get an answer to the problem. Compute the contact forces acting on each wheel normal to the road in the following cases:

a) The vehicle moves at constant speed v along a straight road.

b) The vehicle moves at constant speed v along a circular road (radius ρ) curving to the left. Does the right or left wheel experience the larger force?

Rear end view



problem 12.54:

(Filename:pfigure.blue.90.1)

12.55 A car drives counter-clockwise in circles at constant speed V. The radius of the circle that the left rear wheel travels on is R. The radius of the wheel is r. Just when the wheel is rolling due north a pebble of mass m, which is stuck in the tread, is passing directly above the wheel axle. (You should assume that the wheel axle is horizontal, that the wheel rolls without slip and that the wheel deformation and gravity are negligible.) At this instant, what is the force that the tread causes on the pebble? Resolve this force into east, north and upwards components. [Hint: several intermediate calculations are probably required.]

12.56 A child constructs a spinning top from a solid sphere with radius a and mass m attached to a massless rod of length $2d$. The sphere pivots freely about O. It is set spinning with a constant rate $\dot{\phi}$ and precessing with a constant rate $\dot{\psi}$. The top of the rod is held by a horizontal cord.

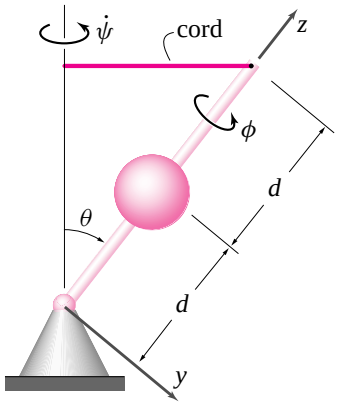
Note: The moments of inertia of a solid sphere of mass m and radius a about any axis through the center is $(2/5)ma^2$. (All products of inertia are zero, due to symmetry.)

a) Using an axis system attached as shown (z lies along the shaft, x is out of the page and y is chosen to produce a right-handed system), write the body's angular velocity $\vec{\boldsymbol{\omega}}$ as well as the angular velocity $\vec{\boldsymbol{\Omega}}$ of the moving coordinate system relative to inertial space.

b) What is the sphere's inertia tensor about O?

c) What is the top's angular momentum about O?

d) Compute the tension in the cord.



problem 12.56:

(Filename:pfigure.blue.110.1)

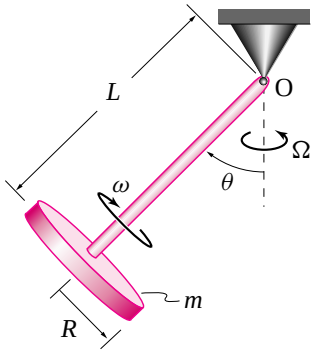
12.57 A thin disk of mass m and radius R is connected by a massless shaft of length L to a free swivel joint at O. The disk spins about the shaft at rate ω and proceeds in a steady slow precession (rate Ω) about the vertical.

a) Write the disk's inertia tensor about O. Specify the axes that you have chosen.

b) Write the disk's angular momentum about O, $\vec{\boldsymbol{H}}_{\mathrm{O}}$, in the same coordinate system.

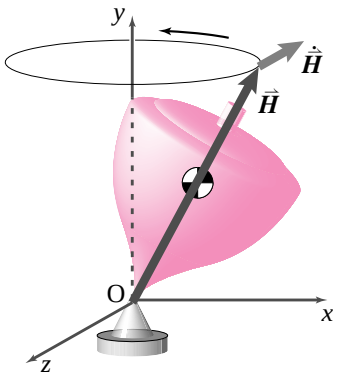
c) Calculate the rate of change of angular momentum about O, $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$.

d) Compute the precession rate Ω for the slow precession case ($\omega \gg \Omega$).



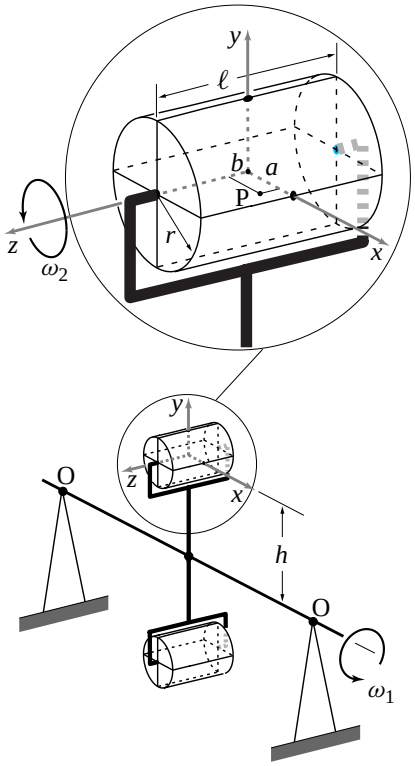
problem 12.57:

(Filename:pfigure.blue.107.1)



problem 12.58:

(Filename:pfigure.s94h14p4)



problem 12.59:

(Filename:pfigure.blue.122.1)

12.58 Spinning and precessing top. (hard question) The picture schematically shows what is going on when a spinning top is in steady precession. The top is, approximately, spinning at constant rate about its symmetry axis. This axis is in turn (so to speak) rotating about the y axis at a constant rate. This rotation about the y axis is called precession. The angular momentum vector thus changes in time as the top precesses and the angular momentum vector also precesses. The picture is a reasonable approximation if the top is rotating much faster than it is precessing. The picture is not accurate if the precession rate is not much smaller than the rotation rate. Here are some questions to help you see why.

a) If the top is precessing as well as spinning what is the direction of the angular velocity vector?

b) What is the direction of the angular momentum vector of the top? Is it parallel to the top's axis? Is it parallel to the angular velocity vector?

c) How does the angular momentum vector differ from that shown in the figure?

d) Does this difference affect any predictions about the precession rate of a spinning top?

e) Assume the top is on a pedestal and is horizontal as it precesses. In this case the correct precession rate *is* predicted by the naive/approximate theory that the angular momentum is just that due to spin and that this vector precesses with the top axis. Why does this simplification happen to work?

12.59 An amusement park ride consists of two cylinder-shaped cars of length $\ell = 8$ ft and radius $r = 3$ ft mounted at opposite ends of a long arm of length $2h = 20$ ft which is pivoted at its center as shown in the figure. The arm is constrained to rotate about the horizontal axle $O - O$. Each car can also rotate about its own axis. Car A, presently at the top of its flight, is spinning (much to the discomfort of its occupants) on its own axis as shown at a constant rate of $\omega_2 = 5$ rad/s relative to the vertical arm. The vertical arm is rotating about axis $O - O$ at the rate $\omega_1 = 0.5$ rad/s in the direction shown, as seen by a ground observer. The point P, located at $x = a = 2$ ft and $z = b = 1$ ft and representing a point in the inner ear of a passenger, is undergoing absolute velocity and acceleration $\vec{\boldsymbol{v}}_P$ and $\vec{\boldsymbol{a}}_P$, respectively. Find the vectors $\vec{\boldsymbol{v}}_P$ and $\vec{\boldsymbol{a}}_P$. (If you are using a moving coordinate system to determine $\vec{\boldsymbol{v}}_P$ and $\vec{\boldsymbol{a}}_P$, state clearly how it moves).

12.60 buzzing coin If you drop a round coin it will often go round and round progressively faster before buzzing to a stop. In terms of parameters you define, what is the frequency of this buzzing as a function of the tip angle? This problem is more subtle than a straightforward calculation as there are many solutions for a coin rolling at a given tip angle in circles. What feature picks out the solutions that are observed with real coins?

12.8 Motion relative to the earth

12.61 Estimate the sideways force (Pounds or Newtons) required for a person to walk a straight marked path over the North pole. Radius of earth $\approx$ 4000 miles $\approx 2 \times 10^7$ feet $\approx 6 \times 10^6$ meters. Make other simplifying approximations so that you can do the arithmetic without a calculator.

12.62 Foucoult pendulum

12.63 Estimating the error of using the earth as a Newtonian frame

12.64 The path of an air mass sliding on a frictionless earth

Answers to *'d problems

2.33) $r_x = \vec{r}\cdot\hat{\imath} = (3\cos\theta + 1.5\sin\theta)$ ft, $r_y = \vec{r}\cdot\hat{\jmath} = (3\sin\theta - 1.5\cos\theta)$ ft.

2.68) No partial credit.

2.69) To get chicken road sin theta.

2.74) $\vec{N} = \frac{1000\,\mathrm{N}}{\sqrt{3}}(\hat{\imath}+\hat{\jmath}+\hat{k})$.

2.77) $d = \sqrt{\frac{3}{2}}$.

2.81a) $\hat{\lambda}_{OB} = \frac{1}{\sqrt{50}}(4\hat{\imath}+3\hat{\jmath}+5\hat{k})$.

b) $\hat{\lambda}_{OA} = \frac{1}{\sqrt{34}}(3\hat{\jmath}+5\hat{k})$.

c) $\vec{F}_1 = \frac{5\,\mathrm{N}}{\sqrt{34}}(3\hat{\jmath}+5\hat{k})$, $\vec{F}_2 = \frac{7\,\mathrm{N}}{\sqrt{50}}(4\hat{\imath}+3\hat{\jmath}+5\hat{k})$.

d) $\angle AOB = 34.45\,\mathrm{deg}$.

e) $F_{1_x} = 0$

f) $\vec{r}_{DO}\times\vec{F}_1 = \left(\frac{100}{\sqrt{34}}\hat{\jmath} - \frac{60}{\sqrt{34}}\hat{k}\right)$ N·m.

g) $M_\lambda = \frac{140}{\sqrt{50}}$ N·m.

h) $M_\lambda = \frac{140}{\sqrt{50}}$ N·m.(same as (7))

2.83a) $\hat{n} = \frac{1}{3}(2\hat{\imath}+2\hat{\jmath}+\hat{k})$.

b) $d = 1$.

c) $\frac{1}{3}(-2, 19, 11)$.

2.94) (0.5 m, −0.4 m)

4.2) $T_1 = Nmg$, $T_2 = (N-1)mg$, $T_N = (1)mg$, and in general $T_n = (N+1-n)mg$

4.3) Suprise! This pendulum is in equilibrium for all values of θ.

4.8) Yes.

4.17a) $\frac{m}{M} = \frac{R\sin\theta}{R\cos\theta+r} = \frac{2\sin\theta}{1+2\cos\theta}$.

b) $T = mg = 2Mg\frac{\sin\theta}{1+2\cos\theta}$.

c) $\vec{F}_C = Mg\left[-\frac{2\sin\theta}{2\cos\theta+1}\hat{\imath}' + \hat{\jmath}'\right]$ (where $\hat{\imath}'$ and $\hat{\jmath}'$ are aligned with the horizontal and vertical directions)

d) $\tan\phi = \frac{\sin\theta}{2+\cos\theta}$. Needs somewhat involved trigonometry, geometry, and algebra.

e) $\tan\psi = \frac{m}{M} = \frac{2\sin\theta}{1+2\cos\theta}$.

4.18a) $\frac{m}{M} = \frac{R\sin\theta}{R\cos\theta-r} = \frac{2\sin\theta}{2\cos\theta-1}$.

b) $T = mg = 2Mg\frac{\sin\theta}{2\cos\theta-1}$.

c) $\vec{F}_C = \frac{Mg}{1-2\cos\theta}\left[\sin\theta\hat{\imath} + (\cos\theta-2)\hat{\jmath}\right]$.

4.21a) $\frac{F_1}{F_2} = \frac{R_o+R_i\sin\phi}{R_o-R_i\sin\phi}$

b) For $R_o = 3R_i$ and $\mu = 0.2$, $\frac{F_1}{F_2} \approx 1.14$.

c) $\frac{F_1}{F_2} = e^{\mu\theta} = e^{0.2\pi} \approx 1.87$.

4.23) (a)$T_{AB} = 30$ N, (b) $T_{AB} = \frac{300}{17}$ N, (c)$T_{AB} = \frac{5\sqrt{26}}{2}$ N

4.25g) $T_{EH} = 0$ as you can find a number of ways.

5.5) $x(3\,\mathrm{s}) = 20\,\mathrm{m}$

5.22) $h_{max} = e^2 h$.

5.34) (a) $m\ddot{x} + kx = F(t)$, (b) $m\ddot{x} + kx = F(t)$, and (c) $m\ddot{y} + 2ky - 2k\ell_0\frac{y}{\sqrt{\ell_0^2+y^2}} = F(t)$

5.37b) $mg - k(x-\ell_0) = m\ddot{x}$

c) $\ddot{x} + \frac{k}{m}x = g + \frac{k\ell_0}{m}$

e) This solution is the static equilibrium position; i.e., when the mass is hanging at rest, its weight is exactly balanced by the upwards force of the spring at this constant position x.

f) $\ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$

g) $x(t) = \left[D - \left(\ell_0 + \frac{mg}{k}\right)\right]\cos\sqrt{\frac{k}{m}}\,t + (\ell_0 + \frac{mg}{k})$

h) period=$2\pi\sqrt{\frac{m}{k}}$.

i) If the initial position D is more than $\ell_0 + 2mg/k$, then the spring is in compression for part of the motion. A floppy spring would buckle.

5.39a) period= $\frac{2\pi}{\sqrt{\frac{k}{m}}} = 0.96\,\mathrm{s}$

b) maximum amplitude=0.75 ft

c) period= $2\sqrt{\frac{2h}{g}} + \sqrt{\frac{m}{k}}\left[\pi + 2\tan^{-1}\sqrt{\frac{mg}{2kh}}\right] \approx 1.64\,\mathrm{s}$.

5.41) LHS of Linear Momentum Balance: $\sum\vec{F} = -(kx + b\dot{x})\hat{\imath} + (N-mg)\hat{\jmath}$.

5.55a) Two normal modes.

b) $x_2 = const * x_1 = const * (A\sin(ct) + B\cos(ct))$,where $const = \pm 1$.

c) $\omega_1 = \sqrt{\frac{3k}{m}}$, $\omega_2 = \sqrt{\frac{k}{m}}$.

5.56a) $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

b) If we start off by assuming that each mass undergoes simple harmonic motion at the same frequency but different amplitudes, we will find that this two-degree-of-freedom system has two natural frequencies. Associated with each natural frequency is a fixed ratio between the amplitudes of each mass. Each mass will undergo simple harmonic motion at one of the two natural frequencies only if the initial displacements of the masses are in the fixed ratio associated with that frequency.

5.58) $\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - k_2(x_B - x_A) + c_1(\dot{x}_D - \dot{x}_B) + k_3(x_D - x_B)]\hat{\imath}$.

5.59) $\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - c_1(\dot{x}_B - \dot{x}_A) + (k_2 + k_3)(x_D - x_B)]$.

5.62a) $\omega = \sqrt{\frac{2k}{m}}$.

5.66a) One normal mode: [1, 0, 0].

b) The other two normal modes: [0, 1, $\frac{1\pm\sqrt{17}}{4}$].

5.72a) $\vec{v}(5\,\text{s}) = (30\hat{\imath} + 300\hat{\jmath})\,\text{m/s}$.

b) $\vec{a}(5\,\text{s}) = (6\hat{\imath} + 120\hat{\jmath})\,\text{m/s}^2$.

5.73) $\vec{r}(t) = \left(x_0 + \frac{u_0}{\Omega} - \frac{u_0}{\Omega}\cos(\Omega t)\right)\hat{\imath} + (y_0 + v_0 t)\,\hat{\jmath}$.

5.81) $\vec{v} = 2t\,\text{m/s}^2\hat{\imath} + e^{\frac{t}{\text{s}}}\,\text{m/s}\hat{\jmath}$, $\vec{a} = 2\,\text{m/s}^2\hat{\imath} + e^{\frac{t}{\text{s}}}\,\text{m/s}^2\hat{\jmath}$.

5.118) $T_3 = 13\,\text{N}$

5.131) Equation of motion: $-mg\hat{\jmath} - b(\dot{x}^2 + \dot{y}^2)\left(\frac{\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$.

5.132a) System of equations:

$$\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= -\frac{b}{m}v_x\sqrt{v_x^2 + v_y^2} \\
\dot{v}_y &= -g - \frac{b}{m}v_y\sqrt{v_x^2 + v_y^2}
\end{aligned}$$

5.143) No. You need to know the angular momenta of the particles relative to the center of mass to complete the calculation, information which is not given.

5.154a) $v_0 = \frac{1}{m}(mv_B + m_Bv_B + m_Av_A)$.

b) $v_1 = \frac{(m+m_B)}{m}v_B$.

c) (1) $E_{loss} = \frac{1}{2}m\left[v_0^2 - \left(\frac{m+m_B}{m}\right)v_B^2\right] - \frac{1}{2}m_Av_A^2$. $E_{loss} = \frac{1}{2}m\left[\frac{(m+m_B)^2}{m}v_B^2 - (m+m_B)v_B^2\right]$.

5.155) $v_A = \sqrt{\frac{m_Bk\delta^2}{m_A^2+m_Bm_A}}$.

6.3) $T_n = \frac{P_t}{v_t}\frac{n}{N}$.

6.9a) $a_B = \left(\frac{m_B-m_A}{m_A+m_B}\right)g$

b) $T = 2\frac{m_Am_B}{m_A+m_B}g$.

6.13) (a) $\vec{a}_A = \vec{a}_B = \frac{F}{m}\hat{\imath}$, where $\hat{\imath}$ is parallel to the ground and pointing to the right., (b) $\vec{a}_A = \frac{2F}{m}\hat{\imath}$, $\vec{a}_B = \frac{4F}{m}\hat{\imath}$, (c) $\vec{a}_A = \frac{F}{2m}\hat{\imath}$, $\vec{a}_B = \frac{F}{4m}\hat{\imath}$, (d) $\vec{a}_A = \frac{F}{m}\hat{\imath}$, $\vec{a}_B = -\frac{F}{m}\hat{\imath}$.

6.15) ratio of the acceleration of A to that of point B, $\frac{a_A}{a_B} = 81$.

6.18a) $\vec{a}_A = \frac{5F}{m}\hat{\imath}$, $\vec{a}_B = \frac{25F}{m}\hat{\imath}$, where $\hat{\imath}$ is parallel to the ground and points to the right.

b) $\vec{a}_A = \frac{g}{(4m_1+m_2)}(2m_2 - \sqrt{3}m_2)\hat{\boldsymbol{\lambda}}_1$, $\vec{a}_B = -\frac{g}{2(4m_1+m_2)}(2m_2 - \sqrt{3}m_2)\hat{\boldsymbol{\lambda}}_2$, where $\hat{\boldsymbol{\lambda}}_1$ is parallel to the slope that mass m_1 travels along, pointing down and to the left, and $\hat{\boldsymbol{\lambda}}_2$ is parallel to the slope that mass m_2 travels along, pointing down and to the right.

6.22) angular frequency of vibration $\equiv \lambda = \sqrt{\frac{64k}{65m}}$.

6.27a) $m\ddot{x} + 4kx = A\sin\omega t + mg$, where x is the distance measured from the unstretched position of the center of the pulley.

b) The string will go slack if $\omega > \sqrt{\frac{4k}{m}\left(1 - \frac{A}{mg}\right)}$.

6.28a) $\vec{a}_A = -\frac{9kd}{m_A}\hat{\imath}$.

b) $v = 3d\sqrt{\frac{k}{m_A}}$

6.33) $T_{AB} = \frac{5\sqrt{39}}{28}m(a_y + g)$

6.36) $a_x > \frac{3}{2}g$

6.39) Can't solve for T_{AB}.

6.51d) Normal reaction at rear wheel: $N_r = \frac{mgw}{2(h\mu+w)}$, normal reaction at front wheel: $N_f = mg - \frac{mgw}{2(h\mu+w)}$, deceleration of car: $a_{car} = -\frac{\mu gw}{2(h\mu+w)}$.

e) Normal reaction at rear wheel: $N_r = mg - \frac{mgw}{2(w-\mu h)}$, normal reaction at front wheel: $N_f = \frac{mgw}{2(w-\mu h)}$, deceleration of car: $a_{car} = -\frac{\mu gw}{2(w-\mu h)}$. Car stops more quickly for front wheel skidding. Car stops at same rate for front or rear wheel skidding if $h = 0$.

f) Normal reaction at rear wheel: $N_r = \frac{mg(w/2-\mu h)}{w}$, normal reaction at front wheel: $N_f = \frac{mg(w/2+\mu h)}{w}$, deceleration of car: $a_{car} = -\mu g$.

g) No. Simple superposition just doesn't work.

h) No reaction at rear wheel.

i) Reaction at rear wheel is negative. Not allowing for rotation of the car in the xy-plane gives rise to this impossibility. In actuality, the rear of the car would flip over the front.

6.52a) $\vec{a} = g(\sin\phi - \mu\cos\phi)\hat{\imath}$, where $\hat{\imath}$ is parallel to the slope and pointing downwards

b) $\vec{a} = g\sin\phi$

c) $\vec{v} = g(\sin\phi - \mu\cos\phi)t\hat{\imath}$, $\vec{r} = g(\sin\phi - \mu\cos\phi)\frac{t^2}{2}$

d) $\vec{v} = g\sin\phi t\hat{\imath}$, $\vec{r} = g\sin\phi\frac{t^2}{2}\hat{\imath}$

6.54a) $\vec{R}_A = \frac{(1-\mu)mg\cos\theta}{2}(\hat{\jmath}' - \mu\hat{\imath}')$.

c) No tipping if $N_A = \frac{(1-\mu)mg\cos\theta}{2} > 0$; i.e., no tipping if $\mu < 1$ since $\cos\theta > 0$ for $0 < \theta < \frac{\pi}{2}$.(Here $\mu = 0.9$)

6.56) braking acceleration$= g(\frac{1}{2}\cos\theta - \sin\theta)$.

6.60a) $v = d\sqrt{\frac{k}{m}}$.

b) The cart undergoes simple harmonic motion for any size oscillation.

6.63a) $\vec{a}_{bike} = \frac{F_pL_c}{MR_f}$.

b) $\max(\vec{a}_{bike}) = \frac{ga}{a+b+2R_f}$.

6.64) $T_{EF} = 640\sqrt{2}\,\text{lbf}$.

6.65a) $T_{BD} = 92.6\,\text{lbm}\cdot\,\text{ft/s}^2$.

b) $T_{GH} = 5\sqrt{61}\,\text{lbm}\cdot\,\text{ft/s}^2$.

6.66b) $T_{EH} = 0$

c) $(R_{C_x} - T_{AB})\hat{\imath} + (R_{C_y} - \frac{T_{GD}}{\sqrt{2}})\hat{\jmath} + (T_{HE} + R_{C_z} + \frac{T_{GD}}{\sqrt{2}})\hat{k} = m\vec{a} = 10\,\text{N}\hat{k}$.

d) $\sum \vec{M}_{\text{cm}} = (\frac{T_{GD}}{\sqrt{2}} - T_{HE} - R_{C_z})\hat{\imath} + (R_{C_z} - \frac{T_{GD}}{\sqrt{2}} - T_{HE})\hat{\jmath} + (T_{AB} + R_{C_x} - R_{C_y} - \frac{T_{GD}}{\sqrt{2}})\hat{k} = \vec{0}$

e)

$$\begin{aligned} R_{C_x} - T_{AB} &= 0 \\ R_{C_y} - \frac{T_{GD}}{\sqrt{2}} &= 0 \\ R_{C_z} + \frac{T_{GD}}{\sqrt{2}} + T_{EH} &= 5\,\text{N} \\ -T_{EH} + \frac{T_{GD}}{\sqrt{2}} - R_{C_z} &= 0 \\ -T_{EH} - \frac{T_{GD}}{\sqrt{2}} + R_{C_z} &= 0 \\ T_{AB} - \frac{T_{GD}}{\sqrt{2}} + R_{C_x} - R_{C_y} &= 0 \end{aligned}$$

f) $R_{C_x} = 5\,\text{N}$, $R_{C_y} = 5\,\text{N}$, $R_{C_z} = 5\,\text{N}$, $T_{GD} = \frac{10}{\sqrt{2}}\,\text{N}$, $T_{EH} = 0\,\text{N}$, $T_{AB} = 5\,\text{N}$.

g) Find moment about CD axis; e.g., $(\sum \vec{M}_{\text{C}} = \vec{r}_{cm/C} \times \text{m}\vec{a}_{cm}) \cdot \hat{\lambda}_{CD}$, where $\hat{\lambda}_{CD}$ is a unit vector in the direction of axis CD.

6.71a) $F_L = \frac{1}{2} m_{\text{tot}} g$.

b) $\vec{a}_P = \frac{1}{m_{\text{tot}}}[2(T - F_D) - D]\,\hat{\imath}$.

c) $\vec{F} = \left[\frac{m_w}{m_{\text{tot}}}(2T - D - 2F_D) - T + F_D\right]\hat{\imath} + (m_w g - F_L)\hat{\jmath}$ and $\vec{M} = (bF_L - am_w g)\hat{\imath} + \left[(bF_D - cT) + a\frac{m_w}{m_{\text{tot}}}(2T - D - 2F_D)\right]\hat{\jmath}$.

6.72) sideways force = $F_B\hat{\imath} = \frac{wma}{2\ell}\hat{\imath}$.

7.3)

- one solution is $\vec{r} = R(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath})$ with R and ω constants,
- another solution is $\vec{r} = [At\hat{\imath} + Bt^2\hat{\jmath}]/\sqrt{1 + 4B^2t^2/A^2}$,
- general solution is any motion where speed = $v = |\vec{v}|$ is a constant.

7.17) $F = 0.52\,\text{lbf} = 2.3\,\text{N}$

7.24b) For $\theta = 0^\circ$,

$$\begin{aligned} \hat{e}_r &= \hat{\imath} \\ \hat{e}_t &= \hat{\jmath} \\ \vec{v} &= \frac{2\pi r}{\tau}\hat{\jmath} \\ \vec{a} &= -\frac{4\pi^2 r}{\tau^2}\hat{\imath}, \end{aligned}$$

for $\theta = 90^\circ$,

$$\begin{aligned} \hat{e}_r &= \hat{\jmath} \\ \hat{e}_t &= -\hat{\imath} \\ \vec{v} &= -\frac{2\pi r}{\tau}\hat{\imath} \\ \vec{a} &= -\frac{4\pi^2 r}{\tau^2}\hat{\jmath}, \end{aligned}$$

and for $\theta = 210^\circ$,

$$\begin{aligned} \hat{e}_r &= -\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath} \\ \hat{e}_t &= \frac{1}{2}\hat{\imath} - \frac{\sqrt{3}}{2}\hat{\jmath} \\ \vec{v} &= -\frac{\sqrt{3}\pi r}{\tau}\hat{\jmath} + \frac{\pi r}{\tau}\hat{\imath} \\ \vec{a} &= \frac{2\sqrt{3}\pi^2 r}{\tau^2}\hat{\imath} + \frac{2\pi^2 r}{\tau^2}\hat{\jmath}. \end{aligned}$$

c) $T = \frac{4m\pi^2 r}{\tau^2}$.

d) Tension is enough.

7.27b) $(\dot{\vec{H}}_{\text{O}})_I = \vec{0}$, $(\dot{\vec{H}}_{\text{O}})_{II} = 0.0080\,\text{N·m}\hat{k}$.

c) Position-A: $(\vec{H}_{\text{O}})_I = 0.012\,\text{N·m} \cdot \text{s}\hat{k}$,$(\vec{H}_{\text{O}})_{II} = 0.012\,\text{N·m} \cdot \text{s}\hat{k}$, Position-B: $(\vec{H}_{\text{O}})_I 0.012\,\text{N·m} \cdot \text{s}\hat{k}$, $(\vec{H}_{\text{O}})_{II} = 0.014\,\text{N·m} \cdot \text{s}\hat{k}$.]

7.29) $r = \frac{kr_o}{k - m\omega_o^2}$.

7.31) $\ell_0 = 0.2\,\text{m}$

7.33b) $T = 0.16\pi^4 N$.

c) $\vec{H}_{\text{O}} = 0.04\pi^2\,\text{kg·m/s}\hat{k}$

d) $\vec{r} = [\frac{\sqrt{2}}{2} - v\cos(\frac{\pi t}{4})]\hat{\imath} + [\frac{\sqrt{2}}{2} + v\sin(\frac{\pi t}{4})]\hat{\jmath}$.

7.35a) $2mg$.

b) $\omega = \sqrt{99g/r}$

c) $r \approx 1\,\text{m}$ ($r > 0.98\,\text{m}$)

7.38) (b) $\ddot{\theta} + \frac{3g}{2L}\sin\theta = 0$

7.41b) The solution is a simple multiple of the person's weight.

7.43a) $\ddot{\theta} = -(g/L)\sin\theta$

d) $\dot{\alpha} = -(g/L)\sin\theta, \quad \dot{\theta} = \alpha$

f) $T_{max} = 30N$

7.44a) $\dot{v} = -\mu\frac{v^2}{R}$.

b) $v = v_0 e^{-\mu\theta}$.

7.47a) The velocity of departure is $\vec{v}_{dep} = \sqrt{\frac{k(\Delta\ell)^2}{m} - 2GR}\hat{\jmath}$, where $\hat{\jmath}$ is perpendicular to the curved end of the tube.

b) Just before leaving the tube the net force on the pellet is due to the wall and gravity, $\vec{F}_{net} = -mg\hat{\jmath} - m\frac{|\vec{v}_{dep}|^2}{R}\hat{\imath}$; Just after leaving the tube, the net force on the pellet is only due to gravity, $\vec{F}_{net} = -mg\hat{\jmath}$.

7.51) $\vec{a}_t = 20\,\text{in/s}^2\hat{\jmath}.$, $\vec{a}_n = -250\,\text{in/s}^2\hat{\imath}$.

7.65) $\omega_{min} = 10\,\text{rpm}$ and $\omega_{max} = 240\,\text{rpm}$

7.76a) 7.85 kW

b) 7.85 kW

c) 750 rev/min

d) 100 N·m.

7.79a) $v_P = 2\,\text{m/s}$.

b) $\vec{v}_P = -2\,\text{m/s}\hat{\imath}$.

c) $\vec{a}_P = -4\,\text{m/s}^2\hat{\jmath}$.

d) $\vec{v}_r = 1\,\text{m/s}\hat{\lambda}_r$, where $\hat{\lambda}_r$ is a unit vector pointing in the direction of the rack, down and to the right.

e) No force needed to move at constant velocity.

7.80a) $P_{in} = 7.33$kilo-watts

b) 500 rpm

c) $M_{out} = 140\,\text{N}\cdot\text{m}$

7.84a) $\alpha_B = 20\,\text{rad/s}^2$ (CW)

b) $a = 4\,\text{m/s}^2$ (up)

c) $T = 280\,\text{N}$.

7.86a) $F_B = 100\,\text{lbf}$.

c) $v_{right} = v$.

7.94b)

(b) $T = 2.29\,\text{s}$

(e) $T = 1.99\,\text{s}$

(b) has a longer period than (e) does since in (b) the moment of inertia about the center of mass (located at the same position as the mass in (e)) is non-zero.

7.98a) $\ddot{\theta} = 0\ \text{rad/s}^2$.

b) $\ddot{\theta} = \frac{\sin\theta}{m\ell}(Dk - mg)$.

7.99b) $-F(t)\ell\cos\phi - mg\ell\sin\phi + T_m = -m\ell^2\ddot{\phi}$.

7.100a) (a) $\vec{\boldsymbol{F}} = 0.33\,\text{N}\hat{\boldsymbol{\imath}} - 0.54\,\text{N}\hat{\boldsymbol{\jmath}}$.

7.101a) $T(r) = \frac{m\omega^2}{2L}(L^2 - r^2)$

b) at $r = 0$; i.e., at the center of rotation

c) $r = L/\sqrt{2}$

7.103) $I_{zz} = 0.125\,\text{kg}\cdot\,\text{m}^2$.

7.104a) $0.2\,\text{kg}\cdot\,\text{m}^2$.

b) 0.29 m.

7.105) At 0.72ℓ from either end

7.106a) $(I_{zz})_{min} = m\ell^2/2$, about the midpoint.

b) $(I_{zz})_{max} = m\ell^2$, about either end

7.107a) C

b) A

c) $I^A_{zz}/I^B_{zz} = 2$

d) smaller, $r_{gyr} = \sqrt{I^C_{zz}/(3m)} = \sqrt{2}\ell$

7.108a) Biggest: I^O_{zz}; smallest: $I^O_{yy} = I^O_{xx}$.

b) $I^O_{xx} = \frac{3}{2}m\ell^2 = I^O_{yy}$. $I^O_{zz} = 3m\ell^2$.

d) $r_{gyr} = \ell$.

7.113a) $\omega_n = \sqrt{\frac{gL(M+\frac{m}{2})+K}{(M+\frac{m}{3})L^2+M\frac{R^2}{2}}}$.

b) $\omega_n = \sqrt{\frac{gL(M+\frac{m}{2})+K}{(M+\frac{m}{3})L^2}}$. Frequency higher than in (a)

7.114a) $I^{cm}_{zz} = 2m\ell^2$

b) $P \equiv A,\ B,\ C$, or D

c) $r_{gyr} = \ell/\sqrt{2}$

7.115) $I^O_{xx} = I^O_{yy} = 0.3\,\text{kg}\cdot\,\text{m}^2$.

7.117a) $I^O_{zz} = \dfrac{2m}{bh}\displaystyle\int_0^b\int_0^{hx/b}(x^2+y^2)\,dy\,dx$.

7.128a) Point at $2L/3$ from A

b) $mg/4$ directed upwards.

7.129c) $T = \frac{2\pi}{\sqrt{g/\ell}}\sqrt{\frac{1}{12(d/\ell)} + \frac{d}{\ell}}$

g) $d = 0.29\,\ell$

7.132a) Net force: $\vec{\boldsymbol{F}}_{net} = -(\frac{3m\omega^2L}{2})\hat{\boldsymbol{\imath}} - (\frac{m\omega^2L}{2})\hat{\boldsymbol{\jmath}}$, Net moment: $\vec{\boldsymbol{M}}_{net} = \vec{\boldsymbol{0}}$.

b) Net force: $\vec{\boldsymbol{F}}_{net} = -(\frac{3m\omega^2L}{2})\hat{\boldsymbol{\imath}} + (2mg - \frac{m\omega^2L}{2})\hat{\boldsymbol{\jmath}}$, Net moment: $\vec{\boldsymbol{M}}_{net} = \frac{3mgL}{2}\hat{\boldsymbol{k}}$.

7.133) $T_{rev} = \sqrt{\frac{2M_O\ell^2\pi}{3FR_p}}$.

7.139) period = $\pi\sqrt{\frac{2m}{k}}$.

7.141a) $\vec{\boldsymbol{\alpha}} = \ddot{\theta}\hat{\boldsymbol{k}} = ?\,\text{rad/s}^2\hat{\boldsymbol{k}}$ (oops).

b) $T = 538\,\text{N}$.

7.142) $\vec{\boldsymbol{v}}_B = \sqrt{\frac{2gh[m_B - 2m_A(\sin\theta+\mu\cos\theta)]}{\left[4m_A+m_B+4m_C\left(\frac{K_C}{R_C}\right)^2\right]}}$.

7.143) $\vec{\boldsymbol{a}}_m = 0.188\,\text{m/s}^2$

8.13a) $\vec{\boldsymbol{r}}_{P/O} = 0.5\,\text{m}\left[(\cos\phi)\hat{\boldsymbol{\imath}} + \sin\phi\,\hat{\boldsymbol{\jmath}}\right]$.

b) $\vec{\boldsymbol{v}}_P = (0.5\omega\sin\phi\hat{\boldsymbol{k}})\,\text{m/s}$, $\vec{\boldsymbol{a}}_P = (-0.5\omega^2\sin\phi\,\hat{\boldsymbol{\jmath}})\,\text{m/s}^2$.

8.23a) $T_{DB} = 0.9\,\text{N}$.

b) $\vec{\boldsymbol{F}}_A = -0.3\,\text{N}\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{F}}_C = -0.6\,\text{N}\hat{\boldsymbol{\imath}}$

c) $\text{Torque}_{motor} = 0$.

8.25a) Angles ϕ for steady circular motion are solutions to $10\tan\phi = 8 + 40\sin\phi$.

b) ϕ = [1.3627 rad, −0.2735 rad, −1.2484 rad, −2, 9813 rad], four solutions, found numerically. Graphically, the solutions are the interns of the curves $f_1(\phi) = 10\tan\phi$ and $f_2(\phi) = 8 + 40\sin\phi$ between $-\pi$ and π.

8.28b) Period of revolution= $\frac{2\pi}{\omega}$, Period of simple pendulum (for small oscillations)= $2\pi\sqrt{\frac{L}{g}}$, where ℓ is the length of the pendulum.

c) $T = m\omega^2L$ (strange that ϕ drops out!), $\vec{\boldsymbol{v}}_{rock} = -\omega L\sin\phi\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{a}}_{rock} = -\omega^2L\sin\phi\,\hat{\boldsymbol{\jmath}}$, $\frac{dH_x}{dt} = -m\omega L^2\sin\phi\cos\phi$.

8.29) $\phi = \cos^{-1}(\frac{g}{\omega^2L})$.

8.32) (a) $T = -69.2\,\text{N}$, (b) $1.64\,\text{rad/s} < \omega < 3.47\,\text{rad/s}$

8.36) $\Delta x = -0.554\,\text{m}$

8.37b) $\dot{\vec{\boldsymbol{H}}}_{\text{O}} = -\frac{1}{12}m\omega^2\ell^2\sin\phi\cos\phi\hat{\boldsymbol{k}}$.

c) $\vec{\boldsymbol{F}}_A = (\frac{1}{12}m\omega^2\ell^2/d\sin\phi\cos\phi)\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{F}}_B = -\vec{\boldsymbol{F}}_A$.

d) $\vec{\boldsymbol{F}}_A = (\frac{1}{12}m\omega^2\ell^2/d\sin\phi\cos\phi + \frac{1}{2}mg)\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{F}}_B \neq -\vec{\boldsymbol{F}}_A$.

8.40) $T = \frac{\sqrt{2}m\omega^2R}{6}$.

8.57a) $\vec{\boldsymbol{\omega}} \times ([\boldsymbol{I}^{\text{G}}]\cdot\vec{\boldsymbol{\omega}}) = \vec{\boldsymbol{0}}\,\text{kg}\cdot\text{m}^2/\text{s}^2$.

b) $\vec{\boldsymbol{\omega}} \times ([\boldsymbol{I}^{\text{G}}]\cdot\vec{\boldsymbol{\omega}}) = -25\hat{\boldsymbol{\imath}}\,\text{kg}\cdot\text{m}^2/\text{s}^2$.

8.66b) $\omega^2 = \frac{3g}{2L\cos\phi}$, $T_{rev} = \frac{2\pi}{\omega}$.

8.67a) $\omega^2 = \frac{3gn}{L(2n+1)\cos\phi}$.

8.68a)

$$[\boldsymbol{I}^{\text{O}}] = mL^2\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix},$$

in terms of the coordinate system shown.

b) $\dot{\vec{\boldsymbol{H}}}_{\text{O}} = 2mL^2\omega(-\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$.

c) $\dot{\vec{\boldsymbol{H}}}_{\text{O}} = -2mL^2\omega^2\hat{\boldsymbol{\imath}}$.

d) $\vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{M}}_{\text{O}} = -2mL^2\omega^2\hat{\boldsymbol{\imath}}$.

8.69c) $\vec{\boldsymbol{R}}_O = -\frac{m\omega^2 bh(b^2-h^2)}{12(b^2+h^2)^{\frac{3}{2}}}\hat{\boldsymbol{k}} = -\vec{\boldsymbol{R}}_C$.

8.70a) $\vec{\boldsymbol{a}}_D = \frac{M_y}{I_{YY}^{cm}}\frac{2dc}{\sqrt{c^2+d^2}}\hat{\boldsymbol{\imath}}$.

b) $\vec{\boldsymbol{B}} = -\frac{M_y I_{YZ}^{cm}}{2I_{YY}^{cm}\sqrt{c^2+d^2}}\hat{\boldsymbol{\imath}}$.

8.79a) $\vec{\boldsymbol{H}}_{\rm O} = (-0.108\hat{\boldsymbol{\imath}}' + 0.126\boldsymbol{k}')\ \text{kg}\cdot\text{m}^2/\text{s}$

b) $\vec{\boldsymbol{H}}_{\rm O}$ is in the $x'z'$ plane at the moment of interest and at an angle $\alpha \approx 41^\circ$, clockwise from the z'-axis.

c) $\vec{\boldsymbol{H}}_{\rm O}$ does not change in magnitude (m, R, ω, β are constants). $\vec{\boldsymbol{H}}_{\rm O}$ rotates with the disk; i.,e., it changes direction.

d) $\dot{\vec{\boldsymbol{H}}}_{\rm O}$ = the rate of change of $\vec{\boldsymbol{H}}_{\rm O}$. The tip of $\vec{\boldsymbol{H}}_{\rm O}$ goes in a circle. At the instant of interest, the change will be in the $\hat{\boldsymbol{\jmath}}'$ or $\hat{\boldsymbol{\jmath}}$ direction. Thus, $\sum \vec{\boldsymbol{M}}_{\rm O} = \dot{\vec{\boldsymbol{H}}}_{\rm O}$ will be in the $\hat{\boldsymbol{\jmath}}$ direction.

8.82a) $\dot{\vec{\boldsymbol{H}}}_{\rm cm} = \frac{9\sqrt{3}}{128}\ \text{N}\cdot\text{m}\hat{\boldsymbol{\jmath}} = 0.122\ \text{N}\cdot\text{m}\hat{\boldsymbol{\jmath}}$

b) $\dot{\vec{\boldsymbol{H}}}_A = -26.89\ \text{N}\cdot\text{m}\hat{\boldsymbol{\jmath}}$

c) $\vec{\boldsymbol{R}}_A = -4.56\ \text{N}\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{R}}_C = -13.439\ \text{N}\hat{\boldsymbol{\imath}},\ M_z = 0.$

8.90b) M_{Az} is always zero.

c) No. See problems 8.68 and 8.37 for counter-examples.

9.1) plot(2): $\theta(t) = br(t)$ and $\vec{\boldsymbol{v}} = \frac{\dot{\theta}}{b}\hat{\boldsymbol{e}}_r + \frac{\theta\dot{\theta}}{b}\hat{\boldsymbol{e}}_t$.

9.1) plot(2): $x = r\cos(br)$ and $y = r\sin br$ or $x = \frac{\theta}{b}\cos(\theta)$ and $y = \frac{\theta}{b}\sin\theta$

9.12) $\vec{\boldsymbol{v}}_P = \omega_1 L(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) + (\omega_1+\omega_2)r\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{a}}_P = -\omega_1^2 L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + (\omega_1+\omega_2)^2\hat{\boldsymbol{\jmath}}$.

9.17a)

$$\begin{aligned}
\hat{\boldsymbol{e}}_t &= \frac{\frac{2t}{\text{s}}\hat{\boldsymbol{\imath}} + e^{\frac{t}{\text{s}}}\hat{\boldsymbol{\jmath}}}{\sqrt{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}}} \\
a_t &= \frac{4t\frac{\text{m}}{\text{s}^3} + e^{\frac{2t}{\text{s}}}\ \text{m/s}^2}{\sqrt{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}}} \\
\vec{\boldsymbol{a}}_n &= \frac{(2-2\frac{t}{\text{s}})e^{\frac{2t}{\text{s}}}\ \text{m/s}^2\hat{\boldsymbol{\imath}} + (4\frac{t^2}{\text{s}^2} - 4\frac{t}{\text{s}})e^{\frac{t}{\text{s}}}\ \text{m/s}^2\hat{\boldsymbol{\jmath}}}{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}} \\
\hat{\boldsymbol{e}}_n &= \frac{e^{\frac{t}{\text{s}}}\hat{\boldsymbol{\imath}} - 2\frac{t}{\text{s}}\hat{\boldsymbol{\jmath}}}{\sqrt{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}}} \\
\rho &= \frac{(4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}})}{(2-2\frac{t}{\text{s}})e^{\frac{t}{\text{s}}}}\ \text{m}
\end{aligned}$$

9.22a) $\vec{\boldsymbol{a}}_{cm} = \frac{F}{m}\hat{\boldsymbol{\imath}}$.

b) $\vec{\boldsymbol{\alpha}} = -\frac{6F}{m\ell}\hat{\boldsymbol{k}}$.

c) $\vec{\boldsymbol{a}}_A = \frac{4F}{m}\hat{\boldsymbol{\imath}}$.

d) $\vec{\boldsymbol{F}}_B = \frac{F}{2}\hat{\boldsymbol{\imath}}$

9.25a) $\vec{\boldsymbol{a}} = \frac{F}{m}\hat{\boldsymbol{\jmath}}$.

b) $\vec{\boldsymbol{\alpha}} = \frac{Fd}{m\frac{R^2}{2}}\hat{\boldsymbol{k}}$.

9.34a) $F_{out} = \frac{3}{7}$ lb.

b) F_{out} is always less than the F_{in}.

9.38a) $\vec{\boldsymbol{\alpha}}_{Disk} = -\frac{4}{3}\ \text{rad/s}^2\hat{\boldsymbol{k}}$.

b) $\vec{\boldsymbol{a}}_A = \frac{8}{3}\ \text{m/s}^2\hat{\boldsymbol{\imath}}$.

9.40a) $\boldsymbol{a}_C = (F/m)(1 - R_i/R_O)\hat{\boldsymbol{\imath}}$.

b) $(-R_i/R_O)F\hat{\boldsymbol{\imath}}$.

9.45a) speed $v = \frac{R\dot{\theta}}{3}$.

b) The energy lost to friction is $E_{fric} = \frac{mR^2\dot{\theta}_0^2}{6}$. The energy lost to friction is independent of μ for $\mu > 0$. Thus, the energy lost to friction is constant for given m, R, and $\dot{\theta}_0$. As $\mu \to 0$, the transition time to rolling $\to \infty$. It is not true, however, that the energy lost to friction $\to 0$ as $\mu \to 0$. Since the energy lost is constant for any $\mu > 0$, the disk will slip for longer and longer times so that the distance of slip goes to infinity. The dissipation rate $\to 0$ since the constant energy is divided by increasing transition time. The energy lost is zero only for $\mu = 0$.

9.46) $V = 2\ \text{m/s}$.

9.47) $v_o = 2\frac{\sqrt{2gR}}{3}$.

9.53) Accelerations of the center of mass, where $\hat{\boldsymbol{\imath}}$ is parallel to the slope and pointing down: (a) $\vec{\boldsymbol{a}}_{cm} = g\sin\theta\hat{\boldsymbol{\imath}}$, (b) $\vec{\boldsymbol{a}}_{cm} = \frac{2}{3}g\sin\theta\hat{\boldsymbol{\imath}}$, (c) $\vec{\boldsymbol{a}}_{cm} = \frac{1}{2}g\sin\theta\hat{\boldsymbol{\imath}}$, (d) $\vec{\boldsymbol{a}}_{cm} = \frac{2}{3}g\sin\theta\hat{\boldsymbol{\imath}}$. So, the block is fastest, all uniform disks are second, and the hollow pipe is third.

9.65) $h = 2L/3]$

10.11) (g) $\vec{\boldsymbol{v}}_P = 8\ \text{m/s}\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{0}}$; (h) $\vec{\boldsymbol{v}}_P = 3\ \text{m/s}\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{0}}$.

10.12) $\vec{\boldsymbol{a}}_b = -(s^2A + a\omega_D^2 + 2\omega_D v_b)\hat{\boldsymbol{\imath}} - a\dot{\omega}_D\hat{\boldsymbol{k}}$.

10.17b) One case: $\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{a}}_{O'}$, where P refers to the bug; the bug stands still on the turntable, the truck accelerates, and the turntable does not rotate.

c) One case: $\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{a}}_{O'} + \vec{\boldsymbol{a}}_{P/\mathcal{D}}$, where $\mathcal{D}$ is the turntable; the bug accelerates along a straight line on the turntable, the truck accelerates, and the turntable does not rotate.

10.18a) $\vec{\boldsymbol{\omega}}_{OA} = \dot{\theta}\hat{\boldsymbol{k}}$.

b) $\vec{\boldsymbol{\alpha}}_{OA} = \ddot{\theta}\hat{\boldsymbol{k}}$.

c) $\vec{\boldsymbol{v}}_A = -R\dot{\theta}\sin\theta\hat{\boldsymbol{\imath}} + R\dot{\theta}\cos\theta\hat{\boldsymbol{\jmath}}$.

d) $\vec{\boldsymbol{a}}_A = -(R\dot{\theta}^2\cos\theta + R\ddot{\theta}\sin\theta)\hat{\boldsymbol{\imath}} + (R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta)\hat{\boldsymbol{\jmath}}$.

e) $\vec{\boldsymbol{\omega}}_{AB} = \frac{-R\dot{\theta}\cos\theta}{\sqrt{\ell^2 - R^2\sin^2\theta}}$.

f) $\theta = (2n+1)\frac{\pi}{2}$, $n = 0, \pm1, \pm2, \ldots$.

10.21) $\vec{\boldsymbol{\omega}}_{DE} = -\frac{\omega}{2}\hat{\boldsymbol{k}} = -1.5\ \text{rad/s}\hat{\boldsymbol{k}}$

10.22) $\vec{\boldsymbol{\alpha}}_{DE} = 3.5\ \text{rad/s}^2\hat{\boldsymbol{k}}$

10.29) One situation: $\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta$;the case of a particle constrained to move in a circle.

10.30) One situation: $\vec{\boldsymbol{a}} = \ddot{R}\hat{\boldsymbol{e}}_R$; the case of no rotation ($\dot{\theta} = \ddot{\theta} = 0$).

10.31) $\vec{\boldsymbol{\omega}}_D = (\omega_1 + \omega_2)\hat{\boldsymbol{k}}$.

10.33a)

$$\hat{\boldsymbol{e}}_t = \frac{\frac{2t}{\text{s}}\hat{\boldsymbol{\imath}} + e^{\frac{t}{\text{s}}}\hat{\boldsymbol{\jmath}}}{\sqrt{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}}}$$

$$
\begin{aligned}
a_t &= \frac{4t\,\frac{\mathrm{m}}{\mathrm{s}^3} + e^{\frac{2t}{\mathrm{s}}}\,\mathrm{m/s^2}}{\sqrt{4\frac{t^2}{\mathrm{s}^2} + e^{\frac{2t}{\mathrm{s}}}}} \\
\vec{a}_n &= \frac{(2-2\frac{t}{\mathrm{s}})e^{\frac{2t}{\mathrm{s}}}\,\mathrm{m/s^2}\hat{\imath} + (4\frac{t^2}{\mathrm{s}^2} - 4\frac{t}{\mathrm{s}})e^{\frac{t}{\mathrm{s}}}\,\mathrm{m/s^2}\hat{\jmath}}{4\frac{t^2}{\mathrm{s}^2} + e^{\frac{2t}{\mathrm{s}}}} \\
\hat{e}_n &= \frac{e^{\frac{t}{\mathrm{s}}}\hat{\imath} - 2\frac{t}{\mathrm{s}}\hat{\jmath}}{\sqrt{4\frac{t^2}{\mathrm{s}^2} + e^{\frac{2t}{\mathrm{s}}}}} \\
\rho &= \frac{(4\frac{t^2}{\mathrm{s}^2} + e^{\frac{2t}{\mathrm{s}}})}{(2-2\frac{t}{\mathrm{s}})e^{\frac{t}{\mathrm{s}}}}\,\mathrm{m}
\end{aligned}
$$

11.9a) $\ddot{R}(t) - \omega_o^2 R(t) = 0$.
b) $\frac{d^2R(\theta)}{d\theta^2} - R(\theta) = 0$.
c) $R(\theta = 2\pi) = 267.7\,\mathrm{ft}$, $R(\theta = 4\pi) = 1.43 \times 10^5\,\mathrm{ft}$.
d) $v = 2378.7\,\mathrm{ft/s}$
e) $E_\mathrm{K} = (2.83 \times 10^6) \times (mass)(\,\mathrm{ft/s})^2$.

11.10a) $\ddot{R} = R_0\omega^2 - 2\mu\dot{R}_0\omega$.
b) $\phi = \sin^{-1}\left[\frac{1}{\sqrt{(\sqrt{1+\mu^2}-\mu)^2+1}}\right]$.

11.19a) $\omega^- = 4\,\mathrm{rad/s}$, where the minus sign '-' means 'just before leaving'.
b) $\omega^+ = 4\,\mathrm{rad/s}$, where the plus sign '+' means 'just after leaving'.
c) $v_r = 3.841\,\mathrm{m/s}$, $v_t = 2.4\,\mathrm{m/s}$.
d) torque=0.072 N·m

11.20) $\vec{F} = -0.6\,\mathrm{N}\hat{\jmath}$.

11.22a) $T_{BC} = m\frac{\sqrt{34}}{13}(2a_x + g)$.
b) $\vec{a} = \frac{1}{34}[(25a_x + 15g)\hat{\imath} + (15a_x - 25g)\hat{\jmath}]$
c) $T_{BC} = m\frac{1}{\sqrt{34}}(5a_x + 3g)$.

11.28a) $\vec{H}_\mathrm{O} = \left[mL^2\omega_1 + \frac{1}{2}mr^2(\omega_1+\omega_2)\right]\hat{k}$.
b) $\dot{\vec{H}}_\mathrm{O} = \vec{0}$.
c) $\vec{H}_\mathrm{C} = \frac{1}{2}mr^2(\omega_1+\omega_2)\hat{k}$.
d) $\dot{\vec{H}}_\mathrm{C} = \vec{0}$.

11.29) $\vec{F} = m\vec{a} = -109.3\,\mathrm{N}\hat{\imath} - 19.54\,\mathrm{N}\hat{\jmath}$.

11.31b) $\ddot{\theta} = -\frac{3g}{2L}\cos\theta$.
c) From the diagram, we see $\dot{\omega}_{AB} = -\ddot{\theta}(t=0)\hat{k} = \frac{3g}{2L}\cos\theta_0$, $\vec{a}_G = \frac{3}{4}g\cos\theta_0[\sin\theta_0\hat{\imath} - \cos\theta_0\hat{\jmath}]$.
d) $\vec{R}_A = \frac{3}{4}mg\sin\theta_0\cos\theta_0\hat{\imath}$, $\vec{R}_B = mg[1 - \frac{3}{4}g\cos^2\theta_0]\hat{\jmath}$.
e) $\vec{a}_B = \frac{3}{2}g\sin\theta_0\cos\theta_0\hat{\imath}$.
f) At $\theta = \frac{\theta_0}{2}$, $\vec{\omega}_{AB} = \sqrt{\frac{3g}{L}(\sin\theta_0 - \sin\frac{\theta_0}{2})}\hat{k}$, $\vec{a}_A = -3g\left[\frac{1}{2}\cos^2\frac{\theta_0}{2} + \sin\frac{\theta_0}{2}(\sin\theta_0 - \sin\frac{\theta_0}{2})\right]\hat{\jmath}$.

11.37) There are many solution methods. $-8\pi^2$ poundals to the left. (Note: 1 poundal = 1 ft · lbm · s^{-2}.)

11.40a) $\Delta = \sqrt{\frac{(M_c+6M_w)v_0^2}{k}}$.
b) Tangential forces point toward the wall!
c) $\Delta = \sqrt{\frac{(M_c+4M_w)v_0^2}{k}}$.

11.45a) $-k(x\hat{\imath} + y\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$.
b) $-kr\hat{e}_r = m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_t$.
c) $\frac{d}{dt}(x\dot{y} - y\dot{x}) = 0$.
d) $\frac{d}{dt}(r^2\dot{\theta}) = 0$.
e) dot equation in (a) with $(x\hat{\jmath} - y\hat{\imath})$ to get $(x\ddot{y} - y\ddot{x}) = 0$ which can be rewritten as (c); dot equation in (b) with $\hat{e}_t$ to get $(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$ which can be rewritten as (d).
f) $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2) = const.$
g) $\frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) + \frac{1}{2}kr^2$=const.
h) dot equation in (a) with $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ to get $m(\ddot{x}\dot{x} + \ddot{y}\dot{y}) + k(\dot{x}x + \dot{y}y) = 0$ which can be rewritten as (f).
i) $x = A_1\sin(\omega t + B_1)$, $y = A_2\sin(\omega t + B_2)$, where $\omega = \sqrt{\frac{k}{m}}$. The general motion is an ellipse.
j) Yes. Consider $B_1 = B_2 = 0$, $A_1 = 1$, and $A_2 = 2$.

11.46a) $0.33\,\mathrm{m}\hat{\imath}$
b) $\vec{v}_C = 1.82\,\mathrm{m/s}\hat{\imath}$.
c) $T \approx 240\,\mathrm{N}$.

11.52b)

$$
\begin{aligned}
\ddot{R} - R\dot{\theta}^2 &= 0 \\
(I_{zz} + mR^2)\ddot{\theta} + 2mR\dot{R}\dot{\theta} &= 0
\end{aligned}
$$

c) The second equation in part (b) can be rewritten in the form $\frac{d}{dt}\left[I_{zz} + mR^2)\dot{\theta}\right] = 0$. The quantity inside the derivative is angular momentum; thus, it is conserved and equal to a constant, say, $(H_{/o})_0$, which can be found in terms of the initial conditions.
d) $\ddot{R} - R\left[\frac{(H_o)_0^2}{(I_{zz}^O + mR^2)^2}\right] = 0$.
e) $E_0 = \frac{1}{2}m\dot{R}^2 + \frac{1}{2}\omega(H_{/o})_0$.
f) The bead's distance goes to infinity and its speed approaches a constant. The turntable's angular velocity goes to zero and its net angle of twist goes to a constant.

11.53a)

$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\frac{2mRv\omega}{I_{zz} + mR^2} \\
\dot{R} &= v \\
\dot{v} &= \omega^2 R
\end{aligned}
$$

11.55) $v_{2_x} = \sqrt{\frac{2g(a-b)\tan\phi}{(1+\frac{1}{n}+\left(\frac{n+1}{n}\right)^2\tan^2\phi)}}$, $v_{2_y} = v_{2_x}\frac{n+1}{n}\tan\phi$, $v_{1_x} = -\frac{v_{2_x}}{n}$, $v_{1_y} = 0$, where 2 refers to the top wedge and 1 refers to the lower wedge.

11.56a) $\vec{F} = m(g + a_0)\left[\frac{\sqrt{3}}{4}\hat{\imath} + \frac{3}{4}\hat{\jmath}\right]$.
b) For $a_0 = -g$, the acceleration of the mass is exactly vertical; d_0, v_0, and t could be anything.

11.57a) angular speed = 8.8 rad/s.
b) displacement = 0.5 ft.

11.68a) For point mass: $\ddot{\phi} = (g/L)\sin\phi - (a_{hand}/L)\cos\phi$.

b) There are many correct solutions. Test your solution with a computer simulation. Show your result with appropriate plots.

11.69e) $\ddot{\phi} - \left[\frac{\ddot{y}+g}{L}\right]\sin\phi = 0$, where $y(t)$ is the vertical displacement of your hand and $\phi(t)$ is the angle of the broom from the vertical.

11.70) $\ddot{\phi}_1 = 0,\ \ddot{\phi}_2 = -3g/2\ell$.

11.71a) $\ddot{\phi} + \left[3(R - \frac{L}{2})\frac{g}{L^2}\right]\phi = 0$.

b) This first-order approximation show that stable standing requires $R > \frac{L}{2}$. $R = \frac{L}{2}$ is neutrally stable (like a wheel).

12.7i) $-(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})$.

12.10) Net force: $\vec{\boldsymbol{F}} = m(-omega_0^2 L)\hat{\boldsymbol{\jmath}}$, Net moment: $\vec{\boldsymbol{M}}_{/G} = \frac{1}{2}mr^2(-\omega_0\omega_1\hat{\boldsymbol{\imath}} + \omega_0\omega_2\hat{\boldsymbol{\jmath}} - \omega_1\omega_2\hat{\boldsymbol{k}})$.

12.11) $\vec{\boldsymbol{a}}_P = 0.732g\hat{\boldsymbol{k}}$.

12.12a) $\vec{\boldsymbol{\omega}}_{car} = \frac{v_o}{R}\hat{\boldsymbol{k}}$.

b) $\vec{\boldsymbol{\omega}}_{tire/car} = -\frac{v_o}{r}\hat{\boldsymbol{e}}_r$, where $\hat{\boldsymbol{e}}_r$ is parallel to the axle of the wheel and moves with the car and k is perpendicular to the ground.

c) $\vec{\boldsymbol{\omega}}_{tire} = \frac{v_o}{R}\hat{\boldsymbol{k}} - \frac{v_o}{r}\hat{\boldsymbol{e}}_r$.

d) total torque, $\vec{\boldsymbol{M}} = \vec{\boldsymbol{0}}$.

e) $\vec{\boldsymbol{M}} = \vec{\boldsymbol{0}}$.

f) $\vec{\boldsymbol{H}}_{\rm cm} = \frac{mv_o r^2}{4}\left[-\frac{2}{r}\hat{\boldsymbol{e}}_r + \frac{1}{R}\hat{\boldsymbol{k}}\right]$.

g) $\dot{\vec{\boldsymbol{H}}}_{\rm cm} = \dot{\vec{\boldsymbol{H}}}_{\rm cm} = -\frac{mv_o^2 r}{2R}\hat{\boldsymbol{e}}_t$.

h) $\sum \vec{\boldsymbol{M}}_{\rm cm} = -\frac{mv_o^2 r}{2R}\hat{\boldsymbol{e}}_t$.

i) $\vec{\boldsymbol{a}}_{bug} = -\frac{v_o^2}{R}\hat{\boldsymbol{e}}_r + \frac{v_o v_g}{r}\hat{\boldsymbol{e}}_t$, $\vec{\boldsymbol{F}}_{bug} = m_{bug}\vec{\boldsymbol{a}}_{bug}$.

12.14a) $\vec{\boldsymbol{a}}_G = \frac{F}{m}\hat{\boldsymbol{k}}$.

b) $\dot{\vec{\boldsymbol{\omega}}} = \frac{3F}{mL}(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}})$.

c) $\vec{\boldsymbol{a}}_B = \frac{4F}{m}\hat{\boldsymbol{k}}$.

12.15a) $\vec{\boldsymbol{a}}_B = \frac{4F}{m}\hat{\boldsymbol{k}}$.

b) $\dot{\vec{\boldsymbol{\omega}}} = (96\hat{\boldsymbol{\imath}} + 48\hat{\boldsymbol{\jmath}})\ \text{rad/s}^2$

12.26a) $\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{F}} = \Omega\hat{\boldsymbol{k}} + \omega\hat{\boldsymbol{\jmath}}$.

b) $\vec{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{F}} = -\Omega\omega\hat{\boldsymbol{\imath}}$.

c) $\vec{\boldsymbol{v}}_B = \Omega R\hat{\boldsymbol{\jmath}} + \frac{1}{2}\omega D\hat{\boldsymbol{\imath}}$.

d) $\vec{\boldsymbol{a}}_B = -\Omega^2 R\hat{\boldsymbol{\imath}} + \Omega\omega D\hat{\boldsymbol{\jmath}} - \frac{1}{2}\omega^2 D\hat{\boldsymbol{k}}$.

12.27) $\vec{\boldsymbol{\omega}}_{\mathcal{D}/\mathcal{F}} = \omega_2\hat{\boldsymbol{\imath}} + \omega_1\hat{\boldsymbol{\jmath}} + \omega_0\hat{\boldsymbol{k}}$.

12.30a) $\vec{\boldsymbol{v}}_B = -\frac{\omega_1 h^2}{\sqrt{h^2+r^2}}\hat{\boldsymbol{k}}$.

b) $\vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{F}} = -\frac{\omega_1\omega_2}{\sqrt{1+\left(\frac{r}{h}\right)^2}}\hat{\boldsymbol{k}}$.

c) $\omega_1 = \frac{\omega_2}{\sqrt{h^2+r^2}}$.

d) $\left|\vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{F}}\right| = \frac{v^2}{hr}\left(1 + \left(\frac{r}{h}\right)^2\right)$.

12.33) ${}^{\mathcal{F}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}} = -\omega_0\omega_1\hat{\boldsymbol{\imath}} + \omega_0\omega_2\hat{\boldsymbol{\jmath}} - \omega_1\omega_2\hat{\boldsymbol{k}}$.

12.36b) $\vec{\boldsymbol{a}}_P = -(\ell\dot{\Omega} + 2r\Omega\omega_s)\hat{\boldsymbol{\imath}} + (2r\Omega\omega_c - \ell\Omega^2)\hat{\boldsymbol{\jmath}} - r(\omega_c^2 + \omega_s^2)\hat{\boldsymbol{k}}$.

12.38a) $\vec{\boldsymbol{H}}_{/G} = \frac{1}{2}mr^2\omega_2\hat{\boldsymbol{\imath}} + \frac{1}{4}mr^2\omega_1\hat{\boldsymbol{\jmath}} + m\omega_0(L^2 + \frac{1}{4}r^2)\hat{\boldsymbol{k}}$, $\dot{\vec{\boldsymbol{H}}}_{/G} = \frac{1}{2}mr^2(-\omega_0\omega_1\hat{\boldsymbol{\imath}} + \omega_0\omega_2\hat{\boldsymbol{\jmath}} - \omega_1\omega_2\hat{\boldsymbol{k}})$.

b) $\vec{\boldsymbol{H}}_{/E} = \frac{1}{2}mr^2\omega_2\hat{\boldsymbol{\imath}} + \frac{1}{4}mr^2\omega_1\hat{\boldsymbol{\jmath}} + \frac{1}{4}mr^2\omega_0\hat{\boldsymbol{k}}$, $\dot{\vec{\boldsymbol{H}}}_{/E} = \dot{\vec{\boldsymbol{H}}}_{/G} = \frac{1}{2}mr^2(-\omega_0\omega_1\hat{\boldsymbol{\imath}} + \omega_0\omega_2\hat{\boldsymbol{\jmath}} - \omega_1\omega_2\hat{\boldsymbol{k}})$.

12.39) $A_y = -3.16\,\text{N},\ B_y = 3.16\,\text{N},\ A_z = B_z = -63.16\,\text{N}$

12.48) $\vec{\boldsymbol{\omega}}_{\mathcal{D}} = 2\pi(2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\ \text{rad/s}$ and $\vec{\boldsymbol{\alpha}}_{\mathcal{D}} = 8\pi^2\hat{\boldsymbol{k}}\ \text{rad/s}^2$

12.50a) $\vec{\boldsymbol{\omega}}_{\mathcal{D}} = (2\hat{\boldsymbol{\imath}} + 4\pi\hat{\boldsymbol{k}})\ \text{rad/s}$

b) $\vec{\boldsymbol{\alpha}}_{\mathcal{D}} = -8\pi\ \text{rad/s}^2\,\hat{\boldsymbol{\jmath}}$

c) $\vec{\boldsymbol{M}} = -\frac{\pi}{2}\,\text{N·m}\hat{\boldsymbol{\jmath}}$

d) $\vec{\boldsymbol{R}}_A = \frac{\pi}{2}\,\text{N}\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{R}}_B = -\vec{\boldsymbol{R}}_A$

12.52) $\vec{\boldsymbol{N}} = \frac{\omega^2 mR}{2}\hat{\boldsymbol{k}}$